Lec 21 - Nov 23rd

Phase Space in Mechanics

From section §2.3 c of The Geometry of Physics by Frankel

In classical Mechanics we describe a system using generalized coordinates,

$q^1, \dots, q^n$ } compactly called q

These form an n-dimensial manifold M that we call configuration space.

The Lagrangian is a function of q and $\dot{q}$, where $\dot{q} = \frac{dq}{dt}$ which also has n- coordinates

These 2n coordinates $q, \dot{q}$ completely specify the state

The $\dot{q}$ are generalized velocities and are in T_pM There fore, $(q, \dot{q})$ is in the tangent bundle, TM

The Lagrangian, $L(q, \dot{q})$, is a map $L: TM \to \mathbb{R}$.

For Hamiltonian mechanics, we need the generalized momenta

$\boxed{p_i(q, \dot{q}) \equiv \frac{\partial L}{\partial \dot{q}^i}}$ → one form, ie in co-tangent space

↳ sub script means p is in cotangent space

To build the Hamiltonian, we need the Lagrangian and a transformation

$$(q, \dot{q}) \to (q, p)$$

This is not simply changing coordinates

To see this suppose we have a change in generalized coordinates

$$q_u \longrightarrow q_v$$

This can be described as

$$q_u = q_v(q_u)$$

prime denotes new coordinates

$$\dot{q}_v^{i'} = \left(\frac{\partial q_v^{i'}}{\partial q_u^j}\right) \dot{q}_u^j$$

Compare this with $\Lambda^{i'}_{\ j} = \frac{\partial y^{i'}}{\partial x^j}$ $\quad V^{i'} = \Lambda^{i'}_{\ j} V^j$ (contravariant) (Vectors do this)

The p's transform as follows:

$$p_{i'}^{v} \equiv \frac{\partial L}{\partial \dot{q}_v^{i'}} = \left(\frac{\partial L}{\partial q_u^j} \underbrace{\frac{\partial q_u^j}{\partial \dot{q}_v^{i'}}}_{0} + \frac{\partial L}{\partial \dot{q}_u^j} \frac{\partial \dot{q}_u^j}{\partial \dot{q}_v^{i'}} \right)$$

$$p_{i'}^{v} = p_j^u \left(\frac{\partial \dot{q}_u^j}{\partial \dot{q}_v^{i'}} \right)$$ This shows this is covariant and must live in the cotangent space

Compare with $\Lambda^{j}_{\ i'} \equiv \frac{\partial x^j}{\partial y^{i'}}$

$\dot{q}$ is in the tangent space (vector)

p is in the cotangent space (one-form)

Hence computing p is not only changing variables but is really a map

$$p: TM \longrightarrow T^*M$$ right hand side could be T_p^*M but if we add q then T^*M

T^*M is the phase space (q, p)

The Hamiltonian is a map H s.t.

$$H: T^*M \longrightarrow \mathbb{R}$$ $H(q,p)$

The Lagrangian: $L(q,\dot{q}) = T(q,\dot{q}) - U(q)$

Where the KE is: $T(q,\dot{q}) = \frac{1}{2} g_{jk} \dot{q}^j \dot{q}^k$ metric

Example: Suppose we have 2 masses in 1D

$M = \mathbb{R}^2$ and $TM = \mathbb{R}^4$

$T = \frac{1}{2} m_1 (\dot{q}_1)^2 + \frac{1}{2} m_2 (\dot{q}_2)^2$

need with $g_{ij} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$

Example: If we have a mass in 2D

$T = \frac{1}{2} m(\dot{x} + \dot{y})^2$ [cartesian]

with $g_{ij} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

Involved q

$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$ $g_{ij} = \begin{bmatrix} m & 0 \\ 0 & r^2 \end{bmatrix}$ [polar coordinates]

In general $p_i \equiv \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial T}{\partial \dot{q}^i} = g_{ij}(q)\dot{q}^j$

∂T can be used to define a Riemannia metric

$$\langle \dot{q}, \dot{q} \rangle = g_{ij}(q)\dot{q}^i\dot{q}^j$$

Kinetic Energy is ½ the length squared of the velocity vector

The generalized momenta p is the covariant version of the generalized velocity.

Example 1: $p_1 = m_1\dot{q}^1$ and $p_2 = m_2\dot{q}^2$

In general $p_i = g_{ij}\dot{q}^j$ and $\dot{q}^i = g^{ij}p_j$

§ 5.4 Hamiltonian Vector Fields

Given a Lagrangian, we can obtain the equations of motion from the Euler Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

Hamiltonian

$$\mathcal{H}(q,p) = p\dot{q} - L$$

Hamiltons' eqns

$$\dot{q}\ \frac{\partial \mathcal{H}}{\partial p} \quad + \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

Phase space is the tangent bundle T^*M, which includes M and T_p^*M

On T^*M, which is a manifold, we define a 2-form,

$$\boxed{\tilde{\omega} = \tilde{d}q \wedge \tilde{d}p} \quad \text{area in phase space}$$

can be called M?

Take a curve on T^*M of the form

$$\{ q = f(t),\ p = g(t) \}$$

Which is a solution to Hamilton's equations. The tangent vector to the curve is,

$$\bar{U} = \frac{d}{dt} = \dot{f}\frac{\partial}{\partial q} + \dot{g}\frac{\partial}{\partial p_i}$$

basis vectors

Theorem: If $\bar{u}$ is a tangent vector to the solution curve then $\mathcal{L}_{\bar{V}}\tilde{\omega} = 0$

Proof: From a formula (4.67)

$$\mathcal{L}_{\bar{V}}\tilde{\omega} = \tilde{d}[\tilde{\omega}(\bar{u})] + \overbrace{(\tilde{d}\tilde{\omega})}^{=0}(\bar{u})$$

The 2^{nd} term is 0 since $\tilde{\omega}$ is a 2-form on a 2 dim Manifold

$$\Rightarrow \mathcal{L}_{\bar{V}}\tilde{\omega} = \tilde{d}[\tilde{\omega}(\bar{u})]$$

$$= \tilde{d}[\tilde{d}q \wedge \tilde{d}p\,(\bar{u})]$$

$$= \tilde{d}[(\tilde{d}q \otimes \tilde{d}p - \tilde{d}p \otimes \tilde{d}q)(\bar{u})]$$

$$= \tilde{d}[(\tilde{d}q(\bar{u})\tilde{d}p - \underbrace{\tilde{d}p(\bar{u})}_{\text{coefficient of 1 form}}\overbrace{\tilde{d}q}^{\text{one form}})]$$

But $\bar{u} = f\frac{\partial}{\partial q} + g\frac{\partial}{\partial p}$ which yields

$$\tilde{d}q(\bar{u}) = \dot{f} \quad \text{and} \quad \tilde{d}p(\bar{u}) = \dot{g}$$

$$\Rightarrow \mathcal{L}_{\bar{V}}\tilde{\omega} = \tilde{d}[\dot{f}\,\tilde{d}p - \dot{g}\,\tilde{d}q]$$

However $\dot{f} = \frac{\partial H}{\partial p}$ and $\dot{g} = -\frac{\partial H}{\partial q}$ from Hamiltons eqn

$$\mathcal{L}_{V}\tilde{\omega} = \tilde{d}\left[\frac{\partial H}{\partial q}\tilde{d}q + \frac{\partial H}{\partial p}\tilde{d}p\right]$$

$$= \tilde{d}[\tilde{d}H] = 0$$

The area in phase space is conserved along solns to Hamiltons equations

A Vector field with $\mathcal{L}_{\bar{V}}\tilde{\omega} = 0$ is a Hamitonian vector field. $\bar{u}$ is tangent to the curves in phase space

The System is conservative (H is constant along Solns)

$$\mathcal{L}_{\bar{V}} H = \frac{dH}{dt} = \dot{q}\frac{\partial H}{\partial q} + \dot{p}\frac{\partial H}{\partial p}$$

$$= \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial H}{\partial p}$$

$$= 0$$