Differentiation 1

Differentiation is a method for calculating the gradient of a curve at a given point. On this sheet, we recap the definition of the derivative of a function in one variable and practice the method of differentiation from first principles.

We will also revise the standard formula for the derivative of a power, and practice using it to differentiate polynomials. The problem set at the end of this resource includes some contextual questions to give a taste of how differentiation can be used to solve problems in the real world.

First Principles Differentiation

Definition 1: The derivative of a function f at a point $P = (a, f(a))$ on the curve $y = f(x)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \qquad (1)$$

The rationale for this formula is the following: suppose Q is a second point on the curve, very close to P.

Provided P and Q are very close together, the line L passing through both of these points will give a good approximation for the tangent line at P.

Suppose that $P = (a, f(a))$ and $Q = (a+h, f(a+h))$, where h is a very small number.

The gradient of the straight line through P and Q is given by the standard formula

$$\text{Gradient of } L = \frac{\text{Change in } y}{\text{Change in } x} = \frac{f(a+h) - f(a)}{h}$$

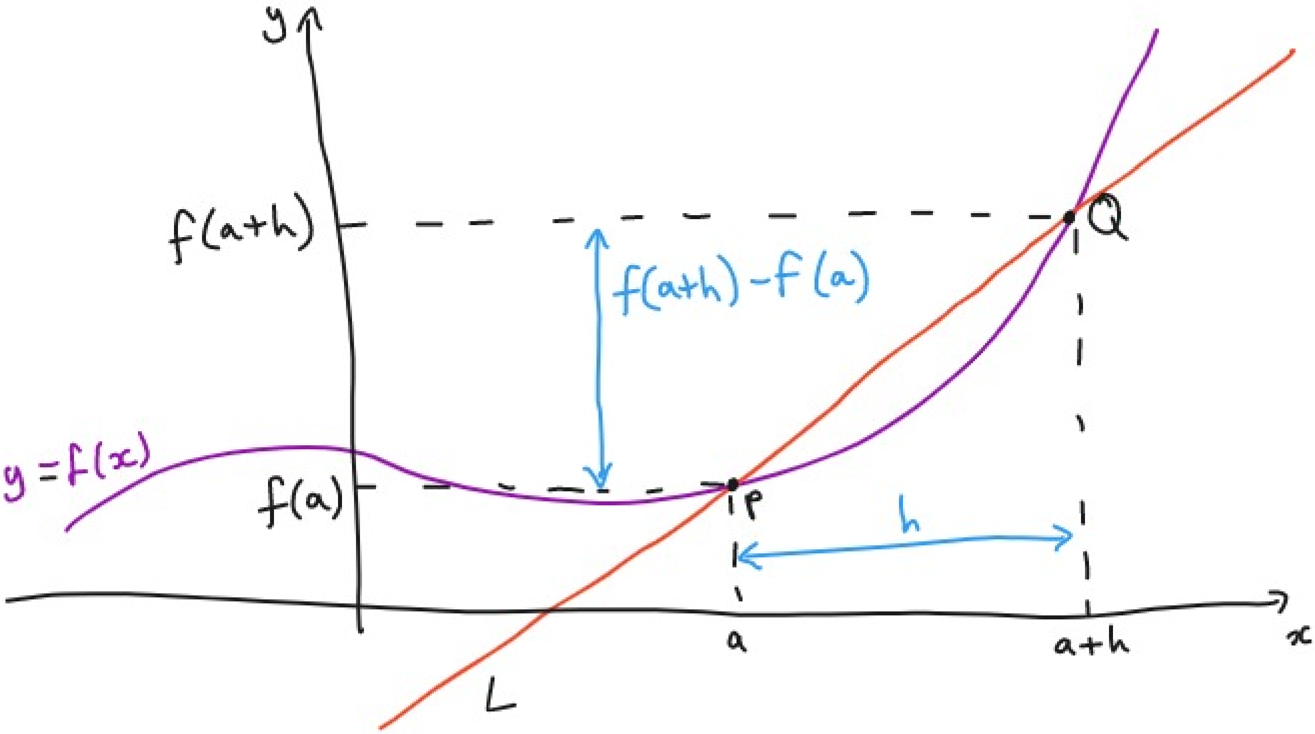

Figure 1: Two nearby points on a curve, connected by a straight line.

As h becomes very small, the point Q approaches P, and L becomes the tangent line at P. The gradient of L becomes the derivative of f at P, given by formula (1).

Differentiation using (1) is sometimes called differentiation by first principles.

Since we want to view the derivative as a function, we tend to use x instead of a in formula (1). Then, the derivative is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (2)$$

Example 2: Differentiate $f(x) = 3x^2$ from first principles.

Solution: We use (2). Before we can evaluate the limit, we calculate $\dfrac{f(x+h)-f(x)}{h}$ for this particular function

$$f(x) = 3x^2.$$

We have
$$\frac{f(x+h)-f(x)}{h} = \frac{3(x+h)^2 - 3x^2}{h}$$
$$= \frac{3(x^2+2xh+h^2) - 3x^2}{h}$$
$$= \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$$
$$= \frac{6xh + 3h^2}{h}$$
$$= 6x + 3h$$

Letting h tend to zero, we then get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + 3h)$$
$$= 6x.$$

Exercise 3: Let $f(x) = x^3$.

(i) Calculate $\dfrac{f(x+h)-f(x)}{h}$

(ii) Hence use (2) to differentiate $f(x)$ from first principles.

Some Properties of Derivatives:

Linearity Property:

If two functions $f(x)$ and $g(x)$ are differentiable, and a and b are constants, then the derivative of $af(x)+bg(x)$ is

$$\frac{d}{dx}\left(af(x)+bg(x)\right) = af'(x)+bg'(x).$$

Differentiating Powers:

General rule: For any non-zero real number r,

$$\frac{d}{dx}\left(x^r\right) = rx^{r-1}. \qquad (3)$$

Equation (3) may used in conjunction with the linearity property for derivatives to differentiate any linear combination of powers of x^r.

Example 4: Use equation (3) to differentiate

$$f(x) = 3x^2 - 2\sqrt{x} + \frac{7}{x^3}.$$

Solution: First we rewrite the expression for $f(x)$ using power notation:

$$f(x) = 3x^2 - 3x^{1/2} + 7x^{-3}.$$

By the linearity property for derivatives, we can calculate $f'(x)$ by differentiating term by term.

Each term can be differentiated using the power rule (3).

Hence,

$$f'(x) = 6x - 3 \cdot \frac{1}{2} x^{-1/2} + 7(-3)x^{-4}$$

$$= 6x - \frac{3}{2\sqrt{x}} - \frac{21}{x^4}.$$

Exercise 5: Use (3) to differentiate $15x - \dfrac{2}{\sqrt[3]{x}} - \dfrac{x^4}{2}$ with respect to x.

Solutions to Exercises:

Exercise (3):

(i) We have $f(x) = x^3$, and so

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h}$$

$$= \frac{3hx^2 + 3h^2x + h^3}{h}$$

$$= 3x^2 + 3hx + h^2$$

(ii) Hence,

$$f'(x) = \lim_{h \to 0} (3x^2 + 3hx + h^2) = 3x^2$$

Exercise (5):

$$\frac{d}{dx}\left(15x - \frac{2}{\sqrt[3]{x}} - \frac{x^4}{2}\right) = \frac{d}{dx}\left(15x - 2x^{1/3} - \frac{1}{2}x^4\right)$$

$$= 15 - 2 \cdot \frac{1}{3} x^{-2/3} - \frac{1}{2} \cdot 4x^3 = 15 - \frac{2}{3}x^{-2/3} - 2x^3.$$