MAS2004/9 Semester 2 Problems

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CONTENTS

Maths is best learned by doing many problems.

This problem booklet serves 3 purposes:

1. **Problems of the Week.** Each week, I will circulate a selection of problems from this booklet that relate most closely to the material covered in lectures that week. I recommend working on these steadily throughout the semester, and keeping a record of any you would like help with.

2. **Tutorials.** The problem set for each tutorial will come from this booklet.

3. **Homework.** Written homework will be a selection of problems from the problem booklet.

See Blackboard[1] for more information.

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0.1 Problems

0.1.1 Preliminary problems

P1. Which of the following statements is the correct definition of convergence of a real sequence (x_n) to a limit $l \in \mathbb{R}$? How would you correct the incorrect statements?

[Note: There is more than one correct answer.]

(i) $\varepsilon > 0$ $N \in \mathbb{N}$ then $|x_n - l| < \varepsilon$.

(ii) $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $|x_n - l| < \varepsilon$ $\forall n \geq N$.

(iii) For all $\varepsilon > 0$ and $N \in \mathbb{N}$, we have $|x_n - l| < \varepsilon$ whenever $n \geq N$.

(iv) $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $|x_n - l| < \varepsilon$ $\forall n \geq N$.

(v) Given any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|x_n - l| < \varepsilon$ whenever $n \geq N$.

P2. *(Homework 1 question).* Prove using the $(\varepsilon - N)$-definition of convergence that the sequence (x_n) given by $x_n = \frac{2n}{3n-1}$ converges to $\frac{2}{3}$ as $n \to \infty$.

P3. *(Homework 1 question).* Using your MAS107 notes or another resource, look up and carefully state the Bolzano–Weierstrass theorem.

Its proof combines two other important results about sequences from MAS107. What do these results say?

[Note: We will use all three of these theorems at various points this semester. Check you are clear on what each of them is saying, and remember your lecturers and tutors are here to help.]

P4.
(i) If $a, b \geq 0$, show that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$.

[Hint: Consider the square of both sides. You may use that the square root function is increasing.]

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[1] Links: MAS2004; MAS2009

(ii) If $a, b \in \mathbb{R}$, deduce that $\left| \sqrt{|a|} - \sqrt{|b|} \right| \leq \sqrt{|a - b|}$.

[Hint: Immitate the proof of Corollary 1.1 in the lecture notes.]

(iii) Hence prove that if the sequence (a_n) converges to l, then $\left(\sqrt{|a_n|} \right)$ converges to $\sqrt{|l|}$.

P5. Look up and carefully define the *supremum* and *infimum* of a set $A \subseteq \mathbb{R}$. What conditions must be met for each of these to exist?

P6. Compute, without proof, the suprema and infima (if they exist) of the following sets:

(i) $\left\{ \frac{m}{n} : m, n \in \mathbb{N} \text{ s.t } m < n \right\}$.

(ii) $\left\{ \frac{(-1)^m}{n} : m, n \in \mathbb{N} \text{ s.t } m < n \right\}$.

(iii) $\left\{ \frac{n}{3n+1} : n \in \mathbb{N} \right\}$.

P7. Let A and B be bounded and non-empty subsets of $\mathbb{R}$. Which of the following statements are true, and which are false? Prove or supply counter-examples, as appropriate.

(i) $\inf A < \sup A$.

(ii) For all $\varepsilon > 0$, there is $x \in A$ such that $x - \inf A < \varepsilon$.

(iii) $A \subseteq B$ implies $\sup A \leq \sup B$.

(iv) If $x > 0$ for all $x \in B$, then $\inf B > 0$.

(v) $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

0.1.2 Limits of functions

1. For each of the following formulas, what is the largest subset X of $\mathbb{R}$ which may be taken as the domain of a function with that formula?

(i) $g_1(x) = \dfrac{x^2 + 2x + 7}{x(x + 1)}$,

(ii) $g_2(x) = \dfrac{(x - 1)(x + 4)}{x^3 + 4x^2 + x - 6}$,

(iii) $g_3(x) = \dfrac{x + 4}{x^2 + 5x + 6}$,

(iv) $g_4(x) = \exp\left(-\dfrac{1}{x - 1} \right)$,

(v) $g_5(x) = \cos\left(\dfrac{1}{\pi x} \right)$.

2. *(Homework 1 question).* For each of the sets $X \subseteq \mathbb{R}$ below, calculate the associated set L of its limit points.

(i) $X = (0, 1) \cup [2, 3) \cup \{4, 5\}$,

(ii) $X = \mathbb{Z}$,

(iii) $X = \mathbb{R} \setminus \mathbb{Z}$,

(iv) $X = \{x \in \mathbb{Q} : 0 < x < 1\}$,

(v) $X = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$.

3. *(Homework 1 question).* For each of the following functions f, calculate $\lim_{x \to 2} f(x)$, and prove your calculation is correct using the $(\varepsilon - \delta)$ criterion (Definition 2.2). Does f converge to a limit as $x \to 0$?

   (i) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = 4x + 7$.

   (ii) $f : \{0\} \cup [1,3] \to \mathbb{R}$; $f(x) = 3x^2 - 1$.

   (iii) $f : (0, \infty) \to \mathbb{R}$; $f(x) = x + \frac{1}{x}$.

4. For $g_2 : X \to \mathbb{R}$ as in part (ii) of the *Problem 1*, investigate each of
$$\lim_{x \to 1} g_2(x), \qquad \lim_{x \to -2} g_2(x) \qquad \text{and} \qquad \lim_{x \to -3} g_2(x).$$
Use the sequential criterion (Theorem 2.1) to prove you are right.

5. If $f : \mathbb{R} \to [0, \infty)$ satisfies $\lim_{x \to a} f(x) = l$, where $l > 0$, show that
$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{l}.$$
Hence calculate $\lim_{x \to 1} \sqrt{\dfrac{x+1}{x^2}}$.

[Hint: For the first part, use the result of *P4 (iii)* from the preliminary exercises.]

6. Why are limits of functions unique?

7. Verify that $\operatorname{sgn}(x) = \dfrac{|x|}{x} = \dfrac{x}{|x|}$, for $x \neq 0$ (see Example 2.3 in the notes for the definition).

   Show that $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist. [Hint: Use a well-chosen sequence and Theorem 2.1.]

   Show that both the left and right limits exist at $x = 0$, and find their values.

8. *(Homework 2 question).* For the following functions, each of which is defined on the whole of $\mathbb{R}$, find every point at which both the left and right limits exist, but are different from each other, and find the values of these limits. Prove you are right using Definition 2.4, or the equivalent definitions involving sequences.

   (i) $f(x) = \begin{cases} 1 - x & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1. \end{cases}$

   (ii) $g(x) = [x]$, where
$$[x] = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0 \\ \lceil x \rceil & \text{if } x < 0 \end{cases} \tag{1}$$
   denotes the integer part of x.

   (iii) $h(x) = 3 - 5\mathbb{1}_{(0,1]}(x) + 7\mathbb{1}_{(1,2]}(x)$. [Here we have used the notation for indicator functions — see Example 2.4 in the notes.]

9. *(Homework 2 question).* What is the largest subset A of $\mathbb{R}$ for which we can specify a function by $f(x) = \sin\left(\frac{1}{x}\right)$? Does $f : A \to \mathbb{R}$ have a limit at $x = 0$? [Hint: Consider sequences whose nth term is $\frac{1}{\theta + 2n\pi}$, and think about good choices for θ.]

10. What is the largest subset A of $\mathbb{R}$ for which we can specify a function by $f(x) = x \sin\left(\frac{1}{x}\right)$? Does $f : A \to \mathbb{R}$ have a limit at $x = 0$?

11. In the notes, we gave meaning to $\lim_{x \to a} f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is a function and $a \in \mathbb{R}$. In this question, you can investigate what happens when x tends to ∞ or $-\infty$.

    (i) Formulate a rigorous definition, similar to Definition 2.2, of what it means for $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ to exist.

[Hint: For the analogue of $(\varepsilon - \delta)$, think very carefully about how you are going to control the behaviour of x. Remember that you cannot treat ∞ as if it were a number!]

(ii) Find an analogue of the sequential criterion for this case, and prove the analogous result to Theorem 2.1.

(iii) Check that you can prove that $\lim\limits_{x \to \infty} \dfrac{1}{x} = \lim\limits_{x \to -\infty} \dfrac{1}{x} = 0$, using either your definition or sequential criterion.

12. (i) Formulate a rigorous $(\varepsilon - \delta)$ definition for $\lim\limits_{x \to \infty} f(x) = \infty$ and write down an analogue of the sequential criterion.

[The cases $\lim\limits_{x \to \infty} f(x) = -\infty$ and $\lim\limits_{x \to -\infty} f(x) = \pm\infty$ can be treated similarly.]

(ii) Let $f : \mathbb{R} \to [0, \infty)$ and $g : \mathbb{R} \to [0, \infty)$, and suppose that $\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} g(x) = l$, where $l > 0$. Define $h(x) = f(x)g(x)$ for all $x \in \mathbb{R}$. Show that $\lim\limits_{x \to \infty} h(x) = \infty$.

(iii) Let $p : \mathbb{R} \to \mathbb{R}$ be an even polynomial of degree m, where the leading coefficient (i.e. the coefficient of x^m) is positive. Show that $\lim\limits_{x \to \infty} p(x) = \lim\limits_{x \to -\infty} p(x) = \infty$. What happens when m is odd?

0.1.3 Continuity

13. Return to *Problem 1*. Consider each function there, defined on the largest subset A of $\mathbb{R}$ which can be taken as its domain, as found in that problem. For each function, what is the maximum subset of A on which it is continuous?

14. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at a point a. Prove that $|f|$ is continuous at a, where $|f|(x) = |f(x)|$, for all $x \in \mathbb{R}$.

    [Hint: Use the formulation of continuity with sequences (Theorem 3.1(ii) and Corollary 1.1 in the notes).]

15. Prove Theorem 3.3 in the notes (i.e. that composition of continuous functions is continuous). [Hint: Use sequences.]

16. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ by $f(x) = \frac{1}{x}$, and $g(x) = 1 + x^2$. Write down the functions $f \circ g$, and $g \circ f$, giving their domains explicitly. What can you say about continuity of these functions?

17. For each of the functions in *Problem 7*:

    (i) Find the maximum subset of $\mathbb{R}$ on which it is continuous.

    (ii) Identify those discontinuities which are jumps, and calculate the size of each jump.

18. (i) Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$f(x) = \frac{(1 + x)^2 - 1}{x}.$$

Find an extension of f to the whole of $\mathbb{R}$ that is continuous there.

(ii) Write down the largest subset of $\mathbb{R}$ which can be taken as the domain A of the function f given by

$$f(x) = \frac{x^3 - 8}{x^2 - 4}$$

and explain why f is continuous at every point of A. The complement A^c of A in $\mathbb{R}$ comprises two points. Show that f may be extended to be continuous at only one of these points, and write down this continuous extension.

19. *(Homework 2 question).* Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is continuous at a with $g(a) > 0$. Show that there exists $\delta > 0$ such that $g(x) > 0$ for all $x \in (a - \delta, a + \delta)$. [Hint: Use the $\varepsilon - \delta$ criterion for continuity.]

20. (i) For any $x, y \in \mathbb{R}$ show that

$$\max\{x, y\} = \frac{1}{2}(x + y) + \frac{1}{2}|x - y|.$$

Hence show that if both $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are continuous at a, then so is $\max\{f, g\}$, where for all $x \in A \cap B$,

$$\max\{f, g\}(x) = \max\{f(x), g(x)\}.$$

(ii) Find a similar expression for $\min\{x, y\}$, and hence prove continuity of $\min\{f, g\}$ at a.

21. The aim of this question is to prove the following: the only functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous at zero and satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ are the linear mappings $f(x) = kx$ for all $x \in \mathbb{R}$, where $k \in \mathbb{R}$ is fixed.

    Begin by considering $f : \mathbb{R} \to \mathbb{R}$, such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

    (i) Prove that $f(0) = 0$.

    (ii) Show that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

    (iii) If f is continuous at zero, prove that it is continuous at every $x \in \mathbb{R}$.

    (iv) If $f(1) = k$, prove that $f(n) = kn$ for all $n \in \mathbb{Z}$.

    (v) If $f(1) = k$, prove that $f\left(\frac{p}{q}\right) = k\frac{p}{q}$ for all $\frac{p}{q} \in \mathbb{Q}$.

    (vi) If $f(1) = k$ and f is continuous at zero, prove that $f(x) = kx$ for all $x \in \mathbb{R}$.

22. *(Homework 2 question).* Show that Dirichlet's "other" function, as discussed in Example 3.10 in the notes, is discontinuous at every rational point in its domain.

    [Hint: Let $a \in [0, 1) \cap \mathbb{Q}$, and use the fact that for all $n \in \mathbb{N}$, there is an irrational number x_n lying strictly between a and $a + \frac{1}{n}$. (See also Cor 2.14 on page 42 of the MAS107/117 Sem 2 lecture notes.)]

23. What can you say about left/right continuity of the function $\mathbb{1}_{(a,b)} : \mathbb{R} \to \mathbb{R}$ at the points a and b?

24. Prove Corollary 3.1. [Hint: Apply the intermediate value theorem to the function $g(x) = f(x) - \gamma$, for $x \in [a, b]$.]

25. Prove the following *fixed point theorem*: if $f : [a, b] \to (a, b)$ is continuous, then there exists $c \in (a, b)$ such that $f(c) = c$.

    [Hint: This is a similar proof to that of *Problem 24*. This time you need to consider a function of the form $g(x) = f(x) - ($*something*$)$. What is *something*?] Give a counter-example to demonstrate that the claim is false if the domain of f is restricted to $(0, 1)$.

26. Complete the proof of the extreme value theorem (Theorem 3.5), i.e. prove that a continuous function $f : [a, b] \to \mathbb{R}$ attains its infimum.

27. Use Corollary 3.3 to prove each of the following statements.

    (i) Every continuous function from $\mathbb{R}$ to $\mathbb{Z}$ is constant.

    (ii) Every continuous function from $\mathbb{R}$ to $\mathbb{Q}$ is constant.

28. Show that if $f : [0, 1] \to \mathbb{R}$ is continuous and 0 is not in the range of f, then the function $\frac{1}{f} : [0, 1] \to \mathbb{R}$ is bounded, where for each $x \in [0, 1]$, $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$.

29. Explain why there are no continuous functions having domain $[0, 1]$ and range $\mathbb{R}$. [Hint: Use Theorem 3.5.]

30. Suppose that the function $f : [0, 1] \to [0, 1]$ is continuous, and fix $0 < r < 1$. Suppose that we are given a sequence (x_n) in $[0, 1]$ such that $f(x_{n+1}) \leq rf(x_n)$ for all $n \in \mathbb{N}$. Show that there exists $c \in [0, 1]$ for which $f(c) = 0$. [Hint: Use the Bolzano-Weierstrass theorem.]

31. Show that for each $n \in \mathbb{N}$, the function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = x^n$ is strictly monotonic increasing.

32. Show that the function $f(x) = \sin(x)$ has a continuous inverse when restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. What goes wrong outside this interval? [Hint: Use an appropriate trigonometric identity.]

33. Find $\lim\limits_{x \to 1} \dfrac{1 - x}{1 - x^{\frac{m}{n}}}$, where $m, n \in \mathbb{N}$.

34. If $f : [a, b] \to \mathbb{R}$ is continuous and bijective, and $f(a) < f(b)$, show that f is strictly monotonic increasing.

35.* Show that if $f, g : \mathbb{R} \to \mathbb{R}$ are monotonic increasing, with $f + g$ continuous, then both f and g are continuous.

36.* (i) Let (x_n) be a sequence in $\mathbb{R}$ for which $x_1 = a > 0$ and $x_{n+1} = \sqrt{x_n}$, for all $n \in \mathbb{N}$. Show that $\lim\limits_{n \to \infty} x_n = 1$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function for which $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Use the result of (i) to show that f is constant.

0.1.4 Differentiation

37. Let $f : A \to \mathbb{R}$ and let $a \in \mathbb{R}$ be such that there is some sequence (x_n) in A with $x_n \neq a$ for all n and $\lim\limits_{n \to \infty} x_n = a$. What does it mean to say that the function f has limit l at the point a? What notation do we use to write this?

    [This question is asking you to recall (or revise if you can't recall) Definition 2.2 from the lecture notes. That's a key definition and we build on it when we define what it means to be differentiable — see the next question.]

38. (i) Give the definition of what it means for a function $f : A \to \mathbb{R}$ to be differentiable at $a \in A$.

    (ii) State carefully what this means in terms of limits of sequences.

    (iii) State carefully what this means in terms of the $\varepsilon - \delta$ criterion. [The first part is just asking you to give Definition 4.1 and the other parts are checking that you know what this means.

    The key points from Chapter 2 are Definition 2.2 of a limit of a function and Theorem 2.1 giving the sequential criterion. But you need to see how to apply these, not just copy them out. Here they have to be applied to the relevant function for the definition of differentiability, not f itself.]

39. Give a rigorous proof that the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is differentiable for all $x \in \mathbb{R} \setminus \{0\}$ from the definition of the derivative and find $f'(x)$ explicitly. Can we extend the function so that it is differentiable on the whole of $\mathbb{R}$ by defining its value at zero to be zero?

40. Let $k \in \mathbb{R} \setminus \{0\}$. Give a rigorous proof from the definition of the derivative that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^{kx}$ is differentiable for all $x \in \mathbb{R}$, and find $f'(x)$ explicitly.

    [Hint: For fixed $k \in \mathbb{R}$, let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(h) = e^{kh} - 1 - kh$. You may use the fact that $\lim\limits_{h \to 0} \dfrac{g(h)}{h} = 0$. (This follows from the series expansion of the exponential function, which you already know, and we will study more later.)]

41. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

    (i) Show that f is differentiable at every $x \neq 0$. (You can use standard derivatives and facts about derivatives, such as the product and chain rules.)

    (ii) Show that f is not differentiable at $x = 0$. (Use the definition of the derivative as a limit. You may assume that $\sin\left(\frac{1}{x}\right)$ has no limit as x tends to 0. This was *Problem 8*.)

42. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is differentiable at every $x \in \mathbb{R}$. What can you say about its second derivative?

[Hint: You will need to consider the case $x = 0$ separately from $x \in \mathbb{R} \setminus \{0\}$. You may assume that the limit as x tends to 0 of $x \sin\left(\frac{1}{x}\right)$ exists and equals 0. This was studied in *Problem 10*.]

43. (i) Sketch the graph of any continuous function $f : [0,1] \to \mathbb{R}$ which is not differentiable at $x = \frac{1}{2}$, but is differentiable at all other points. [You do *not* need to give a formula.]

    (ii) Sketch the graph of any continuous function $f : [0,1] \to \mathbb{R}$ which is not differentiable at $x = \frac{1}{3}$ and is not differentiable at $x = \frac{2}{3}$, but is differentiable at all other points. [You do *not* need to give a formula.]

44. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x - [x]$ for all $x \in \mathbb{R}$, where $[x]$ denotes the integer part of f (see *(1.1)*).

    Explain carefully at which points f is differentiable, and find the value of its derivative there. [It will probably help to sketch the graph!]

45. Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a then

$$\lim_{h \downarrow 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a).$$

By considering $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$, show that the limit on the left-hand side may exist, even when f is not differentiable at a.

46. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \begin{cases} -x^2 & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0. \end{cases}$

    Determine whether each of the following is true or false. Justify your answers.

    (i) f is continuous at 0.

    (ii) $f'(0)$ exists.

    (iii) $f' : \mathbb{R} \to \mathbb{R}$ is continuous at 0.

    (iv) $f''(0)$ exists.

47. (i) Must any differentiable function $f : [a,b] \to \mathbb{R}$ have a maximum and minimum value? Why?

    (ii) If $f : [a,b] \to \mathbb{R}$ is differentiable and $f(a) = f(b)$, must f have a maximum and minimum value in (a,b)?

48. Let $a_0, a_1, \ldots, a_n$ be real numbers such that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Show that there is some $c \in (0,1)$ such that $f(c) = 0$. [Hint: Integrate the function f term–by–term, and think about how to use Rolle's theorem.]

49. (i) Use the mean value theorem to show the following. If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with $f'(c) = 0$ for all $c \in (a,b)$, then f is constant on $[a,b]$.

    (ii) Use part (i) to show that if $g, h : \mathbb{R} \to \mathbb{R}$ are both continuous on $[a,b]$ and differentiable on (a,b) with $h'(x) = g'(x)$ for all $x \in (a,b)$, then there exists $k \in \mathbb{R}$ such that $h(x) = g(x) + k$, for all $x \in [a,b]$.

50. If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and there exist $m, M \in \mathbb{R}$ such that $m \leq f'(c) \leq M$, for all $c \in (a,b)$, show that

$$f(a) + m(b-a) \leq f(b) \leq f(a) + M(b-a).$$

51. If $r > 0$ and $q \in \mathbb{R}$ show that the polynomial $p(x) = x^3 + rx + q$ has exactly one real zero. [You may assume the result of Corollary 3.2: p has at least one real root. So you need to show that there can't be more than one.]

52. Let $r > 1$ and fix $y \in (0,1)$. By applying the mean value theorem to the function $f(x) = x^r$ on $[y, 1]$, show that

$$1 - y^r < r(1-y).$$

53. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable at a with $f'(a) = 0$. If $f''(a) < 0$, show that f has a local maximum at a, while if $f''(a) > 0$, show that f has a local minimum at a.

0.1.5 Sequences and series of functions

54. Consider the sequence of functions (f_n), where $f_n : [0, \pi] \to \mathbb{R}$ is defined by $f_n(x) = \sin^n(x)$ for each $n \in \mathbb{N}$. Show that the sequence (f_n) converges pointwise. Does the sequence (f_n) converge uniformly? Justify your answer.

55. For each of the following sequences of functions (f_n) determine the pointwise limit (if it exists), and decide whether (f_n) converges uniformly to this limit.

    (i) $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = x^{\frac{1}{n}}$.

    (ii) $f_n : \mathbb{R} \to \mathbb{R}$, where $f_n(x) = \begin{cases} 0 & x \leq n, \\ x - n & x \geq n \end{cases}$.

    (iii) $f_n : (1, \infty) \to \mathbb{R}$, $f_n(x) = \frac{e^x}{x^n}$.

    (iv) $f_n : [-1, 1] \to \mathbb{R}$, $f_n(x) = e^{-nx^2}$.

    (v) $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = e^{-x^2} n$.

56. For each of the following sequences of functions (g_n) find the pointwise limit, and determine whether the sequence converges uniformly on $[0, 1]$, and on $[0, \infty)$.

    (i) $g_n(x) = \dfrac{x}{n}$.

    (ii) $g_n(x) = \dfrac{x^n}{1 + x^n}$.

    (iii) $g_n(x) = \dfrac{x^n}{n + x^n}$.

57. For each of the following sequences of functions (h_n), where $h_n : [0, 1] \to \mathbb{R}$, find the pointwise limit, if it exists, and in that case determine whether the sequence converges uniformly.

    (i) $h_n(x) = \left(1 - \frac{x}{n}\right)^2$.

    (ii) $h_n(x) = x - x^n$.

    (iii) $h_n(x) = \sum_{k=0}^{n} x^k$.

58. Define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{n + \cos x}{2n + \sin^2 x}$.

    (i) Find the pointwise limit of the sequence of functions (f_n).

    (ii) Show that the sequence (f_n) converges uniformly.

    (iii) Calculate f_n' and show that $f'(x) = \lim_{n \to \infty} f_n'(x)$. Does (f_n) satisfy the conditions of Theorem 5.2?

59. (i) Let $n \in \mathbb{N}$. Show that we can define a continuous function $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & x = 0, \\ \dfrac{x^{\frac{1}{n}} - 1}{\ln x} & 0 < x < 1, \\ \dfrac{1}{n} & x = 1. \end{cases}$$

   (Note: you only need check continuity at $x = 0$ and $x = 1$.)

   (ii) Does the sequence (f_n) converge uniformly to a limit? Justify your answer. If you wish, you may assume without proof that each function f_n is monotone increasing.

60. Let $f_n : [a, b] \to \mathbb{R}$. Suppose that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f. Show that the sequence (f_n) converges uniformly to the zero function.

61. The functions $c, s : \mathbb{R} \to \mathbb{R}$ are defined by the infinite series $s(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $c(x) = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2n}}{(2n)!}$.

    (i) Prove that these functions are well-defined as pointwise limits, and that their series converge absolutely.

    (ii) Show that when restricted to a closed bounded interval $[-R, R]$, where $R > 0$, both series' converge uniformly.

    (iii) Prove that s and c are differentiable everywhere, with $s'(x) = c(x)$ and $c'(x) = -s(x)$. [Hint: Mimic the approach of Example 5.7 in the notes.]

    (iv) Prove that $c^2 + s^2 = 1$. [Hint: Differentiate the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = c(x)^2 + s(x)^2$ for each x.]

    (v)* Show that $\exp(ix) = c(x) + is(x)$ for all $x \in \mathbb{R}$, where exp is the function from Example 5.7. You should justify any rearrangements of terms in infinite series (MAS107 Theorem 4.18 may help).

62. By using the Weierstrass M-test or otherwise, for each of the following series, determine whether it converges uniformly on $\mathbb{R}$ and whether it converges uniformly on $[0, 1]$.

    (i) $\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$

    (ii) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

    (iii) $\sum_{n=1}^{\infty} \sin(nx)$

    (iv)* $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$

63. (i) Show the series $\sum_{n=1}^{\infty} x^n$ converges uniformly for $x \in [0, a]$ whenever $0 < a < 1$.

    (ii) Does the series converge uniformly on $[0, 1)$~? Explain.

64. Prove that there is a function $f : [0, 2\pi] \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

and that this function is continuous.

65.* This question is about the controversial series presented by Fourier in 1807, when he first attempted to publish his solution to the diffusion equation (for a rectangular lamina).

$$\frac{4}{\pi} \left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \frac{1}{7}\cos\left(\frac{7\pi x}{2}\right) + ... \right] \qquad (2)$$

While a perfectly good series, mathematical understanding in europe at the time was simply not equipped to make any sense of it. Completing this question may give you a sense of why Fourier's contemporaries found it so disturbing.

Disclaimer: I am not a historian. For more on this fascinating subject, I recommend reading Chapter 1 of A Radical Approach to Real Analysis by Bressoud.

(i) Using Desmos or otherwise, determine (or make an educated guess) what function $f : \mathbb{R} \to \mathbb{R}$ is represented by the Fourier series *(1.2)*, and sketch its graph. What is its period?

(ii) Evaluate the series when x is an odd integer. What do you notice? Do you need to adjust your answer to (i)?

(iii) Calculate $\mathrm{Dom}(f')$, and write down a formula for f'.

(iv) Now differentiate the series *(1.2)* term by term. Does this series converge pointwise for any values of x? You may find Desmos or a similar graphing tool helpful for this part.

0.1.6 Integration

66. Let $f : [1,4] \to \mathbb{R}$; $f(x) = \dfrac{1}{x}$. Let P be the partition consisting of points $\{1, \frac{3}{2}, 2, 4\}$.

    (i) Compute $L(f,P)$, $U(f,P)$ and $U(f,P) - L(f,P)$.

    (ii) What happens to the value of $U(f,P) - L(f,P)$ when we add the point 3 to the partition?

    (iii) Find a partition P' of $[1,4]$ for which $U(f,P') - L(f,P') < \frac{2}{5}$. [Hint: Mimick the proof of Theorem 6.1.]

67. Let $g : [0,1] \to \mathbb{R}$; $g(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 2 & \text{for } x = 1 \end{cases}$.

    (i) Show that $L(g,P) = 1$ for every partition P of $[0,1]$.

    (ii) Construct a partition P for which $U(g,P) < 1 + \frac{1}{10}$

    (iii) Given $\varepsilon > 0$, construct a partition P_ε such that $U(g,P_\varepsilon) < 1 + \varepsilon$.

68. Complete the proof of Proposition 6.1. That is, show that for any closed, bounded interval $[a,b]$, the functions $\mathbb{1}_{[a,b]}$, $\mathbb{1}_{[a,b]}$, $\mathbb{1}_{[a,b]}$ and $\mathbb{1}_{[a,b]}$ are Riemann integrable, and

$$\int_a^b \mathbb{1}_{[a,b]}(x)dx = \int_a^b \mathbb{1}_{(a,b)}(x)dx = \int_a^b \mathbb{1}_{[a,b)}(x)dx = \int_a^b \mathbb{1}_{(a,b]}(x)dx = b - a.$$

69. *(Exam question).* Define step functions $r, s \colon \mathbb{R} \to \mathbb{R}$ by

$$r = \mathbb{1}_{[0,1)} + e\mathbb{1}_{[1,2)} + e^4\mathbb{1}_{[2,3]}, \qquad s = e\mathbb{1}_{[0,1]} + e^4\mathbb{1}_{(1,2]} + e^9\mathbb{1}_{(2,3]}.$$

    (i) Evaluate the integrals $\int_0^3 r(x)dx$ and $\int_0^3 s(x)dx$. You may use results from the course to help you, but should indicate clearly when doing so.

    (ii) Why is the function $f \colon [0,3] \to \mathbb{R}$ defined by $f(x) = e^{x^2}$ Riemann integrable?

    (iii) Prove that

$$1 + e + e^4 \le \int_0^3 e^{x^2}\, dx \le e + e^4 + e^9.$$

    [Hint: there's no need to calculate the integral in the middle; use the previous parts to prove the inequalities.]

70. Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions.

    (i) Show that if P is a partition of $[a,b]$, then

$$U(f+g, P) \le U(f,P) + U(g,P)$$

    and

$$L(f+g, P) \ge L(f,P) + L(g,P).$$

    Can you think of a particular example where these inequalities are strict?

    (ii) Let P_1 and P_2 be partitions of $[a,b]$. Show that

$$U(f+g, P_1 \cup P_2) \le U(f, P_1) + U(g, P_2)$$

    and

$$L(f+g, P_1 \cup P_2) \ge L(f, P_1) + L(g, P_2)$$

[Hint: Lemma 6.1 may be helpful.]

(iii) Deduce that

$$L(f+g) \geq \int_a^b f(x)dx + \int_a^b g(x)dx$$

and

$$U(f+g) \leq \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Explain why this means $f + g$ must be integrable, with

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

71. Show that if $k \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ is integrable, then so is $kf : [a, b] \to \mathbb{R}; x \mapsto kf(x)$, and

$$\int_a^b kf(x)dx = k \int_a^b f(x).$$

[Hint: It may help to treat the cases $k \geq 0$ and $k < 0$ separately.]

72. Let f be bounded on a set $A \subseteq \mathbb{R}$, let $M = \sup\{f(x) : x \in A\}$, $m = \inf\{f(x) : x \in A\}$, and let $M' = \sup\{|f(x)| : x \in A\}$, and $m' = \inf\{|f(x)| : x \in A\}$.

(i) Show that $M - m \geq M' - m'$.

(ii) Show that if f is integrable on $[a, b]$, then for any partition P of $[a, b]$,

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P).$$

(iii) Complete the proof of Proposition 6.3(iv) that is, show that $|f|$ is integrable whenever f is.

73. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $a, b : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Prove that

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t)dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

74. Define a function $l : (0, \infty) \to \mathbb{R}$ by $l(x) = \int_1^x \frac{1}{t}dt$.

Show the following directly from the definition of l via an integral (that is, *without* using any properties of the function ln).

(i) l is differentiable, and $l'(x) = \frac{1}{x}$.

(ii) $l(xy) = l(x) + l(y)$ for all $x, y > 0$.

75.* Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Prove that there is a number $x \in [a, b]$ such that

$$\int_a^x f(t)dt = \int_x^b f(t)dt.$$

[Hint: apply the intermediate value theorem to the function $F : [a, b] \to \mathbb{R}$ given by

$$F(x) = \int_a^x f(t)dt - \int_x^b f(t)dt. \qquad]$$

76. (i) Let $f\colon [a, b] \to \mathbb{R}$ be Riemann integrable. Suppose there are $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Prove that there is a number $\mu \in [m, M]$ such that

$$\int_a^b f(x)\,dx = (b-a)\mu.$$

(ii) Let $f\colon [a, b] \to \mathbb{R}$ be continuous. Prove that there is some $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = (b-a)f(c).$$

[Hint: The intermediate value theorem is useful.]

77. Let $f\colon [a, b] \to \mathbb{R}$ be continuous, and let $g\colon [a, b] \to [0, \infty)$ be integrable. Prove that there is some $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

Do we need the assumption $g(x) \geq 0$? Justify your answer.

0.2 Solutions

0.2.1 Preliminary problems

P1. The correct statements are (ii) and (v).

In statement (iv), the quantifiers $\forall$ and $\exists$ are the wrong way round, while in (i) they are missing altogether. Statement (iii) asserts that last part, $|x_n - l| < \varepsilon \ \forall n \geq N$, should hold for all ε and for all N — so again, there is an issue with the quantifiers.

Bonus exercise: prove that (iv) holds if and only if (x_n) is the constant sequence $x_n = l$ for all $n \in \mathbb{N}$.

P2. (Homework 1 question).

Let $\varepsilon > 0$. According to the definition, we must prove there is an $N \in \mathbb{N}$ for which $\left|\frac{2n}{3n-1} - \frac{2}{3}\right| < \varepsilon$ whenever $n \geq N$.

Now,

$$\left|x_n - \frac{2}{3}\right| = \left|\frac{2n}{3n-1} - \frac{2}{3}\right| = \frac{6n - (6n-2)}{3(3n-1)} = \frac{2}{3(3n-1)}.$$

This will be strictly less than ε whenever $3n - 1 > \frac{2}{3\varepsilon}$., or in other words, whenever

$$n > \frac{1}{3}\left(\frac{2}{3\varepsilon} + 1\right) = \frac{2 + 3\varepsilon}{9\varepsilon}.$$

Let N be any integer greater than $\frac{2+3\varepsilon}{9\varepsilon}$. Then $\left|x_n - \frac{2}{3}\right| < \varepsilon$ for all $n \geq N$, and we have proven that $x_n \to \frac{2}{3}$ as $n \to \infty$ using the definition.

P3. (Homework 1 question).

The Bolzano–Weierstrass theorem states that every bounded sequence has a convergent subsequence. Its proof combines the following two results:

- The monotone convergence theorem (Theorem 3.10 in the MAS107 notes), which states that bounded monotone sequences must converge. More specifically, every monotone increasing sequence that is bounded above converges to its supremum, and every monotone decreasing sequence bounded below converges to its infimum.

- Theorem 3.13 from MAS107: Every sequence has a monotone subsequence.

Proof of Bolzano–Weierstrass: If (x_n) is a bounded sequence, then it has a monotone subsequence, (x_{n_k}), by Theorem 3.13. This subsequence must also be bounded since (x_n) is bounded, and hence it converges by the monotone convergence theorem.

P4.(i) Let $a, b \geq 0$. Then,

$$\left(\sqrt{a+b}\right)^2 = a + b,$$

while,

$$\left(\sqrt{a} + \sqrt{b}\right)^2 = a + 2\sqrt{a}\sqrt{b} + b \geq a + b.$$

Therefore

$$\left(\sqrt{a} + \sqrt{b}\right)^2 \geq \left(\sqrt{a+b}\right)^2.$$

Since $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{a+b}$ are all non-negative and the square root function is increasing, we can square root both sides to get

$$\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}.$$

(ii) Note that the inequality we have been asked to prove is symmetric in a and b, in the sense that exchanging the roles of a and b does not affect the value of either side.

This means we can assume without loss of generality that $\sqrt{|a|} \geq \sqrt{|b|}$. Then,

$$\left|\sqrt{|a|} - \sqrt{|b|}\right| = \sqrt{|a|} - \sqrt{|b|},$$

and we need only show that

$$\sqrt{|a|} - \sqrt{|b|} \leq \sqrt{|a-b|}. \tag{1}$$

We use a trick similar to the proof of Corollary 1.1, and write

$$\sqrt{|a|} = \sqrt{|(a-b)+b|}.$$

By the triangle inequality, we have that $|a| = |(a-b)+b| \leq |a-b| + |b|$.

Since the square root function is increasing, we can square root both sides to get

$$\sqrt{|a|} \leq \sqrt{|a-b| + |b|}.$$

Applying the result from (i) to the right-hand side, it follows that

$$\sqrt{|a|} \leq \sqrt{|a-b| + |b|} \leq \sqrt{|a-b|} + \sqrt{|b|}.$$

Equation *(2.1)* now follows by subtracting $\sqrt{|b|}$ from both sides.

(iii) Let (a_n) be a real sequence converging to $l \in \mathbb{R}$. By (ii),

$$\left|\sqrt{|a_n|} - \sqrt{|l|}\right| \leq \sqrt{|a_n - l|}$$

for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. Since $a_n \to l$, there is $N \in \mathbb{N}$ such that $|a_n - l| < \varepsilon^2$ whenever $n \geq N$. Hence for $n \geq N$,

$$\left| \sqrt{|a_n|} - \sqrt{|l|} \right| \leq \sqrt{\varepsilon^2} = \varepsilon.$$

Thus $\lim_{n \to \infty} \sqrt{|a_n|} = \sqrt{|l|}$.

P5. When it exists, the supremum of a set $A \subseteq \mathbb{R}$ is defined to be the unique number $\alpha \in \mathbb{R}$ such that

- α is an upper bound for A, and

- if M is any other upper bound for A, then $\alpha \leq M$.

We write $\sup A$ for the supremum of A, when it exists.

The definition of the infimum is similar: when it exists, the infimum of A is the unique number $\beta \in \mathbb{R}$ such that

(i) β is a lower bound for A, and

(ii) if L is any other lower bound for A, then $\alpha \geq L$.

We write $\inf A$ for the infimum of A, when it exists.

The axiom of completeness for the real numbers says that every non-empty bounded above subset of $\mathbb{R}$ has a supremum. Equivalently, it says that every non-empty bounded below subset of $\mathbb{R}$ has an infimum.

For more details, see page 32 of your Semester 2 MAS107 notes.

P6.(i) $\sup \left\{ \dfrac{m}{n} : m, n \in \mathbb{N} \text{ s.t } m < n \right\} = 1$, $\inf \left\{ \dfrac{m}{n} : m, n \in \mathbb{N} \text{ s.t } m < n \right\} = 0$.

(ii) $\sup \left\{ \dfrac{(-1)^m}{n} : m, n \in \mathbb{N} \text{ s.t } m < n \right\} = \dfrac{1}{3}$, $\inf \left\{ \dfrac{(-1)^m}{n} : m, n \in \mathbb{N} \text{ s.t } m < n \right\} = -\dfrac{1}{2}$.

(iii) $\sup \left\{ \dfrac{n}{3n + 1} : n \in \mathbb{N} \right\} = \dfrac{1}{3}$, $\inf \left\{ \dfrac{n}{3n + 1} : n \in \mathbb{N} \right\} = \dfrac{1}{4}$.

P7.(i) This is false — for a counter-example, take any singleton set. Then $\sup A = \inf A = 1$.

A correct version of the statement would be $\inf A \leq \sup A$.

(ii) True. The analogous statement for sup was called the characteristic property of the supremum in your MAS107 lecture notes — see Lemma 2.8 on page 36.

(iii) True. Since $\sup B$ is an upper bound for B, and A is a subset of B, $\sup B$ must be an upper bound for A. But $\sup A$ is a the least upper bound for A, hence $\sup A \leq \sup B$. The analogous property for inf is that $\inf A \geq \inf B$ whenever $A \subseteq B$.

(iv) False. For a counter-example, take $B = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$, for which $\inf B = 0$.

(v) True. Let $s = \max\{\sup A, \sup B\}$. Then $s \geq \sup A$ and $s \geq \sup B$, so s is an upper bound for both A and B. Therefore, s is an upper bound for $A \cup B$. But $\sup(A \cup B)$ is the least upper bound of $A \cup B$, and so $s \geq \sup(A \cup B)$. Also, since A and B are both subsets of $A \cup B$, we have by statement (iii) that $s = \max\{\sup A, \sup B\} \leq \sup(A \cup B)$. Hence $\max\{\sup A, \sup B) = \sup(A \cup B)$.

0.2.2 Limits of functions

1. One has to identify the real numbers where the given formula does not make sense, usually because of a zero in a denominator somewhere, and exclude them.

(i) $A = \mathbb{R} \setminus \{0, -1\}$.

(ii) $g_2(x) = \dfrac{(x-1)(x+4)}{(x-1)(x+2)(x+3)}$, so $A = \mathbb{R} \setminus \{1, -2, -3\}$.

(iii) $g_3(x) = \dfrac{x+4}{(x+2)(x+3)}$, so $A = \mathbb{R} \setminus \{-2, -3\}$.

(iv) $A = \mathbb{R} \setminus \{1\}$.

(v) $A = \mathbb{R} \setminus \{0\}$.

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2. (Homework 1 question).

(i) $L = [0, 1] \cup [2, 3]$.

(ii) $L = \emptyset$. (All convergent sequences in $\mathbb{Z}$ are eventually constant.)

(iii) $L = \mathbb{R}$.

(iv) $L = [0, 1]$

(v) $L = \{0\}$

---

3. (Homework 1 question).

(i) We are given $f : \mathbb{R} \to \mathbb{R}$; $f(x) = 4x + 7$. Intuition tells us that

$$\lim_{x \to 2} f(x) = 4 \cdot 2 + 7 = 15.$$

To prove this, let $\varepsilon > 0$. We seek $\delta > 0$ such that $0 < |x - 2| < \delta$ implies $|f(x) - 15| < \varepsilon$.

Note that

$$|f(x) - 15| = |4x + 7 - 15| = |4x - 8| = 4|x - 2|.$$

Therefore, putting $\delta = \frac{\varepsilon}{4}$, we get that whenever $0 < |x - 2| < \delta$,

$$|f(x) - 17| = 4|x - 2| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

For the limit as $x \to 0$, we claim that $\lim_{x \to 0} f(x) = 7$. To prove this, note that $|f(x) - 7| = 4|x|$. This means that given $\varepsilon > 0$, if $0 < |x| < \frac{\varepsilon}{4}$ then

$$|f(x) - 7| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

It follows that $\lim_{x \to 0} f(x)$ exists and is equal to 7.

(ii) We have $f : \{0\} \cup [1, 3] \to \mathbb{R}$; $f(x) = 3x^2 - 1$, so using intuition only, $\lim_{x \to 2} f(x) = 3 \cdot 4 - 1 = 11$.

Let $\varepsilon > 0$. We seek $\delta > 0$ such that for all $x \in \{0\} \cup [1, 3]$,

$$0 < |x - 2| < \delta \implies |f(x) - 11| < \varepsilon.$$

Now, for $x \in \{0\} \cup [1, 3]$,

$$|f(x) - 11| = |3x^2 - 12| = 3|x^2 - 4| = 3|x - 2||x + 2| \leq 15|x - 2|.$$

(Here, we have used the fact that $|x + 2| = x + 2 \leq 5$, for $x \in \{0\} \cup [1, 3]$.)

Hence we can let $\delta := \frac{\varepsilon}{15}$ and conclude that if $x \in \{0\} \cup [1, 3]$ and $0 < |x - 2| < \frac{1}{15}$, then

$$|f(x) - 11| < 15 \cdot \frac{\varepsilon}{15} = \varepsilon.$$

For this f, $\lim_{x \to 0} f(x)$ is not defined, since 0 is not a limit point of the domain of f.

(iii) Finally, let $f : (0, \infty) \to \mathbb{R}$; $f(x) = x + \frac{1}{x}$. The limit of $f(x)$ as $x \to 2$ ought to be $2 + \frac{1}{2} = \frac{5}{2}$.

Let $\varepsilon > 0$, and consider

$$\left| f(x) - \frac{5}{2} \right| = \left| x + \frac{1}{x} - \frac{5}{2} \right| = \left| \frac{2x^2 + 2 - 5x}{2x} \right| = \left| \frac{(x - 2)(2x - 1)}{2x} \right| = |x - 2| \left| 1 - \frac{1}{2x} \right|.$$

Observe that if $x \in (0, \infty)$ and $|x - 2| < 1$, then $0 < 1 - \frac{1}{2x} < 1$, and so

$$\left| f(x) - \frac{5}{2} \right| = |x - 2| \left| 1 - \frac{1}{2x} \right| < |x - 2|.$$

To make sure the above inequality holds **and** is bounded above by ε, let $\delta := \min\{\varepsilon, 1\}$.

Suppose that $x \in (0, \infty)$ with $0 < |x - 2| < \delta$. Then by our choice of δ, $0 < |x - 2| < 1$ and $0 < |x - 2| < \varepsilon$ both hold simultaneously. It follows that

$$\left| f(x) - \frac{5}{2} \right| < |x - 2| < \varepsilon,$$

and hence $\lim_{x \to 2} f(x) = \frac{5}{2}$ is proved.

For the last part, note that $f(x) = x + \frac{1}{x}$ does not converge to a finite limit as $x \to 0$. In fact, $\lim_{x \to 0} f(x) = \infty$. One way to prove this is to show that $f(x)$ surpasses any possible bound as $x \to 0$. Note that

$$f(x) = x + \frac{1}{x} > \frac{1}{x}.$$

Therefore, given an arbitrary number $K > 0$, we can take $0 < x < \frac{1}{K}$ to ensure that

$$f(x) = x + \frac{1}{x} > \frac{1}{x} > K.$$

This proves that $\lim_{x \to 0} f(x) = \infty$, for this choice of f.

---

4. We had $A = \mathbb{R} \setminus \{1, -2, -3\}$, and $f_2 : A \to \mathbb{R}$; $g_2(x) = \dfrac{(x + 4)}{(x + 2)(x + 3)}$.

It is considerably easier to use the sequential criterion for functional limits (Theorem 2.1) than it is to proceed directly using the Definition 2.1. We include both methods, for completeness.

Method 1: Sequences: Let (x_n) be a sequence in A converging to 1. Then, using algebra of limits for real sequences,

$$\lim_{n \to \infty} f_2(x_n) = \lim_{n \to \infty} \frac{(x_n + 4)}{(x_n + 2)(x_n + 3)} = \frac{(1 + 4)}{(1 + 2)(1 + 3)} = \frac{5}{12},$$

where the last equality is using algebra of limits. So $\lim_{x \to 1} g_2(x) = \dfrac{5}{12}$.

Let $x_n = -2 + \frac{1}{n}$, so that (x_n) is a sequence in A converging to -2. Then we see that $(f_2(x_n))$ diverges to $+\infty$ and so $\lim_{x \to -2} g_2(x)$ does not exist.

Similarly, considering $x_n = -3 + \frac{1}{n}$, we see that $\lim_{x \to -3} g_2(x)$ does not exist.

Method 2: $(\varepsilon - \delta)$ **criterion**: Intuitively speaking, the limit as $x \to 1$ "should" be $\frac{(1+4)}{(1+2)(1+3)} = \frac{5}{12}$.

We prove $\lim_{x \to 1} f_2(x) = \frac{5}{12}$.

Let $\varepsilon > 0$. We wish to find $\delta > 0$ so that if $x \in A$ and $0 < |x - 1| < \delta$, then $\left| f_2(x) - \frac{5}{12} \right| < \varepsilon$.

Now,

$$
\begin{aligned}
\left| f_2(x) - \frac{5}{12} \right| &= \left| \frac{(x+4)}{(x+2)(x+3)} - \frac{5}{12} \right| \\
&= \left| \frac{12(x+4) - 5(x+2)(x+3)}{(x+2)(x+3)} \right| \\
&= \left| \frac{18 - 5x^2 - 13x}{(x+2)(x+3)} \right| \\
&= \left| \frac{(1-x)(5x+18)}{(x+2)(x+3)} \right| = |x - 1| \cdot \frac{|5x+18|}{|x+2||x+3|}.
\end{aligned}
$$

We are going to constrain x to be very close to 1. In particular, we can assume $0 < x < 2$, so

$$
\left| f_2(x) - \frac{5}{12} \right| = |x-1| \cdot \frac{|5x+18|}{|x+2||x+3|} < |x-1| \frac{5(2)+18}{(2)(3)} = \frac{14}{3}|x-1|.
$$

Finally, we choose $\delta := \frac{3}{14}\varepsilon$. Then, for $0 < |x-1| < \delta$,

$$
\left| f_2(x) - \frac{5}{12} \right| < \frac{14}{3}|x-1| < \frac{14}{3} \cdot \frac{3}{14}\varepsilon = \varepsilon.
$$

---

5. The first part follows by using the definition of the limit of a function in terms of limits of sequences, and then applying the result of *P4 (iii)..*

To be precise let (x_n) be any sequence in $\mathbb{R} \setminus \{a\}$ that converges to a. Then since we are given that $\lim_{x \to a} f(x) = l$, we must have that $\lim_{n \to \infty} f(x_n) = l$. But then by *P4 (iii).*, we have $\lim_{n \to \infty} \sqrt{f(x_n)} = \sqrt{l}$. So by definition of the limit of a function, $\lim_{x \to a} \sqrt{f(x)} = \sqrt{l}$.

Using algebra of limits, $\lim_{x \to 1} \frac{x+1}{x^2} = \frac{1+1}{1^2} = 2$ and then by the first part $\lim_{x \to 1} \sqrt{\frac{x+1}{x^2}} = \sqrt{2}$.

---

6. Let $f : A \to \mathbb{R}$ and suppose that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = l'$. Then given any sequence (x_n) in $A \setminus \{a\}$ that converges to a, we have $\lim_{n \to \infty} f(x_n) = l$ and also $\lim_{n \to \infty} f(x_n) = l'$. But then $l = l'$, by uniqueness of limits.

---

7. When $x > 0$, $\frac{|x|}{x} = \frac{x}{|x|} = \frac{x}{x} = 1 = \text{sgn}(x)$, and when $x < 0$, $\frac{|x|}{x} = \frac{x}{|x|} = -\frac{x}{x} = -1 = \text{sgn}(x)$. So $\text{sgn}(x) = \frac{|x|}{x} = \frac{x}{|x|}$, when $x \neq 0$.

Consider the sequences $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$. If $\lim_{x \to 0} \text{sgn}(x)$ existed, then we would have

$$
\lim_{n \to \infty} \text{sgn}(x_n) = \lim_{n \to \infty} \text{sgn}(y_n) = \lim_{x \to 0} \text{sgn}(x).
$$

But in fact, $\text{sgn}(x_n) = 1$ and $\text{sgn}(y_n) = -1$ for all $n \in \mathbb{N}$, so

$$\lim_{n \to \infty} \text{sgn}(x_n) = 1 \neq -1 = \lim_{n \to \infty} \text{sgn}(y_n).$$

So sgn does not converge as $x \to 0$.

For the left and right limits at 0:

- The left limit is $\lim_{x \to 0^-} \text{sgn}(x) = -1$, since for any sequence (x_n) approaching 0 from the left, we have $\text{sgn}(x_n) = -1$ for all n.

- The right limit is $\lim_{x \to 0^+} \text{sgn}(x) = 1$, since for any sequence (x_n) approaching 0 from the right, we have $\text{sgn}(x_n) = 1$ for all n.

---

8. (Homework 2 question).

(i) For $a \neq 1$, the left and right limits exist and are both equal to $f(a)$, using the algebra of limits. The only point at which left and right limits disagree is $a = 1$, with

$$\lim_{x \to 1-} f(x) = 0, \quad \text{and} \quad \lim_{x \to 1+} f(x) = 1.$$

(ii) In this case, left and right limits disagree at every $n \in \mathbb{Z}$, with $\lim_{x \to a-}[x] = n - 1$, $\lim_{x \to n+}[x] = n$. At every other real number, the left and right limits exist and are both equal to the value of the function there.

(iii) Here, left and right limits disagree at $x = 0, 1$ and 2. We have

$$\lim_{x \to 0^-} h(x) = 3, \quad \lim_{x \to 1-} h(x) = -2, \quad \lim_{x \to 2-} h(x) = 10,$$

$$\lim_{x \to 0^+} h(x) = -2, \quad \lim_{x \to 1+} h(x) = 10, \quad \lim_{x \to 2+} h(x) = 3.$$

---

9. (Homework 2 question). The largest subset of $\mathbb{R}$ for which the formula $f(x) = \sin\left(\frac{1}{x}\right)$ makes sense is $A = \mathbb{R} \setminus \{0\}$.

Observe that for any real number $\theta \in \mathbb{R}$,

$$f\left(\frac{1}{\theta + 2\pi n}\right) = \sin(\theta + 2\pi n) = \sin(\theta).$$

To prove f has no limit at 0, we need only choose two values of θ for which sine takes distinct values. We choose $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.

By choosing the right values for θ, we can use this to construct two distinct sequences (x_n) and (y_n) that both converge to 0, but for which $\sin(x_n)$ and $\sin(y_n)$ have different limits.

We use $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ (other choices possible).

For each $n \in \mathbb{N}$, let $x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$ and $y_n = \frac{1}{3\frac{\pi}{2} + 2\pi n}$. Then

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0,$$

while for all $n \in \mathbb{N}$,

$$f(x_n) = \sin\left(\frac{\pi}{2}\right) = 1 \quad \text{and} \quad f(y_n) = \sin\left(\frac{3\pi}{2}\right) = -1.$$

So $\lim_{n \to \infty} f(x_n) = 1$, but $\lim_{n \to \infty} f(y_n) = -1$. Since these limits are not equal, $\lim_{x \to 0} f(x)$ does not exist.
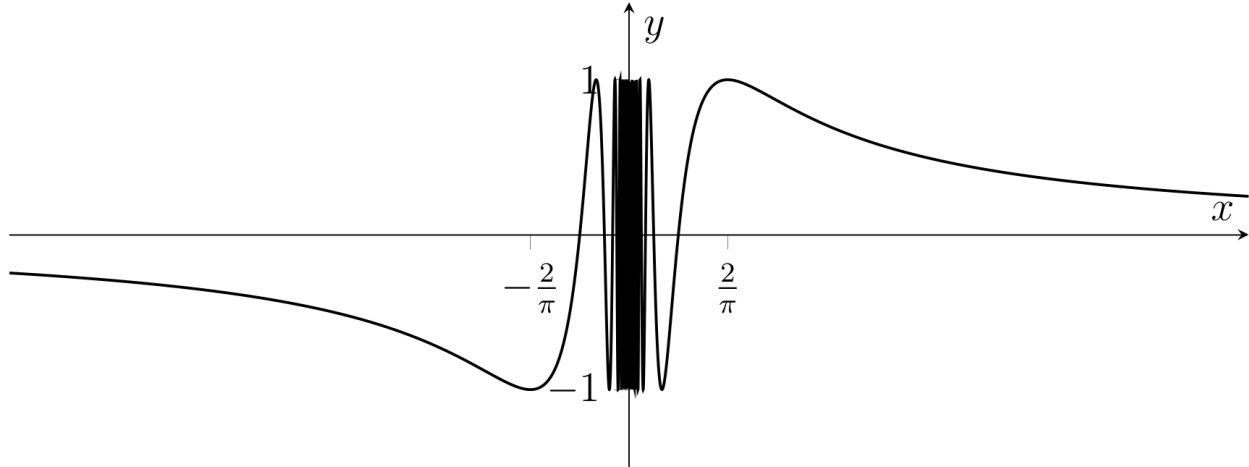

Fig. 1: Graph of the function $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \sin\left(\frac{1}{x}\right)$ (Problem 9).

Of course, we could have chosen any values of θ we liked, and constructed a sequence (x_n) with limit 0 for which $\lim_{n \to \infty} f(x_n)$ is equal to any real number we liked in the interval $[-1, 1]$. The graph of this function (Fig. 1) may give some intuition as to how this is possible.

---

10. The largest subset of $\mathbb{R}$ for which the formula $f(x) = x \sin\left(\frac{1}{x}\right)$ makes sense is $A = \mathbb{R} \setminus \{0\}$.

We claim that $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

To see this, let (x_n) be an arbitrary sequence in A that converges to 0. The sine function takes values only in the interval $[-1, 1]$, and so

$$\left| x_n \sin\left(\frac{1}{x_n}\right) \right| \leq |x_n|$$

for each $n \in \mathbb{N}$. Put another way,

$$-|x_n| \leq x_n \sin\left(\frac{1}{x_n}\right) \leq |x_n|$$

for all $n \in \mathbb{N}$. Since $\lim_{n \to \infty} x_n = 0$, we have $\lim_{n \to \infty} |x_n| = 0$. Therefore, but the sandwich rule, $\lim_{n \to 0} x_n \sin\left(\frac{1}{x_n}\right) = 0$.

Since (x_n) was arbitrary, $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

You should contrast Fig. 2 with Fig. 1 that for the previous question.

---

11. We'll just do $\lim_{x \to \infty} f(x)$ here, as $\lim_{x \to -\infty} f(x)$ is so similar.

(i) Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. We say that $\lim_{x \to \infty} f(x) = l$ if for all $\varepsilon > 0$ there exists $K > 0$ such that for all $x \in X$, $x > K$ implies $|f(x) - l| < \varepsilon$.

(ii) The analogue of the sequential criterion is:

$\lim_{x \to \infty} f(x) = l$ if for any sequence (x_n) in X that diverges to infinity, we have $\lim_{x \to \infty} f(x_n) = l$.
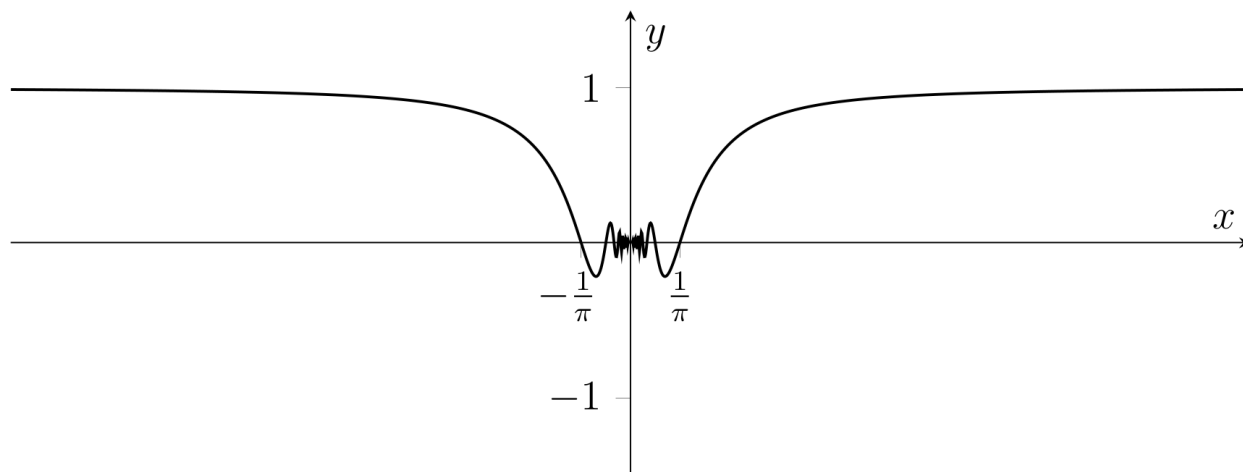
Fig. 2: Graph of the function $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x \sin\left(\frac{1}{x}\right)$ (Problem 10).

For the proof, imitate the proof of Theorem 2.1. Suppose the $(\varepsilon - K)$ criterion stated in part (i) holds. Let $\varepsilon > 0$, and choose $K > 0$ such that for all $x \in X$, $x > K$ implies $|f(x) - l| < \varepsilon$. Let (x_n) be any sequence in X that diverges to infinity. Then using the same $K > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $x_n > K$. But then, for all $n \geq N$, $x_n > K$, which in turn implies $|f(x_n) - l| < \varepsilon$. Hence $\lim_{x \to \infty} f(x_n) = l$, as required.

For the converse, we prove by contrapositive. Suppose the $(\varepsilon - K)$ criterion does not hold. Then there exists $\varepsilon > 0$ such that for all $K > 0$, there is $x \in X$ for which $x > K$ and $|f(x) - l| \geq \varepsilon$. Apply this successively to $K = 1, 2, 3, \ldots$. For $K = n$, we get $x_n \in X$ with $x_n > n$, and $|f(x_n) - l| \geq \varepsilon$ for each $n \in \mathbb{N}$. But then, (x_n) is a sequence in X diverging to infinity, and such that $f(x_n) \nrightarrow l$. This shows that the sequential criterion fails.

(iii) Using the definition: Given any $\varepsilon > 0$, choose $K = \frac{1}{\varepsilon}$. Then $x > K \Rightarrow \frac{1}{x} < \varepsilon$, and so $\lim_{x \to \infty} \frac{1}{x} = 0$.

Using sequences: Let (x_n) be any sequence in $\mathbb{R} \setminus \{0\}$ diverging to infinity as $n \to \infty$. Then, by algebra of limits, $\frac{1}{x_n} \to 0$. Since the sequence was chosen arbitrarily, it follows that $\lim_{x \to \infty} \frac{1}{x} = 0$.

The case where $x \to -\infty$ is similar.

---

12. (i) Let $f : X \to \mathbb{R}$ be a function, where $X \subseteq \mathbb{R}$. We say that $\lim_{x \to \infty} f(x) = \infty$ if:

Given any $M > 0$, there exists $K > 0$ such that $x > K$ implies $f(x) > M$.

The analogue of the sequential criterion is:

For any sequence (x_n) in X which diverges to infinity, we also have that $(f(x_n))$ diverges to infinity.

The other cases are similar.

(ii) Let $f : \mathbb{R} \to [0, \infty)$, $g : \mathbb{R} \to [0, \infty)$, and suppose $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to \infty} g(x) = l$, where $l > 0$.

Let $M > 0$. Since $\lim_{x \to \infty} f(x) = \infty$, there exists $K_1 > 0$ such that

$$f(x) > \frac{2M}{l} \qquad \forall x > K_1.$$

Since $\lim_{x \to \infty} g(x) = l$, for all $\varepsilon > 0$ we can find $K_2 > 0$ such that $x > K_2$ implies $|g(x) - l| < \varepsilon$. In other words,

$$l - \varepsilon < g(x) < l + \varepsilon, \qquad \forall x > K_2.$$

This is true for all ε. We apply it specifically to $\varepsilon = \frac{l}{2}$. Then, there is $K_2 > 0$ such that

$$\frac{l}{2} < g(x) < \frac{3l}{2}, \qquad \forall x > K_2.$$

Let $K = \max\{K_1, K_2\}$. If $x > K$, then $x > K_1$ and $x > K_2$, so $f(x) > \frac{2M}{l}$ and $\frac{l}{2} < g(x) < \frac{3l}{2}$.

Hence for $x > K$,

$$f(x)g(x) > \frac{2M}{l}\frac{l}{2} = M.$$

That is, $\lim_{x \to \infty} f(x)g(x) = \infty$.

(iii) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial of even degree m, with positive leading coefficient. Write $m = 2n$ and let

$$p(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$$
$$= x^{2n}\left(a_{2n} + \frac{a_{2n-1}}{x} + \cdots + \frac{a_1}{x^{2n-1}} + \frac{a_0}{x^{2n}}\right).$$

We use part (ii). Take $f(x) = x^{2n}$ and $g(x) = a_{2n} + \frac{a_{2n-1}}{x} + \cdots + \frac{a_1}{x^{2n-1}} + \frac{a_0}{x^{2n}}$.

Observe that $\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} g(x) = a_{2n}$. Hence by part (ii),

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} f(x)g(x) = \infty.$$

If n is odd, $\lim_{x \to \infty} p(x) = \infty$, but $\lim_{x \to -\infty} p(x) = -\infty$.

0.2.3 Continuity

13. For each of (i) to (v), the function is continuous at each point of its domain. For (i), (ii) and (iii), we have rational functions, which are continuous on all points of A (which is the subset of $\mathbb{R}$ where the denominator is non-zero). For (iv) and (v), we have compositions of rational functions with the exponential and cosine functions, which are continuous on the whole of $\mathbb{R}$. Again, the functions are continuous on all points of A (which is the subset of $\mathbb{R}$ where the denominator of the rational function is non-zero).

---

14. If (x_n) is any sequence that converges to a, we know that $f(x_n)$ converges to $f(a)$ as f is continuous at a, and we need to show that $(|f|(x_n))$ converges to $|f|(a)$.

By Corollary 1.1, if (x_n) is any sequence that converges to a,

$$0 \le ||f|(a)| - |f|(x_n)| = ||f(a)| - |f(x_n)||$$
$$\le |f(a) - f(x_n)| \to 0 \text{ as } n \to \infty,$$

as f is continuous. So by the sandwich rule, $\lim_{n \to \infty} |f|(x_n) = |f|(a)$, and so f is continuous at a.

---

15. Let (x_n) be a sequence in A that converges to a. Since f is continuous at a we have $\lim_{n \to \infty} f(x_n) = f(a)$. And then, since g is continuous at $f(a)$, we have $\lim_{n \to \infty} g(f(x_n)) = g(f(a))$, and the result follows.

---

16. $f \circ g : \mathbb{R} \to \mathbb{R}$ given by $(f \circ g)(x) = \frac{1}{1+x^2}$. It is continuous on $\mathbb{R}$ by Theorem 3.2(iv).

$g \circ f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ given by $(g \circ f)(x) = 1 + \frac{1}{x^2}$. It is continuous on $\mathbb{R} \setminus \{0\}$ by Theorem 3.2(iv).

---

17. (i) Continuous on $\mathbb{R} \setminus \{1\}$. Jump discontinuity at 1 with $J_f(1) = 1$. (ii) Continuous on $\mathbb{R} \setminus \mathbb{Z}$. Jump discontinuity at n with $J_g(n) = 1$ for all $n \in \mathbb{Z}_+$. (iii) Continuous at $\mathbb{R} \setminus \{0, 1, 2\}$. Each of $0, 1, 2$ is a jump discontinuity and we have $J_h(0) = -5, J_h(1) = 12, J_h(2) = -7$.

18.

(i) For $x \neq 0$, $f(x) = x + 2$. Since $\lim_{x \to 0} f(x) = 2$, the required continuous extension is $\tilde{f}$ where

$$\tilde{f}(x) = \begin{cases} \dfrac{(1+x)^2 - 1}{x} & \text{if } x \neq 0 \\ \\ 2 & \text{if } x = 0. \end{cases}$$

(ii) The largest domain is $A = \mathbb{R} \setminus \{-2, 2\}$ (the set where the denominator is non-zero). For $x \neq 2, -2$,

$$f(x) = \frac{(x-2)(x^2 + 2x + 4)}{(x-2)(x+2)} = \frac{x^2 + 2x + 4}{x+2}.$$

This is a rational function, and hence continuous everywhere the denominator is non-zero by Theorem 3.2(iv).

We claim that $\lim_{x \to -2} f(x)$ does not exist. To see this, consider the sequence (x_n), whose nth term is $-2 + \frac{1}{n}$, and check that

$$f(x_n) = n\left(-4 - \frac{2}{n} + \frac{1}{n^2}\right) \to -\infty \text{ when } n \to \infty.$$

Thus f has no continuous extension to the point $x = -2$.

But on the other hand,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 + 2x + 4}{x+2} = \frac{12}{4} = 3,$$

and so f has a continuous extension

$$\tilde{f} : \mathbb{R} \setminus \{-2\} \to \mathbb{R}; \quad \tilde{f}(x) = \frac{x^2 + 2x + 4}{x+2}.$$

We can see this behaviour in the graph of the function (Fig. 3)

---

19. (Homework 2 question). Assume g is continuous and $g(a) > 0$. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, $|x - a| < \delta$ implies $|g(x) - g(a)| < \varepsilon$. Let $\varepsilon = \frac{g(a)}{2}$. Then for some $\delta > 0$, we have $|g(x) - g(a)| < \frac{g(a)}{2}$. But then $\frac{g(a)}{2} < g(x) < \frac{3g(a)}{2}$, and in particular, $g(x) > \frac{g(a)}{2} > 0$.

Alternate solution using sequences: Suppose, for a contradiction, that there is no $\delta > 0$ such that $g(x) > 0$ for all $x \in (a - \delta, a + \delta)$. Then, in particular, for all $n \in \mathbb{N}$ there exists $x_n \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$ such that $g(x_n) \leq 0$. By the sandwich rule, we have $\lim_{n \to \infty} x_n = a$. So, by continuity of g at a, we have $\lim_{n \to \infty} g(x_n)$ exists and equals $g(a)$. But $g(x_n) \leq 0$ for all $n \in \mathbb{N}$, and so

$$g(a) = \lim_{n \to \infty} g(x_n) \leq 0,$$

which is a contradiction.

---

20. (Homework 2 question). (i)

$$\max\{a, b\} = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } a < b. \end{cases}$$
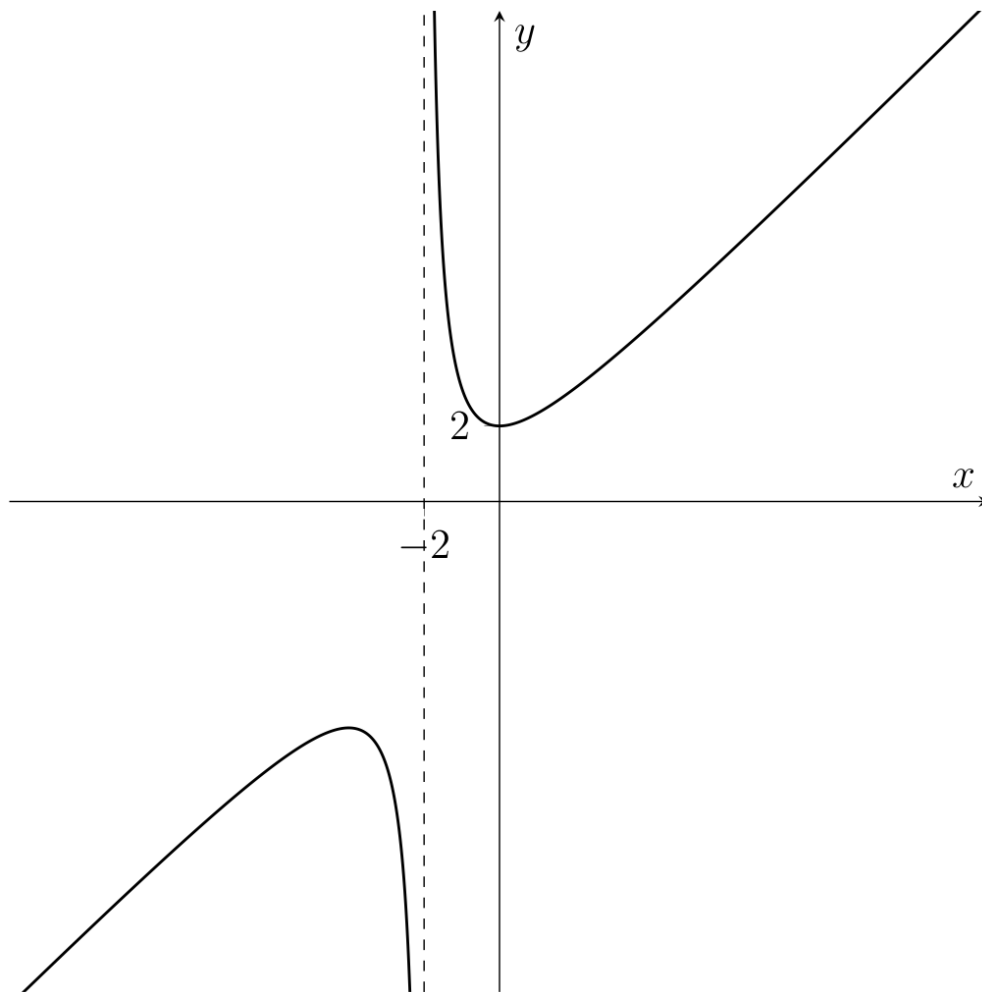
Fig. 3: Graph of the function $\tilde{f} : \mathbb{R} \setminus \{-2\} \to \mathbb{R}$; $f(x) = \frac{x^2+2x+4}{x+2}$ (Problem 18).

On the other hand,

$$\frac{1}{2}(a+b) + \frac{1}{2}|a-b| = \begin{cases} \frac{1}{2}(a+b) + \frac{1}{2}(a-b) & = a & \text{if } a \geq b \\ \frac{1}{2}(a+b) + \frac{1}{2}(b-a) & = b & \text{if } a < b, \end{cases}$$

and the result follows.

Then for all $x \in A \cap B$,

$$\max\{f,g\}(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|,$$

and continuity follows by Theorem 3.2(i) and (iii) and *Problem 14*.

(ii) Check that for all $a, b \in \mathbb{R}$,

$$\min\{a,b\} = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|,$$

and then argue as in (i).

Alternatively, one could derive (ii) from (i) by using $\min\{f,g\} = -\max\{-f,-g\}$.

---

21. (i) $f(0) = f(0 + 0) = f(0) + f(0) = 2f(0)$, hence $f(0) = 0$.

(ii) By (i), $0 = f(0) = f(x + -x) = f(x) + f(-x)$. So $f(-x) = -f(x)$.

(iii) If $a \neq 0$ then any sequence (x_n) which converges to a can be written as $x_n = a + y_n$ where (y_n) converges to zero. If f is continuous at 0, then $\lim_{n \to \infty} f(y_n)$ exists and equals $f(0)$, which is 0 by part (i). So

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(a + y_n) = f(a) + \lim_{n \to \infty} f(y_n) = f(a) + f(0) = f(a).$$

(iv) Firstly, assume $n \in \mathbb{N}$ and use induction. It is true for $n = 1$. Assume it holds for some $n \in \mathbb{N}$, then

$$f(n + 1) = f(n) + f(1) = nk + k = (n+1)k.$$

So by induction, the result holds for all $n \in \mathbb{N}$. Combine this with part (ii), to extend to all $n \in \mathbb{Z}$.

(v) Consider $\frac{p}{q} \in \mathbb{Q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. By (iv),

$$pk = f(p) = f\left(q \cdot \frac{p}{q}\right) = qf\left(\frac{p}{q}\right),$$

and the result follows.

(vi) We have already proved this for rational x, so suppose that x is irrational. Then we can find a sequence $\left(\frac{p_n}{q_n}\right)$ of rational numbers that converges to x (as we did in the solution to Example 3.7). Then using (iii), (v) and algebra of limits, we have

$$f(x) = \lim_{n \to \infty} f\left(\frac{p_n}{q_n}\right) = \lim_{n \to \infty} k\frac{p_n}{q_n} = k \lim_{n \to \infty} \frac{p_n}{q_n} = kx.$$

---

22. Suppose that $x \in \mathbb{Q}$. Then as in the solution to Example 3.7 we can find a sequence (x_n) of irrationals that converges to x and then

$$\lim_{n \to \infty} g(x_n) = 0 \neq g(x),$$

and so g is not continuous at x.

23.

The function $\mathbb{1}_{(a,b)}$ is left continuous at a (but not right continuous), and right continuous at b (but not left continuous). To prove the left continuity at a, let (x_n) be any sequence in $\mathbb{R}$ which converges to a with $x_n < a$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \mathbb{1}_{(a,b)}(x_n) = 0 = \mathbb{1}_{(a,b)}(a)$. On the other hand to see that it is not right continuous at a, let (y_n) be any sequence in $\mathbb{R}$ which converges to a with $a < y_n < b$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \mathbb{1}_{(a,b)}(y_n) = 1 \neq \mathbb{1}_{(a,b)}(a)$. The other assertion is proved similarly.

[Contrast this with $\mathbb{1}_{[a,b]}$, which was discussed in the notes.]

---

24. Since f is continuous on $[a,b]$, so is g. We have $g(a) = f(a) - \gamma < 0$ and $g(b) = f(b) - \gamma > 0$. Hence by the intermediate value theorem (Theorem 3.4), there exists $c \in (a,b)$ with $g(c) = 0$, i.e. $f(c) = \gamma$, as was required.

---

25. (Homework 3 question?)

Something is x. That is, define $g : [a,b] \to \mathbb{R}$ given by $g(x) = f(x) - x$. Then g is continuous (because f is and the function $h(x) = x$ is, and using algebra of limits). Since the range of f is contained in (a,b), we have $f(a) > a$ and $f(b) < b$, and so $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. So we can apply the intermediate value theorem to the function g on $[a,b]$. This says that there exists $c \in (a,b)$ such that $g(c) = 0$, i.e. $f(c) = c$.

For the counter–example, consider $f(x) = x^2$. It is continuous on $(0,1)$ but there is no $c \in (0,1)$ for which $c^2 = c$.

---

26. Define $\gamma = \inf_{x \in [a,b]} f(x)$ and assume that it is not attained, so $\gamma < f(x)$ for all $x \in [a,b]$. Then consider the function $h : [a,b] \to \mathbb{R}$ given by $h(x) = \dfrac{1}{f(x) - \gamma}$. This is continuous, and hence bounded on $[a,b]$. So there exists $K \geq 0$ such that $|h(x)| \leq K$ for all $x \in [a,b]$. By Problem 16(ii), given any $\varepsilon > 0$, there exists $x \in [a,b]$ such that $f(x) < \gamma + \varepsilon$. Now take $\varepsilon = \frac{1}{K}$ to deduce that $h(x) > K$, which yields the required contradiction.

---

27. (i) Let $f : \mathbb{R} \to \mathbb{Z}$ be continuous, and suppose f is not a constant function.

Then there must be $a, b \in \mathbb{R}$ such that $a \neq b$ and $f(a) \neq f(b)$. Without loss of generality, assume $a < b$.

Then f is continuous on $[a,b]$, and since $f(a) \neq f(b)$, f is also non-constant on $[a,b]$. By Corollary 3.3, $f([a,b]) = [m, M]$. This contradicts f having codeomain $\mathbb{Z}$.

It follows that f must be a constant function.

(ii) Argue as in (i), using the fact that between any two rational numbers, we can find an irrational number.

---

28. By algebra of limits, $\frac{1}{f}$ is continuous on $[0,1]$ and so is bounded by Theorem 3.5.

---

29. (Homework 3 question?)

If f is continuous on $[0,1]$, then it is bounded by Theorem 3.5, and so there exists $L \geq 0$ such that $|f(x)| \leq L$ for all $x \in [0,1]$. Hence the range of f is a subset of $[-L, L]$ and cannot be all of $\mathbb{R}$.

---

30. Since (x_n) is bounded, by the Bolzano–Weierstrass theorem it has a convergent subsequence (x_{n_k}). Let $c = \lim_{k\to\infty} x_{n_k}$ and note that $c \in [0,1]$, since $x_{n_k} \in [0,1]$ for all k.

A straight-forward induction argument shows that $f(x_{n_k}) \le r^{n_k - 1} f(x_{n_1})$ for all $k \in \mathbb{N}$. But then, since the range of f is contained in $[0,1]$, we have

$$0 \le f(x_{n_k}) \le r^{n_k - 1}$$

for all $k \in \mathbb{N}$. Since $r < 1$, $\lim_{k\to\infty} r^{n_k - 1} = 0$. By the sandwich rule, $\lim_{k\to\infty} f(x_{n_k}) = 0$ and by continuity of f at c, $f(c) = \lim_{k\to\infty} f(x_{n_k})$, so $f(c) = 0$.

---

31. For all $x > y$,

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-1} + y^{n-1}) > 0.$$

---

32. For all $-\frac{\pi}{2} \le x < y \le \frac{\pi}{2}$,

$$\sin(y) - \sin(x) = 2\sin\left(\frac{y-x}{2}\right)\cos\left(\frac{y+x}{2}\right) > 0,$$

so the function is strictly monotonic increasing on this interval. It is also continuous (stated in notes), and so by Theorem 3.6, it is has a continuous inverse $f^{-1}(x) = \arcsin(x)$ (or $\sin^{-1}(x)$) defined on $[-1, 1]$. Outside the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the function f might fail to be strictly increasing. In fact it is strictly decreasing on each interval of the form $\left((4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2}\right)$, and strictly increasing on each interval of the form $\left((4n-1)\frac{\pi}{2}, (4n+1)\frac{\pi}{2}\right)$, where $n \in \mathbb{Z}_+$.

---

33. We seek a continuous extension of the mapping $x \mapsto \frac{1-x}{1-x^{\frac{m}{n}}}$ to a domain that includes 1. Then, we can use continuity to evaluate the limit as $x \to 1$.

Observe that, if we write $y = x^{\frac{1}{n}}$, then

$$\frac{1-x}{1-x^{\frac{m}{n}}} = \frac{1-y^n}{1-y^m} = \frac{\frac{1}{1-y^m}}{\frac{1}{1-y^n}} = \frac{1 + y + y^2 + \dots + y^{m-1}}{1 + y + y^2 + \dots + y^{n-1}},$$

by the geometric sum formula. Note that the left hand side is not defined at $x = 1$, but the right hand side is.

Let $f : [0, \infty) \to [0, \infty)$; $f(x) = x^{\frac{1}{n}}$, and let

$$g : \mathbb{R} \to \mathbb{R}; \quad g(y) = \frac{1 + y + y^2 + \dots + y^m}{1 + y + y^2 + \dots + y^m}.$$

Both f and g are continuous at every point in their domain (for f, this was proven in Example 3.9). Hence by the composition theorem for continuous functions (Theorem 3.3), the map

$$g \circ f : [0, \infty) \to \mathbb{R}; \quad g \circ f(x) = \frac{1 + x^{\frac{1}{n}} + x^{\frac{2}{n}} + \dots + x^{\frac{m-1}{n}}}{1 + x^{\frac{1}{n}} + x^{\frac{2}{n}} + \dots + x^{\frac{n-1}{n}}}$$

is continuous. Also, by our initial calculation,

$$g \circ f(x) = \frac{1 - x}{1 - x^{\frac{m}{n}}}$$

whenever $x \ge 0$ and $x^{\frac{m}{n}} \ne 1$. Therefore,

$$\lim_{x\to 1} \frac{1-x}{1-x^{\frac{m}{n}}} = \lim_{x\to 1} g \circ f(x) = g \circ f(1) = \frac{1 + 1 + \dots + 1}{1 + 1 + \dots + 1} = \frac{m}{n},$$

by continuity of $g \circ f$.

---

34. We claim that if $a < x < b$, we have $f(a) < f(x) < f(b)$. To see this, note that if $f(a) \geq f(x)$ then either $f(a) = f(x)$, or $f(a) > f(x)$.

If $f(a) = f(x)$, then f cannot be bijective, which is a contradiction.

Suppose that $f(a) > f(x)$. Applying Corollary 3.3 to $f|_{[x,b]}$, the image of $[x, b]$ under f must an interval, $[m, M]$, say. Note that $f(a) \in (m, M)$: indeed, $f(a) > f(x) > M$, and $f(a) < f(b) < m$. Therefore, by the intermediate value theorem (or Corollary 3.1 more specifically), there exists $c \in (x, b)$ such that $f(c) = f(a)$, and this again violates the injectivity of the mapping f. A similar argument can be used to show that we cannot have $f(b) \leq f(x)$.

Finally, applying our initial result to $f|_{[a,y]}$, we see that if $a < x < y < b$, then $f(a) < f(x) < f(y)$. Hence f is strictly monotonic increasing.

---

35. * First observe that $f + g$ is monotonic increasing since both f and g are. Choose $a \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $\delta > 0$ so that if $x > a + \delta$ then

$$|f(x) + g(x) - f(a) - g(a)| = f(x) + g(x) - f(a) - g(a) < \varepsilon,$$

and so

$$|f(x) - f(a)| = f(x) - f(a) < \varepsilon + g(a) - g(x) < \varepsilon,$$

as g is increasing. This proves that g is right–continuous at a. A similar argument (interchanging the roles of a and x) proves that it is left–continuous, and hence continuous at a. Then $g = (f + g) - f$ is the difference of two continuous functions, and hence is itself continuous.

---

36. * (i) If $0 < a < 1$, then $0 < a < \sqrt{a} < 1$. So (x_n) is monotonic increasing and bounded above by 1, and hence converges to some limit $0 \leq l \leq 1$. By continuity of the square root function, $\sqrt{l} = \lim_{n \to \infty} \sqrt{x_n}$. But $\sqrt{x_n} = x_{n+1} \to l$ as $n \to \infty$. By uniqueness of limits, $\sqrt{l} = l$, and so $l = 1$.

If instead $a > 1$, then $a > \sqrt{a} > 1$, and so (x_n) is monotonic decreasing and bounded below by 1. So the sequences converges to a limit, and by an identical argument to above, this limit has to be 1.

(ii) For all $a > 0, n \in \mathbb{N}, f(a) = f\left(a^{\frac{1}{2}}\right) = \cdots = f\left(a^{\frac{1}{2^{n-1}}}\right)$. That is, $f(a) = f(x_n)$ for all $n \in \mathbb{N}$. By continuity,

$$f(a) = \lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right) = f(1).$$

If $a < 0$, then $a^2 > 0$ and so $f(a) = f\left(a^2\right) = f(1)$. Finally, by continuity,

$$f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} f(1) = f(1),$$

and we have shown that $f(x) = f(1)$ for all $x \in \mathbb{R}$.

0.2.4 Differentiation

37. This is Definition 2.2 in the notes:

The function f has limit l at the point a if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$,

$$0 < |x - a| < \delta \text{ implies } |f(x) - l| < \varepsilon.$$

We write $\lim_{x \to a} f(x) = l$.

---

38. (Homework 4 question?)

(i) This is Definition 4.1 in the notes: we say that f is *differentiable* at $a \in A$ if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. Or, equivalently, $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

That is, we fix $a \in A$ and we consider the function g given by $g(h) = \frac{f(a+h) - f(a)}{h}$. Then f is differentiable at a if the limit of g as h goes to zero exists.

(ii) Using the sequential criterion (Theorem 2.1 in the notes):

f is differentiable at a if there exists some number $f'(a) \in \mathbb{R}$ for which, given any sequence (h_n) that converges to 0, with $h_n \neq 0$ for all n, we have

$$\frac{f(a + h_n) - f(a)}{h_n} \to f'(a) \quad \text{as} \quad n \to \infty.$$

(iii) This is asking for the equivalent description of limit of a function given by the $(\varepsilon - \delta)$ criterion (Theorem 2.1). Again it needs to be applied to g as above not f itself:

We say f is differentiable at a if there exists a number $f'(a)$ for which, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |h| < \delta$, then

$$\left| \frac{f(a + h) - f(a)}{h} - f'(a) \right| < \varepsilon.$$

---

39. Let $x \in \mathbb{R} \setminus \{0\}$. Then

$$\frac{f(x + h) - f(x)}{h} = \frac{1}{h} \left(\frac{1}{x + h} - \frac{1}{x} \right)$$

$$= -\frac{1}{x(x + h)} \to -\frac{1}{x^2}, \text{ as } h \to 0.$$

Thus f is differentiable at all $x \in \mathbb{R} \setminus \{0\}$ and $f'(x) = -\frac{1}{x^2}$ for all $x \in \mathbb{R} \setminus \{0\}$.

The extended function (where we define its value at $x = 0$ to be zero) is not continuous at $x = 0$, since $\lim_{x \to 0^+} \frac{1}{x} = \infty$. Therefore it is not differentiable at $x = 0$, by Theorem 4.1.

---

40. As in the hint, let $g(h) = e^{kh} - 1 - kh$. Then

$$\frac{f(x + h) - f(x)}{h} = \frac{1}{h}(e^{k(x+h)} - e^{kx})$$

$$= e^{kx} \frac{1}{h}(e^{kh} - 1) = e^{kx} \left(k + \frac{g(h)}{h} \right) \to ke^{kx}, \text{ as } h \to 0,$$

using the given fact that $\lim_{h \to 0} \frac{g(h)}{h} = 0$. Thus f is differentiable at each $x \in \mathbb{R}$ and $f'(x) = ke^{kx}$.

---

41. (i) Using the product and chain rules for differentiation, and that sin is differentiable with derivative cos, if $x \neq 0$, f is differentiable at x with $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$. (ii) But $\frac{f(x) - f(0)}{x} = \sin\left(\frac{1}{x}\right)$ has no limit as $x \to 0$ (see *Problem 9*), so f is not differentiable at 0.

42. Again using the product and chain rules and standard derivatives, for $x \neq 0$, f is differentiable at x with $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$. At $x = 0$,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0,$$

by *Problem 10*. So f is differentiable at 0 with $f'(0) = 0$. For the second derivative, with $x \neq 0$, we have

$$f''(x) = 2\sin\left(\frac{1}{x}\right) - \frac{2}{x}\cos\left(\frac{1}{x}\right) - \frac{1}{x^2}\sin\left(\frac{1}{x}\right).$$

But f'' doesn't exist at $x = 0$, as in *Problem 41*.

---

43. We give sketches of simple examples of functions as described. There should be some kind of "corner" or "cusp" at the relevant points, so that there is clearly no well-defined gradient there. But the functions are required to be continuous, so there should be no jump or other discontinuity.

(i) *Fig. 4*
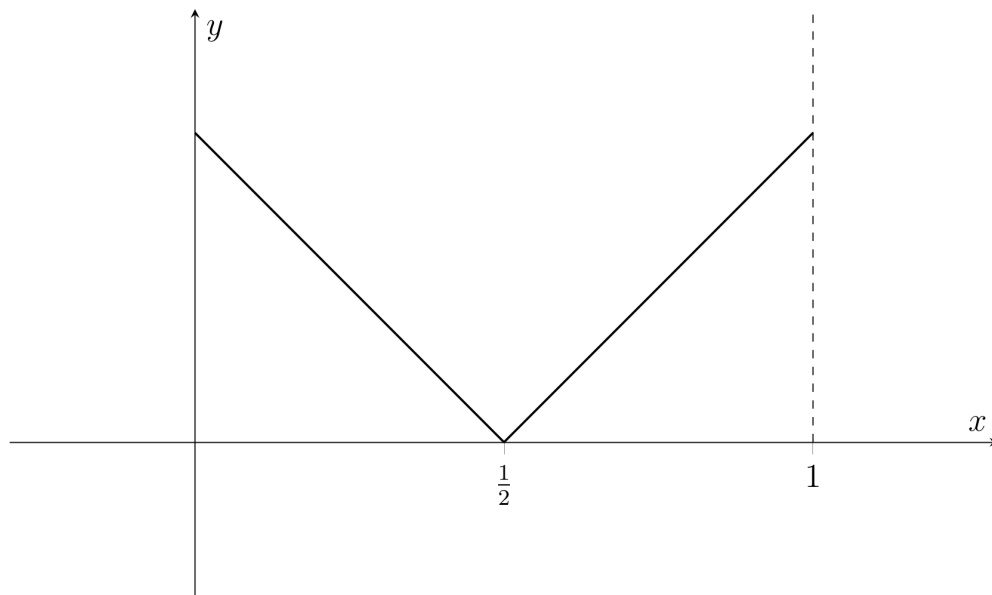
Fig. 4: Graph of a function $f : [0, 1] \to \mathbb{R}$ that is not differentiable at $\frac{1}{2}$, but is everywhere else (Problem 43(i)).

(ii) *Fig. 5*

---

44. It helps to sketch the graph. On the interval $[0, 1]$, we have $[x] = 0$ and so the graph resembles $y = x$. This pattern then repeats, and we have the following graph for $f(x) = x - [x]$ — see *Fig. 6*

The function f is differentiable for all $x \in \mathbb{R} \setminus \mathbb{Z}$. For such points, taking $h > 0$ sufficiently small, we have $[x + h] = [x]$ and so

$$\frac{f(x + h) - f(x)}{h} = \frac{(x + h) - [x + h] - x + [x]}{h} = \frac{h}{h} = 1.$$
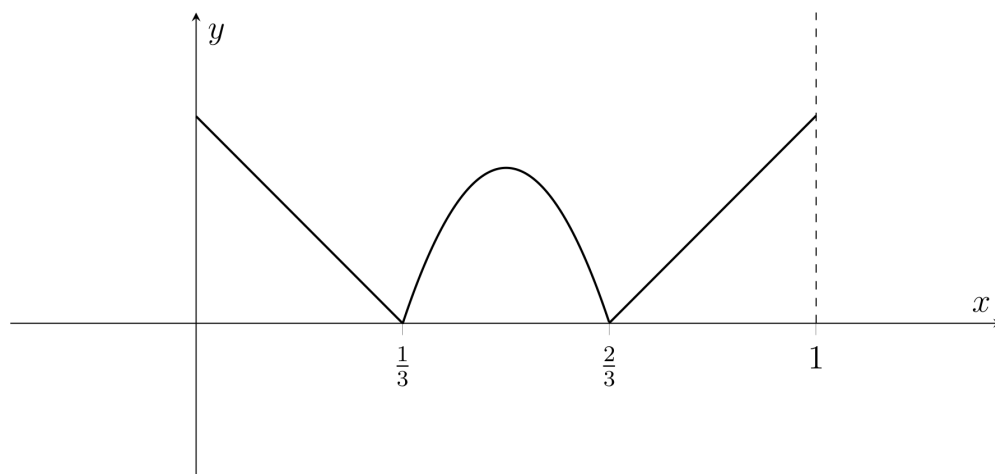
Fig. 5: Graph of a function $f : [0, 1] \to \mathbb{R}$ that is differentiable at all points in its domain apart from $\frac{1}{3}$ and $\frac{2}{3}$ (Problem 43(ii)).
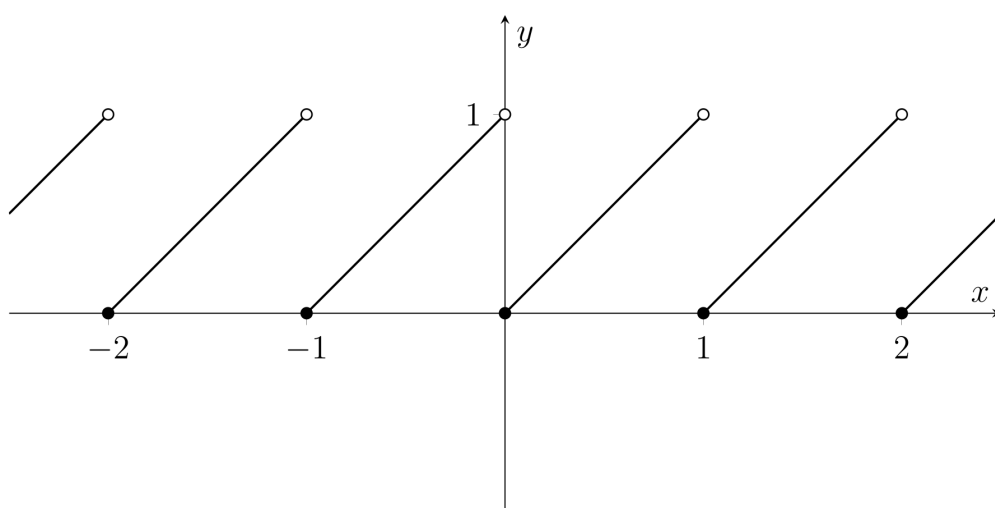

Fig. 6: Graph of the function $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x - [x]$ (Problem 44).

Thus $f'(x) = 1$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$. If $x \in \mathbb{Z}$, then f is not continuous at x (it has a jump discontinuity of -1 there) and so cannot be differentiable there (by Theorem 4.1).

---

45. First observe that if f is differentiable at a, then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a) - f(a-h)}{h}.$$

So

$$\frac{f(a+h) - f(a-h)}{2h} = \frac{1}{2} \left(\frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h} \right)$$

$$\to \frac{1}{2} 2f'(a) = f'(a), \text{ as } h \to 0.$$

On the other hand, by Example 5.2.5, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is not differentiable at $x = 0$. But

$$\lim_{h \to 0^+} \frac{|h| - |-h|}{2h} = 0.$$

---

46. (i) True: f is continuous at zero as $f(0) = 0 = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

(ii) True: $f'(0)$ exists and is zero. To see this compute

$$f'_+(0) = \lim_{h \to 0^+} \frac{h^2}{h} = \lim_{h \to 0^+} h = 0.$$

and

$$f'_-(0) = \lim_{h \to 0^-} \frac{-h^2}{h} = \lim_{h \to 0^-} -h = 0.$$

(iii) True: f' is continuous at zero since $f' : \mathbb{R} \to \mathbb{R}$ is given by

$$f'(x) = \begin{cases} -2x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 2x & \text{if } x > 0 \end{cases},$$

so $f'(0) = \lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x) = 0$.

(iv) False: $f''_+(0) = \lim_{h \to 0^+} \frac{2h - 0}{h} = 2$, $f''_-(0) = \lim_{h \to 0^-} \frac{-2h - 0}{h} = -2$, and so $f''(0)$ does not exist.

---

47. (i) Yes: f is differentiable on $[a, b]$ and hence continuous on $[a, b]$ by Theorem 4.1, so it attains its sup and inf on $[a, b]$ by the extreme value theorem, Theorem 3.5, and these are the maximum and minimum (respectively).

(ii) No: the maximum or minimum could be $f(a) = f(b)$. If $f(a)$ is not the maximum value, then this must occur in (a, b). Similarly for the minimum. If $f(a) = f(b)$ is both the maximum and minimum value, then f is constant, and the value occurs in (a, b).

---

48. Following the hint, define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_n}{n+1} x^n.$$

Then g is a polynomial, so it is differentiable on $[0, 1]$ and using standard derivatives, $g'(x) = f(x)$. Also $g(0) = g(1) = 0$. So by Rolle's theorem, there exists $c \in (0, 1)$ such that $g'(c) = 0$, i.e. $f(c) = 0$.

---

49. (i) Let x, y be such that $a \leq x < y \leq b$. We want to show that $f(x) = f(y)$.

Apply the mean value theorem to the restriction of f to the interval $[x, y]$, to find there exists $d \in (x, y)$ for which

$$f(y) - f(x) = f'(d)(y - x).$$

By assumption, $f'(c) = 0$ for all $c \in (a, b)$, so $f'(d) = 0$.

Thus $f(x) = f(y)$. Since this holds for all $x, y \in [a, b]$, f is constant on $[a, b]$.

(ii) Define $f : \mathbb{R} \to \mathbb{R}$ by $f = h - g$. Then the function f is continuous on $[a, b]$ and differentiable on (a, b), because g and h are, and $f'(x) = h'(x) - g'(x) = 0$ for all $x \in (a, b)$. Thus f satisfies all the conditions of (i). By part (i), f is constant on $[a, b]$. That is, there exists some $k \in \mathbb{R}$, such that $f(x) = k$ for all $x \in [a, b]$. Thus $h(x) = g(x) + k$ for all $x \in [a, b]$.

---

50. By the mean value theorem, $f(b) = f(a) + f'(c)(b - a)$ for some $c \in (a, b)$. So, since $m \leq f'(c) \leq M$, for all $c \in (a, b)$,

$$f(a) + m(b - a) \leq f(b) \leq f(a) + M(b - a).$$

---

51. The polynomial p is of odd degree so it has at least one real root by Corollary 3.2. Also p is differentiable with $p'(x) = 3x^2 + r > 0$, for all $x \in \mathbb{R}$, so p is strictly monotonic increasing on any closed interval $[a, b]$, and hence on the whole of $\mathbb{R}$, by Corollary 4.1. Then by the inverse function theorem (Theorem 3.6), p is invertible and hence injective, and so there is exactly one zero.

---

52. Let $r > 1$ and fix $y \in (0, 1)$. Apply the mean value theorem to the function $f : [y, 1] \to \mathbb{R}$ given by $f(x) = x^r$. Then there exists $c \in (y, 1)$ such that

$$\frac{f(1) - f(y)}{1 - y} = f'(c) = rc^{r-1}.$$

Now, $0 < c < 1$, and $r > 1$, so $c^{r-1} < 1$. Thus

$$1 - y^r = rc^{r-1}(1 - y) < r(1 - y).$$

---

53. Consider the first case. Here we have

$$f''(a) = \lim_{h \to 0} \frac{f'(a + h) - f'(a)}{h} = \lim_{h \to 0} \frac{f'(a + h)}{h} < 0.$$

In particular, $f''_+(a) = \lim_{h \to 0^+} \frac{f'(a+h)}{h} < 0$, and so $f'(a + h) < 0$ for sufficiently small positive h, (say $0 < h < \delta$), in which case, by Corollary 4.1, f is strictly decreasing on $[a, a + \delta]$. We also have $f''_-(a) = \lim_{h \to 0^-} \frac{f'(a+h)}{h} < 0$, and so $f'(a + h) > 0$ for sufficiently small negative h, (say $-\delta_1 < h < 0$), in which case, by Corollary 4.1, f is strictly increasing on $[a - \delta_1, a]$. Then it follows that f has a local minimum at a. The other case is similar.

0.2.5 Sequences and series of functions

54. For $x \in [0, \pi]$ we have $0 \leq \sin x < 1$ except when $x = \frac{\pi}{2}$, where $\sin x = 1$. Thus $f_n(x)$ converges pointwise to the function $f \colon [0, \pi] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \neq \frac{\pi}{2}, \\ 1 & x = \frac{\pi}{2}. \end{cases}$$

Each function f_n is continuous. The function f is not continuous. So the sequence (f_n) does not converge uniformly (otherwise we would have a contradiction to Theorem 5.1 — the uniform limit theorem).

---

55. (i) Observe that (f_n) converges pointwise to the function

$$f(x) = \begin{cases} 0 & x = 0, \\ 1 & x > 0. \end{cases}$$

Each function (f_n) is continuous, but the function f is not continuous at 0, so the convergence cannot be uniform.

(ii) Let $x \in \mathbb{R}$. Then for any $n \geq x$, we have $f_n(x) = 0$, so the sequence (f_n) converges pointwise to the zero function.

On the other hand

$$M_n = \sup\{|f_n(t) - 0| : t \in \mathbb{R}\} = \infty,$$

so certainly (M_n) does not converge to zero, and the sequence (f_n) therefore does not converge uniformly to the zero function.

(iii) For $x > 1$, $\frac{1}{x^n} \to 0$ as $n \to \infty$. Hence for each $x \in (1, \infty)$, $\frac{e^x}{x^n} \to 0$ as $n \to \infty$, and the sequence (f_n) converges pointwise to the zero function.

But $\frac{e^x}{x^n} \to \infty$ as $x \to \infty$, so

$$M_n = \sup\{|f_n(t) - 0| : t \in (1, \infty)\} = \infty.$$

So certainly (M_n) does not converge to zero, and the sequence (f_n) therefore does not converge uniformly to the zero function.

(iv) We know that $e^{-k} \to 0$ as $k \to \infty$, so $f_n(x) \to 0$ as $n \to \infty$ if $x \neq 0$. If $x = 0$, then $f_n(x) = 1$ for all n. So (f_n) converges pointwise to the function

$$f(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0. \end{cases}$$

Each function (f_n) is continuous, but the function f is not continuous at 0, so the convergence cannot be uniform.

(v) The sequence (f_n) clearly does not converge pointwise.

---

56. We consider $g_n \colon D \to \mathbb{R}$, and consider the cases $D = [0, 1]$ and $D = [0, \infty)$, with $g_n(x)$ given by different expressions in each part below.

By Proposition 5.2, (g_n) will converge uniformly to a function g if and only if

$$M_n := \sup\{|g_n(t) - g(t)| : t \in D\} \to 0 \text{ as } n \to \infty.$$

(i) If $g_n(x) = \dfrac{x}{n}$, then for each x, $g_n(x) \to 0$ as $n \to \infty$, so the sequence (g_n) converges pointwise to the zero function.

On the interval $[0, 1]$,

$$M_n = \sup\{|g_n(t) - 0| : t \in [0, 1]\} = \frac{1}{n}$$

Note that $M_n \to 0$ as $n \to \infty$, so on the interval $[0, 1]$, the sequence (g_n) converges uniformly to the zero function. On the other hand, on $[0, \infty)$,

$$M_n = \sup\{|g_n(t) - 0| : t \in [0, \infty)\} = \infty,$$

so the sequence (g_n) does not converge uniformly on $[0, \infty)$.

(ii) Now consider $g_n(x) := \dfrac{x^n}{1 + x^n}$. Observe that

$$g_n(x) = \frac{1}{1 + \frac{1}{x^n}} \to \begin{cases} 0 & x < 1, \\ \frac{1}{2} & x = 1, \\ 1 & x > 1. \end{cases}$$

Thus the pointwise limit function is not continuous on $[0, 1]$ or on $[0, \infty)$, so convergence is not uniform on either interval.

(iii) Let $g_n(x) = \dfrac{x^n}{n + x^n}$.

If $0 \leq x \leq 1$, then $\frac{n}{x^n} \to \infty$ as $n \to \infty$, so $g_n(x) \to 0$ as $n \to \infty$. If $x > 1$, then $\frac{n}{x^n} \to 0$ as $n \to \infty$, so $g_n(x) \to 1$ as $n \to \infty$. So (g_n) converges pointwise to

$$g(x) = \begin{cases} 0 & x \leq 1, \\ 1 & x > 1. \end{cases}$$

On the interval $[0, 1]$,

$$M_n = \sup\{|g_n(t) - 0| : t \in [0, 1]\} = \frac{1}{1 + n} \to 0$$

as $n \to \infty$. So the sequence (g_n) converges uniformly to the zero function on $[0, 1]$.

On the other hand the above pointwise limit g is not continuous on $[0, \infty)$, whereas each function g_n is continuous on $[0, \infty)$, so convergence is not uniform on $[0, \infty)$.

---

57. (i) Observe that, for all $x \in [0, 1]$, $h_n(x) \to 1$ as $n \to \infty$. So (h_n) converges pointwise to the constant function $h : [0, 1] \to \mathbb{R}$, $h(x) = 1$ for all x.

Let

$$M_n = \sup\{|h_n(x) - 1| : x \in [0, 1]\} = 1 - \left(1 - \frac{1}{n}\right)^2 = \frac{1}{n^2} - \frac{2}{n}.$$

So $M_n \to 0$ as $n \to \infty$. So (h_n) converges uniformly to the constant function h.

(ii) We know that $x^n \to 0$ as $n \to \infty$ if $0 \leq x < 1$, and $x^n \to 1$ as $n \to \infty$ if $x = 1$. Hence (h_n) converges pointwise to the function h, where

$$h(x) = \begin{cases} x & 0 \leq x < 1, \\ 0 & x = 1. \end{cases}$$

Each function h_n is continuous, and this limit is not continuous, so convergence is not uniform.

(iii) The sequence $(h_n(x))$ has pointwise limit $\frac{1}{1-x}$ if $0 \leq x < 1$ (geometric series), but it does not converge if $x = 1$. Because it does not converge when $x = 1$, the sequence of functions (h_n) does not converge pointwise on $[0, 1]$ (and so it certainly does not converge uniformly on $[0, 1]$).

---

58. (i) For each t, we have

$$f_n(t) = \frac{n + \cos x}{2n + \sin^2 x} = \frac{1 + \frac{\cos x}{n}}{2 + \frac{\sin^2 x}{n}} \to \frac{1}{2}$$

as $n \to \infty$. So the sequence (f_n) converges pointwise to the constant function $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = \frac{1}{2}$ for all $x \in \mathbb{R}$.

(ii) Note that, for all $x \in \mathbb{R}$,

$$\left| f_n(x) - \frac{1}{2} \right| = \left| \frac{2n + 2\cos x - 2n - \sin^2 x}{2(2n + \sin^2 x)} \right| = \left| \frac{2\cos x - \sin^2 x}{2(2n + \sin^2 x)} \right|$$
$$\leq \frac{2 + 1}{2(2n)} = \frac{3}{4n} \to 0$$

as $n \to \infty$.

It follows that (f_n) converges to f uniformly.

(iii) Using the quotient rule,

$$f_n'(x) = \frac{-\sin(x)(2n + \sin^2(x)) - (n + \cos(x)) \cdot 2\sin(x)\cos(x)}{\left(2n + \sin^2(x)\right)^2}$$
$$= \frac{-\sin(x)\left(2n + \sin^2(x) + 2n\cos(x) + 2\cos^2(x)\right)}{\left(2n + \sin^2(x)\right)^2}$$
$$= \frac{-\sin(x)\left(2n + 1 + 2n\cos(x) + \cos^2(x)\right)}{\left(2n + \sin^2(x)\right)^2}.$$

Since f is a constant function, $f'(x) = 0$ for all $x \in \mathbb{R}$. We show that $f_n'(x) \to 0$ as $n \to \infty$, for each $x \in \mathbb{R}$.

We have:

$$|f_n'(x)| = \frac{|\sin(x)|\left|2n + 1 + 2n\cos(x) + \cos^2(x)\right|}{\left(2n + \sin^2(x)\right)^2}$$
$$\leq \frac{2n + 1 + 2n|\cos(x)| + |\cos(x)|^2}{(2n)^2},$$

(using the triangle inequality and the fact that $\sin^2(x) \geq 0$), so

$$|f_n'(x)| \leq \frac{4n + 2}{4n^2} = \frac{2n + 1}{2n^2} \to 0 \text{ as } n \to \infty.$$

Therefore $f_n'(x) \to 0$ as $n \to \infty$, for all $x \in \mathbb{R}$.

The conditions of Theorem 5.2 will be satisfied if we can prove that $f_n' \to 0$ uniformly.

We have already shown that $|f_n'(x)| \leq \frac{2n+1}{2n^2}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, so most of the work is already done here. We have

$$M_n := \sup\{|f_n(x)| : x \in \mathbb{R}\} \leq \frac{2n + 1}{2n^2} \to 0 \text{ as } n \to \infty,$$

so by Proposition 5.2, $f_n' \to 0$ uniformly. So (f_n) satisfies the conditions of Theorem 5.2.

59. (i) As indicated in the question, we need to check continuity at $x = 0$ and $x = 1$.

Let $0 < x < 1$, so

$$f_n(x) = \frac{x^{\frac{1}{n}} - 1}{\ln x}.$$

We have $x^{\frac{1}{n}} - 1 \to -1 \neq 0$ as $x \to 0$, and $\ln x \to -\infty$ as $x \to 0$. So clearly

$$\frac{x^{\frac{1}{n}} - 1}{\ln x} \to 0 = f_n(0)$$

as $x \to 0$, and f_n is continuous at 0.

On the other hand,

$$x^{\frac{1}{n}} - 1 \to 1 - 1 = 0 \qquad \text{and} \qquad \ln x \to 0$$

as $x \to 1$, so by L'H^opital's rule,

$$\lim_{x \to 1} \frac{x^{\frac{1}{n}} - 1}{\ln x} = \lim_{x \to 1} \frac{\left(\frac{1}{n}\right) x^{\frac{1}{n} - 1}}{\frac{1}{x}} = \frac{1}{n} = f_n(1),$$

so f_n is continuous at 1.

(ii) Since we can assume the function f_n is increasing, we have

$$|f_n(x)| \leq \frac{1}{n}$$

for all $x \in [0, 1]$. But $\frac{1}{n} \to 0$ as $n \to \infty$, so it follows that (f_n) converges uniformly to the zero function.

---

60. Let $s_n(x) = \sum_{i=1}^{n} f_i(x)$. Then (s_n) converges uniformly to f, and by Proposition 5.2, this is equivalent to $\sup\{|s_n(x) - f(x)|\} \to 0$ as $n \to \infty$. So

$$\begin{aligned}
\sup\{|f_n(x)|\} &= \sup\{|s_n(x) - s_{n-1}(x)|\} \\
&\leq \sup\{|s_n(x) - f(x)| + |f(x) - s_{n-1}(x)|\} \\
&\leq \sup\{|s_n(x) - f(x)|\} + \sup\{|f(x) - s_{n-1}(x)|\} \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

So by Proposition 5.2, (f_n) converges uniformly to 0.

---

61. For each $n \in \mathbb{N}$, define $f_n, g_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $g_n(x) = \frac{(-1)^n x^{2n}}{(2n)!}$.

(i) To establish absolute pointwise summability of (f_n), observe that for each **fixed** $x \in \mathbb{R}$,

$$\frac{|f_{n+1}(x)|}{|f_n(x)|} = \frac{x^{2n+3}(2n+1)!}{(2n+3)!x^{2n+1}} = \frac{x^2}{(2n+3)(2n+1)} \to 0,$$

as $n \to \infty$. Therefore by the ratio test, $s(x) = \sum_{n=0}^{\infty} f_n(x)$ is absolutely convergent for each $x \in \mathbb{R}$.

A similar argument applied instead to the seqence (g_n) will show absolute pointwise convergence of the series associate with c (make sure you can write out the details yourself, though!).

(ii) Let $R > 0$, and restrict each f_n defined above to the interval $[-R, R]$. Let $M_n = \frac{R^{2n+1}}{(2n+1)!}$ for each $n \in \mathbb{N}$. Then

$$|f_n(x)| = \frac{x^{2n+1}}{(2n+1)!} \leq M_n$$

for all $x \in [-R, R]$ and $n \in \mathbb{N}$. By the same ratio test argument as above, (M_n) is a summable sequence of real numbers, and so by the Weierstrass M-test the series $s := \sum_{n=0}^{\infty} f_n$ converges uniformly on $[-R, R]$.

Applying the same argument to (g_n) establishes the result for c.

(iii) Observe that f_n and g_n are differentiable everywhere for all $n \in \mathbb{N}$, and for each $x \in \mathbb{R}$,

$$f_n'(x) = \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} = \frac{(-1)^n x^{2n}}{(2n)!} = g_n(x) \qquad \text{for } n = 0, 1, 2, ...$$

and

$$g_n'(x) = \frac{(-1)^n (2n) x^{2n-1}}{(2n)!} = \frac{(-1)^n x^{2n-1}}{(2n-1)!} = -f_{n-1}(x) \qquad \text{for } n = 1, 2, ...$$

Also, $g_0(x) = 1$, so $g_0'(x) = 0$ for all x.

We can now apply Theorem 5.4 to the sequences (f_n) and (g_n) to obtained the desired results for s and c. The uniform summability of (f_n') and (g_n') follows from that of (g_n) and (f_n), respectively. Clearly each term f_n' and g_n' are also continuous, and so by Theorem 5.4, $s = \sum_{n=0}^{\infty} f_n$ and $c = \sum_{n=0}^{\infty} g_n$ are both differentiable on $(-R, R)$, and can be differentiated term by term. But R was arbitrary, and so in fact we have shown that s and c are differentiable everywhere, with derivatives given by

$$s'(x) = \sum_{n=0}^{\infty} g_n(x) = c(x)$$

and

$$c'(x) = \sum_{n=1}^{\infty} (-f_{n-1}(x)) = -s(x),$$

for all $x \in \mathbb{R}$.

(iv) The real crux of this part of this question is that the series' for exp, $s(x)$ and $c(x)$ are absolutely convergent for each $x \in R$. This allows us to rearrange their terms without disturbing any convergence properties — this was the content of Theorem 4.18 in MAS107.

Using rearrangement, we have for each $x \in \mathbb{R}$,

$$\begin{aligned}
\exp(ix) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} &= \sum_{n \text{ even}}^{\infty} \frac{(ix)^n}{n!} + \sum_{n \text{ odd}}^{\infty} \frac{(ix)^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(ix)^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i(-1)^k x^{2k+1}}{(2k+1)!} \\
&= c(x) + is(x).
\end{aligned}$$

---

62. (i) Let $M_n = \frac{1}{n}^2$. Note that

$$\frac{1}{n^2 + x^2} \le \frac{1}{n^2}$$

for all $x \in \mathbb{R}$, and the series $\sum_{n=1}^{\infty} M_n$ converges. So by the Weierstrass M-test, the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$ converges uniformly on $\mathbb{R}$.

It also converges uniformly on $[0, 1]$ by the same argument.

(ii) Observe that, for each $n \in \mathbb{N}$,

$$\sup\left\{\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right| : x \in \mathbb{R}\right\} = \infty$$

so the series cannot converge uniformly on $\mathbb{R}$. (The series converging uniformly means the sequence of partial sums s_n converging uniformly. But then you would have $s_{n+1} - s_n$ converging uniformly to the zero function, which is not the case.)

But let

$$M_n = \sup\left\{\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right| : x \in [0, 1]\right\} = \frac{1}{(2n+1)!}.$$

Then the terms of the series are bounded above by M_n, and $\sum_{n=0}^{\infty} M_n$ converges. So the series converges uniformly on $[0, 1]$.

(iii) $\sin(nx)$ does not converge to 0 as $n \to \infty$ at certain values of x, (for example $x = \frac{\pi}{4} \in [0, 1]$), so the series does not converge, let alone uniformly, either on $[0, 1]$ or $\mathbb{R}$.

(iv)* The series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ does not converge uniformly on $[0, 1]$. So it does not converge uniformly on $\mathbb{R}$ either. To see this, suppose it does converge uniformly. Then the limit $f(x)$ would be a continuous function on $[0, 1]$. Since f is continuous at 0, we have

$$0 = f(0) = \lim_{N \to \infty} f(\frac{\pi}{N}) = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\sin\left(\frac{n\pi}{N}\right)}{n}.$$

On the other hand,

$$\sum_{n=1}^{N} \frac{\sin\left(\frac{n\pi}{N}\right)}{n} \geq \frac{1}{N}\left(\sum_{n=1}^{N} \sin\left(\frac{n\pi}{N}\right)\right)$$

$$= \frac{1}{N}\left(\frac{\sin(\frac{N}{2}\frac{\pi}{N})\sin(\frac{N+1}{2}\frac{\pi}{N})}{\sin(\frac{\pi}{2N})}\right).$$

As $N \to \infty$, this has limit

$$\lim_{N \to \infty} \frac{1}{N}\left(\frac{\sin(\frac{(1+\frac{1}{N})\pi}{2})}{\sin(\frac{\pi}{2N})}\right) = \lim_{x \to 0} \frac{2}{\pi}\frac{x}{\sin(x)} = \frac{2}{\pi},$$

giving a contradiction.

---

63. (i) Let $M_n = a^n$. Then for $x \in [0, a]$, we have $|x^n| \leq M_n$, and since $|a| < 1$, the series $\sum_{n=1}^{\infty} M_n$ converges.

Hence, by the Weierstrass M-test, the series converges uniformly for $x \in [0, a]$.

(ii) Let

$$S_n(x) = 1 + x + x^2 + \cdots + x^n.$$

Let

$$A_n = \sup\{S_n(x) : x \in [0, 1)\} = S_n(1) = 1 + 1 + \cdots + 1 = n.$$

The sequence (A_n) does not converge. It follows that the sequence of partial sums $(S_n(x))$ does not converge uniformly on $[0, 1)$. Hence the series does not converge uniformly on $[0, 1)$.

---

64. Let $f_n : [0, 2\pi] \to \mathbb{R}$ be given by

$$f_n(x) = \frac{\sin(nx)}{n^2}.$$

Then f_n is continuous. Note that

$$|f_n(x)| \leq \frac{1}{n^2}$$

for all $x \in [0, 2\pi]$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Hence, by the Weierstrass M-test, the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly for $x \in [0, 2\pi]$. The uniform limit of a series of continuous functions is continuous, so f is therefore continuous.

65. * Note this question is starred, meaning it is intended more for interest and deviates more from the standard benchmark of question for this module

(i) Entering the series partial sums $\frac{4}{\pi} \sum_{k=0}^{N} \frac{(-1)^k}{2k+1} \cos\left(\frac{(2k+1)\pi x}{2}\right)$ into Desmos for a large value of N yields the following (see Fig. 7).
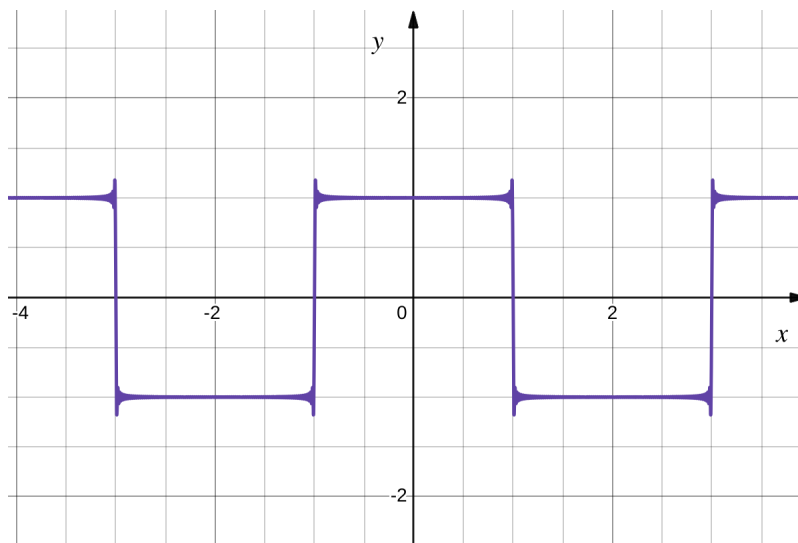

Fig. 7: Desmos graph for $\frac{4}{\pi} \sum_{k=0}^{1} 00 \frac{(-1)^k}{2k+1} \cos\left(\frac{(2k+1)\pi x}{2}\right)$ (Problem 65).

This graph was created with $N = 100$, but any $N > 50$ looks similar.

The graph suggests that the infinite series $\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left(\frac{(2k+1)\pi x}{2}\right)$ is the Fourier series of the following square wave (see Fig. 8).

In Fig. 8, we have been deliberately vague about the value of $f(x)$ when x is an odd integer. In fact, direct calculation using the series shows that $f(x) = 0$ for all odd integer values of x.

In particular,

$$f(x) = \begin{cases} 0 & \text{for } x \text{ an odd integer,} \\ 1 & \text{for } 4k - 1 < x < 4k + 1, \ k \in \mathbb{Z}, \\ -1 & \text{for } 4k + 1 < x < 4k + 3 \ k \in \mathbb{Z} \end{cases}$$
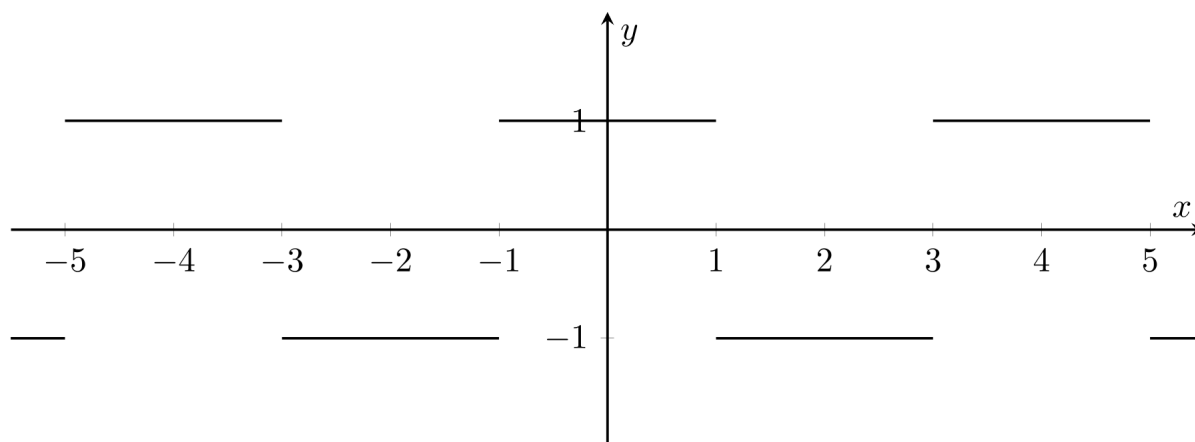
Fig. 8: Square wave $f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left(\frac{(2k+1)\pi x}{2}\right)$ (Problem 65).
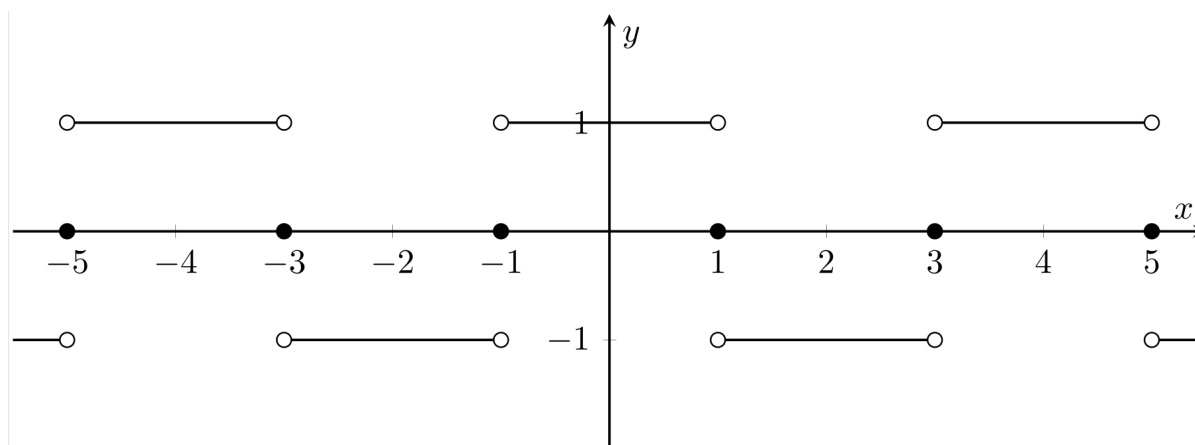

Fig. 9: Improved graph of the square wave $f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left(\frac{(2k+1)\pi x}{2}\right)$ (Problem 65).

Of course, f is a periodic function with period 2.

(ii) We already showed directly in (i) that $f(x) = 0$ whenever x is an odd integer, but this might not have been obvious from the Desmos graph alone.

(iii) f is differentiable at all points in $\mathbb{R}$ except for the odd integers. So $\text{Dom}(f') = \mathbb{R} \setminus \{2k + 1 : k \in \mathbb{Z}\}$. For $x \in \text{Dom}(f')$, $f'(x) = 0$.

(iv) Differentiating the formal series

$$\frac{4}{\pi}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \frac{1}{7}\cos\left(\frac{7\pi x}{2}\right) + \dots\right]$$

term-by-term yields

$$-2\left[\sin\left(\frac{\pi x}{2}\right) - \sin\left(\frac{3\pi x}{2}\right) + \sin\left(\frac{5\pi x}{2}\right) - \sin\left(\frac{7\pi x}{2}\right) + \dots\right].$$

When x is an even integer, this series converges and equals 0. A quick plug into Desmos suggests the series diverges for all other values of x.

Extension: Can you prove this?

> **ⓘ Historical note**
>
> In *Problem 65*, we found that at all points aside from the odd integers, the function
>
> $$f(x) = \frac{4}{\pi}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \frac{1}{7}\cos\left(\frac{7\pi x}{2}\right) + \dots\right]$$
>
> is differentiable with derivative equal to 0. However, differentiating term-by-term yields a series that diverges for all values of x except the even integers.
>
> It was apparent contradictions such as this that created such a stir among the contemporaries of Fourier, and meant his 1807 paper was never accepted for publication. As modern mathematicians, we have access to subtle ideas like uniform and absolute convergence (not to mention a functioning definition of differentiability) that allow us to make sense of the strange behaviour exhibited by this series. Mathematicians of the early 19th century could not reconcile the obvious usefulness of the method Fourier had showcased with everything they knew or believed to be true about functions, series and derivatives.
>
> Following these events was a huge collective effort of the whole mathematical community to properly understand and develop rigorous foundations for calculus — or as it came to be known, real analysis.
>
> **Disclaimer:** I am not a historian. For more on this fascinating subject, I recommend reading Chapter 1 of A Radical Approach to Real Analysis by Bressoud.

0.2.6 Integration

66. Let $f : [1, 4] \to \mathbb{R}$; $f(x) = \frac{1}{x}$, and let $P = \{1, \frac{3}{2}, 2, 4\}$.

(i) Below (Fig. 10) is a sketch of the graph of f, the partition P, and the associated upper and lower Riemann sums.

We have $U(f, P) = \sum_{k=1}^{4} M_k(x_k - x_{k-1})$, where $x_0 = 1$, $x_1 = \frac{3}{2}$, $x_2 = 2$, $x_3 = 4$, and

$$M_1 = \sup\left\{\frac{1}{x} : 1 \le x \le \frac{3}{2}\right\} = 1, \quad M_2 = \sup\left\{\frac{1}{x} : \frac{3}{2} \le x \le 2\right\} = \frac{2}{3},$$

$$M_3 = \sup\left\{\frac{1}{x} : 2 \le x \le 4\right\} = \frac{1}{2}.$$

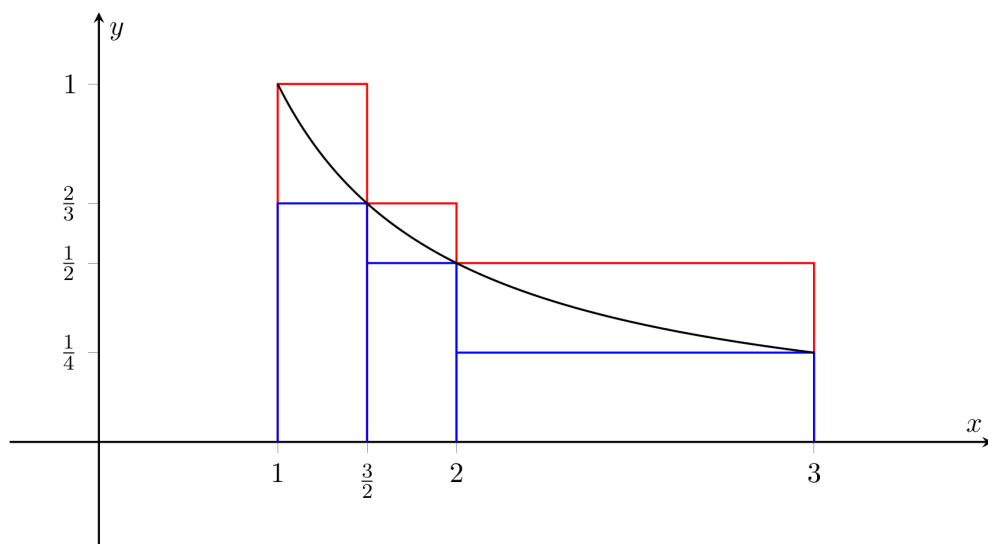Fig. 10: $y = \frac{1}{x}$, with upper and lower sums (Problem 66(i)).

Therefore,

$$U(f,P) = 1 \cdot \left(\frac{3}{2} - 1\right) + \frac{2}{3} \cdot \left(2 - \frac{3}{2}\right) + \frac{1}{2} \cdot (4 - 2)$$

$$= \frac{1}{2} + \frac{1}{3} + 1 = \frac{11}{6}.$$

Similarly, $L(f,P) = \sum_{k=1}^{4} m_k(x_k - x_{k-1})$, where

$$m_1 = \inf\left\{\frac{1}{x} : 1 \le x \le \frac{3}{2}\right\} = \frac{2}{3}, \quad m_2 = \inf\left\{\frac{1}{x} : \frac{3}{2} \le x \le 2\right\} = \frac{1}{2},$$

and

$$m_3 = \inf\left\{\frac{1}{x} : 2 \le x \le 4\right\} = \frac{1}{4}.$$

Hence

$$L(f,P) = \frac{2}{3} \cdot \left(\frac{3}{2} - 1\right) + \frac{1}{2}\left(2 - \frac{3}{2}\right) + \frac{1}{4}(4 - 2)$$

$$= \frac{1}{3} + \frac{1}{4} + \frac{1}{2} = \frac{13}{12},$$

and so

$$U(f,P) - L(f,P) = \frac{11}{6} - \frac{13}{12} = \frac{9}{12} = \frac{3}{4}.$$

(ii) Adding the point P has the following effect (Fig. 11)

We now have $x_0 = 1$, $x_1 = \frac{3}{2}$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 4$. M_1, M_2, m_1 and m_2 are not affected by the additional point, while

$$M_3 = \sup\left\{\frac{1}{x} : 2 \le x \le 3\right\} = \frac{1}{2}, \quad M_4 = \sup\left\{\frac{1}{x} : 3 \le x \le 4\right\} = \frac{1}{3},$$

$$m_3 = \inf\left\{\frac{1}{x} : 2 \le x \le 3\right\} = \frac{1}{3}, \quad \text{and} \quad m_4 = \inf\left\{\frac{1}{x} : 3 \le x \le 4\right\} = \frac{1}{4}.$$
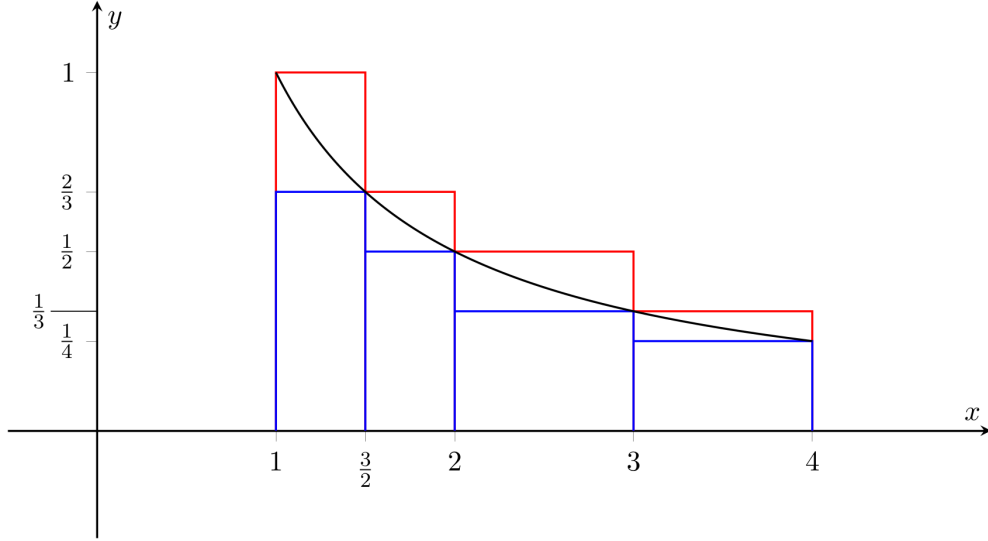
Fig. 11: Upper and lower sums with additional point 3 (Problem 66(ii)).

Therefore

$$U(f,P) = 1\left(\frac{1}{2}\right) + \frac{2}{3}\left(\frac{1}{2}\right) + \frac{1}{2}(1) + \frac{1}{3}(1) = \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{5}{3},$$

$$L(f,P) = \frac{2}{3}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{3}(1) + \frac{1}{4}(1) = \frac{1}{3} + \frac{1}{4} + \frac{1}{3} + \frac{1}{4} = \frac{7}{6},$$

and

$$U(f,P) - L(f,P) = \frac{5}{3} - \frac{7}{6} = \frac{1}{2}.$$

So the size of $U(f,P) - L(f,P)$ has decreased after adding the point 3 to P.

(iii) Mimicking the proof of Theorem 6.1, let $x_k = 1 + \frac{3}{n}k$ for $k = 0, \dots, n$. Then, $P = \{x_0, \dots, x_n\}$ partitions $[1,3]$ uniformly into n intervals, each of width $\frac{3}{n}$. Since $\frac{1}{x}$ is decreasing,

$$M_k = \sup\left\{\frac{1}{x} : x \in [x_{k-1}, x_k]\right\} = \frac{1}{x_{k-1}}$$

and

$$m_k = \inf\left\{\frac{1}{x} : x \in [x_{k-1}, x_k]\right\} = \frac{1}{x_k}$$

for each k. So,

$$U(f,P) - L(f,P) = \sum_{k=1}^{n}\left(\frac{1}{x_k} - \frac{1}{x_{k-1}}\right)\frac{3}{n}$$

$$= \frac{3}{n}\left(\frac{1}{x_n} - \frac{1}{x_0}\right) = \frac{3}{n}\left(1 - \frac{1}{4}\right) = \frac{9}{4n}.$$

We have been asked to find P so that $U(f,P) - L(f,P) < \frac{2}{5}$, so we choose n so that $\frac{9}{4n} < \frac{2}{5}$. That is, $n > \frac{45}{8}$. In fact, $n = 6$ will do.

So $x_k = 1 + \frac{1}{2}k$ for $k = 0, \dots, 6$, and $P = \left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$.
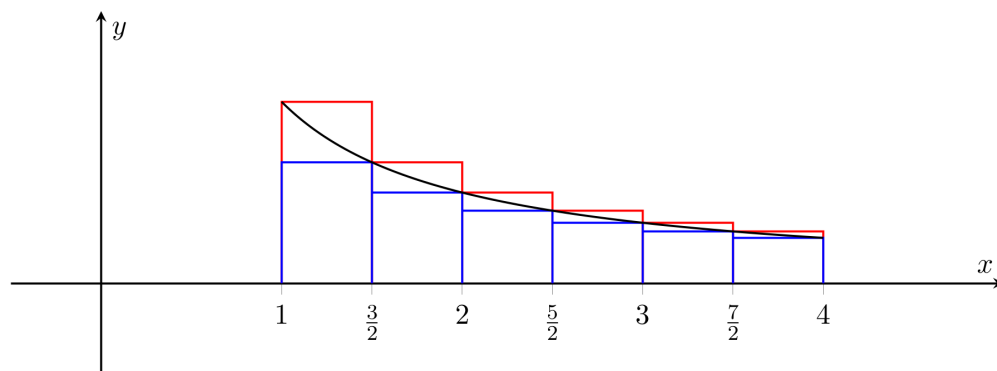
Fig. 12: Q66: A partition for which $U(f, P) - L(f, P) < \frac{2}{5}$ (Problem 66(iii)).

67. Let $g : [0, 1] \to \mathbb{R}$; $g(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 2 & \text{for } x = 1 \end{cases}$.

(i) Let $P = \{x_0, x_1, \ldots x_n\}$ be any partition $[0, 1]$, with $0 = x_0 < x_1 < \ldots < x_n = 1$. Then for $k = 1, \ldots n$,

$$m_k = \inf\{g(x) : x \in [x_{k-1}, x_k\} = 1.$$

Therefore

$$L(g, P) = \sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0 = 1.$$

(ii) Since g is constant on $[0, 1)$, we may as well take a partition consisting of 3 points — the end points and a single point in the middle.

Let $\delta > 0$ and consider the partition $P = \{0, 1 - \delta, 1\}$ of $[0, 1]$ (see Fig. 13).

Making δ smaller should reduce the size of $U(g, P)$. We have

$$M_1 = \sup\{g(x) : x \in [0, 1 - \delta]\} = 1, \quad \text{and} \quad M_2 = \sup\{g(x) : x \in [1 - \delta, 1]\} = 2,$$

so

$$U(g, P) = 1 \cdot (1 - \delta) + 2 \cdot \delta = 1 + \delta.$$

Taking $\delta = \frac{1}{10}$ will give $U(g, P) = 1 + \frac{1}{10}$. So our choice of partition should be $P = \{0, \frac{9}{10}, 1\}$.

(iii) By (ii), taking $P = \{0, 1 - \frac{\varepsilon}{2}, 1\}$ will give $U(g, P) = 1 + \frac{\varepsilon}{2} < 1 + \varepsilon$.

By (i), $L(g, P) = 1$, and so we've found a partition P so that $U(g, P) - L(g, P) < \varepsilon$. This proves g is Riemann integrable (by Proposition 6.1).

68. For $\mathbb{1}_{[a,b]}$, this is just a special case of Example 6.2, and there is nothing to show.

Note also that for any partition P of $[a, b]$,

$$U(\mathbb{1}_{(a,b]}), P) = U(\mathbb{1}_{(a,b))}, P) = U(\mathbb{1}_{(a,b])}, P) = b - a$$

This is because each of these functions have supremum 1 on any subinterval of $[a, b]$.
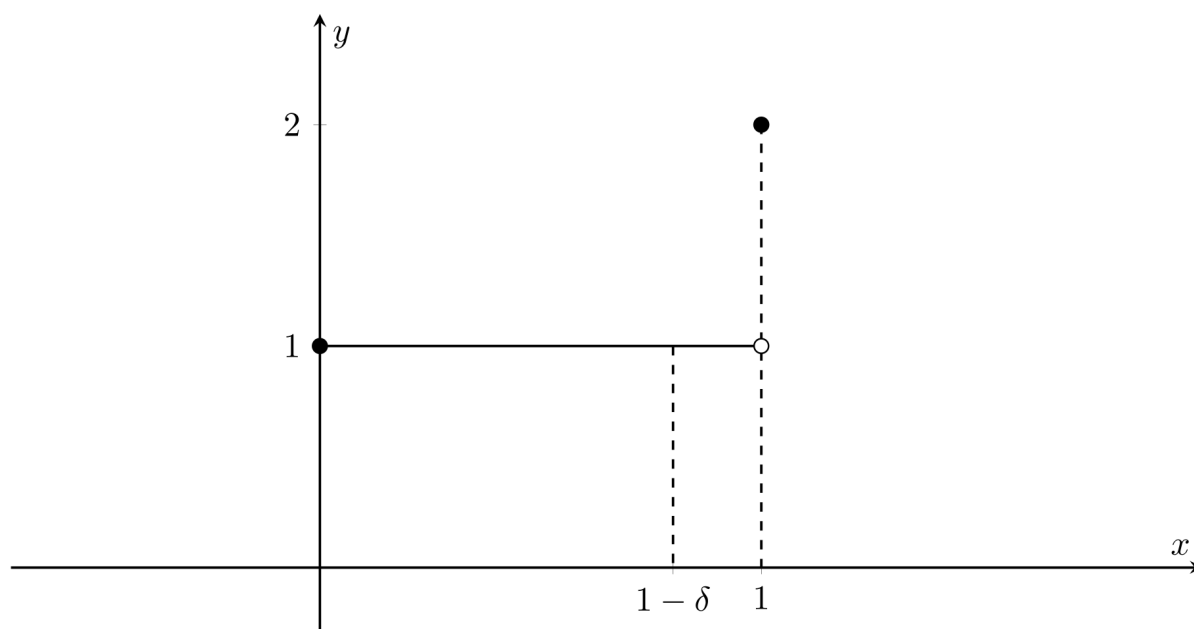
Fig. 13: Graph of g with partition $P = \{0, 1 - \delta, 1\}$ (Problem 67).
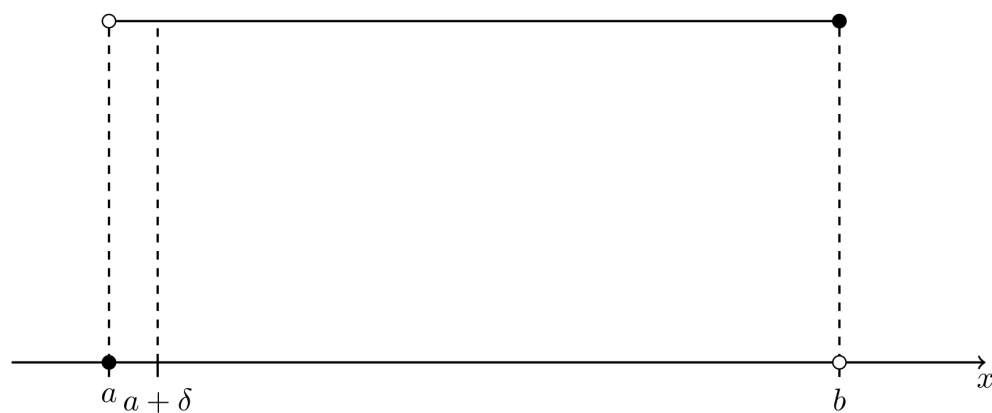

Fig. 14: Graph of $\mathbb{1}_{(a,b]}$, with partition $P = \{a, a + \delta, b\}$ (Problem 68).

For the lower integrals, let's first consider $\mathbb{1}_{(a,b]}$. Consider the partition $P = \{a, a + \delta, b\}$ (see Fig. 14).

Then $U(\mathbb{1}_{(a,b]}, P) = b - a$ as before, while

$$L(\mathbb{1}_{(a,b]}, P) = 0 \cdot (\varepsilon) + 1 \cdot (b - a - \delta) = b - a - \delta.$$

Therefore, $U(\mathbb{1}_{(a,b]}, P) - L(\mathbb{1}_{(a,b]}, P) = \delta$ for this partition P.

We can now use Proposition 6.1: given $\varepsilon > 0$, the partition $P_\varepsilon := \{a, a + \frac{\varepsilon}{2}, b\}$ gives

$$U(\mathbb{1}_{(a,b]}, P_\varepsilon) - L(\mathbb{1}_{(a,b]}, P_\varepsilon) = \frac{\varepsilon}{2} < \varepsilon,$$

and so $\mathbb{1}_{(a,b]}$ is Riemann integrable.

To evaluate $\int_a^b \mathbb{1}_{(a,b]}(x)dx$, note that by our work above,

$$L(\mathbb{1}_{(a,b]}, P_\varepsilon) = b - a \leq \int_a^b \mathbb{1}_{(a,b]}(x)dx < b - a + \varepsilon = U(\mathbb{1}_{(a,b]}, P_\varepsilon).$$

Since this is true for all ε, the only option is for $\int_a^b \mathbb{1}_{(a,b])}(x)dx = b - a$.

The proof that $\mathbb{1}_{[a,b)}$ is integrable with $\int_a^b \mathbb{1}_{[a,b)}(x)dx = b - a$ is nearly identical — just consider the partitions of the form $P = \{a, b - \delta, b\}$ instead.

For $\mathbb{1}_{(a,b)}$, let $\delta > 0$, and consider $P = \{a, a + \delta, b - \delta, b\}$. As before, $U(\mathbb{1}_{(a,b)}, P) = b - a$, and direct calculation gives

$$L(\mathbb{1}_{(a,b)}, P) = 0 \cdot \delta + 1 \cdot (b - \delta - a - \delta) + 0 \cdot \delta = b - a - 2\delta.$$

Then, given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{4}$. We get

$$U(\mathbb{1}_{(a,b)}, P) - L(\mathbb{1}_{(a,b)}, P) = b - a - \left(b - a - 2 \cdot \frac{\varepsilon}{4}\right) = \frac{\varepsilon}{2} < \varepsilon,$$

and so $\mathbb{1}_{(a,b)}$ is integrable, by Proposition 6.1. Moreover,

$$b - a \leq \int_a^b \mathbb{1}_{(a,b)}(x)dx < b - a + \varepsilon$$

and since this is true for all $\varepsilon > 0$, we must have $\int_a^b \mathbb{1}_{(a,b)}(x)dx = b - a$.

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69. (i) By results from the course (specifically, Propositions 6.2 and Proposition 6.3(ii)), r and s are both linear combinations of integrable functions, and hence are integrable with

$$\int_0^3 r(x)dx = \int_0^1 1dx + \int_1^2 edx + \int_2^3 e^4 dx = 1 + e + e^4,$$

and

$$\int_0^3 s(t)\,dt = \int_0^1 edt + \int_1^2 e^4 dt + \int_2^3 e^9 dt = e + e^4 + e^9.$$

(ii) Continuous functions are integrable, by another result from the course (specifically, by Theorem 6.2). So $f \colon [0,3] \to \mathbb{R}$ defined by $f(x) = e^{x^2}$ is integrable since it is a continuous function.

[Alternative answer: by the fact f is monotonic increasing (c.f. Theorem 6.1).]

(iii) Note that $r(t) \leq f(t) \leq s(t)$ for all t, and so by Proposition 6.3(i),

$$1 + e + e^4 = \int_0^3 r(t)\,dt \leq \int_0^3 f(t)\,dt \leq \int_0^3 s(t)\,dt = e + e^4 + e^9.$$

---

70. Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions.

(i) For any subset $A \subset [a, b]$,

$$\sup\{f(x) + g(x) : x \in A\} \leq \sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\},$$

and similarly

$$\inf\{f(x) + g(x) : x \in A\} \geq \inf\{f(x) : x \in A\} + \inf\{g(x) : x \in A\}.$$

It is then immediate from the definition of the upper and lower sums that

$$U(f + g, P) \leq U(f, P) + U(g, P)$$

and

$$L(f + g, P) \geq L(f, P) + L(g, P).$$

For the example where this is strict, a good strategy is to look for functions with features that "cancel out" in some sense. For example, you could take

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ -1 & \text{if } 1 \leq x \leq 2 \end{cases}$$

and

$$g(x) = \begin{cases} -1 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 \leq x \leq 2. \end{cases}$$

Then $f + g$ is the zero function, so $L(f + g, P) = U(f + g, P) = 0$. On the other hand

$$U(f, P) + U(g, P) = 1 + 1 = 2,$$

and

$$L(f, P) + L(g, P) = -1 - 1 = -2.$$

(ii) Let P_1 and P_2 be partitions of $[a, b]$. Then

$$\begin{aligned} U(f + g) &\leq U(f + g, P_1 \cup P_2) && \text{(by definition of } U(f + g)) \\ &\leq U(f, P_1 \cup P_2) + U(g, P_1 \cup P_2) && \text{(by part (i))} \\ &\leq U(f, P_1) + U(g, P_2), \end{aligned}$$

by Lemma 6.1, since $P_1 \cup P_2$ is a refinement of both partitions P_1 and P_2.

The proof of the lower sum statement is similar:

$$\begin{aligned} L(f + g) &\leq L(f + g, P_1 \cup P_2) && \text{(definition of } L(f + g)) \\ &\geq L(f, P_1 \cup P_2) + L(g, P_1 \cup P_2) && \text{(part (i))} \\ &\leq L(f, P_1) + L(g, P_2) && \text{(Lemma 6.1).} \end{aligned}$$

(iii) We just proved

$$U(f + g, P_1 \cup P_2) \leq U(f, P_1) + U(g, P_2),$$

for all partitions P_1 and P_2 of $[a, b]$.

Taking inf over P_1,

$$U(f + g) \leq U(f) + U(g, P_2),$$

and then taking inf over P_2,

$$U(f + g) \leq U(f) + U(g).$$

Since f and g are integrable, $\int_a^b f(x)dx = U(f)$ and $\int_a^b g(x)dx = U(g)$. This shows that

$$U(f + g) \leq \int_a^b f(x)dx + \int_a^b g(x)dx.$$

A similar argument shows that

$$L(f + g) \geq \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Hence

$$\int_a^b f(x)dx + \int_a^b g(x)dx \leq L(f + g) \leq U(f + g) \leq \int_a^b f(x)dx + \int_a^b g(x)dx.$$

The only option is

$$L(f + g) = U(f + g) = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

That is, $f + g$ is integrable and

$$\int_a^b (f(t) + g(t))dt = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

---

71. Let P be the partition $a = x_0 < x_1 < ... < x_n = b$. If $k \geq 0$, then

$$\sup_{x \in [x_{k-1}, x_k]} kf(x) = k \sup_{x \in [x_{k-1}, x_k]} f(x)$$

and

$$\inf_{x \in [x_{k-1}, x_k]} kf(x) = k \inf_{x \in [x_{k-1}, x_k]} f(x).$$

Therefore $U(kf, P) = kU(f, P)$ and $L(kf, P) = kL(f, P)$, and taking infima/suprema, $U(kf) = kU(f)$ and $L(kf) = kL(f)$.

Since f is integrable, $U(f) = L(f) = \int_a^b f(x)dx$, and hence

$$U(kf) = kU(f) = kL(f) = L(kf) = k \int_a^b f(x)dx.$$

It follows that kf is integrable, with $\int_a^b kf(x)dx = k\int_a^b f(x)dx.$

To extend this result to include negative k, it is sufficient to prove the $k = -1$ case. Note that

$$\sup_{x\in[x_{k-1},x_k]}(-f(x)) = -\inf_{x\in[x_{k-1},x_k]}f(x),$$

and

$$\inf_{x\in[x_{k-1},x_k]}(-f(x)) = -\sup_{x\in[x_{k-1},x_k]}f(x).$$

Therefore, $U(-f, P) = -L(f, P)$ and $L(-f, P) = -U(f, P)$, and so

$$\begin{aligned}
U(-f) &= \inf\{U(-f, P) : \text{ partitions } P \text{ of } [a, b]\} \\
&= \inf\{-L(f, P) : \text{ partitions } P \text{ of } [a, b]\} \\
&= -\sup\{L(f, P) : \text{ partitions } P \text{ of } [a, b]\} = -L(f).
\end{aligned}$$

Similarly,

$$\begin{aligned}
L(-f) &= \sup\{L(-f, P) : \text{ partitions } P \text{ of } [a, b]\} \\
&= \sup\{-U(f, P) : \text{ partitions } P \text{ of } [a, b]\} \\
&= -\inf\{U(f, P) : \text{ partitions } P \text{ of } [a, b]\} = -U(f).
\end{aligned}$$

Therefore $U(-f) = -L(f) = -U(f) = L(-f)$, so $-f$ is integrable, and

$$\int_a^b (-f(x))dx = -\int_a^b f(x)dx.$$

---

72. We have: $M = \sup\{f(x) : x \in A\}$, $m = \inf\{f(x) : x \in A\}$, and $M' = \sup\{|f(x)| : x \in A\}$, and $m' = \inf\{|f(x)| : x \in A\}$.

(i) Case 1: $m \geq 0$. Then f is a non-negative function, so $M = M'$, $m = m'$, and $M - m = M' - m'$.

Case 2: $m \leq M \leq 0$. Then $M' = -m$, $m' = -M$, and so again $M' - m' = M - m$.

Case 3: $m < 0 < M$. Then $M' = \max\{M, -m\} \leq M + (-m)$, since both M and $-m$ are positive numbers. Then,

$$M' - m' \leq M' \leq M - m,$$

since $m' \geq 0$.

(ii) Let $f : [a, b] \to \mathbb{R}$ be integrable, and let P be the partition $a = x_0 < x_1 < ... < x_n = b$. For $k = 1, ..., n$. let

$$M_k = \sup_{x\in[x_{k-1},x_k]}f(x), \ m_k = \inf x \in [x_{k-1}, x_k]f(x),$$

$$M_k' = \sup_{x\in[x_{k-1},x_k]}|f(x)|, \ \text{ and } \ m_k' = \inf x \in [x_{k-1}, x_k]|f(x)|.$$

Then by part (i), $M_k - m_k \geq M_k' - m_k'$ for each k. Therefore,

$$\begin{aligned}
U(|f|, P) - L(|f|, P) &= \sum_{k=1}^n (M_k' - m_k')(x_k - x_{k-1}) \\
&\leq \sum_{k=1}^n (M_k - m_k)(x_k - x_{k-1}) = U(f, P) - L(f, P).
\end{aligned}$$

(iii) Since f is integrable, for all $\varepsilon > 0$, we can choose the a partition P so that $U(f, P) - L(f, P) < \varepsilon$. Using the result of part (ii). it follows that $U(|f|, P) - L(|f|, P) < \varepsilon$ for this partition, and so $|f|$ is integrable.

Since $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$, Proposition 6.3(iii) implies that

$$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx.$$

That is,

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

---

73. We have

$$\int_{a(x)}^{b(x)} f(t)dt = \int_0^{b(x)} f(t)dt - \int_0^{a(x)} f(t) \; dt.$$

Write $F : \mathbb{R} \to \mathbb{R}$; $F(x) = \int_0^x f(t)dt$. Then by the fundamental theorem of calculus (Theorem 6.3), F is differentiable, and $F'(x) = f(x)$ for all $x \in \mathbb{R}$. Also,

$$(F \circ b)(x) = F(b(x)) = \int_0^{b(x)} f(t)dt = F(b(x)),$$

and b is differentiable. Therefore, by the chain rule (Theorem 4.4), $F \circ b$ is also differentiable, with

$$(F \circ b)'(x) = \frac{d}{dx} \int_0^{b(x)} f(t)dt = F'(b(x))b'(x) = f(b(x))b'(x),$$

for all $x \in \mathbb{R}$. Similarly,

$$\frac{d}{dx} \int_0^{a(x)} f(y)dt = f(a(x))a'(x).$$

Subtracting these

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t)dt = f(b(x))b'(x) - f(a(x))a'(x),$$

as required.

---

74. (i) It follows immediately from the fundamental theorem of calculus that l is differentiable, with $l'(x) = \frac{1}{x}$.

(ii) By Proposition 6.3 in the lecture notes,

$$l(xy) = \int_1^{xy} \frac{1}{t} \, dt = \int_1^x \frac{1}{t} \, dt + \int_x^{xy} \frac{1}{t} \, dt = l(y) + \int_x^{xy} \frac{1}{t}dt.$$

In the second integral, set $a(x) = xy$, $b(x) = x$ and use *Problem 72*. We have $a'(x) = y$ and $b'(x) = 1$, so

$$\frac{d}{dx} \int_x^{xy} \frac{1}{t}dt = \frac{1}{b(x)}b'(x) - \frac{1}{a(x)}a'(x) = \frac{1}{x} - \frac{y}{xy} = 0.$$

Therefore $\int_x^{xy} \frac{1}{t}\,dt$ is constant as a function of x. In particular, it coincides with its value when $x = 1$:

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{t}\,dt = l(y),$$

and $l(xy) = l(x) + l(y)$ follows.

75.* Define $G \colon [a, b] \to \mathbb{R}$ by

$$G(x) = \int_a^x f(t)\,dt$$

Then G is a continuous function. This was mentioned in the lectures. To see why, let $h > 0$. Since the function f is Riemann integrable, it is bounded, so we have $M \in \mathbb{R}$ such that $|f(t)| \le M$ for all $t \in [a, b]$. Then

$$|G(x + h) - G(x)| = \left| \int_x^{x+h} f(t)\,dt \right| \le \int_x^{x+h} |f(t)|\,dt \le Mh.$$

Similarly $|G(x - h) - G(x)| \le Mh$.

So $G(x + h) \to G(x)$ as $h \to 0$, and G is continuous.

Thus we can define a continuous function $F \colon [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt - \int_x^b f(t)\,dt.$$

Observe that

$$F(a) = -\int_a^b f(t)\,dt, \qquad F(b) = \int_a^b f(t)\,dt = -F(a).$$

If $\int_a^b f(t)\,dt = 0$, then we can take $x = a$ and we have the required result. Otherwise, either $F(a) < 0$ and $F(b) > 0$ or $F(a) > 0$ and $F(b) < 0$.

Either way, by the intermediate value theorem, we have $x \in [a, b]$ such that $F(x) = 0$, that is to say

$$\int_a^x f(t)\,dt = \int_x^b f(t)\,dt.$$

---

76. (i) Let $\mu = \frac{1}{b-a} \int_a^b f(x)\,dx$. Then $\int_a^b f(x)\,dx = (b - a)\mu$, and we need only show that $\mu \in [m, M]$.

Consider the step functions $m\mathbf{1}_{[a,b]}$ and $M\mathbf{1}_{[a,b]}$ with $m\mathbf{1}_{[a,b]}(x) \le f(x) \le M\mathbf{1}_{[a,b]}(x)$ for all $x \in [a, b]$. Thus we have

$$m(b - a) = \int_a^b m\mathbf{1}_{[a,b]}(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\mathbf{1}_{[a,b]}(x)\,dx = M(b - a).$$

But $\int_a^b f(x)\,dx = (b - a)\mu$, and so $\mu \in [m, M]$.

(ii) We are now given that $f : [a, b] \to \mathbb{R}$ is continuous. By Corollary 3.3, $f([a, b]) = [m, M]$, where

$$m = \inf\{f(x) : x \in [a, b]\} \quad \text{and} \quad M = \sup\{f(x) : x \in [a, b]\}.$$

In particular, $m < f(x) < M$ for all $x \in [a, b]$, and so by part (i),

$$\int_a^b f(x)\,dx = (b-a)\mu,$$

for some $\mu \in [m, M]$. As $\mu \in [m, M] = f([a, b])$ and f is continuous, there is some $\xi \in [a, b]$ such that $f(\xi) = \mu$. Hence

$$\int_a^b f(x)\,dx = (b-a)f(\xi).$$

---

77. Let $f : [a, b] \to \mathbb{R}$ be continuous, and $m = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$ as before. Then $m \le f(x) \le M$ for all $x \in [a, b]$. As $g(x) \ge 0$, we have

$$mg(x) \le f(x)g(x) \le Mg(x)$$

for all $x \in [a, b]$. By Proposition 6.3(i),

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M\int_a^b g(x)\,dx.$$

Let

$$I = \int_a^b g(x)\,dx.$$

Either $I = 0$ or $I > 0$. If $I = 0$, then the above tells us that

$$\int_a^b f(x)g(x)\,dx = 0$$

and any $\xi \in [a, b]$ satisfies

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

If $I > 0$, then

$$m \le \frac{1}{I}\int_a^b f(x)g(x)\,dx \le M.$$

By Corollary 3.3, $f([a, b]) = [m, M]$, and so there is $\xi \in [m, M]$ such that

$$f(\xi) = \frac{1}{I}\int_a^b f(x)g(x)\,dx$$

or in other words

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

We do need the assumption $g(x) \ge 0$. To see this, consider the function $g(x) = x$ for $x \in [-1, 1]$. Then

$$\int_{-1}^1 g(x)\,dx = 0,$$

so the result would say that for any continuous function $f : [-1, 1] \to \mathbb{R}$ we have

$$\int_{-1}^1 xf(x)\,dx = 0.$$

But this is clearly not the case, taking for example $f(x) = x$.