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Differentiation 1

Differentiation is a method for calculating the gradient of a curve at a given point. On this sheet, we recap the definition of the derivative of a function in one variable and practice the method of differentiation from first principles.

We will also revise the standard formula for the derivative of a power, and practice using it to differentiate polynomials. The problem set at the end of this resource includes some contextual questions to give a taste of how differentiation can be used to solve problems in the real world.

First Principles Differentiation

Definition 1. The derivative of a function f at a point $P = (a, f(a))$ on the curve $y = f(x)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \quad (1)$$

The rationale for this formula is the following: suppose Q is a second point on the curve, very close to P.

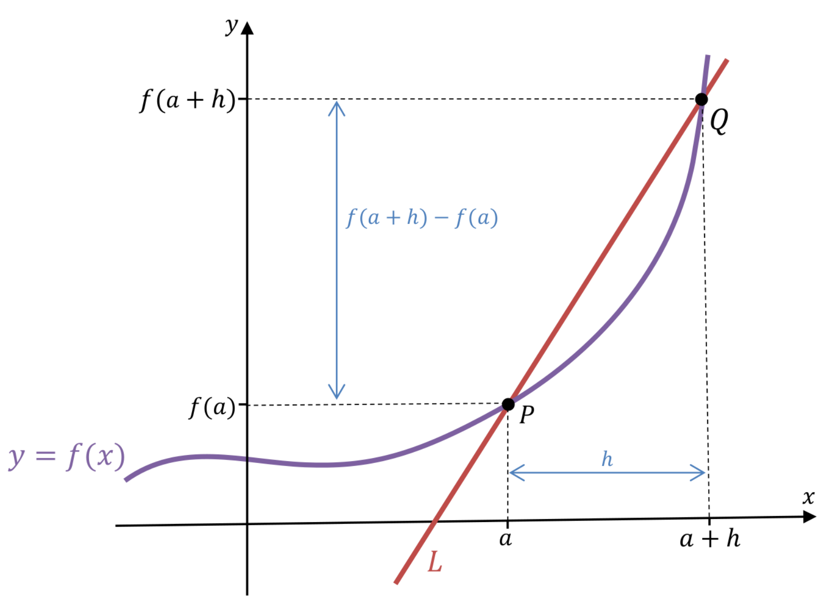

Figure 1: Two nearby points on a curve, connected by a straight line.

Provided P and Q are very close together, the line L passing through both of these points will give a good approximation for the tangent line at P.

Suppose that $P = (a, f(a))$ and $Q = (a+h, f(a+h))$, where h is a very small number.

The gradient of the straight line through P and Q is given by the standard formula

$$\text{Gradient of } L = \frac{\text{Change in } y}{\text{Change in } x} = \frac{f(a+h) - f(a)}{h}$$

As h becomes very small, the point Q approaches P, and L becomes the tangent line at P. The gradient of L becomes the derivative of f at P, given by formula (1).

Differentiation using (1) is sometimes called *differentiation by first principles*.

Since we want to view the derivative as a funciton, we tend to use x instead of a in formula (1). Then, the derivative is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

Example 2. Differentiate $f(x) = 3x^2$ from first principles.

Solution. We use (2). Before we can evaluate the limit, we calculate $\frac{f(x+h)-f(x)}{h}$ for this particular function $f(x) = 3x^2$.

We have

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h)^2 - 3x^2}{h} \\ &= \frac{3(x^2 + 2xh + h^2) - 3x^2}{h} \\ &= \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} \\ &= \frac{6xh + 3h^2}{h} = 6x + 3h.\end{aligned}$$

Letting h tend to zero, we then get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + h) = 6x.$$

Exercise 3. Let $f(x) = x^3$.

(i) Calculate $\dfrac{f(x+h) - f(x)}{h}$.

(ii) Hence use (2) to differentiate $f(x)$ from first principles.

Some Properties of Derivatives

Linearity Property

If two functions $f(x)$ and $g(x)$ are differentiable, and a and b are constants, then the derivative of $af(x) + bg(x)$ is

$$\frac{d}{dx}\left(af(x) + bg(x)\right) = af'(x) + bg'(x).$$

Differentiating Powers

General rule: For any non-zero real number r,

$$\frac{d}{dx}\left(x^r\right) = rx^{r-1}. \tag{3}$$

Equation (3) may be used in conjunction with the linearity property for derivatives to differentiate any linear combination of powers of x^r.

Example 4. Use equation (3) to differentiate $f(x) = 3x^2 - 2\sqrt{x} + \frac{7}{x^3}$.

Solution. First we rewrite the expression for $f(x)$ using power notation:

$$f(x) = 3x^2 - 3x^{\frac{1}{2}} + 7x^{-3}.$$

By the linearity property for derivatives, we can calculate $f'(x)$ by differentiating term by term. Each term can be differentiating using the power rule (3). Hence

$$\begin{aligned} f'(x) &= 6x - 3 \cdot \frac{1}{2}x^{-\frac{1}{2}} + 7(-3)x^{-4} \\ &= 6x - \frac{3}{2\sqrt{x}} - \frac{21}{x^4}. \end{aligned}$$

Exercise 5. Use (3) to differentiate $15x - \dfrac{2}{\sqrt[3]{x}} - \dfrac{x^4}{2}$ with respect to x.

Solutions to Exercises

Solution to Exercise 3:

(i) We have $f(x) = x^3$, and so

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^3 - x^3}{h} \\ &= \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\ &= \frac{3hx^2 + +3h^2x + h^3}{h} = 3x^2 + 3hx + h^2\end{aligned}$$

(ii) Hence

$$f'(x) = \lim_{h\to 0}\left(3x^2 + 3hx + h^2\right) = 3x^2.$$

Solution to Exercise 5:

$$\begin{aligned}\frac{d}{dx}\left(15x - \frac{2}{\sqrt[3]{x}} - \frac{x^4}{2}\right) &= \frac{d}{dx}\left(15x - 2x^{\frac{1}{3}} - \frac{1}{2}x^4\right) \\ &= 15 - 2\cdot\frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{2}\cdot 4x^3 = 15 - \frac{2}{3}x^{-\frac{2}{3}} - 2x^3.\end{aligned}$$