Differential Equation

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This note contains parts that I learnt from the Differential equation course of Rice university in edx.org. The code for rmarkdown can be obtained *https://raw.githubusercontent.com/Roshann-Rai/differential-equation/main/differential.Rmd*

1. Differential Equation

A differential equation of simple form $\frac{dy}{dt} + P(t)y = f(t)$ explains how the change in one variable (independent variable) affects the other dependent variable. It shows the direction of movement as well as the magnitude of the movement of dependent variable with respect the indpendent variable. It can be (i) ordinary differential equation that has 1 independent variable, (ii) partial differential equation that has at least 2 independent variables.

1.1 Solution of differential equation

a. General Solution includes all the possible solutions that typically includes arbitrary constant. For eg. $y(t) = t^3 + c$ is a general solution.
b. Particular Solution includes the solution without arbitrary constant. Consider the intial condition:

$$y(t_0) = 0$$

$$c = 0$$

So,

$$y(t) = t^3$$

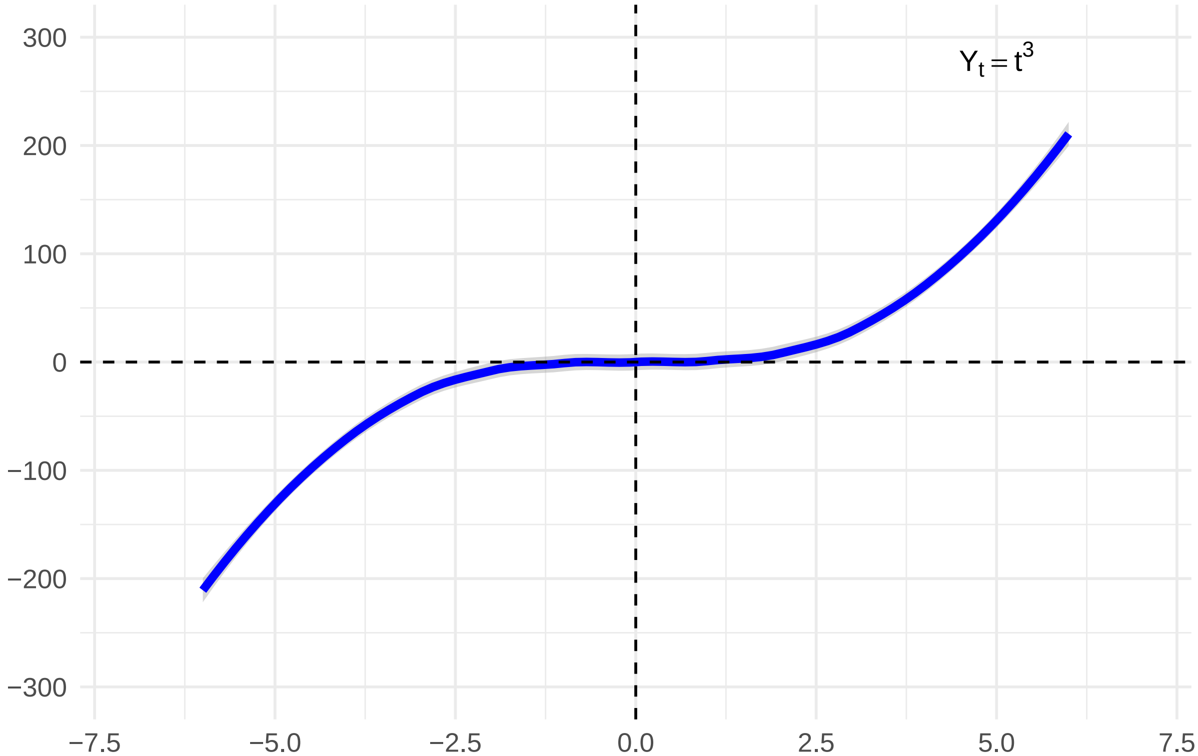


Figure 1: Plot of $y = t^3$

1.2 Order of Differential Equation

The order of the differential equation is the highest derivative of the dependent variable that exists in the equation.

$$\frac{d^n y}{dt} = f(t, y^1, y^2, ..., y^{n-1}, y^n)$$

is the $n - th$ order differential equation.
First order differential equation is

$$\frac{dy}{dt} = f(t, y)$$

1.3 Directional Fields

Directional field also known as slope field is the graphical representation of the solutions to the first order differential equation. Consider the differential equation,

$$\frac{dy}{dt} = (y - 2)(y + 1)(1 - y)^2$$

To determine the directional field, we equate the above equation equals to 0,

$$(y-2)(y+1)(1-y)^2 = 0$$

So, $y = 2, \pm 1$. The graph is divided into four regions i.e. $y < -1$, $-1 < y < 1$, $1 < y < 2$ and $y > 2$.
For, $y < -1$, $\frac{dy}{dt} = 36$ when $y = -2$,
For, $-1 < y < 1$, $\frac{dy}{dt} = -1.05$ when $y = -0.9$
For, $1 < y < 2$, $\frac{dy}{dt} = -0.0189$ when $y = 1.1$
For, $y > 2$, $\frac{dy}{dt} = 0.3751$ when $y = 2.1$

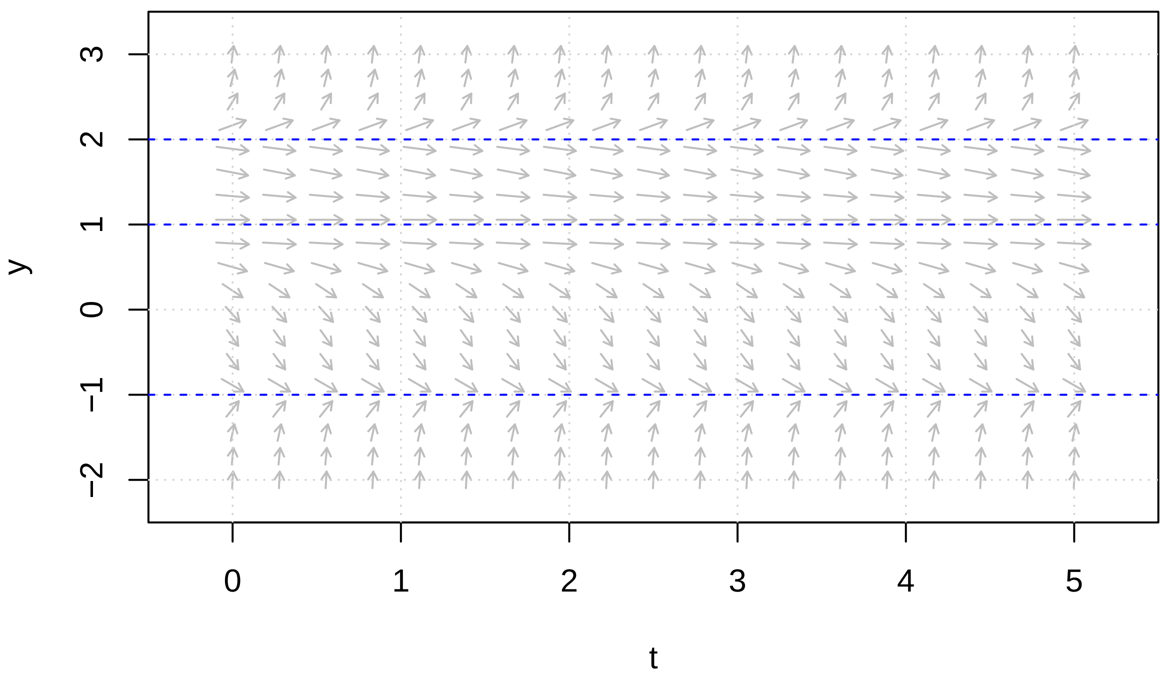
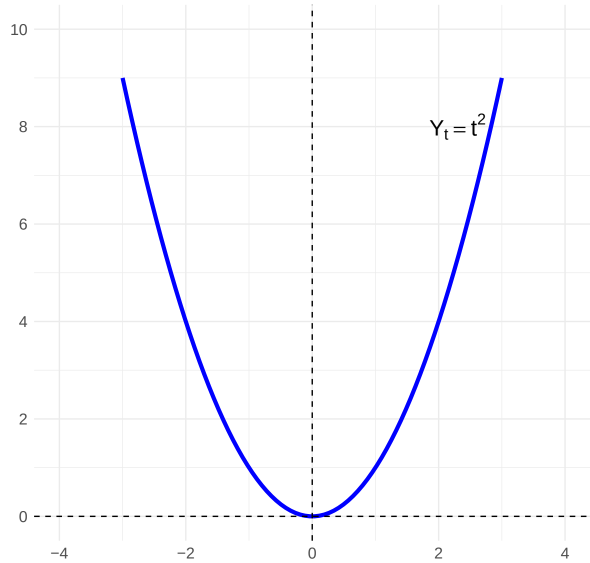


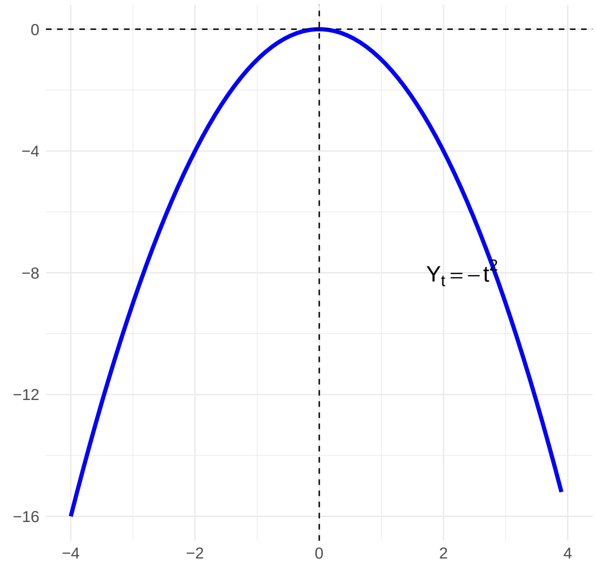
Figure 2: Directional field of $\frac{dy}{dt} = (y-2)(y+1)(1-y)^2$

1.4 Concavity

The graph of $f(x)$ is concave up if $f'(x)$ is increasing i.e. $f' > 0$ and concave down if $f'(x)$ is decreasing i.e. $f'(x) < 0$.



(a) Convex function (Concave upward)



(b) Concave function (Concave downward)

1.5 Separable equations

Differential equation is separable if $y' = f(t)g(y)$.

Example:

$$\frac{dy}{dt} = 3t^2(1+y)$$

$$\frac{1}{1+y}dy = 3t^2dt \tag{1}$$

$$\int \frac{1}{1+y}dy = \int 3t^2dt$$

$$ln|1+y| = t^3 + c$$

$$e^{ln|1+y|} = e^{t^3+c}$$

$$|1+y| = e^{t^3}.e^c$$

$$1 + y = \pm e^c e^{t^3} \tag{2}$$

where $e^c > 0$ in equation (1).

$$y = Ke^{t^3}$$

$$y = -1 + Ke^{t^3}$$

$$y = -1 + Ce^{t^3} \tag{3}$$

In equation (3),

a. If $C = 0$, $y = -1$ is equilibrium solution.

b. If $C \neq 0$, it gives all other possible solutions.

Example:

$$\frac{dy}{dt} = \frac{3t^2 + 1}{1 + 2y}$$

and $y(0) = 1$

The equation is in the form $y' = f(t)g(y)$. So the equation can be separated.

Does the equation has equilibrium solution?

Set, $g(y) = 0$, $\implies \frac{1}{1+2y} = 0$. So, no equilibrium solution.

Now,

$$\int (1 + 2y)dy = \int (3t^2 + 1)dt$$

$$y + y^2 = t^3 + t + c \tag{1}$$

Put $y(0) = 1$, then $C = 2$. So,

$$y + y^2 = t^3 + t + 2$$

$$(y^2 + y) - (t^3 + t + 2) = 0 \tag{2}$$

Equation (2) is in the form of $ax^2 + bx + c = 0$, where $a = 1$, $b = 1$ and $c = -(t^3 + t + 2)$.

$$y = \frac{-1 \pm \sqrt{1 + 4(t^3 + t + 2)}}{2}$$

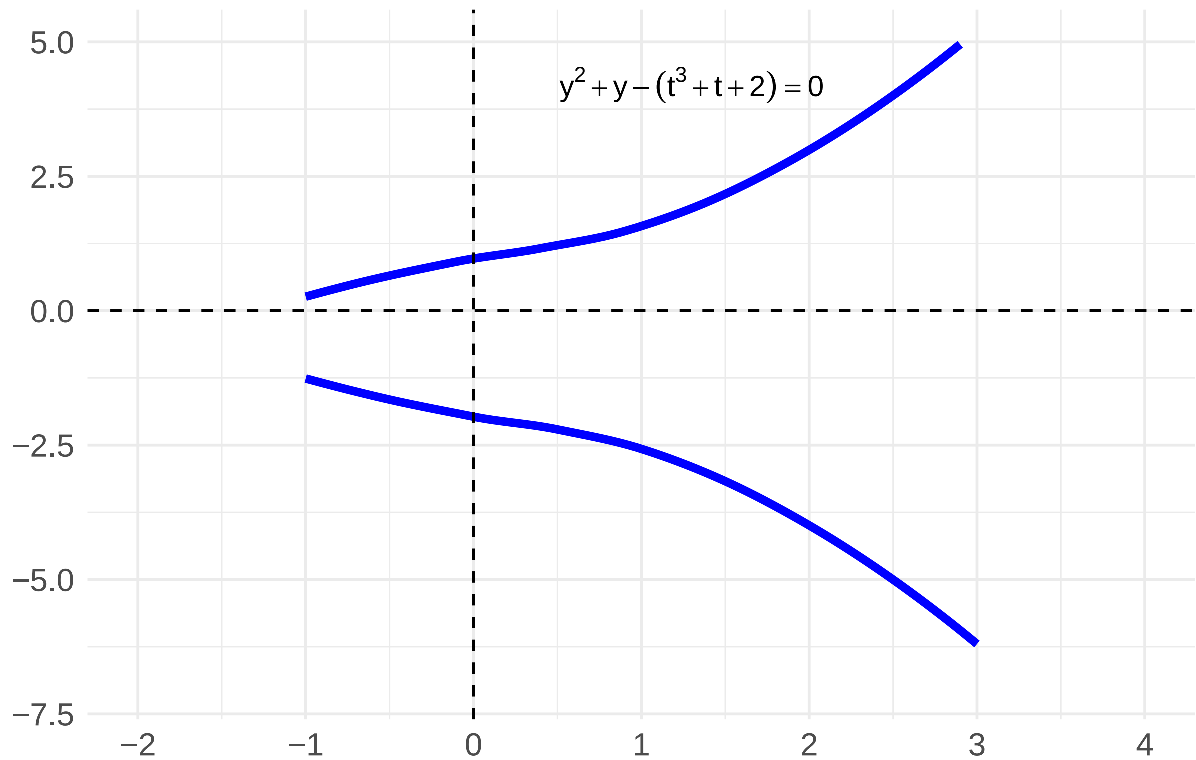


Figure 3: Plot of $y^2 + y - (t^3 + t + 2) = 0$

1.6 Picard's Theorem

For $y' = f(t, y)$ with $y(t_0) = y_0$ to have solution:

- Condition I: f(t,y) should be continuous function[1] in a neighborhood of t_0, y_0.
- Condition II: The solution is unique if $\frac{\delta f}{\delta y}$ is also continuous in neighbourhood of this initial condition t_0 and y_0.

[1] A function $f(x)$ is continous at $x = a$ if a. $f(a)$ is defined, b. $\lim_{x \to a} f(x)$ exists and c. $\lim_{x \to a} f(x) = f(a)$

1.7 Linearity vs Non-linearity

A linear differential equation is in the form $y' + p(t)y = f(t)$ which is a first order linear differential equation. Non-linear differential equation can be $y'y = cost$. Properties of linear equations are:

- **Superposition principle**: If y_1 and y_2 are homogeneous solutions, their linear combinations $c_1y_1 + c_2y_3$ is also homogeneous.
- **Non homogeneous principle**: General solution for non-homogeneous equation is $y(t) = y_P + y_H$ where y_P is solution for non-homogeneous equation and y_H is solution for homogeneous equation.

2. First order Differential Equation

2.1 First order linear non-homogeneous Equation: Variation of Parameters

Consider the following first order ODE:

$$y' + p(t)y = f(t) \tag{1}$$

For equation (1),

$$y_H = Ce^{-\int p(t)dt} \tag{2}$$

$$y_p = vy_H \tag{3}$$

Can v be constant?

No, v can't be constant because if v is constant then y_p is scalar multiplicative of y_H. This won't solve the non-homogeneous equation. v is non-constant function of t i.e. $v(t)$.

From equation (3),

$$y_p' = (vy_H)' \longrightarrow v'y_H + vy_H'$$

So, in equation (1),

$$v'y_H + vy_H' + p(t)vy_H = f(t)$$

$$v'y_H + v[y_H' + p(t)y_H] = f(t)$$

where $y_H' + p(t)y_H = 0$

$$v'y_H = f(t)$$

$$v' = \frac{f(t)}{y_H}$$

$$v = \int \frac{f(t)}{y_H} dt \tag{4}$$

Equation (4) gives v.

Example:

$$y' + \frac{1}{1+t}y = 2$$

$$y(0) = 0, t \geq 0$$

$$y_H = Ce^{-\int \frac{1}{1+t}dt}$$

$$y_H = Ce^{-ln|1+t|}$$

$$y_H = Ce^{ln|1+t|^{-1}}$$

$$y_H = C|1+t|^{-1}$$

$$y_H = \frac{C}{1+t} \tag{1}$$

Here we only take +ve sign because $t \geq 0$.

$$y_P = v y_H$$

$$v' = \frac{f}{y_H}$$

$$v' = \frac{f}{\frac{1}{1+t}}$$

$$v' = 2(1+t)$$

where $f = 2$

$$v = \int 2(1+t)dt$$

$$v = 2t + t^2 + c \tag{2}$$

Now,

$$y_P = (2t + t^2 + c)y_H$$

$$y_P = \frac{(t^2 + 2t + c)}{1+t} \tag{3}$$

Here, we don't need to write c because we will get the constant from y_H. General Solution is

$$y = y_P + y_H$$

$$y = \frac{2t + t^2}{t+1} + \frac{C}{t+1} \tag{4}$$

We know, $y(0) = 0$ so,

$$0 = \frac{2 * 0 + 0^2}{0+1} + \frac{c}{0+1}$$

$$c = 0$$

Hence,

$$y = \frac{2t + t^2}{t+1} \tag{5}$$

Eqn (5) is case when $t \neq -1$.

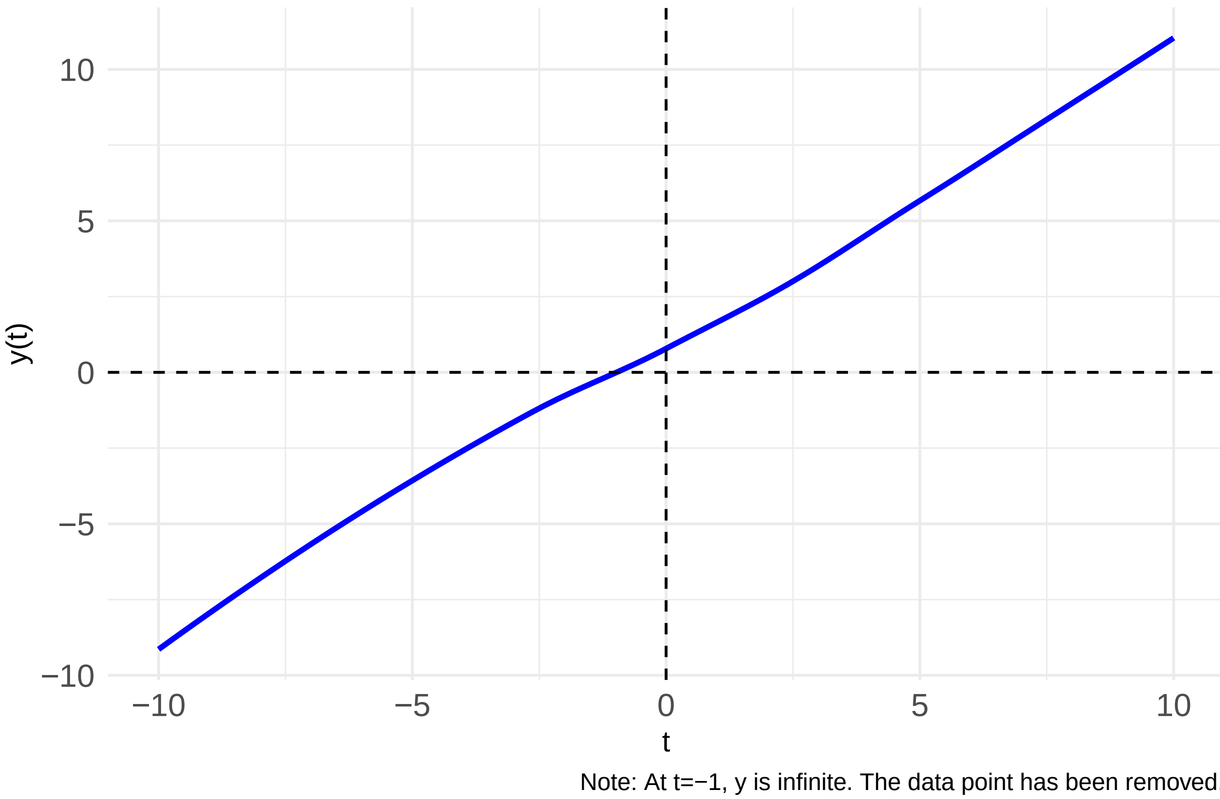


Figure 4: Plot of $y = (2t + t^2)/(t + 1)$

2.2 First order linear non-homogeneous equation: Integrating Factors

Consider the following

$$y' + p(t)y = f(t) \tag{1}$$

Multiply equation (1) by μ,

$$\mu y' + p(t)y\mu = f(t)\mu$$

Assume, $\mu p(t) = \mu'$
Now,

$$\mu y' + \mu' y = \mu f(t)$$

$$(\mu y)' = \mu f(t)$$

$$\int (\mu y)' dt = \int \mu f(t) dt$$

$$\mu y + c = \int \mu f(t) dt$$

$$y = \frac{\int \mu f(t) dt - c}{\mu} \tag{2}$$

In eqn (2), since c is unknown constant, we change the sign -ve to +ve.

$$y = \frac{\int \mu f(t) dt + c}{\mu} \tag{3}$$

From our assumption,

$$\mu p(t) = \mu' \longrightarrow p(t) = \frac{\mu'}{\mu}$$

On right hand side, it is simply the natural log of μ. So,

$$p(t) = (ln\mu)'$$

$$\int p(t) dt = \int (ln\mu)' dt$$

$$\int p(t) dt + k = \ln \mu$$

$$\mu = e^{\int p(t)dt + k}$$

$$\mu = e^k e^{\int p(t)dt}$$

$$\mu = K e^{\int p(t)dt} \tag{4}$$

where $K = e^k$.
If we set the value of μ in equation 3, we get y,

$$y = \frac{\int K e^{\int p(t)dt} f(t)dt + c}{K e^{\int p(t)dt}} \tag{5}$$

Example

$$y' + \frac{1}{t+1} = 2$$

and $y(0) = 0, t \geq 0$
We Know,

$$\mu = e^{\int p(t)dt}$$

$$\mu = e^{\int \frac{1}{1+t} dt}$$

$$\mu = e^{ln|1+t|}$$

$$\mu = |1+t|$$

$$\mu = t+1$$

We only take positive sign because $t \geq 0$.

$$(\mu y)' = \mu f(t)$$

$$(\mu y)' = 2\mu$$

$$[(1+t)y]' = 2(1+t)$$

$$\int [(1+t)y]' dt = \int 2(1+t)dt$$

$$(1+t)y = t^2 + 2t + c$$

$$y = \frac{t^2+2t}{t+1} + \frac{c}{t+1} \tag{1}$$

Since, $y(0) = 0$ so, $c = 0$ in equation (1). Then,

$$y(t) = \frac{t^2+2t}{t+1} \tag{2}$$

3. Second order Differential Equation

General form of Second order differential equation is in the form

$$ay'' + by' + cy = f(t)$$

3.1 With constant coefficient

To find $y_H = Basis\ y_1, y_2$ where y_1, y_2 are two linearly independent functions.

$$y(t) = e^{\lambda t}$$

$$y'(t) = \lambda e^{\lambda t}$$

$$y''(t) = \lambda^2 e^{\lambda t}$$

Then,

$$a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0$$

$$e^{\lambda t}(a\lambda^2 + b\lambda + c) = 0$$

Since, $e^{\lambda t} \neq 0$ so,

$$a\lambda^2 + b\lambda + c = 0$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- $b^2 - 4ac > 0 \implies$ Real distinct roots
- $b^2 - 4ac = 0 \implies$ Repeated roots
- $b^2 - 4ac < 0 \implies$ Complex Roots

Example:

$$y" + 5y' + 6y = 0$$

Consider $y(t) = e^{\lambda t}$, then,

$$\lambda^2 + 5\lambda + 6 = 0$$

$$\lambda_{1,2} = \frac{-5 \pm \sqrt{25 - 4 * 6}}{2}$$

$$\lambda_{1,2} = -3, -2$$

So, $y_1 = e^{-3t}$ and $y_2 = e^{-2t}$

$y_H =$ span (y_1, y_2) for which linear independency should be treated

Example:

$$y'' + 4y' + 4y = 0$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4*4}}{2}$$
$$\lambda_{1,2} = -2$$

It is a case of Repeated Real roots.

$y_1 = e^{-2t}$ and $y_2 = ?$

For repeated real roots, we us variation of parameter

$$y_2 = vy_1 \tag{1}$$

Can v be constant?

V is not constant. We want to generate y_H that is coming from linearly independent solution so v is function of t i.e. $v(t)$.

$$y_2' = v'y_1 + y_1'v$$
$$y_2'' = v'y_1 + y_1'v' + y_1"v + v'y_1'$$

$y_1 = e^{-2t}$, $y_1' = -2e^{-2t}$ and $y_1'' = 4e^{-2t}$ So,

$$y'' + 4y' + 4y = 0$$

$$v''e^{-2t} - 2v'e^{-2t} + 4ve^{-2t} - 2v'e^{-2t} + 4(v'e^{-2t} - 2ve^{-2t}) + 4ve^{-2t} = 0$$
$$v''e^{-2t} = 0$$

Since, $e^{-2t} \neq 0$, $v'' = 0$. If $v = at + b$, $y_2 = vy_1 = (at + b)y_1$. Here by_1 gives dependent piece so, we don't need b. For a, when span of y_1 and y_2 are taken all the linear combination of y_1 and y_2 are handled by constant. So, $v(t) = t$.

Example:

$$y'' + 2y' + 4y = 0$$
$$\lambda^2 + 2\lambda + 4 = 0$$
$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4*4}}{2}$$
$$\lambda_{1,2} = -1 \pm \sqrt{3}i$$
$$y(t) = C_1 e^{(-1-\sqrt{3}i)t} + C_2 e^{(-1+\sqrt{3}i)t}$$

$$y(t) = C_1 e^{-t} e^{-\sqrt{3}it} + C_2 e^{-t} e^{\sqrt{3}it}$$

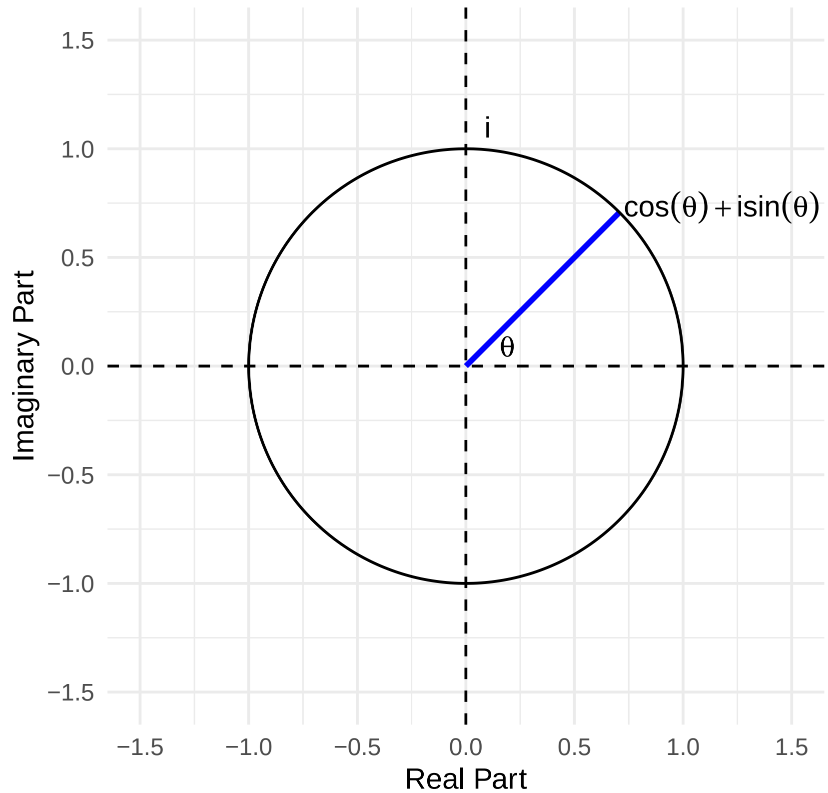
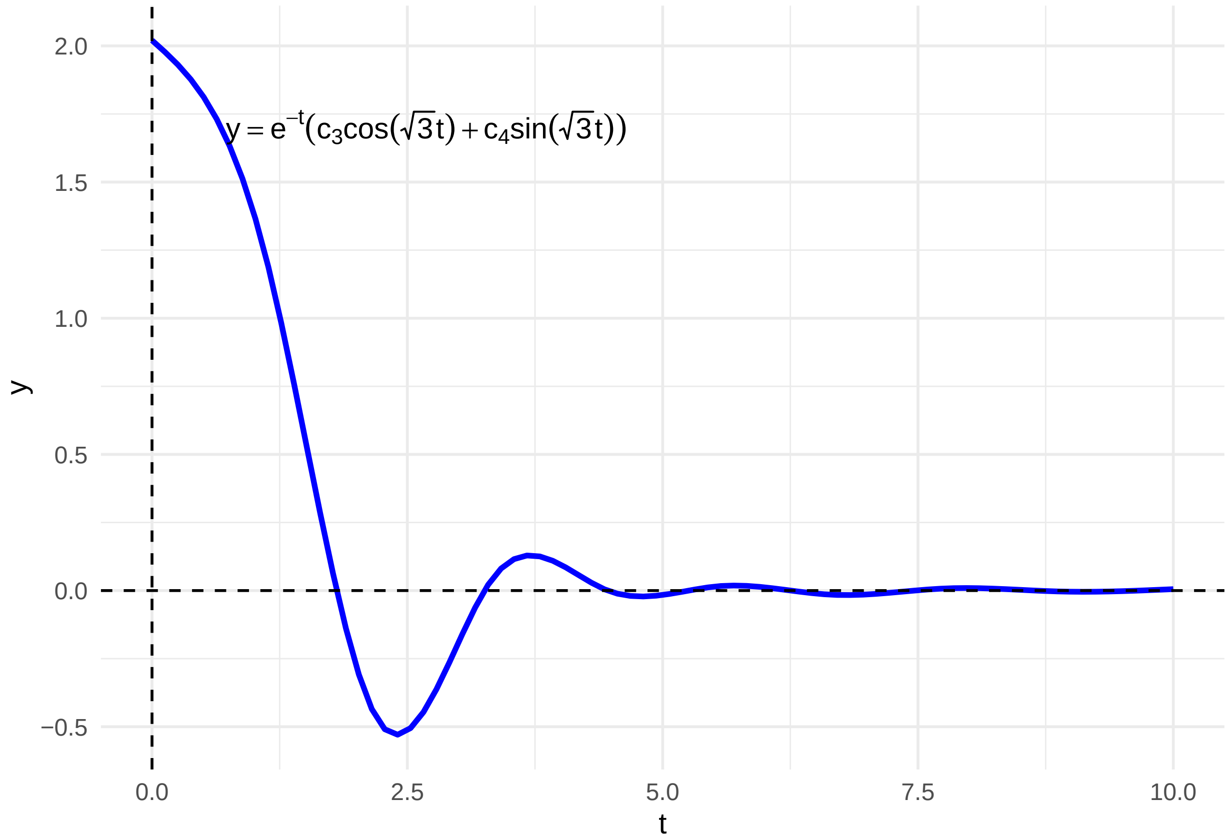


Figure 5: Complex Exponential Function $e^{i\theta}$

$$
\begin{gathered}
y(t) = e^{-t}(C_1 e^{-\sqrt{3}it} + C_2 e^{\sqrt{3}it}) \\
y(t) = e^{-t}[C_1(cos(-\sqrt{3}t) + sin(-\sqrt{3}t)) + C_2(cos(\sqrt{3}t) + sin(\sqrt{3}t))] \\
y(t) = e^{-t}[C_1(cos(\sqrt{3}t) - sin(\sqrt{3}t)) + C_2(cos(\sqrt{3}t) + sin(\sqrt{3}t))] \\
y(t) = e^{-t}[C_1 cos(\sqrt{3}t) - C_1 sin(\sqrt{3}t)) + C_2 cos(\sqrt{3}t) + C_2 sin(\sqrt{3}t)] \\
y(t) = e^{-t}[(C_1 + C_2)cos(\sqrt{3}t) + (C_2 - C_1)sin(\sqrt{3}t)] \\
y(t) = e^{-t}[C_3 cos(\sqrt{3}t) + C_4 sin(\sqrt{3}t)] \qquad (2)
\end{gathered}
$$


$y = e^{-t}(c_3\cos(\sqrt{3}t) + c_4\sin(\sqrt{3}t))$
2.0
1.5
1.0
0.5
0.0
−0.5
y
0.0
2.5
5.0
7.5
10.0
t

4.Linear Independency of Functions

Theorem: A set of functions $\{f_1, f_2, ..., f_n\}$ is linearly independent set if there is no linear combination functions i.e $= 0$ for all $t \in I$. $\{f_1, f_2, ..., f_n\}$ is linearly dependent such that $c_1 f_1 + ... + c_n f_n = 0$ i.e. satisfy non-trivial solution for some $t \in I$.

Taking derivative, $c_1 f_n' + ... + c_n f_n' = 0$ which indicates that some of the functions will vanish i.e. some will be equal to 0. But as long as O is included in function set, it will be linearly dependent. When we derivative a function equal to 0, the linear dependency is still preserved for some $t \in I$.

$$c_1 f_1^{n-1} + ... + c_n f_n^{n-1} = 0$$

gives us non-trivial expression assuming $c_1, ..., c_n \neq 0$ or $f_1, ..., f_n \neq 0$.

If we take derivative then $c_1 f_1' + ... + c_n f_n' = 0$. Some of the functions will vanish i.e. some will be equal to 0. But as long as 0 is included in function set, it will be linearly dependent i.e. $\{f_1, ..., f_n, 0\}$. If we take derivative of a function equal to 0, the linear dependency is still preserved for some $t \in I$ i.e. $c_1 f_1^{(n-1)} + ... + c_n f_n^{(n-1)} = 0$ gives us non-trivial expression assuming $c_1, ..., c_n \neq 0$ or $f_1, ..., f_n \neq 0$.

$$\begin{bmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & \vdots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

For any linearly dependent set, derivatives up to any degree will also be linearly dependent.

If $\{f_1, ..., f_n\}$ is linearly dependent set then Wronskian of $\{f_1, ..., f_n\}$ is equal to 0 i.e. $W[f_1, ..., f_n] = 0$. So,

$$\begin{vmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & \vdots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = 0$$

Theorem: If $\{y_1, ..., y_n\}$ are solutions to n^{th} order linear differential equation and $W[y_1, ..., y_n] = 0$ for some $t \in I$ then $\{y_1, ..., y_n\}$ solutions are linearly dependent.