Lec 1 Sep 7th

at the end of this course ask the q: is the gradient a vector?

This course is useful in 2 aspects
1. General Relativity needs diff geo because of curved space and time
2. learn powerful and beautiful tools to describe physics in any geometry

Chapter 1 - on some basic Mathematics

We need some maths to be able to define a manifold (Thursday of next week)

1.1 $\mathbb{R}^n$ and its topology

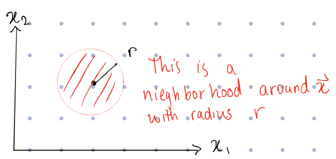
A point in $\mathbb{R}^n$ is an n-tuple $(x_1, \dots, x_n)$ w $\forall x_i \in \mathbb{R}$. The idea of continuous is that any 2 points in $\mathbb{R}^n$ have a line connecting them that exists in $\mathbb{R}^n$.

ex. Integers are not continuous (discrete)

The continuity of a space defines its topology. Here we focus on local vs global topology. We use distance to define the topology.

Recall, the distance between $\vec{x}, \vec{y} \in \mathbb{R}^n$ is $d(x,y) = [(x_1-y_1)^2 + \dots + (x_n-y_n)^2]^{1/2}$

A neighborhood of radius r of $\vec{x} \in \mathbb{R}^n$ is the set of points s.t. $N_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^n, d(\vec{x},\vec{y}) < r\}$



A set of points in $\mathbb{R}^n$ are discrete if there exists a neighbourhood about each point that contains no other points.

A set of points $S \in \mathbb{R}^n$ is open $\forall x \in S$ $\exists$ a neighbourhood all in S

Example ① $S = \{x \mid a < x < b\}$ is open

② $S = \{x \mid a \le x < b\}$ is not open. because $x = a$ does not have a neighbourhood all with in S

Note: Open sets cannot contain boundry points.

$\mathbb{R}^n$ has the hasdorff property, which means that any 2 points in $\mathbb{R}^n$ have neighbourhoods that do not intersect.

$d(\vec{x},\vec{y})$ induces a topology on $\mathbb{R}^n$, which says that d determines whether a set is open or not.

Open sets have the following propeties:
1. empty set $\emptyset$ and the whole set S are open
2. If O_1 & O_2 are open sets the $O_1 \cap O_2$ is open
3. The union of open sets (finite number) is open

The topology of a set consits of the set and all the open sets in that set.

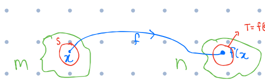
Any distance function induces the natural topology in $\mathbb{R}^n$

For example, $d'(\vec{x},\vec{y}) = [4(x_1-y_1)^2 + (x_2-y_2)^2 + \dots (x_n-y_n)^2]^{1/2}$ has the same induced topology as any other distance fcn

You can define a topology without distance.

§1.2 Mappings

A map from M to N associates an elemet $x \in M$ to a unique $y \in N$



$f: M \to N$
$f: x \longmapsto f(x)$

S is a subset of M and the image of S under f is $f(S) = T$. The inverse image of T is $f^{-1}(T) = S$

f can be many to one. If all points in f(S) have a unique inverse in S then f is 1-1 and $\exists$ a one to one map f^{-1} called the inverse of f.

example: $\sin(x)$ is many to one b/c $\sin(x) = \sin(x+2\pi)$

Notation: $f: M \to N$ f maps M to N
$f: x \mapsto y$ f maps x to y

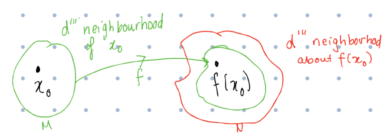
Given $f: M \to N$ and $g: N \to P$ then $\exists$ a composition map $g \circ f: M \to P$ such that $(g \circ f)(x) = g(f(x))$. If $f: M \to N$ then
f defined $\forall$ points in $M \Rightarrow$ f maps M into N
f defined $\forall$ points in $N \Rightarrow$ f maps M onto N
If f is both 1-1 and onto the f is a bijection. If f have an inverse, then f is 1-1

New [A map $f: M \to N$ is continuous at $x \in M$, if any open set in N containing $f(x)$ contains ~~the~~ image of an open set M contains x.

f is continuous on M if it is continuous $\forall x \in M$

old [Look at how this is related to continous defined in calculus. Recall, f is continuous at x_0 if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t
$$|f(x) - f(x_0)| < \varepsilon \text{ if } |x - x_0| < \delta$$

We define $d'''(x, x_0) = |x - x_0|$ then our definition can be rewritten as follows: f is continuous at x_0 if $\forall$ d'''-neighbourhoods of $f(x_0)$ contains the image of a d''' neighbourhood of x_0



Theorem: $f: M \to N$ is continuous iff the inverse image of every open set is open in M

Lec 2 Sep 12

Offical Assignment will be released tonight. Crowdmark link will be sent out. AMATH 433?

Today's Topics: ① Real Analysis ② Group theory ③ Linear Algebra ④ Algebra of Square Matrices

§1.3 Real Analysis

$f(x)$ is analytic at $x = x_0$ if it has a Taylor expansion about x_0 with a non-zero radius of convergence

Analytic functions (C^ω) which is a subset of C^∞

We will assume functions are analytic, but we'll often say smooth (C^∞)

An operator A of functions is a map that takes a function and yeilds another function

Examples: $A(f) = g(f)$, g is a function

$D(f) = \frac{df}{dx}$ where f is C^1

The commutator of 2 operators A, B on f is $[A,B](f) = (AB - BA)f$ OR $A(B(f)) - B(A(f))$. If $[A,B] = 0$ $\forall$ functions then A an B commute

Example $A = \frac{d}{dx}$ and $B = x\frac{d}{dx}$

$$[A,B](f) = \frac{d}{dx}\left(x\frac{df}{dx}\right) - x\frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= x\frac{d^2f}{dx^2} + \frac{df}{dx} - x\frac{d^2f}{dx^2} = \frac{df}{dx}$$ hence A & B do not commute

[function spaces dont have a good intuition on the commutator but in other contexts there is, such as later in the course]

§1.4 Group theory

A set of elements G with a binary operation $\cdot$ is a group if

[G_i] Associative: $x\cdot(y\cdot z) = (x\cdot y)\cdot z$

[G_{ii}] Identity: $\exists\, e \in G$ such that $x\cdot e = e\cdot x = x \quad \forall x \in G$

[G_{iii}] Inverse: $\forall x \in G\ \exists\ x^{-1} \in G$ s.t. $x\cdot x^{-1} = x^{-1}\cdot x = e$

And is closed under the operation

A group is abelion (commutative) if

[G_{iv}] $x\cdot y = y\cdot x$

Example: ① Set of permutation of n objects

② Rotations of a regular polygon

Aside: the Inverse is unique, and the identity is unique

A subgroup is simply a group that is contained within the group

Example: a set of permutation of n objects where the first element is unchanged.

This is (identical) similar to the permutations of (n-1) objects [isomorphism]

§ 1.5 Linear Algebra

A set V is a vector space (over $\mathbb{R}$) if it has a binary operation $+$ where it is an abelian group and satisfies the following under multiplication. Let $\vec{x}, \vec{y} \in V$, $a, b \in \mathbb{R}$

[V_i] $a \cdot (\vec{x} + \vec{y}) = (a \cdot \vec{x}) + (a \cdot \vec{y})$
[V_{ii}] $(a+b) \cdot \vec{x} = (a \cdot \vec{x}) + (b \cdot \vec{x})$
[V_{iii}] $(ab) \cdot \vec{x} = a \cdot (b \cdot \vec{x})$
[V_{iv}] $1 \cdot \vec{x} = \vec{x}$

The identity under addition is $\vec{0} = 0$

Example ① $n \times n$ matrices
② Continuous real function on $a \leq x \leq b$

✲ Dual spaces become critical in the next 2 weeks ✲

Notation: we often drop $\cdot$ & ( ) and write $a\vec{x} + b\vec{y}$

A set $\{x_1, x_2, \dots, x_n\}$ is linearly dependent if $\exists \{a_1, \dots, a_n\}$ $a_i \neq 0$ s.t. $a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_n\vec{x}_n = 0$

if $\forall a_i = 0$ then the set is linearly independent

A vector space has a basis, which of the dimension of V, and allows us to generate any element of V

If $\vec{x}_i$, $i = 1, \dots, n$ is a basis of V then $\forall \vec{y} \in V$, $\exists a_i$'s such that

$$\vec{y} = \sum_{i=1}^{n} a_i x_i$$

A set of vectors $\{\vec{y}_1, \dots, \vec{y}_n\}$ generates a subspace of V with $a_1\vec{y}_1 + \dots + a_m\vec{y}_m$ $a_i \in \mathbb{R}$ $i = 1, \dots, m$

if $m < n \Rightarrow$ proper subspace

A normed vector space is one w a mapping from V to $\mathbb{R}$ s.t. $a \in \mathbb{R}$, $\vec{x}, \vec{y} \in V$

[N_i] $n(\vec{x}) \geq 0$ & $n(\vec{x}) = 0$ iff $\vec{x} = 0$

[N_{ii}] $n(a\vec{x}) = |a| \, n(\vec{x})$

[N_{iii}] $n(\vec{x} + \vec{y}) \leq n(\vec{x}) + n(\vec{y})$

Examples If $V = \mathbb{R}^n$ then
$n(x) = d(\vec{x}, 0) = [x_1^2 + \dots + x_n^2]^{1/2}$

$n'(\vec{x}) = d'(\vec{x}, 0) = [4x_1^2 + \dots + x_n^2]^{1/2}$

$n'''(\vec{x}) = d'''(\vec{x}, 0) = \max(|x_1|, \dots, |x_n|)$

All 3 satisfy N_i, N_{ii}, N_{iii}. In addition, some norms satisfy the parallelgram rule

[N_{iv}] $[n(\vec{x} + \vec{y})]^2 + [n(\vec{x} - \vec{y})]^2 = 2(n(\vec{x}))^2 + 2(n(\vec{y}))^2$

n, n' satisfy N_{iv} by n''' does not

If we have all 4 propreties then we can define a bilinear symmetric inner product

$$\vec{x}\cdot\vec{y} = \frac{1}{4}[n(x+y)]^2 - \frac{1}{4}[n(\vec{x}-\vec{y})]^2$$

bilinear : $(a\vec{x}+b\vec{y})\cdot z = a(\vec{x}\cdot\vec{z}) + b(\vec{y}\cdot z)$
$\vec{z}\,(a\vec{x}+b\vec{y}) = a(\vec{z}\cdot\vec{x}) + b(\vec{z}\cdot\vec{y})$

Symmetry : $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$

Positive definite $\vec{x}\cdot\vec{x}\geq 0$ and $\vec{x}\cdot\vec{x}=0$ iff $\vec{x}=\vec{0}$

$n(\vec{x})$ on $\mathbb{R}^n$ is the Euclidean Norm, $\mathbb{R}^n$ with the Euclidean norm is denoted E^n

A pseudonorm is a norm that violates N_i & N_{iii}, This occurs in special relativity What is the History of the Psuedo Norm

§1.6 Algebra of Square Matrices

A linear transformation T on a vectorspace is a map from V onto V which is Linear

$$T(a\vec{x}+b\vec{y}) = a\,T(\vec{x}) + b\,T(\vec{y})$$

If $\{\vec{e}_i\}$ $i=1,..,n$ is a basis for V, then $\vec{x} = \sum_{i=1}^{n} a_i\vec{e}_i$
and $T(\vec{x}) = T(\sum_i^n a_i\vec{e}_i) = \sum_i^n a_i T(\vec{e}_i)$ The $T(\vec{e}_i)$ can be expressed as $T_{ij}\vec{e}_j$
$= \sum_i^n a_i \sum_j^n T_{ij}\vec{e}_j$

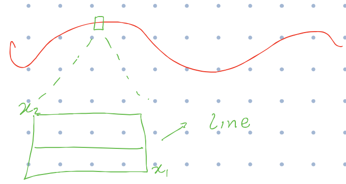
Where T_{ij} are the components of transformation T and are often written in matrix form.

If $\vec{A},\vec{C}$ ar vectors and B is a matrix then $\vec{A}^T B\vec{C} = \sum_{i,j}^{n} A_i B_{ij} C_j$ Strongly encourage to write in this form and not switch indices

Lec 3 Sep 14

Newton's law can be expressed without any coordinates. The manifold is locally like $\mathbb{R}^n$

example:



§ 2.1 Definition of a Manifold

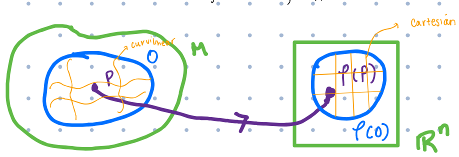
Idea: Any smooth curve / surface / volume [any dimension] looks locally like $\mathbb{R}^n$

A set of points M is a manifold if

$\forall x \in M$ has an open neighbourhood that has a continuous map which is 1-1 onto to map (bijective) open set of $\mathbb{R}^n$. then M has dimension n.

In this framework, there is no measure of length on M. Distance is a global property and we will discuss this later.

Points in M look like $\mathbb{R}^n$, not $\mathcal{E}^n$, unless we have a metric



What is the difference between map, chart, atlas?

$\varphi: O \to \mathbb{R}^n\ [x^1, x^2, \dots x^n]$

O is an open set

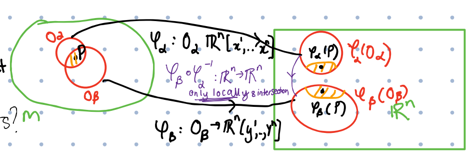
(O, φ) is a chart

$(x^1, x^2, \dots, x^n)$ are the coordinates of $\mathbb{R}^n$

There can be multiple charts at a given point on a manifold. These chart must overlap

Q: WTF is para-compact

Q: is picture's same w closed sets?



Q: if they maintain intersection, does that mean area & and angle persved, is possible? $\Rightarrow$ metric

A collection of charts $(O_\alpha, \varphi_\alpha)$ is an atlas. This covers the manifold.

Note: $\varphi_\alpha : O_\alpha \to \mathbb{R}^n$ & $\varphi_\beta : O_\beta \to \mathbb{R}^n$ are homeomorphic (Defn continuous and bijective) $\Rightarrow$ $\varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and is a coordinate transformation

$y^i = y^i(x^1, \dots, x^n)$ $i = 1, \dots, n$

We say φ_α φ_β are C^k related if all the partial derivatives of order k are continuous

If all $p \in M$, for all charts in M, is C^k-related then M is a C^k- manifold.

We assume M is a C^∞- manifold [differentiable]

Examples: $\mathbb{R}^n$ is a n- differentiable manifold

① $\mathbb{R}^n$ has the natural topology

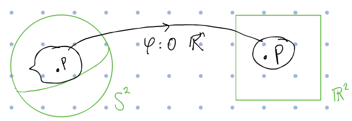
② Can use identity map for the charts [any point in $\mathbb{R}^n$ maps to itself]

§2.2 The Sphere is a Manifold

The two-sphere in $\mathbb{R}^3$ is denoted by S^2 and defined by

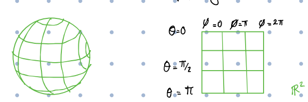
$$(x^1)^2 + (x^2)^2 + (x^3)^2 = \text{constant}$$

A one- sphere is a circle



We can map small neighbourhoods of P to a disc in $\mathbb{R}^2$. This map does not perserve length or angles

Another way to do this mapping is to use spherical coordinates



$\theta = x^1$ $\quad 0 \leq x^1 \leq \pi$ colatitude

$\phi = x^2$ $\quad 0 \leq x^2 \leq 2\pi$ longitude

The map has problem at:

$\theta = 0, \pi$ the line is mapped to a point

$\phi = 0$, this line gets mapped to 2π

A solution restricted to $0 < x^1 < \pi$ $\quad 0 < x^2 < 2\pi$ yields a chart for almost the entire sphere.

Another chart could be a similar system but where $\phi = 0$ at the equator and then go from $\phi = -\pi/2$ to $\phi = \frac{\pi}{2}$

Assignment 2 will have a Q on stereographic projects.

§2.3 Other examples of Manifolds

A set M that can be parameterized continuously is a manifold and its dimension is the number of indepedent parameters

① Set of rotations of a Rigid Object of 3D. Dimension 3 (Euler Angles)

② All (pure boosts) Lorentz transformations is a manifold of dimension 3. The parameters are the components of Velocity

③ N particles in 3D, 3N dimensions for the position and 3N for Velocity. This is a manifold of dim 6N

④ An algebraic or differential equation for a dependent variable y in terms of an indep var x, The set (y, x) is a manifold

⑤ A vector space V over $\mathbb{R}$ is a manifold. Suppose V is n-dim with basis $\{\vec{e}_1, \dots, \vec{e}_n\}$. Any $y \in V$ can be written as $\vec{y} = a^1\vec{e}_1 + \dots + a^n\vec{e}_n$

We have a mapping $\vec{y} \to (a^1, \dots a^n)$ is from V to $\mathbb{R}^n$. It turns out that V is identical (isomorphic) to $\mathbb{R}^n$

Lec 4 Sep 19

§ 2.5 curves

A map from a C^∞ manifold to another manifold N that is C^∞ and a bijection is a C^∞ diffeomorphism from M to N [How does this compare to a homeomorphism, a homeomorphism is continuous (C^0) and a bijection]

Diffeomorphism's in a specialized case of a homeomorphism [a subset / subspace?]

A differentiable Manifold is a set M such that all points in M have an open set that has a map (diffeomorphism) to an open set in $\mathbb{R}^n$, we say it has dimension N

A curve is a differentiable map, say γ, (gamma) from an open set of $\mathbb{R}$ into M.

$$\gamma : [a, b] \to M \quad \text{or} \quad \gamma \longmapsto \gamma(\lambda) \in M$$

We parameterize the curve with lambda λ.

Two curves with the same image but different parameterizations are different.

Quote:
Thank you for paying attention more so than me

Suppose the image of the curve is in the open set O with chart φ:

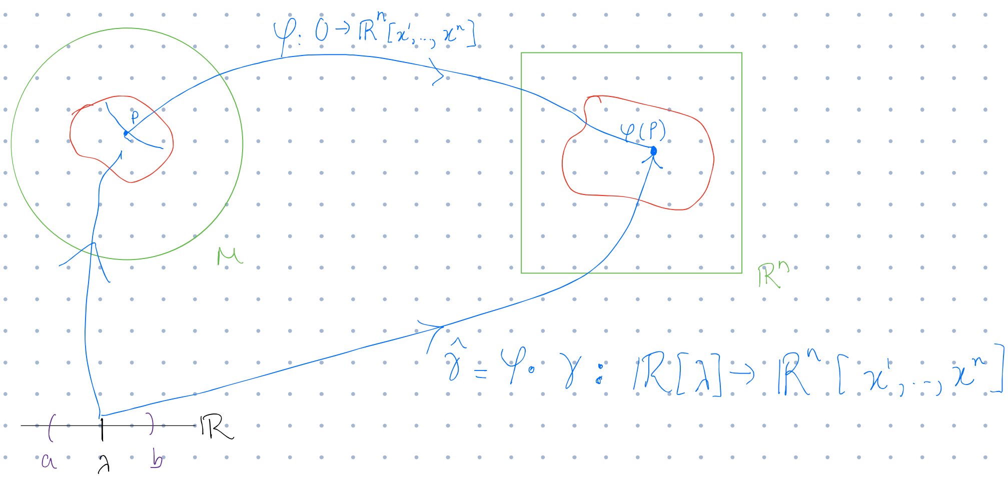
$$\varphi : O \to \mathbb{R}^n [x^1, \dots, x^n]$$

We obtain a coordinate representation of the curve:

$$\hat{\gamma} = \varphi \circ \gamma : \mathbb{R}[\lambda] \longrightarrow \mathbb{R}^n [x^1, \dots, x^n]$$

OR

$$\lambda \longmapsto (x^1(\lambda), \dots x^2(\lambda)) \equiv [x^1(\gamma(\lambda)), \dots, x^n(\gamma(\lambda))]$$



γ is differentiable if M is a C^∞ manifold

§ 2.6 Functions

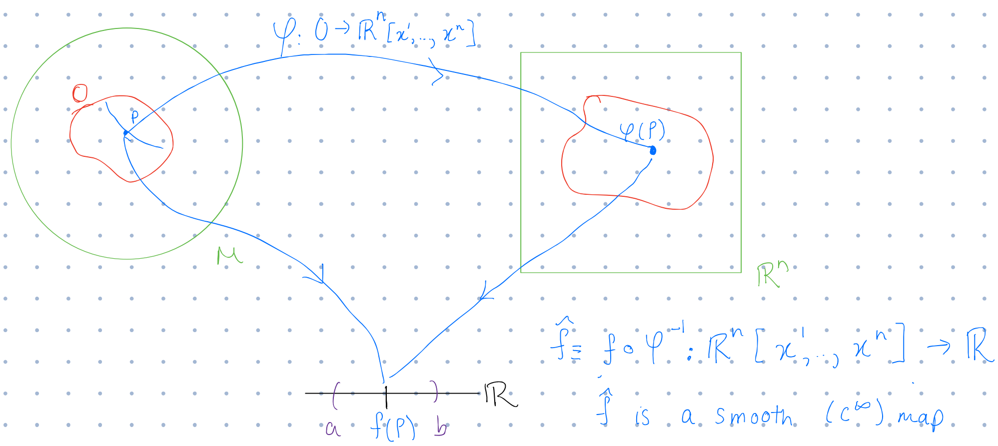
A Function, say f, on M is a smooth map from M to $\mathbb{R}$,

$$f: M \to \mathbb{R} \quad \text{or} \quad x \longmapsto f(x) \in \mathbb{R}$$

With chart $\varphi: O \to \mathbb{R}^n[x^1, \ldots, x^n]$ we get a coordinate representation of f:

$$\hat{f} = f \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\text{OR} \quad (x^1, \ldots, x^n) \longmapsto \hat{f}(x^1, \ldots, x^n)$$



On a manifold we always have coordinates but we don't always mention them explicitly

§ 2.7 Vectors and Vector Field

In a manifold, no magnitude for a vector.

Vectors typically have a direction and magnitude. In our definition, we will have a direction but no magnitude because we only have a local desciption about each point.

Suppose we have a curve, γ, that passes through the point $P \in M$ with coordinates $\{x^i\}$ $i = 1, \ldots, n$ and also a smooth function for M

$$\gamma: [a, b] \to M$$
$$f: M \to \mathbb{R}$$

We can evaluate f on the curve

$$g \equiv f \circ \gamma = f \circ \varphi^{-1} \circ \varphi \circ \gamma = \hat{f} \circ \gamma : [a, b] \to \mathbb{R}$$

OR

$$\lambda \longmapsto f(x^1(\lambda), \ldots, x^n(\lambda))$$

OR

$$f(x^i(\lambda)) : [a, b] \to \mathbb{R}$$

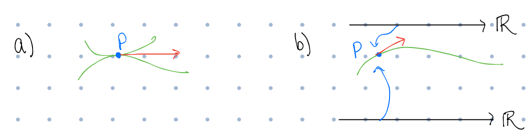
Since f and γ are both differentiable, so is g. We can differentiate g w.r.t x using the chain rule

This is a directional derivative. What is a directional derivative?

$$\boxed{\frac{dg}{d\lambda} = \sum_{i=1}^{n} \frac{dx^i}{d\lambda}\frac{\partial f}{\partial x_i}} \Rightarrow \left[\frac{d}{d\lambda} = \sum_{i=1}^{n} \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}\right]$$ This is an operator

Observe that the above is a directional derivative of f in the direction of $\left\{\frac{dx^i}{d\lambda}\right\}$. These are the components of the tangent to the curve

Note that each tangent vector $\left\{\frac{dx^i}{d\lambda}\right\}$ has an infinite number an infinite number c curves that are tangent to it.



Example:

At the point $P = x^i(0)$, consider the curve

$$\hat{\gamma}_1 = (\varphi \circ \gamma_1)(\lambda) = \lambda a^i \quad [a^i \text{ constants}]$$

The Tangent is

$$\left.\frac{dx^i}{d\lambda}\right|_{\lambda=0} = a_i$$

Now consider a different curve at P

$$\hat{\gamma}_2 = (\varphi \circ \gamma_2)(\mu) = \mu^2 b^i + \mu a^i = x^i, \quad a^i\ b^i \text{ constants}$$

The tangent vector is $\left.\frac{dx^i}{d\mu}\right|_{\mu=0} = 2\mu b^i + a^i = a^i$

The curves γ_1 and γ_2 have the same tangent vector at P

It can be shown that tangents to the curve form a Vector space. We can show all the properties of a vector space are satisfied but we only show closure.

Proof: Suppose $a, b \in \mathbb{R}$ and we have curves $\gamma_1(\lambda)$ and $\gamma_2(\mu)$

$$\text{From } \gamma_1 : \frac{d}{d\lambda} = \sum_{i=1}^{n} \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}$$

$$\text{From } \gamma_2 : \frac{d}{d\mu} = \sum_{i=1}^{n} \frac{dx^i}{d\mu}\frac{\partial}{\partial x^i}$$

Consider the linear Super position of the two,

$$a\frac{d}{d\lambda} + b\frac{d}{d\mu} = \sum_{i=1}^{n}\left(a\frac{dx^i}{d\lambda} + b\frac{dx^i}{d\mu}\right)\frac{\partial}{\partial x^i}$$

we introduce a new parameter ϕ such that

$$\frac{d}{d\phi} = \sum_{i=1}^{n}\left(a\frac{dx^i}{d\lambda} + b\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x_i} = \sum_{i=1}^{n}\frac{\partial x^i}{\partial \phi}\frac{\partial}{\partial x^i}$$

This proves closure, which means the Scalar sum of 2 tangent vectors is also a tangent vector

$\blacksquare$

If you consider the tangent vectors along the coordinate lines, $\{x^i\}$, we get $\left\{\frac{\partial}{\partial x_i}\right\}$ This forms a basis to the vector space

In our equations for $\frac{d}{d\lambda}$, $\left\{\frac{dx^i}{d\lambda}\right\}$ are the components in the vector space.

Lec 5- Sep 21

All future assignments will be due on Thursdays @ 5pm

Previously, we define $g = f\circ\gamma : (a,b) \to \mathbb{R}$, $\frac{dg}{d\lambda} = \sum_{i=1}^{n}\frac{dx^i}{d\lambda}\frac{\partial f}{\partial x^i} \Rightarrow \frac{d}{d\lambda} = \sum_{i=1}^{n}\frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}$ where $\frac{dx^i}{d\lambda}$ are the components of the tangent to the curve

With Coordinates $\{x^i\}$, we get coordinate lines $\left\{\frac{\partial}{\partial x^i}\right\}$ which form a basis.

Since there is a 1 to 1 correspondnce between the tangent vectors at P and the space of partial derivatives at P, we use $\frac{\partial}{\partial x^i}$ to denote the tangent vectors to the curve

§ 2.8 Basis vectors and basis vector fields

Every point P in manifold M has a tangent space denoted by T_p or T_pM, which is a vectorspace, $\bar{w}$ same dim (dim(M) = n.

We need n linearly independent vectors in T_pM to form a basis. A coordinate system $\{x_i\}$ at P has a coordinate basis of $\left\{\frac{\partial}{\partial x^i}\right\}$ of T_pM for all $P \in M$

Suppose $\{\bar{e}_i\}$ $i = 1, \dots, n$ is another basis of T_pM. Any vector in T_pM, say $\bar{V}$, can be written as

$$\bar{V} = \sum_{i=1}^{n} V^i\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} V^{j'}e_j$$

$\{V^i\}$ & $\{V^{j'}\}$ are the coordinates w.r.t. $\left\{\frac{\partial}{\partial x^i}\right\}$ and $\{\bar{e}_i\}$ These coordinates are functions on M.

Note: In $\left\{\frac{\partial}{\partial x^i}\right\}$ i is a superscript but appears in the denominator & ∴ is considered a subscript

For Vectors we use subscripts for the basis and superscripts for coordinates

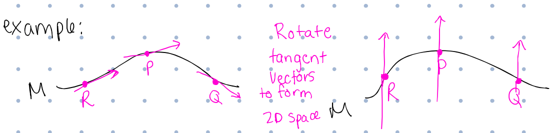
A vector is an object that lives in T_pM. A vector field are mappings that defines a vector for all $P \in M$

A vector field is diff'able if its coordinates are differentiable. The basis $\{\frac{\partial}{\partial x^i}\}$ are linearly indepent if $\{x^i\}$ are (proper) coordinates.

§2.9 Fiber Bundles

A manifold M with a tangent space $T_P M$ can be combined to form a tangent Bunde (TM)

1D example:



The tangent Bunde is a 2D space. (in general 2n) is also a manifold. We can define a projection to get the point from the tangent bundle. The tangent bundle is an example of a fiber bundle.

§2.12 Vector Feilds and Integral Curves

Any curve on a manifold has a tangent vector at every point. Since this is true $\forall$ points on the curve this defines a vector field. It can be shown that any smooth vector field has a curve associated with it this is an Integral Curve.

Vector fields coorespond to a system of first order ODEs and the integral curve is the solution.

Supose we have a smooth vector field $\bar{V} \in T_P M$ with components $V^i(P)$ with coordinates $\{x^i\}$. Then if we write $V^i(P) = v^i(x^1, \dots, x^n)$. Then we get a system of DEs

$$\boxed{\frac{dx^i}{d\lambda} = V^i(x^1, \dots, x^n)}$$

If V^i is C^1 $\forall i$ then $\exists$ a soln to the System which is the Integral Curves.

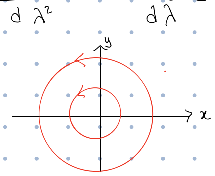
Example:

$$\bar{V} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$\frac{dx}{d\lambda} = -y \qquad \frac{dy}{d\lambda} = x$$

If you differentiate the first

$$\frac{d^2 x}{d\lambda^2} = -\frac{dy}{d\lambda} = -x \quad \text{Harmonic Oscillator}$$



§2.13 Exponentiation of the Operator $\frac{d}{d\lambda}$

Suppose we have an analytic (smooth) manifold (C^∞) with coordinates $\{x^i(\lambda)\}$ along Integral Curves

Then $\bar{Y} = \frac{d}{d\lambda}$ are analytic functions of λ and we can $\therefore$ Taylor Expand. We Taylor Expand $\{x^i\}$ about λ_0,

$$x^i(\lambda_0 + \epsilon) = x^i(\lambda_0) + \epsilon\left(\frac{dx^i}{d\lambda}\right)\Big|_{\lambda_0} + \frac{1}{2}\epsilon^2\left(\frac{d^2x^i}{dx^2}\right)\Big|_{\lambda_0} + \dots$$

$$= \left(1 + \epsilon\frac{d}{d\lambda} + \frac{\epsilon^2}{2}\frac{d^2}{d\lambda} + \dots\right)x^i\Big|_{\lambda_0}$$

$$\equiv \exp\left[\epsilon\frac{d}{d\lambda}\right]x^i\Big|_{\lambda_0} \quad [\text{THIS IS NOTATION}]$$

We have the exponentiation of the operator, which is short hand for the above expression.

Note: $\exp\left(\epsilon\frac{d}{d\lambda}\right) = e^{\epsilon\frac{d}{d\lambda}} = e^{\epsilon\bar{Y}}$ This is trying to give us a sense of distance

§ Lie brackets and noncoordinate basis

Suppose $\{x^i\}$ is a coordinate system and $\{\frac{\partial}{\partial x^i}\}$ is a basis of vector fields

We know that any n linearly independent vectors form a basis but can a basis form a coordinate system? No

By construction $\frac{\partial}{\partial x^i}$ & $\frac{\partial}{\partial x^j}$ commute for all i,j

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} = 0$$

Suppose we have $\bar{V} = \frac{d}{d\lambda}$ and $\bar{W} = \frac{d}{d\mu}$ we can show that they need not always commute

$$\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right] = \frac{d}{d\lambda}\left(\frac{d}{d\mu}\right) - \frac{d}{d\mu}\left(\frac{d}{d\lambda}\right)$$

$$= \left(\sum_{i=1}^{n} V^i\frac{\partial}{\partial x^i}\right)\left(\sum_{j=1}^{n} W^j\frac{\partial}{\partial x^j}\right) - \left(\sum_{j=1}^{n} W^j\frac{\partial}{\partial x^j}\right)\left(\sum_{i=1}^{n} V^i\frac{\partial}{\partial x^i}\right)$$

$$= \sum_{i,j}\Big\{V^i\frac{\partial W^j}{\partial x^i}\frac{\partial}{\partial x^j} + V^iW^j\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$$

$$- W^j\frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial x^i} - W^jV^i\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\Big\}$$

$$\boxed{\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right] = \sum_{i,j} V^i\frac{\partial W^j}{\partial x^i}\frac{\partial}{\partial x^j} - W^j\frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial x^i}}$$

If This is non zero then $\frac{d}{d\lambda}$ & $\frac{d}{d\mu}$ form a non-coordinate basis

The Lie braket of $\bar{V} = \frac{d}{d\lambda}$ and $\bar{W} = \frac{d}{d\mu}$ is $\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right]$

Lec 6 - Sep 26

Q1] just show they are inverses

A1 marks very latest next thursday

Q4) using C^∞ argue that commutator is 0

§ 2.14 Lie brackets & non-coordinate basis

The Lie bracket is defined as the comutator of two vectors: $\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right]$

Geometric Interpretation:

Consider a coordinate basis $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$ where $\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right] = 0$

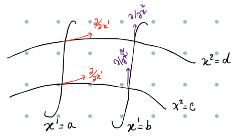
The integral curve of $\frac{\partial}{\partial x^1}$ [that are tangent to $\frac{\partial}{\partial x^1}$]

$\frac{dx^1}{d\lambda} = 1$ and $\frac{dx^2}{d\lambda} = 0$

$\Rightarrow$ $x^1 = \lambda + c$ and $x^2 = \text{constant}$.

Integral Curves of $\frac{\partial}{\partial x^2}$ are $x^1 = \text{constant}$, $x^2 = \lambda + \text{constant}$

HOW DOES THIS RELATE TO DYNAMIC SYSTEMS and Vector fields discussed in Calc 4

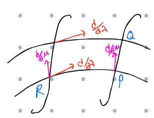


Locally things can be curvy but on a cure only 1 parameter changes

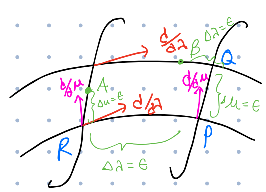
Along each integral curve all the x^i's are constant except for one that changes

Next consider a non-coordinate basis $\bar{V} = \frac{d}{d\lambda}$, $\bar{W} = \frac{d}{d\mu}$ with $\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right] \neq 0$

On the Integral Curves of $\frac{d}{d\lambda}$, λ increases and μ can also change.



Suppose we start a P and move $\Delta\lambda = \epsilon$ and then $\Delta\mu = \epsilon$ to end up at A



We can also start at P, move $\Delta\mu = \epsilon$ then $\Delta\lambda = \epsilon$ and end up at B

Find the approximate distance between A + B:

1st path, first move to R: $x^i(R) = \exp\left[\epsilon \frac{d}{d\lambda}\right] x^i \Big|_P$

Then to A,

$$x^i(A) = \exp\left[\epsilon \frac{d}{d\mu}\right] \exp\left[\epsilon \frac{d}{d\lambda}\right] x^i \Big|_P$$

Similarily, we move from P to Q to B

$$x^i(B) = \exp\left[\epsilon \frac{d}{d\lambda}\right] \exp\left[\epsilon \frac{d}{d\mu}\right] x^i \Big|_P$$

The difference between the two is

$$x^i(B) - x^i(A) = \left[\exp\left(\epsilon \frac{d}{d\lambda}\right), \exp\left(\epsilon \frac{d}{d\mu}\right)\right] x^i \Big|_P$$

$$= \dots \quad [\text{Assignment } 2\ 4(b)?]$$

$$= \epsilon^2 \left[d/d\lambda, \frac{d}{d\mu} \right] + O(\epsilon^3)$$

$\bar{V}, \bar{W}$ are in a coordinate basis iff $[\bar{V}, \bar{W}] = 0$ $\bar{V}$ and $\bar{W}$ are vector fields

§2.16 One - forms [co vectors]

T_pM is the space of tangent Vectors at $P \in M$. A one-form is a Linear, real valued function of Vectors

$$\tilde{\omega} : V \longrightarrow a \in \mathbb{R}$$

The space of one forms is the dual space to the tangent space T_pM

Suppose $\tilde{\omega}$ is a one form and $\bar{V}$ is a vector, both at P, then we have an operation

$$\boxed{\tilde{\omega}(\bar{V}) \in \mathbb{R}}$$

One forms are linear with ($a, b \in \mathbb{R}$, $\tilde{\sigma}$ is a one form)

1. $\tilde{\omega}(a\bar{V} + b\bar{W}) = a\tilde{\omega}(V) + b\,\tilde{\omega}(\bar{W})$
2. $(a\tilde{\omega})(\bar{V}) = a(\tilde{w}(V))$
3. $(\tilde{\omega} + \tilde{\sigma})(V) = \tilde{\omega}(V) + \tilde{\sigma}(\bar{V})$

These proprieties ensure that one forms at P forms a vector space This is called the dua space of T_pM called T_p^*M

Vectors are Linear, real-valued functions of one forms and hence T_pM is the dual of T_p^*M

$$T_p^{**}M = T_pM \qquad [\text{this is always the case}]$$

Example: $(a\tilde{\omega} + b\tilde{\sigma})(\bar{v}) = (a\tilde{\omega})(\bar{v}) + (b\tilde{\sigma})(\bar{v})$
$= a(\tilde{\omega}(\bar{v})) + b(\tilde{\sigma}(\bar{v}))$

Notation: $\tilde{\omega}(\bar{v})$ or $\bar{V}(\tilde{\omega}) = \langle \tilde{\omega}, \bar{V} \rangle$ these are all called contraction

coordinates have superscripts

Vectors are sometimes called Contravariant; One-forms (covectors) are covariant → coordinates have subscripts

§2.17 Examples of One-forms

example matrix algebra
colomn vectors are vectors
row vectors are one-forms

$$(a,b) : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto (a,b) \begin{pmatrix} x \\ y \end{pmatrix} = ax + by \in \mathbb{R}$$

§2.18 dirac delta function

C^∞ functions are an abelian group under addition and a Vector Space under multiplication.

The dual space of the functions are one forms and called distributions

$$\delta(x) : f(x) \mapsto \int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

§2.19 The gradient and the pictorial representation of a one-form

A vector field has a unique vector at every point. A field of one-form has a unique one form at every point.

Differentiabily of one forms will be determined in terms of diff'abitly of vectors and functions

A Tangent Bundle, TM contains $M \,\&\, T_pM$. A cotangent Bundle contains T^*M contains $M \,\&\, T_p^*$
Both are fiber bundles

a function

We will show that the gradient of f, denoted $\tilde{d}f$ is a one-form anddefined as:

$$\boxed{\tilde{d}f\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda} = \sum_{n=1}^{i} \frac{dx^i}{d\lambda}\frac{\partial f}{\partial x^i}} \in \mathbb{R}$$

gradient "exists" in Dual space and takes in from the Tangent Space

$\tilde{d} f$ is an element of $T_p^* M$ and the contraction with $\frac{dx^i}{d\lambda}$ yields the directional derivative of f alo
a curve tangent to $\bar{V}$

Check $\tilde{d} f$ is a one form:

$$\tilde{d} f \left(a \frac{d}{d\lambda} + b \frac{d}{d\mu} \right) = \left(a \frac{d}{d\lambda} + b \frac{d}{d\mu} \right) f \quad \text{by above definition}$$

$$= a \frac{df}{d\lambda} + b \frac{df}{d\mu} = a \, \tilde{d} f \left(\frac{d}{d\lambda} \right) + b \, \tilde{d} f \left(\frac{d}{d\mu} \right)$$

we see $\tilde{d} f$ is a linear operator on vectors.

$\frac{df}{d\lambda}$ at P is computed from $\frac{\partial f}{\partial x^i}$ at P and this forms the components of $\tilde{d} f$

Note $\frac{\partial f}{\partial x^i}$ has a lower index, opposite to be basis of vectors.

Lec 7 - Sep 28th

§ 2.21 Basis 1-forms and components of 1 forms

Any n linearly independent one-forms are a basis of $T_p^* M$ [cotangent space]

Given a basis of $T_p M$ say $\{\bar{e}_i, i=1,\ldots,n\}$ this induces a dual basis to $T_p^* M$ $\{\tilde{\omega}^i, i=1\ldots n\}$

If $\bar{V} \in T_p M$ then the dual basis $\tilde{\omega}^i$ is defined by

$$\boxed{\tilde{\omega}^i(\bar{V}) = V^i} \qquad \bar{V} = \sum_{i=1}^{n} V^i \bar{e}_i$$

and

$$\boxed{\tilde{\omega}^i(\bar{e}_j) = \delta^i_j \text{ [Kroneker delta]}}$$

We show that $\{\tilde{\omega}^j\}$ are linearly undepent and form a basis of $T_p^* M$

Consider any one-form $\tilde{q}$

$$\tilde{q}(\bar{V}) = \tilde{q}\left(\sum_{j=1}^{n} V^j \bar{e}_j\right)$$

$$= \sum_{j=1}^{n} V^j \tilde{q}(\bar{e}_j) \qquad \text{Linear}$$

Define $q_j = \tilde{q}(\bar{e}_j)$ to be the components of $\tilde{q}$ on the dual basis to $\{\bar{e}_j\}$

Also $\tilde{q}(\bar{V}) = \sum_j q_j \tilde{\omega}^j(\bar{V}) \Rightarrow \tilde{q} = \sum_j q_j \tilde{\omega}^j$

Note: compare with $\bar{V} = \sum_j V^j \bar{e}_j$

Therefore, $\{\tilde{\omega}_j\}$ are a basis since there are n of them and we can generate any $\tilde{q}$ with This decomposition

It follows,

$$\boxed{\tilde{q}(\bar{V}) = \sum_j q_j V^j} \qquad \text{This is a Contraction}$$

If $\{\bar{e}_i\}$ is a basis of $T_p M$ $\forall$ points $U \subset M$ then $\{\tilde{\omega}_j\}$ is a basis $T_p^* M$ $\forall$ points $U \subset M$

The coordinate basis $\{x^i\}$ on $\mathcal{U}$ defines a natural vector field $\{\partial/\partial x^i\}$ (a bais of $T_p M$)

and this defines a natural basis of one forms $\{\tilde{d} x^i\}$ [at each point]

With this notation, $\tilde{d}x^i\left(\partial/\partial x^j\right) = \delta^i_j$

§ 2.21 Index Notation

Components of vectors V^i

Components of one-forms ω_j

Vector basis: $\bar{e}_i$

One form basis $\tilde{\omega}^j$

Coordinate basis; (1-forms) $\tilde{d}x^j$

(Vectors) $\partial/\partial x^i$

Example: $\tilde{\omega}(\bar{V}) = \sum_j V^j \omega_j = V^j \omega_j$ [einstein's Summation Notation]

An index that occurs twice is summed over if one is a subscript and on is a superscript

Example: $\tilde{\omega} = \sum_j \omega_j \tilde{d}x^j \rightarrow \omega_j \tilde{d}x^j$

Examples with no sum: $V^j V^k$ (not repeated index no sub) $V^j \omega_i$ (no repeated index) $V^j W^j$ (no subscripts)

§ 2.22 Tensor and Tensor fields

We build on vectors and 1-forms to get tensors

At $P \in M$ a tensor of Type $\binom{N}{N'}$ is a Linear Map (operator) that takes N- 1 forms and N' vectors and Yeilds a real number

Example: F is a $\binom{2}{2}$ tensor

We can write this as:

$$F(\tilde{\omega}, \tilde{\sigma}; \bar{V}, \bar{w})$$

Since it is Linear in all arguments

$$F(a\tilde{\omega} + b\tilde{\lambda}, \tilde{\sigma}; \bar{V}, \bar{\omega}) = a F(\tilde{\omega}, \tilde{\sigma}; \bar{V}, \bar{\omega}) + b F(\tilde{\lambda}, \tilde{\sigma}; \bar{V}, \bar{W})$$

$$F(\tilde{\omega}, \tilde{\sigma}; a\bar{V} + b\bar{U}, \bar{W}) = aF(\tilde{\omega}, \tilde{\sigma}; \bar{V}, \bar{W}) + bF(\tilde{\omega}, \tilde{\sigma}; \bar{U}, \bar{\omega})$$

§ 2.23 Examples of Tensors

In Linear Algebra, column vectors are vectors are $\binom{1}{0}$ tensors, Row "Vectors" are One-forms or $\binom{0}{1}$ tensors, and Matrix is a $\binom{1}{1}$ tensor

§ 2.24 Components of Tensors and the outer product

Consider 2 Vectors $\bar{V}, \bar{W}$. We can form a $\binom{2}{0}$ Tensor with the outer product.

Chosen / Variable

$$\bar{V} \otimes \bar{W} (\tilde{p}, \tilde{q}) = a \in \mathbb{R}$$
$$\equiv \bar{V}(\tilde{p})\, \bar{W}(\tilde{q})$$

Outer product
direct product
tensor product

If $\tilde{p}, \tilde{q}$ are one forms we can form a $\binom{0}{2}$ tensor

$$\tilde{p} \otimes \tilde{q} (\bar{V}, \bar{W}) = \tilde{p}(\bar{V})\, \tilde{q}(\bar{W})$$

chosen / variable

$\otimes$ is the outer/direct/tensor product

The outer product of an $\binom{N}{M}$ tensor and an $\binom{N'}{M'}$ tensor is a tensor of order $\binom{N+N'}{M+M'}$

The components of a tensor are the values it takes when it has basis vectors and 1-form as arguments.

Example: If S is a $\binom{3}{2}$ tensor, then on the basis $\{\bar{e}_i\}$ and $\{\tilde{\omega}_i\}$ has components

$$S^{ijk}{}_{lm} \equiv S(\tilde{\omega}^i, \tilde{\omega}^j, \tilde{\omega}^k; \bar{e}_l, \bar{e}_m)$$

Lec 8 Oct 3

§ 2.25 Contractions

$\bar{V} \otimes \tilde{\omega}$ is a $\binom{1}{1}$ tensor and is written as $V^i \omega_j$.

Consider examples with

S^i_{jk} is a $\binom{1}{2}$ tensor

P^{lm} as a $\binom{2}{0}$ tensor

These can be contracted in various ways

$\underbrace{S^i_{jk} P^{jm}}_{\text{contracted}}$ is a $\binom{2}{1}$ tensor

$S^i_{jk} P^{ks}$ is a $\binom{2}{1}$ tensor

In general, the 2 above differ [unless P is symetric]

Proprety: Contractions are independent of the basis

IDEA of Proof:

① $\tilde{q}(\bar{V}) = \cdots = q_i V^i$ see assignment 2 for the details

② Consider A a $\binom{2}{0}$ tensor and B a $\binom{0}{2}$ tensor. A contraction of A and B is $A^{ij} B_{jk}$
$= C^i_k$ where C is a $\binom{1}{1}$ tensor

Consider C applied to $\tilde{\sigma}$ and $\bar{V}$, $C(\tilde{\sigma}; \bar{V}) = C^i_k \sigma_i V^k$

$= A^{ij} B_{jk} \sigma_i V^k$

$= \sigma_i A^{ij} B_{jk} V^k$ } components to operators?

$= \sigma_i A(\tilde{\omega}^i, \tilde{\omega}^j) B(\bar{e}_j, \bar{e}_k) V^k$

using tensors are linear

$= A(\sigma_i \tilde{\omega}^i, \tilde{\omega}^j) B(\bar{e}_j, V^k \bar{e}_k)$

$= A(\tilde{\sigma}, \tilde{\omega}^j) B(\bar{e}_j, \bar{V})$

$= A(\tilde{\sigma}, B(\bar{e}_j, \bar{V}) \tilde{\omega}^j)$

Aside: $B(\bar{e}_j, \bar{V}) \tilde{\omega}^j = B_{lm} \tilde{\omega}^l \tilde{\omega}^m \bar{e}_j V^n \bar{e}_n \tilde{\omega}^j$

$= B_{lm} \underbrace{(\tilde{\omega}^l \bar{e}_j)}_{= \delta^l_j} \underbrace{(\tilde{\omega}^m \bar{e}_n)}_{= \delta^m_n} (V^n \tilde{\omega}^j)$

$\Rightarrow B(\bar{e}_j, \bar{V}) \tilde{\omega}^j = B_{jn} V^n \tilde{\omega}^j$

$= B_{jn} \tilde{\omega}^j V^n$

$= B(\ , \bar{V})$

$\therefore C(\tilde{\sigma}; \bar{V}) = A(\tilde{\sigma}, B(\underbrace{\ }_{\text{Empty space}}, \bar{V}))$ → completely independent from basis.

Aside: $B(\bar{e}_j, \bar{V})\tilde{\omega}^j = (B_{\ell m}\tilde{\omega}^\ell \tilde{\omega}^m)\ \bar{e}_j V^p \bar{e}_p \tilde{\omega}^j$

$= B_{\ell m} \underbrace{(\tilde{\omega}^\ell \bar{e}_j)}_{\delta^\ell_j} \underbrace{(\tilde{\omega}^m \bar{e}_p)}_{\delta^m_p} V^p \tilde{\omega}^j$

$= B_{jp} V^p \tilde{\omega}^j$

$\boxed{\text{Recall: } \tilde{\omega}^i \bar{e}_j = \delta^i_j}$

This is independent of basis and indices

§2.26 Basis Transformations

Recall, A tensor of type $\binom{N}{N'}$ is a linear function that takes N 1-forms and N' Vectors as arguments

This definition is modern. Previously tensors were defined as how components change under a change of Basis

At $P \in M$ suppose $\{\bar{e}_i\}$, $\{\bar{e}_{j'}\}$ are bases to T_pM. There is a linear transformation matrix Λ with $\bar{e}_{j'} = \Lambda^i_{j'} \bar{e}_i$

$\Lambda^i_{j'}$ is non-singular

Recall: One forms have a basis defined by $\tilde{\omega}^i(\bar{e}_k) = \delta^i_k$

We will determine how $\tilde{\omega}^i$ basis transforms. To do this, we multiply the definition by $\Lambda^k_{j'}$

$$\Lambda^k_{j'}\, \tilde{\omega}^i(\bar{e}_k) = \Lambda^k_{j'}\, \delta^i_k$$

$$\tilde{\omega}^i(\Lambda^k_{j'}\, \bar{e}_k) = \Lambda^i_{j'}$$

↓ by above

$$\tilde{\omega}^i(\bar{e}_{j'})$$

$$\Rightarrow \tilde{\omega}^i(\bar{e}_{j'}) = \Lambda^i_{j'}$$

We define the inverse of $\Lambda^i_{j'}$ to be $\Lambda^{k'}_{j}$

$$\boxed{\begin{gathered} \Lambda^{k'}_{j}\Lambda^{j}_{i'} = \delta^{k'}_{i'} \\ \Lambda^{k'}_{j}\Lambda^{i}_{k'} = \delta^{i}_{j} \end{gathered}}$$

We multiple the previous eqn by $\Lambda^{k'}_i$ to get $\Lambda^{k'}_i\, \tilde{\omega}(\bar{e}_{j'}) = \Lambda^{k'}_i \Lambda^i_{j'} = \delta^{k'}_{j'}$

$$\Lambda^{k'}_i\, \tilde{\omega}^j(\bar{e}_{j'}) = \delta^{k'}_{j'} = \tilde{\omega}^{k'}(\bar{e}_{j'})$$

The functions in front of $\bar{e}_{j'}$ must be equal.

$$\boxed{\tilde{\omega}^{k'} = \Lambda^{k'}_i \omega^i}$$

$\tilde{\omega}^{k'}$ transforms with $\Lambda^{k'}_i$

Compare with

$$\bar{e}_{j'} = \Lambda^{i}{}_{j'} \bar{e}_i$$ $\bar{e}_{j'}$ transforms with $\Lambda^{i}{}_{j'}$

1-forms transform with the inverse of Λ. we can also transform coordinates

$$V^{i'} = \tilde{\omega}^{i'}(\bar{V}) = \Lambda^{i'}{}_j \tilde{\omega}^{j}(\bar{V}) = \Lambda^{i'}{}_j V^j$$

and Similarly

$$q_{k'} = \tilde{q}(\bar{e}_{k'}) = \tilde{q}(\Lambda^{j}{}_{k'} \bar{e}_j) = \Lambda^{j}{}_{k'} \tilde{q}(\bar{e}_j) = \Lambda^{j}{}_{k'} q_j$$

Summary: V^i and $\tilde{\omega}^i$ Transform with $\Lambda^{i'}{}_j$

q_i and $\bar{e}_i$ transform with $\Lambda^{i}{}_{k'}$

Its because of these difference that

$V^i \bar{e}_i$ & $V^j \bar{e}_j$ are basis independent.

Vector Coordinates (Superscipts) are contra variant since they transforms "oposite" ~~to~~ How there basis transforms

1-form Coordinates (subscripts) are covariant since they transform like $\bar{e}_i$

§ Coordinate Transformations

Suppose $U \subset M$ has coordinates $\{x^i, i=1,\ldots,n\}$. Introduce new coordinates $\{y^i, i=1,\ldots,n\}$

$$y^{j'} = f(x^1, \ldots, x^n) \quad j' = 1,\ldots,n$$
$$y^{j'} = f^{j'}(x^j)$$

The coordinate transformation is $\frac{\partial y^{j'}}{\partial x^i}$ has a non-zero determinant in U Jacobian?

ALL $P \in V$ can be described with $\{x^i\}$ or $\{y^{j'}\}$. has coordinat vector basis

$$\left\{ \frac{\partial}{\partial x^i} \right\} \text{ and } \left\{ \frac{\partial}{\partial y^{j'}} \right\}$$

These must be related by

$$\frac{\partial}{\partial y^{j'}} = \frac{\partial x^{i}}{\partial y^{j'}} \frac{\partial}{\partial x^{i}}$$

Compare with what we saw previously

$$\bar{e}_{j'} = \Lambda^{i}{}_{j'} \bar{e}_{i}$$ explict expression for lamba

$$\therefore \boxed{\Lambda^{i}{}_{j'} = \frac{\partial x^{i}}{\partial y^{j'}}}$$ covariant

Similarly $\boxed{\Lambda^{k'}{}_{j} = \frac{\partial y^{k'}}{\partial x^{j}}}$ Contra variant

Since $\frac{\partial x^{i}}{\partial y^{j'}} \frac{\partial y^{j'}}{\partial x^{k}} = \frac{\partial x^{i}}{\partial x^{k}} = \delta^{i}{}_{k}$

Lec 9 Oct 5, 2023

§ 2.27 Tensor Operations on Components

Given a tensor T with components $\{T^{i\cdots}{}_{j\cdots}\}$ on a basis, the following is basis invariant.

$$aT \quad \text{or} \quad \{a T^{i\cdots}{}_{j\cdots}\}$$

This can be denoted as $T \to aT$

The Outer Product is also basis invariant

$$A, B \to A \otimes B$$

OR

$$\{A^{i\cdots}{}_{j\cdots}\}, \{B^{k\cdots}{}_{l\cdots}\} \to \{A^{i\cdots}{}_{j\cdots} B^{k\cdots}{}_{l\cdots}\}$$

A tensor operation is one where operations on components produces components that are the tensor, independent of the basis. This include:

1. Addition
2. Scalar Mutiplication
3. Outer Products
4. Contractions

§ 2.28 Functions and Scalars

A scalar is a $\binom{0}{0}$ tensor, which is a function on M independent of the basis

Example: $V^i \tilde{\omega}_i$ is a scalar

V^i is not a scalar

§ 2.29 The metric tensor on a Vector Space

An inner product is a rule that associates a number with 2 vectors and it is a $\binom{0}{2}$ tensor

It is also reffered to as a metric tensor, $g_{\bar{i}}$ bar not one

$$\boxed{g_1(\bar{v}, \bar{w}) = g_1(\bar{w}, \bar{v}) \equiv \bar{w} \cdot \bar{v}}$$

g_1 is a symmetric tensor with components

$$\boxed{g_{ij} = g_1(\bar{e}_i, \bar{e}_j)}$$

We will require that g_1 has an inverse.

If $g_{ij} = \delta_{ij}$ then it is the Euclidean metric, and the vector space is Euclidean space, $\mathcal{E}^n$

Given any g_{ij}, we can change to a new basis, say $\{\bar{e}_{j'}\}$ such that

$$g_{i'j'} = \Lambda^k{}_{i'} \Lambda^\ell{}_{j'} \, g_{k\ell}$$

We can pick a basis where the metric tensor is diagonal and only has +1 or −1 as enteries

The convention is to list the −1's first then the +1's:

$$\boxed{g_{i'j'} = \text{diag}(-1, \ldots, -1, 1, \ldots 1)}$$

This is written in an orthanormal basis

The sum [trace] of these elements is the signature.

We classify the metric tensor as

positive definite if all +1's

negative definite if all −1's

in definte if we have both 1's and −1's

Example: Minkowski metric in special relativity

$(-1, 1, 1, 1)$ or $(1, 1, 1, -1)$

This is indefinite

Euclidean Space, $\mathcal{E}^n$ is called Cartesian, with

$$g_{ij} = \delta_{ij} \quad \text{or} \quad g_1 = I$$

The minkowski metric has the Lorentz basis

$$\eta = \mathrm{diag}(-1, 1, 1, 1)$$

A transformation matrix from one Lorentz basis to another can be written as

$$\eta = \Lambda_L^T \, \eta \, \Lambda_L$$

The metric tensors maps vectors to 1- forms.

$$\tilde{V} = g_1(\bar{V}, \underbrace{\_}_{\text{Blank}})$$

In terms of Components,

$$\begin{aligned} V_i &= \tilde{V}(\bar{e}_i) \\ &= g_1(\bar{V}, \bar{e}_i) \\ &= g_1(V^j \bar{e}_j, \bar{e}_i) \\ &= V^j g_1(\bar{e}_j, \bar{e}_i) \\ &= V^j g_{ji} \\ &= g_{ij} V^j \end{aligned}$$

Symmetry in g_{ij}

The inverse matrix g_{ij} is called g^{ij} with

$$g^{ij} g_{jk} = \delta^i_k \quad \text{or} \quad g_{ij} g^{jk} = \delta^k_i$$

you can show

This yields,

$$\begin{aligned} & g^{ki} V_i \\ &= g^{ki} g_{ij} V^j \\ &= \delta^k_j V^j \end{aligned}$$

This recovers what we had before

Summary: $V_i = g_{ij} V^j$

and

$V^k = g^{ki} V_i$

A $\binom{2}{0}$ tensor, A, can map to a $\binom{1}{1}$ tensor

$$A^i{}_j = g_{jk} A^{ik}$$

This can be mapped to $\binom{0}{2}$ tensor

$$A_{\ell j} = g_{\ell m} A^m{}_j$$

This can be inverted $A^{ik} = g^{i\ell} g^{km} A_{\ell m}$

Skip the proof

This is called index raising and lowering with a metric tensor, there is much less diff difference betwen $\binom{N}{N'}$ and $\binom{N-1}{N'-1}$ and $\binom{N+1}{N'+1}$

Hence why we often refer to them as tensors of order $N + N'$

In a Euclidean vector space, a cartesian basis is $g_{ij} = g^{ij}$

There is no difference between super scripts and subscripts. and hence we often only use subscripts.

§ 2.30 The metric tensor field on a manifold

A metric tensor g, on a manifold is a $\binom{0}{2}$ symmetric tensor and has an inverse at every point.

For all points in the manifold, g serves as a metric on T_pM and has all the propreties mentioned previously, but there's more

Using g, we can define distance and curvature

We can use g_1 to define length on M

Suppose a curve has a tangent vector $\bar{V} = \frac{d\bar{x}}{d\lambda}$

then
$$dl^2 = d\bar{x} \cdot d\bar{x} = (\bar{V} d\lambda) \cdot (\bar{V} d\lambda)$$
$$= \bar{V} \cdot \bar{V} (d\lambda)^2$$
$$= g_1 (\bar{V}, \bar{V}) d\lambda^2$$

d is infinitesmal and not gradient

If g_1 is positive definite then dl^2 is positive and

$$\boxed{dl = (g_1(\bar{V}, \bar{V}))^{\frac{1}{2}} d\lambda}$$

If g_1 is indefinite then curves can have
dl positive (space like) or negative (time like)

dl is the proper distance for space like curves and the proper time for time-like curves

§ 2.31 Special Relativity

$\mathbb{R}^4$ with a metric with signature $+2$ is a manifold called Minkowski space time from special relativity

We can define coordinates $(\Delta t, \Delta x, \Delta y, \Delta z)$ then
$$\Delta s^2 = -c^2 (\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

c is the speed of light

Define $x^0 = ct$, $x^1 = x$ $x^2 = y$, $x^3 = z$ then

$$\Delta s^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$

or $$\Delta s^2 = \eta_{\alpha\beta} \Delta x^\alpha \Delta x^\beta$$

This is a psuedo Norm and satisfies

(N iii) factor out a Scalar
(N iv) parallelgram rule

These are what we need to define an inner product

$$\bar{V} \cdot \bar{W} = \eta_{\alpha\beta} V^{\alpha} W^{\beta}$$

Midterm: covers lec 1-9, but none of the special Relativity

To study: make sure you understand all the lectures and assigments [soln's on Sunday]

Francis will give a description of questions to expect

Will post sample formula Sheet on Monday.