Lec 11 ~ Oct 19

§ 3.1 Intro: How a vector field maps a manifold to itself

Recall: A vector field $\bar{V}$ induces an integral curve

$$\boxed{\frac{dx^i}{d\lambda} = v^i(x^j)}$$ → all possible cordinates

Propreties of integral curves:

① $\exists$ a unique curve through each $P \in M$

② These curves fill the manifold M

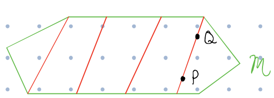
to cover the entire manifold curve has 1 parameter, need $(n-1)$ more to get n-manifold

If M is n dimensonal then the set of integral curves are $(n-1)$ dimensional

Curves like this that fill the manifold is a congruence.

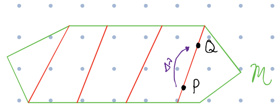
These integral curves provide a natural mapping from M to M along $\bar{V}$. If $\bar{V}$ is C^∞, then the mapping is diffeomorphic

Such a mapping is Lie dragging



§ 3.2 Lie dragging a function.

Suppose f is a function on a manifold M.



P and Q are on the same curve

We define $\boxed{f(P) = f^*_{\Delta\lambda}(Q)}$ → How to get to P from Q along an integral curve

Star denotes new

$\Delta\lambda$ is very small, but could be big, If $f(Q) = f^*_{\Delta\lambda}(P)$ then f is invariant under the map.

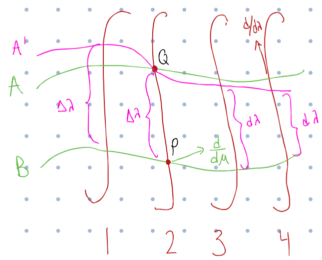
If f is invariant $\forall \Delta\lambda$, then it is said to be Lie dragged

If function f that is Lie dragged must satisfy, $\frac{df}{d\lambda} = 0$

§ 3.3 Lie dragging a Vector field

A vector field can be defined by a congruence of curves for which it is tangent.

We now show how to Lie drag a vector field



1, 2, 3, 4 are Integral curves of $\frac{d}{d\lambda}$
(Congruence in λ)

A, B are Integral curves of $\frac{d}{d\mu}$
They form a congruence of μ

The points along A (μ congruence) are dragged along $\Delta\lambda$ to the curve A', A' need not be a congruence of μ

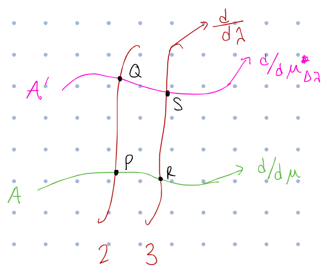
A' defines a new congruence with parameter $\mu^*_{\Delta\lambda}$

This has a tangent vector field $\frac{d}{d\mu^*_{\Delta\lambda}}$, which is the image of $\frac{d}{d\mu}$ under Lie dragging

In general $\mu^*_{\Delta\lambda}$ congruence differs from μ congruence. If they are the same then $\boxed{\frac{d}{d\mu^*_{\Delta\lambda}} = \frac{d}{d\mu}}$ every where

We say a vector field and congruence are invariant under the map

If it is invariant $\forall \Delta\lambda$, then the curves are said to be Lie dragged by $d/d\lambda$.



If the distances are infinitesemal and if $\frac{d}{d\mu}$ stretches from P to R on the curve A then:
$\frac{d}{d\mu^*_{\Delta\lambda}}$ streches from Q to S on A'

If $\frac{d}{d\mu}$ is Lie draged then B conincides with A and

$$\left(\frac{d}{d\mu^*}_{\Delta\lambda}\right)_Q = \left(\frac{d}{d\mu}\right)_Q$$

This implies $\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right] = 0$

A vector field is Lie dragged iff $\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right] = 0$

§ 3.4 Lie derivative

The derivative of a scalar valued function $\mathbb{R}$ is:

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \Big\} \text{ newton's quotient}$$

To compute this we need to compare the function at different points and divide by the distance between them. This has 2 problems:

1. We don't always have distance but we have the parameter λ along 2 integral curves
2. Must compare function at different points. We do this be Lie dragging

① function:

Method ① evaluate f at $\lambda_0 + \Delta\lambda$, $f(\lambda_0 + \Delta\lambda)$ and drag it back to λ_0

② Evaluate f at λ_0.

③ Find the difference, ÷ by $\Delta\lambda$ and take the limit as $\Delta\lambda \to 0$

For ① define f^* such that $\frac{df^*}{d\lambda} = 0$, hence $f^*(\lambda_0) = f(\lambda_0 + \Delta\lambda)$

Hence we get:

$$\lim_{\Delta\lambda\to 0} \frac{f^*(\lambda_0) - f(\lambda_0)}{\Delta\lambda}$$

$$= \lim_{\Delta\lambda\to 0} \frac{f(\lambda_0 + \Delta\lambda) - f(\lambda_0)}{\Delta\lambda} = \left.\frac{df}{d\lambda}\right|_{\lambda_0}$$

The lie derivative of a function is $\boxed{\mathcal{L}_{\overline{V}} f = \overline{V}(f) = \frac{df}{d\lambda}}$

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§ 3.4 Lie derivates (Continued)

The lie derivative of a function $f: M \to \mathbb{R}$ along a vector field $\bar{V}$ is computed using Lie dragging

Recall, $f(P) = f^*_{\Delta\lambda}(Q)$ and $f(\lambda_0) = f^*_{\Delta\lambda}(\lambda_0 + \Delta\lambda)$

If we $\Delta\lambda \longrightarrow -\Delta\lambda$ and $\lambda_0 \longrightarrow \lambda_0 + \Delta\lambda$ then

$$f(\lambda_0) = f^*_{-\Delta\lambda}(\lambda_0 - \Delta\lambda)$$

$$f(\lambda_0 + \Delta\lambda) = f^*_{-\Delta\lambda}(\lambda_0)$$

Then the Lie derivative of f along $\bar{V}$ is

$$(\mathcal{L}_{\bar{V}} f)_{\lambda_0} = \lim_{\Delta\lambda \to 0} \frac{f^*_{-\Delta\lambda}(\lambda_0) - f(\lambda_0)}{\Delta\lambda}$$

$$= \lim_{\Delta\lambda \to 0} \frac{f(\lambda_0 + \Delta\lambda) - f(\lambda_0)}{\Delta\lambda}$$

$$\Rightarrow (\mathcal{L}_{\bar{V}} f)_{\lambda_0} = \left[\frac{df}{d\lambda}\right]_{\lambda_0} = \bar{V}(f)_{\lambda_0}$$

In component form $(\mathcal{L}_{\bar{V}} f)_{\lambda_0} = \left(\frac{dx^i}{d\lambda} \frac{\partial f}{\partial x^i}\right)_{\lambda_0}$

This generalizes to any differentiable manifold

Most text book's use push back / pull forwards.

what is a derivative along a vector?

To Compute the Lie derivative of a vector field, consider $\bar{u} = \frac{d}{d\mu}$ and $\bar{Y} = \frac{d}{d\lambda}$ and consider an arbitrary function f

At λ_0 and $\lambda_0 + \Delta\lambda$ we know that the Lie derivative is

$$(\mathcal{L}_{\bar{u}} f)_{\lambda_0} = \left(\frac{df}{d\mu}\right)_{\lambda_0} = (\bar{U}(f))_{\lambda_0}$$

$$(\mathcal{L}_{\bar{u}} f)_{\lambda_0 + \Delta\lambda} = \left(\frac{df}{d\mu}\right)_{\lambda_0 + \Delta\lambda} = (\bar{U}(f))_{\lambda_0 + \Delta\lambda}$$

Now, we can Lie drag $\bar{u}(\lambda_0 + \Delta\lambda)$ to λ_0 with $\bar{U}^*_{-\Delta\lambda}(\lambda_0) = \bar{U}(\lambda_0 + \Delta\lambda) = \frac{d}{d\mu^*_{-\Delta\lambda}}$

With $[\bar{U}^*_{-\Delta\lambda}, \bar{V}] = 0$ because of what Francis said in lecture [transcribe later]

We taylor expand $\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0}$

$$\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0 + \Delta\lambda} = \left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0} + \Delta\lambda\left[\frac{d}{d\lambda}\left(\frac{d}{d\mu^*_{-\Delta\lambda}} f\right)\right]_{\lambda_0} + O\left(\Delta\lambda^2\right)$$

If we solve for the first term on the RHS,

$$\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0} = \left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0 + \Delta\lambda} - \Delta\lambda\left[\frac{d}{d\lambda}\left(\frac{d}{d\mu^*_{-\Delta\lambda}} f\right)\right]_{\lambda_0}$$

BUT $\frac{d}{d\mu^*_{-\Delta\lambda}} = \frac{d}{d\mu}$

$$\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0} = \left[\frac{df}{d\mu}\right]_{\lambda_0+\Delta\lambda} - \Delta\lambda\left[\frac{d}{d\lambda}\left(\frac{d}{d\mu^*_{-\Delta\lambda}} f\right)\right]_{\lambda_0}$$

↳ Taylor Expand this term

Change order because $[\bar{u}^*_{-\Delta\lambda}, \bar{v}] = 0$

$$\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0} = \left[\frac{df}{d\mu}\right]_{\lambda_0} + \Delta\lambda\left[\frac{d}{d\lambda}\left(\frac{df}{d\mu}\right)\right]_{\lambda_0} - \Delta\lambda\left[\frac{d}{d\mu^*_{-\Delta\lambda}}\left(\frac{df}{d\lambda}\right)\right]_{\lambda_0} + O(\Delta\lambda^2)$$

We define the Lie derivative of $\bar{u}$ along $\bar{v}$ is

$$(\mathcal{L}_{\bar{v}}\bar{u})(f) = \lim_{\Delta\lambda\to 0}\left[\frac{\bar{u}_{-\Delta\lambda}(\lambda_0) - \bar{u}(\lambda_0)}{\Delta\lambda}\right](f)$$

$$= \lim_{\Delta\lambda\to 0}\left[\left(\frac{df}{d\mu^*_{-\Delta\lambda}}\right)_{\lambda_0} - \left(\frac{df}{d\mu}\right)_{\lambda_0}\right] / \Delta\lambda$$

Aside ✱

$$\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0} - \left[\frac{df}{d\mu}\right]_{\lambda_0} = \Delta\lambda\left[\frac{d}{d\lambda}\left(\frac{df}{d\mu}\right)\right]_{\lambda_0} - \Delta\lambda\left[\frac{d}{d\mu^*_{-\Delta\lambda}}\left(\frac{df}{d\lambda}\right)\right]_{\lambda_0} + O(\Delta\lambda^2)$$

Substitute ✱

$$(\mathcal{L}_{\bar{v}}\bar{u})(f) = \lim_{\Delta\lambda\to 0}\left[\frac{d}{d\lambda}\frac{d}{d\mu}f - \frac{d}{d\mu^*_{-\Delta\lambda}}\frac{d}{d\lambda}f\right] + O(\Delta\lambda)$$

This is $O(\Delta\lambda)$ b/c $\bar{w}$ divided by $\Delta\lambda$ in the limit

$$(\mathcal{L}_{\bar{v}}\bar{u})(f) = \left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right](f) \quad \text{or} \quad = [\bar{v}, \bar{u}](f)$$

Lie bracket is anti symmetric

This is equivalent to the directional derivative of $\bar{u}$ in the direction of $\bar{v}$

§ 3.5 Lie derivative of a one-form

We can determine the Lie derivative of a 1-form in terms of the Lie derivative of a function and Vector field

The Lie derivative of a One form can be computed as follows:

Libentz rule is Product Rule

$$\mathcal{L}_{\bar{v}}[\tilde{\omega}(\bar{w})] = \frac{d}{d\lambda}[\tilde{\omega}(\bar{w})]$$

$$= \frac{d\tilde{\omega}}{d\lambda}(\bar{w}) + \tilde{\omega}\left(\frac{d\bar{w}}{d\lambda}\right)$$

OR

$$\mathcal{L}_{\bar{v}}[\tilde{\omega}(\bar{w})] = (\mathcal{L}_{\bar{v}}\tilde{\omega})\bar{w} + \tilde{\omega}(\mathcal{L}_{\bar{v}}\bar{w})$$

This is the Operator, there is an imaginary function at the end of all these terms

This method extends to the outer product of tensors

$$\mathcal{L}_{\bar{v}}(A\otimes B) = \mathcal{L}_{\bar{v}}(A)\otimes B + A\otimes\mathcal{L}_{\bar{v}}(B)$$

or,

$$\mathcal{L}_{\bar{v}}(T(\tilde{\omega},\ldots;\bar{v},\ldots)) = (\mathcal{L}_{\bar{v}}T)(\tilde{\omega},\ldots;\bar{v},\ldots) + T(\mathcal{L}_{\bar{v}}\tilde{\omega},\ldots;\bar{v},\ldots) + \ldots + T(\tilde{\omega},\ldots;\mathcal{L}_{\bar{v}}(\bar{v}),\ldots) + \ldots$$

for all components of the tensor

This is the Product or Leibniz Rule

§ 3.6 Submanifold

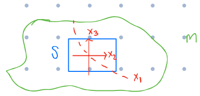
The idea is that a submanifold S of a manifold M is a subset of M which is itself a manifold.

Ex $\mathbb{R}^3$ is a manifold, a smooth surface is a submanifold.
a smooth curve is a submanifold

A m-dimensional submanifold S of a n-dimensional is a subset of M with the proprety that in some neighbourhood of P

$P \in S \subset M$, there exists a coordinate system for M in which the points of S can be written as

$$x^1 = x^2 = \dots = x^{n-m} = 0$$



Solns to the system of DEs $y_i' = f_i(x_1', \dots, x^n)$ with $i = 1, \dots, p$ and with coordinates $\{x_1', \dots, x^n\}$. This is a submanifold with coordinates $\{y_1, \dots, y_n, x_1', \dots, x^n\}$

Suppose $P \in S \subset M$ with $\dim S = m$ and $\dim M = n$. A curve in S through P is a curve in both M & S, Through P.

$T_p S$: Tangent space at P in S (dim m)

$T_p M$: Tangent space at P in M (dim n)

$T_p S$ is a vector subspace of $T_p M$ and a Submanifold

A tangent vector at P is both in $T_p S$ and $T_p M$

$T_p^* S$: cotangent space at P in S
$T_p^* M$: cotangent space at P in M

Any $\tilde{\omega} \in T_p^* M$ yields a $\tilde{\omega} \in T_p^* S$ if we restrict the domain to $T_p S$ instead of $T_p M$.

However, $\tilde{\omega} \in T_p^* S$ does not yield a unique $\tilde{\omega}$ in $T_p^* M$

Summary: $\bar{V} \in T_p S$ is also a vector in $T_p M$ and $\tilde{\omega} \in T_p^* M$ is also a one form in $T_p^* S$

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Lie derivatives

$(\mathcal{L}_{\vec{v}}\vec{u})^i = V^r \frac{\partial}{\partial x^r} u^i - u^r \frac{\partial}{\partial x^r} V^i$

$(\mathcal{L}_{\vec{v}}\tilde{\omega})_i = V^r \frac{\partial}{\partial x^r}\omega_i + \omega_r \frac{\partial}{\partial x^i} V^r$

indices inbetween

In general for $T^{i\ldots j}{}_{k\ldots l}$

$(\mathcal{L}_{\vec{v}} T)^{i\ldots j}{}_{k\ldots l} = V^r \frac{\partial}{\partial x^r} T^{i\ldots j}{}_{k\ldots l} - T^{r\ldots j}{}_{k\ldots l}\frac{\partial}{\partial x^r}V^i - \ldots - T^{i\ldots r}{}_{k\ldots l}\frac{\partial}{\partial x^r}V^j + T^{i\ldots j}{}_{r\ldots l}\frac{\partial}{\partial x^k}V^r + \ldots + T^{i\ldots j}{}_{k\ldots r}\frac{\partial}{\partial x^l}V^r$

§ 3.7 Frobenius Thm (Vector Field Version)

Suppose a coordinate patch of $S \subset M$ has coordinates y^a $a = \{1, \ldots, n\}$ with basis vectors $\{\frac{\partial}{\partial y^a}\}$ for vector fields on S, with $[\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}] = 0$ $\forall a \neq b$. Since it is a coordinate basis

It can be shown that in general for a non coordinate basis, the Lie bracket of any of these two vector fields, yields a vector field tangent to S

The next theorem says something about the sub manifold if we know a proprety of the Lie bracket of a vector field.

Frobenius' theorem (Vector Field Version)

If a set of m smooth vector feilds in $U \subset M$ (subset) have Lie brackets which is a linear combinations of the m vector field, then the integral curves of the fields mesh to form a family of sub manifolds

mesh: means combined

Implications

Dim of the submanifold is $\leq m$

Each point in $\mathcal{U}$ is on one and only on submanifold. This family of sub manifold is a foiliation of $\mathcal{U}$ and fills $\mathcal{U}$ like the congruence curve do. Each sub manifold is a leaf.

§ 3.9 An Example: the generation of S^2

Consider a ϕ-based vector in spherical coordinates called $\bar{e}_\phi = -y\,\bar{e}_x + x\,\bar{e}_y$

Using our notion, this becomes

$$\frac{\partial}{\partial \phi} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \vec{l}_z$$

angular momentum operator in the z direction

Similarily
$$\bar{l}_x = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$
$$\bar{l}_y = -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}$$

It can be shown
$$[\bar{l}_x, \bar{l}_y] = -\bar{l}_z$$
$$[\bar{l}_y, \bar{l}_z] = -\bar{l}_x$$
$$[\vec{l}_3, \bar{l}_x] = -\bar{l}_y$$

Checking $[\bar{l}_x, \bar{l}_y] = \left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)\left(-x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right) - \left(-x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right)\left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)$

$$\vdots$$
$$= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = -\bar{l}_z$$

Since $\{\bar{l}_x, \bar{l}_y, \bar{l}_3\}$ have Lie brackets that are a linear combination of the set, Frobenius' theorem yields integral curves that form a Submanifold

Since we have 3 vector fields, we might think the dimension of this set is 3, it turns out the dim is 2.

To see this consider $r = (x^2 + y^2 + z)^{1/2}$

We can show that $\tilde{d}r$ is the gradient of r

$$\tilde{d}r(\bar{l}_x) = \tilde{d}r(\bar{l}_y) = \tilde{d}r(\bar{l}_3) = 0$$

check: $\tilde{d}r(\bar{l}_x) = \bar{l}_x(r)$

$$= \left(z\frac{\partial}{\partial y} + y\frac{\partial}{\partial 3}\right)\sqrt{x^2 + y^2 + z^2}$$

$$= -z\left(\frac{y}{r}\right) + y\frac{z}{r} = 0$$

B/c of symmetry of the operators, these all equal zero.

Since the gradients are 0 these exist in the tangent Space $\Rightarrow$ dim 2

We can consider $\tilde{d}r$ to be a set of surfaces of constant r, S^2. Since $\bar{l}_x, \bar{l}_y, \bar{l}_3$ are orthogonal to the gradient, they must all lie in the tangent, which is 2 dim $\therefore \{\bar{l}_x, \bar{l}_y, \bar{l}_z\}$ is only dimenson 2

§ 3.10 Invariance

Lie derivatives are often used to show that a tensor is invariant in a direction

We say T is invariant under a vector field $\bar{V}$ if

$$\mathcal{L}_{\bar{V}} T = 0$$

T could be
1. metric tensor
2. a Scalar field for PE of a partical
3. Vectors $\bar{V}$ under which T is invariant are important.

§ 3.11 killing Vector fields

Metric tensor can be invariant with respect to a vector field. These Vector fields are important.

A killing Vector field is a vector field, $\bar{V}$, such that

$$\mathcal{L}_{\bar{V}} g_1 = 0$$

From excersize 3.4, you deduce that $(\mathcal{L}_{\bar{V}} g_1)_{ij} = V^k \frac{\partial}{\partial x^k} g_{ij} + g_{kj} \frac{\partial}{\partial x^i} V^k + g_{ik} \frac{\partial}{\partial x^j} V^k = 0$

For a killing vector field

Pick coordinates such that the integral Curve are in the x^1 direction, then

$$V^i = \delta^i_1$$

Then the above simplifies,

$$(\mathcal{L}_{\bar{V}} g_1)_{ij} = \frac{\partial}{\partial x^1} g_{ij} = 0$$

Therefore the metric tensor is invariant with respect to the killing Vector

Example: Consider $\mathbb{R}^3$ in the different coordinates

1. Euclidean space $g_{ij} = \delta_{ij}$

This form is independent of x, y, z and $\therefore$ $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are all killing vectors

2. Spherical coordinats $g_{rr} = \frac{\partial}{\partial r} \cdot \frac{\partial}{\partial r} = 1$

$$g_{\theta\theta} = \frac{\partial}{\partial \theta} \cdot \frac{\partial}{\partial \theta} = r$$

$$g_{\phi\phi} = \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} = r^2 \sin^2\theta$$

g_1 is a $\binom{0}{2}$ tensor and only diagonal is non zero

g_1 is independent of ϕ hence $\bar{l}_z$ is a killing vector

It can be shown that $\bar{l}_x$ and $\bar{l}_y$ are also killing vectors

These 6 killing vectors $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \bar{l}_x, \bar{l}_y, \bar{l}_z$ are the only killing vectors possible

§ 3.12 killing Vectors and Conserved quantities in partical dynamics.

In classical mechanic s it follow that:

1. if the P.E function is axially symmetric, then the angular momentum is conserved

② If the PE is independent of say x, the the x component of moment is conserved

these symmetries in the PE energy function give rise to conserved quantities.

However if another symmetry is found in PE function does that mean something else is conserved?

Conserved quantities dont just require the PE is invariant w.r.t to a variable but we also require that is a killing vector

Idea: Newton's 2^{nd} Law

$$m\dot{\vec{V}} = -\vec{\nabla}\Phi \quad \text{or} \quad m\dot{V}^i = -\vec{\nabla}^i\Phi \quad \text{or} \quad m\dot{V}^i = -g^{ij}\frac{\partial}{\partial x^j}\Phi$$

Any invariance of this equation that both Φ and g_1^{-1} are invariant w.r.t coordinate

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Claim: most abstract concepts have been introduced. Chapter 4 is on differential forms

§4 Differential forms

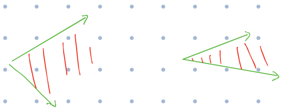
Now, we develop calculus of differential forms or often called exterior calculus or differentialble manifolds

A The algebraic & Integral calculus of forms

§ 4.1 Defn of Volume and the geometric role of differential forms

we now study a class of tensors that enable us to define Volume on elements (with out an inner product)

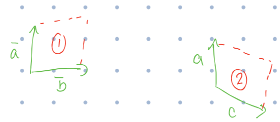
A pair of non parallel vectors in euclidean space defines an infinitesmial area



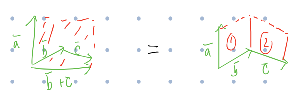
In our definition of area (volume), we do not need to know the length of the vector or the angle between them

Consider a 2D manifold and suppose we have two linearly independent infinitesmial vectors, They form a parralelogram

We want to find the area between $\bar{a}, \bar{b}$. Our definition of area must satisfy the following $\bar{a}\ \bar{b}\ \bar{c}$



$$\boxed{\text{area}(\bar{a},\bar{b}) + \text{area}(\bar{a},\bar{c}) = \text{area}(\bar{a}, \bar{b}+\bar{c})}$$



Since area(,) takes 2 vectors and yields a number it must be a $\binom{0}{2}$ tensor

Observe, $\boxed{\text{area}(\bar{a},\bar{a}) = 0 \quad \text{for all } \bar{a}}$

Excersize 4.1 If B is a $\binom{0}{2}$ tensor with $B(\bar{u},\bar{u}) = 0 \ \forall \bar{u}$ then $B(\bar{u},\bar{w}) = -B(\bar{w},\bar{u})$

Proof $B(\bar{u}+\bar{w}, \bar{u}+\bar{w}) = 0$

B/c Linear operator

$$B(\cancel{\bar{u},\bar{u}}) + B(\bar{w},\bar{u}) + B(\bar{u},\bar{w}) + B(\cancel{\bar{w},\bar{w}})^{0} = 0$$

$$B(\bar{u},\bar{w}) = -B(\bar{w},\bar{u})$$

Note ① that the area function must satisfy this anti symmetry proprety
② Area is not non negative

Recall for linear algebra, we can find area using

$$\text{aree} = \det \begin{vmatrix} V_x & V_y \\ W_x & W_y \end{vmatrix}$$ ← Anti-symetric

§ 4.2 Notation and definitions for anti symmetric tensors

A $\binom{0}{2}$ tensor is anti symetric if

$$\tilde{\omega}(\bar{u},\bar{v}) = -\tilde{\omega}(\bar{v},\bar{u})$$

A $\binom{0}{3}$ tensor is anti symetric if it changes signs when we exchange any 2 elements

$$\tilde{\omega}(\bar{u},\bar{v},\bar{w}) = -\tilde{\omega}(\bar{v},\bar{u},\bar{w})$$
$$= -\tilde{\omega}(\bar{w},\bar{v},\bar{u})$$
$$= -\tilde{\omega}(\bar{v},\bar{w},\bar{v})$$

Given any tensor we can build an antisymmetric version of it.

Ex. If $\tilde{\omega}$ a $\binom{0}{2}$ tensor then

$$\tilde{\omega}_A(\bar{u},\bar{v}) = \frac{1}{2!}\left[\tilde{\omega}(\bar{u},\bar{v}) - \tilde{\omega}(\bar{v},\bar{u})\right]$$

This is the anti symmetric part of $\tilde{\omega}$

If $\tilde{p}$ is a $\binom{0}{3}$ tensor then

$$\tilde{p}_A(\bar{u},\bar{v},\bar{w}) = \frac{1}{3!}\left[\tilde{p}(\overset{1}{\bar{u}},\overset{2}{\bar{v}},\overset{3}{\bar{w}}) + \tilde{p}(\overset{2}{\bar{v}},\overset{3}{\bar{w}},\overset{1}{\bar{u}}) + \tilde{p}(\overset{3}{\bar{w}},\overset{1}{\bar{u}},\overset{2}{\bar{v}}) - \tilde{p}(\overset{3}{\bar{w}},\overset{2}{\bar{v}},\overset{1}{\bar{u}}) - \tilde{p}(\overset{2}{\bar{v}},\overset{1}{\bar{u}},\overset{3}{\bar{w}}) - \tilde{p}(\overset{1}{\bar{u}},\overset{3}{\bar{w}},\overset{2}{\bar{v}})\right]$$

Normalizing by number of terms

$\tilde{p}_A$ is the anti symmetric part of $\tilde{p}$

Notation:

$$(\tilde{\omega}_A)_{ij} = \frac{1}{2!}(\omega_{ij} - \omega_{ji}) \equiv \omega_{[ij]}$$ ↙ this denotes antisymmetric part of $\tilde{\omega}$

$$(\tilde{p}_A)_{ijk} = \frac{1}{3!}(P_{ijk} + P_{jki} + P_{kij} - P_{kji} - P_{jik} - P_{ikj}) \equiv P_{[ijk]}$$

$[i \ldots k]$ denotes a completely antisymmetric set of indicies

Notation: we use ~ to denot a complety anti symetric part of a tensor.

example: T is only at tensor & $\tilde{T}$ is the anti symmetric version of T

Also we say a one form is anti symmetric

Property: For an n-dimensional vector space, a completely antisymmetric $\binom{0}{p}$ tensor $(p \leq n)$ has at most

How many ways can we pick p directions in n space? $C^n_p = \frac{n!}{p!(n-p)!} = \binom{n}{p}$ "n choose p" independent components

why be anti symmetric? will learn later on!

ex In $\mathbb{R}^3$ $n=3$ $C^3_1 = 3$, $C^3_2 = 3$, $C^3_3 = 1$

x, y, z   xy, xz, yz   x, y, z

§ 4.3 Differential forms

A p-form $(p \geq 2)$ is a completely anti symmetric tensor of type $\binom{0}{p}$

A one-form is a $\binom{0}{1}$ tensor (by convention anti symmetric)

A zero-form is a $\binom{0}{0}$ tensor (scalar) ???

$\Rightarrow$ p is the degree

Using $\otimes$ (outer product) can take 2 $\binom{0}{1}$ forms to yield a $\binom{0}{2}$ tensor.

A wedge product takes two one forms and yields a 2-form.

i claim this is anti symmetric $\rightarrow$ $\boxed{\tilde{p} \wedge \tilde{q} = \tilde{p} \otimes \tilde{q} - \tilde{q} \otimes \tilde{p}} = -\tilde{q} \wedge \tilde{p}$

$\tilde{q} \wedge \tilde{p} = \tilde{q} \otimes \tilde{p} - \tilde{p} \otimes \tilde{q}$

Proprety: If $\{\vec{e}_i = i = 1, \dots, n\}$ is a basis of T_pM and $\{\tilde{\omega}^j\}$ is the dual basis of $T_p{}^*M$

then $\{\tilde{\omega}^j \wedge \tilde{\omega}^k, j, k = 1 \dots n\}$ is a basis for the vector space of two-forms

We can build two-forms in a similar way

$$\tilde{p} \wedge (\tilde{q} \wedge \tilde{R}) = (\tilde{p} \wedge \tilde{q}) \wedge \tilde{R} = \tilde{p} \wedge \tilde{q} \wedge \tilde{r}$$

$$\tilde{p} \wedge \tilde{q} \wedge \tilde{R} = \tilde{p} \otimes \tilde{q} \otimes \tilde{r} + \tilde{q} \otimes \tilde{r} \otimes \tilde{p} + \tilde{r} \otimes \tilde{p} \otimes \tilde{q} - \tilde{r} \otimes \tilde{q} \otimes \tilde{p} - \tilde{q} \otimes \tilde{p} \otimes \tilde{r} - \tilde{p} \otimes \tilde{r} \otimes \tilde{q}$$

We can define the wedge product of a p-form and a q-form

§ 4.4 Manipulating differential forms

Commutation rule of form: $\tilde{p} \wedge \tilde{q} = \tilde{q} \wedge \tilde{p} \, (-1)^{pq}$

idea: if $\tilde{p} = \tilde{\omega}^i \wedge \cdots \wedge \tilde{\omega}^j$ p factors

$\tilde{q} = \tilde{\omega}^k \wedge \cdots \wedge \tilde{\omega}^l$ q factors

$\tilde{p} \wedge \tilde{q} = (\tilde{\omega}^i \wedge \cdots \wedge \omega^j) \wedge (\tilde{\omega}^k \wedge \cdots \wedge \tilde{\omega}^l)$

Lecture 15 - Nov 2, 2023

§ 4.4 Manipulation of differential forms

Commutation rule: $\tilde{p} \wedge \tilde{q} = (-1)^{pq}\, \tilde{q} \wedge \tilde{p}$

Interior Product / Contraction of a vector with a form

If $\tilde{\alpha}$ is a p-form and $\bar{V}$ is a vector, then $\tilde{\alpha}$ requires p vectors

$\tilde{\alpha}(\bar{V}) \equiv \tilde{\alpha}(\bar{V}, \underbrace{\ldots}_{p-1 \text{ arguments left}})$ or $\alpha_{ij\cdots k} V^i$ → text books notation

$i_{\bar{V}}(\tilde{\alpha}) = \alpha_{ij\cdots k} V^i$ → other text books notation

This is an inner product

Example:

$i_{\bar{V}}(\tilde{\omega}^i \wedge \tilde{\omega}^j \wedge \ldots \wedge \tilde{\omega}^k) = V^i \tilde{\omega}^j \wedge \ldots \wedge \tilde{\omega}^k - V^j \tilde{\omega}^i \wedge \ldots \wedge \tilde{\omega}^k + \ldots$ → more terms check textbook

§ 4.5 Restrictions to forms

Suppose W is a subspace of a Vector field V. A p-form, $\tilde{\alpha}$, is a $\binom{0}{p}$ tensor that is (completely) antisymmetric and its arguments could be

$$\underbrace{V \times V \times \cdots \times V}_{p\text{-times}}$$

The restriction of $\tilde{\alpha}$ to the subspace $W \subset$ is the same p-form but with the domain restricted to W,

$$\boxed{\tilde{\alpha}\big|_W (\bar{X}, \ldots, \bar{Y}) = \tilde{\alpha}(\bar{X}, \ldots, \bar{Y}) \quad \text{where} \quad \bar{X}, \ldots, \bar{Y} \text{ are in } W}$$

If $m = \dim W < p$ then $\tilde{\alpha}|_W$ is 0

If $m = p$ then $\tilde{\alpha}|_W$ has one component, $C^p_p = 1$

Restricted a form is called sectioning

A form is anulled by a Vector space if its restriction to it vanishes

§ 4.6 Fields of forms

A field of p-forms on manifold M gives a p-form ∀ points on the manifold M.

A Sub manifold $S \subset M$ picks a sub space $T_P S$ for all $P \in S$ and we define the restriction of the p - form $\tilde{\alpha}$ to S by restricting $\tilde{\alpha}$ at P to $T_P S$.

§ 4.7 Handedness and Orientiability

In a n-dimensional manifold there is a 1-dimensional space of n-forms ($C^n_n = 1$)

Suppose that $\tilde{\omega}$ is an n-form field that we can use to find the Volume If we have $\{\bar{e}_1, \dots, \bar{e}_n\}$ is a Vector basis of $T_P M$ that is linearly independent

It follows that $\tilde{\omega}(\bar{e}_1, \dots, \bar{e}_n) \neq 0$ iff $\tilde{\omega} \neq 0$ at P

Aside: $\tilde{\omega} \Rightarrow \omega_{i \dots k} \tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge \dots \wedge \tilde{\omega}_n$

Consider

$$\tilde{\omega}(\bar{e}_1, \dots, \bar{e}_n) = \omega_{i \dots k} (\tilde{\omega}^1 \wedge \tilde{\omega}^2 \wedge \dots \wedge \tilde{\omega}^n)(\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n)$$

$$= \omega_{i \dots k} \tilde{\omega}^1(\bar{e}_1) \wedge \tilde{\omega}^2(\bar{e}_2) \wedge \dots \tilde{\omega}^n(\bar{e}_n)$$

$$= \omega_{i \dots k}$$

$\tilde{\omega}$ seperates the vector bases in to 2 classes

① $\tilde{\omega}(\bar{e}_1, \dots, \bar{e}_n) > 0$ (right hand)

② $\tilde{\omega}(\bar{e}_1, \dots, \bar{e}_n) < 0$ (left hand)

This seperation is unique to any n-form.

This manifold is said to be orientable if we define the handedness consistently (all positive or all negative) on the manifold

Example $\mathbb{R}^n$ is orientable

Mobius is not orientable

We only consider orientable manifold

§ 4.8 Volumes and Integration on Oriented Manifolds

A set of n linearly independent vectors (infinetesimal) or an n-dim manifold, can define a non-zero volume. this forms in n-dim a parallel epiped

Integration a function f on M requires multiply f by an infinitesimal Volume then adding this up over all of M

Suppose $\tilde{\omega}$ is a n-form on an open set U in M with coordinates

$$\{x^1, \dots, x^n\}$$

Since n-forms at $P \in M$ form a 1-D vector space. There exists a function $f(x^1, \dots, x^n)$ such that

$$\tilde{\omega} = f\, \tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n$$

We integrate over U by first dividing $\mathcal{U}$ into regions (cells) spaned by n-tuples of vectors

$$\{\Delta x^1 \frac{\partial}{\partial x^1}, \Delta x^2 \frac{\partial}{\partial x^2}, \dots, \overset{\text{constant}}{\Delta x^n} \frac{\partial}{\partial x^n}\}$$

where Δx^i's are infinitesimal.

The Integral of f over a region is f multiplied by the following

$$\boxed{\underbrace{\Delta x^1 \, \Delta x^2 \dots \Delta x^n}_{\text{This looks like } dV} = (\tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n)(\Delta x^1 \frac{\partial}{\partial x^1}, \dots, \Delta x^n \frac{\partial}{\partial x^n})}$$

The Integral of f over a cell is written as

$$\int_{\text{cell}} f(x^1, \dots, x^n)\, d^n x \approx \tilde{\omega}$$

locally near p

Add over all the cells and we get the integral

$$\boxed{\int \overset{\text{n-form}}{\tilde{\omega}} \equiv \int f(x^1, \dots, x^n)\, dx^1 \dots dx^n}$$

Integrating over region of U

we will show this is coordinate independent.

<u>Example</u> In 2D with coordinates (λ, μ) the above yields

$$\int \tilde{\omega} = \int f(\lambda, \mu)\, \tilde{d}\lambda \wedge \tilde{d}\mu \quad \to \text{calculus of manifold}$$

$$= \int f(\lambda, \mu)\, d\lambda\, d\mu \quad \to \text{calculus of } \mathbb{R}^2$$

Check transformation of coordinate $(\lambda, \mu) \to (x, y)$

$$\tilde{d}\lambda = \tilde{d}\lambda(x,y) = \frac{\partial \lambda}{\partial x} \tilde{d}x + \frac{\partial \lambda}{\partial y} \tilde{d}y$$

$$\tilde{d}\mu = \frac{\partial \mu}{\partial x} \tilde{d}x + \frac{\partial \mu}{\partial \lambda} \tilde{d}y$$

We build the form:

$$\tilde{d}\lambda \wedge \tilde{d}\mu = \left(\frac{\partial \lambda}{\partial x} \tilde{d}x + \frac{\partial \lambda}{\partial y} \tilde{d}x\right)\left(\frac{\partial \mu}{\partial x} \tilde{d}x + \frac{\partial \mu}{\partial x} \tilde{d}y\right)$$

$$= \frac{\partial \lambda}{\partial x}\frac{\partial \mu}{\partial x} \overset{\nearrow 0}{\tilde{d}x \wedge \tilde{d}x} + \frac{\partial \lambda}{\partial x}\frac{\partial \mu}{\partial y} \tilde{d}x \wedge \tilde{d}y + \frac{\partial \lambda}{\partial y}\frac{\partial \mu}{\partial x} \tilde{d}y \wedge \tilde{d}x + \frac{\partial \lambda}{\partial x}\frac{\partial \mu}{\partial y} \overset{\nearrow 0}{\tilde{d}y \wedge \tilde{d}y}$$

$$= \left(\frac{\partial \lambda}{\partial x} \frac{\partial \mu}{\partial \lambda} - \frac{\partial \lambda}{\partial y} \frac{\partial \mu}{\partial x} \right) \tilde{d}x \wedge \tilde{d}y$$

$$= \frac{\partial(\lambda, \mu)}{\partial(x, y)} \tilde{d}x \wedge \tilde{d}y$$ Jacobian of the transformation

For a n- dimensiond manifold we can integrate an n- form to get a non-zero result

For a sub manifold of order p, we can integrate a p- form to get a non-zero result

Lec 16 - online

the lectures are online since francis is out of town. Therefore i will not record sound

§ 4.9 N- Vectors, duals and the symbol $\epsilon_{ij\cdots k}$

A completely anti symmetric $\binom{p}{0}$ tensor is a p vector. On a n- dimensional manifold this has dim C^n_p

The following spaces all have the same dimension

p- forms
(n-p) forms
p - vectors
(n-p) vectors

$\Big\}$ size $\binom{n}{p} = \binom{n}{n-p}$

Example
if $n=3$

	p=1	p=2	p=3
dim	3	3	1

We could use the metric tensor, (inner product) to map a $\binom{p}{0}$ tensor to a $\binom{0}{p}$ tensor and backwards

It can be shown that since the metric tensor is symmetric, this process preserves anti symmetry

Even without a metric, the volume n-form, $\tilde{\omega}$, yields a mapping from p-vectors to (n-p)-forms
this map is the dual map or the hodge -star map [not mentioned in text book

Suppose T is a 2-vector, with components

$$T^{i\ldots k} = T^{[i\ldots k]}$$

with $\tilde{\omega}$, we can defin a tensor $\tilde{A}$ such that

$$\boxed{\underbrace{A_{j\ldots \ell}}_{(n-q)\text{ form}} = \frac{1}{q!}\, \underbrace{\omega_{i\cdots k j\cdots \ell}}_{n\text{-form}}\, \underbrace{T^{i\ldots k}}_{q-\text{vector}}} \quad \text{or} \quad \tilde{A} = \tilde{\omega}(T)$$

→ indepent of coordinates

<u>Notation</u>: $\tilde{A} = * T$ and say that $\tilde{A}$ is the dual of T w.r.t. $\tilde{\omega}$

This is invertable and therefore we can bring p- forms to (n-p) vectors

Examples: Consider $\mathbb{E}^3$ in terms of cartesian coordinates

$\bar{U}$, $\bar{V}$ are both vectors

In cartesian coordinates, the elements of the associated (end of ch 2?) 1-forms are equal (b/c of δ)

$$\tilde{U} = g_1(\bar{u}) \quad \text{or} \quad U_i = g_{ij} U^j = \delta_{ij} U^j$$

and

$$\tilde{V} = g_1(\bar{V}) \quad \text{or} \quad V_i = g_{ij} V^j = \delta_{ij} V^j$$

Typically, we write these as

basis for 1-forms

$$\tilde{U} = U_1 \tilde{d}x^1 + U_2 \tilde{d}x^2 + U_3 \tilde{d}x^3 = U_i \tilde{d}x^i$$
$$\tilde{V} = V_1 \tilde{d}x^1 + V_2 \tilde{d}x^2 + V_3 \tilde{d}x^3 = V_i \tilde{d}x^i$$

With $\tilde{U}$ and $\tilde{V}$ we can use the wedge to find the following 2-form

$$\tilde{U} \wedge \tilde{V} = (U_1 \tilde{d}x^1 + U_2 \tilde{d}x^2 + U_3 \tilde{d}x^3) \wedge (V_1 \tilde{d}x^1 + V_2 \tilde{d}x^2 + V_3 \tilde{d}x^3)$$

0 b/c symmetry

$$= \underbrace{U_1 V_1 \tilde{d}x^1 \wedge \tilde{d}x^1}_{0 \text{ b/c of symmetry}} + U_1 V_2 \tilde{d}x^1 \wedge \tilde{d}x^2 + U_1 V_3 \tilde{d}x^1 \wedge \tilde{d}x^3 + U_2 V_1 \tilde{d}x^2 \wedge \tilde{d}x^1 + U_2 V_2 \tilde{d}x^2 \wedge \tilde{d}x^2 + U_2 V_3 \tilde{d}x^2 \wedge \tilde{d}x^3$$

b/c wedge product is anti-symmetric

0 b/c of symmetry

$$+ U_3 V_1 \tilde{d}x^3 \wedge \tilde{d}x^1 + U_3 V_2 \tilde{d}x^3 \wedge \tilde{d}x^2 + U_3 V_3 \tilde{d}x^3 \wedge \tilde{d}x^3$$

Switching some bases :

$$\tilde{U} \wedge \tilde{V} = (U_1 V_2 - U_2 V_1) \tilde{d}x^1 \wedge \tilde{d}x^2$$
$$+ (U_2 V_3 - U_3 V_2) \tilde{d}x^2 \wedge \tilde{d}x^3$$
$$+ (U_3 V_1 - U_1 V_3) \tilde{d}x^3 \wedge \tilde{d}x^1$$

Coefficents are similar to cross product

Find the dual of this expression:

$$* (\tilde{U} \wedge \tilde{V}) = \frac{1}{2!} \omega_{ijk} (\tilde{U} \wedge \tilde{V})^{ij}$$

Francis claims in lecture, in piazza i'm not sure this is true

$$= \frac{1}{2!} \Big[\omega_{123} (\tilde{U} \wedge \tilde{V})^{12} + \omega_{132} (\tilde{U} \wedge \tilde{V})^{13}$$
$$+ \omega_{231} (\tilde{U} \wedge \tilde{V})^{23} + \omega_{213} (\tilde{U} \wedge \tilde{V})^{21}$$
$$+ \omega_{321} (\tilde{U} \wedge \tilde{V})^{32} + \omega_{312} (\tilde{U} \wedge \tilde{V})^{31} \Big]$$

$$= \frac{1}{2} \Big[(U_1 V_2 - U_2 V_1)^* \tilde{d}x^1 \wedge \tilde{d}x^2 + (U_3 V_1 - U_1 V_3)^* \tilde{d}x^3 \wedge \tilde{d}x^1 + (U_2 V_3 - U_3 V_2)^* \tilde{d}x^2 \wedge \tilde{d}x^3 + (U_1 V_2 - U_2 V_1)^* \tilde{d}x^1 \wedge \tilde{d}x^2$$
$$+ (U_3 V_1 - U_1 V_3)^* \tilde{d}x^3 \wedge \tilde{d}x^1 + (U_2 V_3 - U_3 V_2)^* \tilde{d}x^2 \wedge \tilde{d}x^3 \Big]$$

$$= (U_1 V_2 - U_2 V_1) * (\tilde{d}x^1 \wedge \tilde{d}x^2) + (U_2 V_3 - U_3 V_2) * (\tilde{d}x^2 \wedge \tilde{d}x^3) + (U_3 V_1 - U_1 V_3) * (\tilde{d}x^3 \wedge \tilde{d}x^1)$$

How do we find the dual of the 2-forms?

Note that
$$* (\tilde{d}x^1 \wedge \tilde{d}x^2) = \partial/\partial x^3$$
$$* (\tilde{d}x^2 \wedge \tilde{d}x^3) = \partial/\partial x^1$$
$$* (\tilde{d}x^3 \wedge \tilde{d}x^1) = \partial/\partial x^2$$

$\boxed{\text{Dual of a 2 form is a one vector}}$

dual of a 2-form

Check: $* (\tilde{d}x^2 \wedge \tilde{d}x^3) = \frac{\partial}{\partial x^1}$, $* \left(\frac{\partial}{\partial x^1}\right) = \tilde{d}x^2 \wedge \tilde{d}x^3$

Coefficent in x-direction $= 1$

proof $\frac{1}{1!} \omega_{ijk} T^1 = \omega_{1jk} 1 = A_{jk}$ (2-form)

$\Rightarrow A_{23} = 1$ with basis $\tilde{d}x^2 \wedge \tilde{d}x^3$

or $A_{32} = -1$ with basis $\tilde{d}x^3 \wedge \tilde{d}x^2$

$$* (\tilde{U} \wedge \tilde{V}) = (U_1 V_2 - U_2 V_1) \frac{\partial}{\partial x^3} + (U_2 V_3 - U_3 V_2) \frac{\partial}{\partial x^1} + (U_3 V_1 - U_1 V_3) \frac{\partial}{\partial x^2}$$

Compare with the cross product

$$\bar{U} \times \bar{V} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{bmatrix} U_2 V_3 - U_3 V_2 \\ U_3 V_1 - U_1 V_3 \\ U_1 V_2 - U_2 V_1 \end{bmatrix}$$

Looks like $\frac{\partial}{\partial x^1}$ $\partial/\partial x^2$ $\frac{\partial}{\partial x^3}$

$$\therefore \; * (\tilde{U} \wedge \tilde{V}) = (\bar{U} \times \bar{V}) \Leftrightarrow (\tilde{U} \wedge \tilde{V}) = * (\bar{U} \times \bar{V})$$

This result is unique to $\mathbb{R}^3$.

The map betwe T and $*T$ is invertiable.

Levi - Civita Symbols

$$\epsilon_{ij\ldots k} = \epsilon^{ij\ldots k} = \begin{cases} +1 & \text{if } ij\ldots k \text{ is an even permutation} \\ -1 & \text{if } ij\ldots k \text{ is an odd permutation} \\ 0 & \text{otherwize} \end{cases}$$

example: $\bar{U} \times \bar{V} = \epsilon_{ijk} U^j V^k$

B the differential calculus of form and it applications

Single variable Calculus States

$$\int_a^b df = f(b) - f(a)$$

We want to derive a derivative operator that reduces to this in the simple case, but is more general

§ 4.14 The exterior derivative

If M is a 1D manifold, $\tilde{d}$: $\underbrace{0}_{\text{takes a function}}$-Form $\to$ 1-form. It will agree withe the above

If $\tilde{\alpha}$ is a p-form a $\tilde{\beta}$, $\tilde{\gamma}$ are q-forms we requre

(i) $\tilde{d}(\tilde{\beta} + \tilde{\gamma}) = (\tilde{d}\tilde{\beta}) + (\tilde{d}\tilde{\gamma})$ Distributive

(2) $\tilde{d}(\tilde{\alpha} \wedge \tilde{\beta}) = (\tilde{d}\tilde{\alpha}) \wedge \tilde{\beta} + (-1)^p \tilde{\alpha} \wedge (\tilde{d}\tilde{\beta})$ Anti derivation

(3) $\tilde{d}(\tilde{d}\tilde{\alpha}) = 0$

These 3 properties uniquely define $\tilde{d}$

$\tilde{d}$ is called the exterior derivative

Property (ii) is almost Leibniz but there is an extra $(-1)^p$ to bring $\tilde{d}$ across the p-form

Property (iii) Seems odd but is essential

ex) if $\tilde{\alpha}$ is a function then $\tilde{d}f$ is a one form with component $\frac{\partial f}{\partial x^i}$

$\tilde{d}(\tilde{d}f)$ has components of the form $\frac{\partial^2 f}{\partial x^j \partial x^i}$. But this must be a 2-form and since it must be anti symmetric, it must be 0.

§ 4.15 Notation for derivatives

Partial derivatives: $\frac{\partial f}{\partial x^i} = f_{,i}$ 1st derivative

$\frac{\partial V^i{}_j}{\partial x^k} \quad V^i_{j,k}$ 1st derivative of a $\binom{1}{1}$ tensor

$\frac{\partial^2 f}{\partial x^k \partial x^i} = f_{,ik}$ 2nd Derivative

Recall that a partial derivative is not a tensor operation in general [which assignment went over this?]

example ① $V^i{}_{j,k}$ need not be $\Lambda^i{}_{a'}\Lambda^{b'}{}_j\underbrace{\Lambda^{c'}{}_k}_{\text{this part breaks down ... is it coordinate depenedent}} A^{a'}{}_{b';c'}$ What is it?

② $\tilde{d}f = f_{,i}$ (1-form) and is a tensor operation

③ $[\bar{u},\bar{v}]^i = u^j v^i{}_{,j} - v^j u^i{}_{,j}$ → Lie bracket this is a tensor operator

Each term on the RHS is not a tensor operator but the whole RHS is

Excerse 4.14 on an assigment

a) $\tilde{d}(f\tilde{d}g) = \tilde{d}f \wedge \tilde{d}g$ [with 3rd property means $(-1)^p \tilde{d}\tilde{d}g = 0$]

b) if $\tilde{\alpha} = \frac{1}{p!}\alpha_{i\cdots j}\tilde{d}x^i \wedge \cdots \wedge \tilde{d}x^j$ is a p-form

the $\tilde{d}\tilde{\alpha} = \frac{1}{p!}\frac{\partial}{\partial x^k}(\alpha_{i\cdots j})\tilde{d}x^k \wedge \tilde{d}x^i \wedge \cdots \wedge \tilde{d}x^j$

and

$$(\tilde{d}\alpha)_{ki\cdots j} = (p+1)\frac{\partial}{\partial x}[k\alpha_{i\cdots j}]$$

or $(\tilde{d}\tilde{\alpha})_{ki\cdots j} = (p+1)\ \alpha_{[i\cdots j,k]}$

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§ 4.16 Familiar examples of exterior derivatives

We can revisit some old friends with a new perspective

① $\tilde{d}$ of a 1-form $\tilde{a}$ in 3D:

components

$\tilde{d}\tilde{a} = \tilde{d}(a_1 \tilde{d}x^1 + a_2 \tilde{d}x^2 + a_3 \tilde{d}x^3)$

one-form   two-form   all the components that are not zero due to anti symmetry

$= a_{1,j}\, \tilde{d}x^j \wedge \tilde{d}x^1 + a_{2,j}\, \tilde{d}x^j \wedge \tilde{d}x^2 + a_{3,j}\, \tilde{d}x^j \wedge \tilde{d}x^3$

new index

$= a_{1,2}\, \tilde{d}x^2 \wedge \tilde{d}x^1 + a_{1,3}\, \tilde{d}x^3 \wedge \tilde{d}x^1 + a_{2,1}\, \tilde{d}x^1 \wedge \tilde{d}x^2 + a_{2,3}\, \tilde{d}x^3 \wedge \tilde{d}x^2 + a_{3,1}\, \tilde{d}x^1 \wedge \tilde{d}x^3 + a_{3,2}\, \tilde{d}x^2 \wedge \tilde{d}x^3$

$= (a_{3,2} - a_{2,3})\, \tilde{d}x^2 \wedge \tilde{d}x^3 + (a_{1,3} - a_{3,1})\, \tilde{d}x^3 \wedge \tilde{d}x^1 + (a_{2,1} - a_{1,2})\, \tilde{d}x^1 \wedge \tilde{d}x^2$ → combing like terms by switching the two bases in the wedge

Consider the dual, or hodge-star

$* \tilde{d}\tilde{a} = (a_{3,2} - a_{2,3}) * (\tilde{d}x^2 \wedge \tilde{d}x^3) + (a_{1,3} - a_{3,1}) * (\tilde{d}x^3 \wedge \tilde{d}x^1) + (a_{2,1} - a_{1,2}) * (\tilde{d}x^1 \wedge \tilde{d}x^2)$

From before $*(\tilde{d}x^2 \wedge \tilde{d}x^3) = \frac{\partial}{\partial x^1}$ $*(\tilde{d}x^3 \wedge \tilde{d}x^1) = \frac{\partial}{\partial x^2}$ $*(\tilde{d}x^1 \wedge \tilde{d}x^2) = \frac{\partial}{\partial x^3}$

Hence, $* \tilde{d}\tilde{a} = (a_{3,2} - a_{2,3}) \frac{\partial}{\partial x^1} + (a_{1,3} - a_{3,1}) \frac{\partial}{\partial x^2} + (a_{2,1} - a_{1,2}) \frac{\partial}{\partial x^3}$ → Looks like the curl

or

$* \tilde{d}\tilde{a} = \epsilon_{ijk} \frac{\partial}{\partial x^j} a_k = \epsilon_{ijk} a_{k,j}$

The RHS is the curl since $\bar{\nabla} \times \bar{a} = \epsilon_{ijk} a_{k,j}$

Summary $* \tilde{d} =$ curl in 3D when applied to 1-forms, when exterior derivative is applied to a scalar it gives the gradient, as seen in ③

② First take the dual of a vector then $\tilde{d}$

Suppose $\bar{a}$ is a vector field

$*(\bar{a}) = *(a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + a^3 \frac{\partial}{\partial x^3}) = a^1 *(\frac{\partial}{\partial x^1}) + a^2 *(\frac{\partial}{\partial x^2}) + a^3 *(\frac{\partial}{\partial x^3})$

$= a^1 (\tilde{d}x^2 \wedge \tilde{d}x^3) + a^2 (\tilde{d}x^3 \wedge \tilde{d}x^1) + a^3 (\tilde{d}x^1 \wedge \tilde{d}x^2)$

Only 1 value for each j that would make the wedges non-zero

Apply $\tilde{d}$

$\tilde{d}(*\bar{a}) = a^1{}_{,j} (\tilde{d}x^{j} \wedge \tilde{d}x^2 \wedge \tilde{d}x^3) + a^2{}_{,j} (\tilde{d}x^{j} \wedge \tilde{d}x^3 \wedge \tilde{d}x^1) + a^3{}_{,j} (\tilde{d}x^{j} \wedge \tilde{d}x^1 \wedge \tilde{d}x^2)$

($j=1$, $j=2$, $j=3$)

$= (a^1{}_{,1} + a^2{}_{,2} + a^3{}_{,3})\, \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3$

Note $\tilde{d} * \tilde{a} = (\bar{\nabla} \cdot \bar{a}) \tilde{\omega}$ (divergence)

③ $\tilde{d}f = f_{,i}\, \tilde{d}x^i$ (gradient)

§4.17 Integrability conditions for PDEs

Consider the system of 2 PDEs:

$$\frac{\partial f}{\partial x} = g(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = h(x,y)$$

$\boxed{\text{More interested in when a solution exists}}$

Let (x,y) be coordinates of a manifold. Further, define $\underbrace{a_x = g \text{ and } a_y = h}_{\text{vector components}}$ and then we can re write the system compactly as

one form notation

$$\boxed{f_{,i} = a_i} \quad i = 1,2$$

A coordinate - independent version of this is

$$\tilde{d}f = \tilde{a}$$ → why $\tilde{a}$? i thought a_x, a_y was a vector

This holds because $\frac{\partial f}{\partial x}\tilde{d}x + \frac{\partial f}{\partial y}\tilde{d}y = g\tilde{d}x + h\tilde{d}y$

If f is a soln then it must follow that

$$\tilde{d}(\tilde{d}f) = \tilde{d}\tilde{a} = 0$$

In component form this becomes

$$\tilde{d}\tilde{a} = \tilde{d}(g\tilde{d}x + h\tilde{d}y)$$

$$= \frac{\partial g}{\partial y}\tilde{d}y \wedge \tilde{d}x + \frac{\partial h}{\partial x}\tilde{d}x \wedge \tilde{d}y$$

→ $\frac{\partial g}{\partial x} = 0$ b/c of anti symmetry

$$= \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right)\tilde{d}x \wedge \tilde{d}y = 0$$ → Greens theorem?

or $$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$a_{2,1} - a_{1,2} = 0 \quad \text{or} \quad a_{[i,j]} = 0$$

→ where does this notation come from?

This condition is necessary and later we will show it is sufficient for a solution to exist

§4.18 Exact forms

Observe, if $\tilde{\alpha} = \tilde{d}\tilde{\beta}$ the $\tilde{d}\tilde{\alpha} = \tilde{d}(\tilde{d}\beta) = 0$

If $\tilde{d}\tilde{\alpha} = 0$ then $\tilde{\alpha}$ is closed

If $\tilde{\alpha} = \tilde{d}\tilde{\beta}$ then $\tilde{\alpha}$ is exact

Clearly an exact form is closed. It can be shown that any closed form is exact

§ 4.20 Lie derivatives of forms

If $\tilde{\omega}$ is a p-form then

$$\underbrace{\mathcal{L}_{\bar{V}}\tilde{\omega}}_{\text{p-form}} = \underbrace{\tilde{d}[\underbrace{\tilde{\omega}(\bar{V})}_{\text{p-1 form}}]}_{\text{p-form}} + \underbrace{(\underbrace{\tilde{d}\tilde{\omega}}_{\text{p-1 form}})(\bar{V})}_{\text{p-form}}$$

Idea of proof:

case 1 $\tilde{\omega}$ is a 0-form, $\tilde{\omega} = f$

LHS $\mathcal{L}_{\bar{V}}\tilde{\omega} = \mathcal{L}_{\bar{V}} f = \bar{V}(f) = \frac{df}{d\lambda}$

(RHS1) does not make sense, this term is ignored we cant have a function acting on a vector field

(RHS2) $(\tilde{d}\tilde{\omega})(\bar{V}) = (\tilde{d}f)(\bar{V}) = \frac{\partial f}{\partial x^i}\tilde{d}x^i\left(V^j\frac{\partial}{\partial x^j}\right)$

$$(\tilde{d}\tilde{\omega})(\bar{V}) = V^j f_{,i}\, \underbrace{\tilde{d}x^i\left(\frac{\partial}{\partial x^j}\right)}_{=\delta^i_j}$$

$$= V^i f_{,i} = \frac{df}{d\lambda}$$

Case 2: $\tilde{\omega}$ is a 1-form $\tilde{\omega} = \omega_i \tilde{d}x^i$ [one form notation]

(RHS1) $\tilde{d}(\tilde{\omega}(\bar{V})) = \tilde{d}(\omega_i V^i) = (\omega_i V^i)_{,j}\,\tilde{d}x^j$

(RHS2) $(\tilde{d}\tilde{\omega})(\bar{V}) = (\tilde{d}(\omega_i \tilde{d}x^i))(\bar{V})$

$= (\omega_{i,j}\tilde{d}x^j \wedge \tilde{d}x^i)(\bar{V})$ → which formula does this use?

new index for V

$= \omega_{i,j}(\tilde{d}x^j \otimes \tilde{d}x^i - \tilde{d}x^i \otimes \tilde{d}x^j)\left(V^k\frac{\partial}{\partial x^k}\right)$ → translation of $\wedge$ to outer product

$= \omega_{i,j}\left(\tilde{d}x^j\left(V^k\frac{\partial}{\partial x^k}\right) \otimes \tilde{d}x^i - \tilde{d}x^i\left(V^k\frac{\partial}{\partial x^k} \otimes \tilde{d}x^j\right)\right)$ → distributing $V^k\frac{\partial}{\partial x^k}$

$= \delta^j_k$ $\quad = \delta^i_k$ → where did the cross go?

$= \omega_{i,j}(V^j\tilde{d}x^i - V^i\tilde{d}x^j)$

RHS $= (\omega_i V^i)_{,j}\,\tilde{d}x^j + \omega_{i,j}(V^j\tilde{d}x^i - V^i\tilde{d}x^j)$

product rule

$= \cancel{\omega_{i,j}V^i\tilde{d}x^j} + \omega_i V^i_{,j}\tilde{d}x^j + \omega_{i,j}(V^j\tilde{d}x^i - \cancel{V^i\tilde{d}x^j})$

$= \omega_{i,j}V^j\tilde{d}x^i + \omega_j V^j_{,i}\tilde{d}x^i$

$= (\omega_{i,j}V^j + \omega_j V^j_{,i})\tilde{d}x^i = (\mathcal{L}_{\bar{V}}\tilde{\omega})_i$

Can fully prove using induction

Lec 18 – Nov 14

§4.21 Lie derivatives and Exterior derivative commute

Woohoo Stokes thereom!!!

Thm: $\mathcal{L}_{\bar{V}}$ and $\tilde{d}$ commute

We will prove this for a one-form. or n-form?

proof: we need a formula from §4.20

$$\mathcal{L}_{\bar{V}}\tilde{\omega} = \tilde{d}[\tilde{\omega}(\bar{V})] + (\tilde{d}\tilde{\omega})(\bar{V}) \Big\}$$ Still have n-form

Gives tensor of same space?

but replace $\tilde{\omega}$ with $\tilde{d}\tilde{\omega}$ → true for n+1, exact form

this is not zero b/c of how $[\tilde{d}\tilde{\omega}(\bar{V})]$ gets calculated

$$\mathcal{L}_{\bar{V}}\tilde{d}\tilde{\omega} = \tilde{d}[\tilde{d}\tilde{\omega}(\bar{V})] + \cancel{(\tilde{d}(\tilde{d}\tilde{\omega})}(\bar{V})$$ B/c of properties of $\tilde{d}$

From the first equation we know

$$\tilde{d}\tilde{\omega}(\bar{V}) = \mathcal{L}_{\bar{V}}\tilde{\omega} - \tilde{d}[\tilde{\omega}(\bar{V})]$$

Substitute into the RHS of the previous equation

$$\mathcal{L}_{\bar{V}}\tilde{d}\tilde{\omega} = \tilde{d}[\mathcal{L}_{\bar{V}}\tilde{\omega} - \cancel{\tilde{d}[\tilde{\omega}(\bar{V})]}]$$ the [ ] around $\tilde{\omega}(\bar{V})$ make an n-1 form, hence $\tilde{d}$ is acting on an n-1 form and $\tilde{d}\tilde{d}(\tilde{\omega}(v))$ is zero

$$= \tilde{d}\mathcal{L}_{\bar{V}}\tilde{\omega} \quad \therefore \text{commutes}$$

§4.22 Stokes thm

We show that the exterior derivative is the inverse of integration, in particular

$$\int_U \tilde{d}\tilde{\omega} = \int_{\partial U} \tilde{\omega}$$ Different from $\tilde{\omega}$ we use

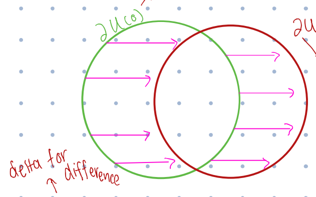
We can integrate n-forms over n-dimension. and we can integrate n-1 forms over n-1 dimenson.

If U is n-dimensional then its boundary is n-1 dimensonial. The boundary is the exterior of U, and why this is called exterior calculus

Assume, U is a smooth, orientable volume on M that is connected. then ∂U is a submanifold of M. also $\vec{\xi}$ is a vector Field on M

Suppose $\mathcal{U} = \mathcal{U}(0)$ is a region on M with boundary $\partial\mathcal{U} = \partial\mathcal{U}(0)$, and $\mathcal{U}(\epsilon)$ is the Lie dragged region along $\vec{\xi}$

for Ex 2dim — Ex 2 dim — Ex 1 dim — Ex. 2 dim



$\delta\mathcal{U}(\epsilon)$ is the area between $\mathcal{U}(0)$ and $\mathcal{U}(\epsilon)$, which is dim 2

or $\delta\mathcal{U}(\epsilon) \equiv \mathcal{U}(\epsilon) - \mathcal{U}(0)$

To find the change of an integral from $\mathcal{U}(0)$ to $\mathcal{U}(\epsilon)$ we compute the following

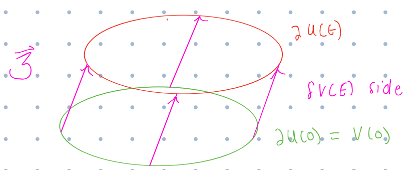
$$\int_{\mathcal{U}(\epsilon)} \tilde{\omega} - \int_{\mathcal{U}(0)} \tilde{\omega} = \int_{\delta\mathcal{U}(\epsilon)} \tilde{\omega} \quad *$$

n-form

$\tilde{\omega}$ is an n-form, the n-form that gives us volume. anti-symmetric and $\binom{0}{N}$ tensor

Suppose V is a coordinate patch of $\partial\mathcal{U}(0)$ with coordinates $\{x^2, \dots, x^n\}$ → No x^1!!

If we Lie drag $\partial\mathcal{U}(0)$ along $\vec{\xi}$ a distance of ϵ, then we have coordinates $\{x^1 = \epsilon, x^2, \dots, x^n\}$



This part has a patch, the $\epsilon = x^1$ coordinate patch gives us a way to describe the boundary of the cylinder

$\vec{\xi}$ is no where tangent to $\partial\mathcal{U}(0)$

We investigate the integral of $\delta V(\epsilon)$ and then extend this to $\delta\mathcal{U}(\epsilon)$

We introduce, $\boxed{\tilde{\omega} = f(x^1, \dots, x^n)\, \tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n}$

If $\epsilon \ll 1$, then the integral over $\delta V(\epsilon)$ is,

This is like Riemann Sums

$$\int_{\partial V(\epsilon)} \tilde{\omega} = \int_{V(0)} \left[\int_0^{\epsilon} f\, dx^1 \right] dx^2 \dots dx^n$$

$$= \epsilon \int_{V(0)} f(0, x^2, \dots, x^n)\, dx^2 \dots dx^n + O(\epsilon) \longrightarrow \text{Linearizing since } \epsilon \ll 1$$

smaller than ϵ

$$= \epsilon \int_{V(0)} \tilde{\omega}(\vec{\xi}) \Big|_{\partial\mathcal{U}(0)} + O(\epsilon)$$ This is a change to Calculus on manifold

Note: $\bar{\xi} = \frac{\partial}{\partial x^1}$ by design

$$\tilde{\omega}\left(\frac{\partial}{\partial x^1}\right) = f(x^1, \dots, x^n)\, \tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n \left(\frac{\partial}{\partial x^1}\right)$$

$$= f(0, x^2, \dots, x^n)\, \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n$$

and $\displaystyle\int_0^\epsilon f\, dx^1 \approx \epsilon\, f(0, x^2, \dots, x^n) + o(\epsilon)$

Summary, $\displaystyle\int_{\partial V(\epsilon)} \tilde{\omega} = \epsilon \int_{V(0)} \tilde{\omega}(\bar{\xi})\Big|_{\partial u} + o(\epsilon)$ ✱✱

Now consider,

$$\frac{d}{d\epsilon} \int_{U(\epsilon)} \tilde{\omega} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_{U(\epsilon)} \tilde{\omega} - \int_{U(0)} \tilde{\omega} \right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\delta U(\epsilon)} \tilde{\omega}$$ From first eqn ✱

If $\delta U(\epsilon) \approx \delta V(\epsilon)$, which is true if $\epsilon \ll 1$, then

$$\frac{d}{d\epsilon} \int_{U(\epsilon)} \tilde{\omega} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\epsilon \int_{V(0)} \tilde{\omega}(\bar{\xi}) \Big|_{\partial u} + o(\epsilon) \right)$$

$$\boxed{\frac{d}{d\epsilon} \int_{U(\epsilon)} \tilde{\omega} = \int_{\partial U(0)} \tilde{\omega}(\bar{\xi}) \Big|_{\partial U(0)}}$$

But using linerization we can approximate the integrand on the LHS using

$$\tilde{\omega} = \epsilon \mathcal{L}_{\bar{\xi}} \tilde{\omega} + o(\epsilon)$$? Francis doesn't get where this comes from

on $U(0)$

Sub into LHS

$$\frac{d}{d\epsilon} \int_{U(\epsilon)} \tilde{\omega} = \frac{d}{d\epsilon} \int_{U(0)} \epsilon\, \mathcal{L}_{\bar{\xi}} \tilde{\omega} + o(\epsilon)$$

$$\frac{d}{d\epsilon} \int_{U(\epsilon)} \tilde{\omega} = \int_{U(0)} \mathcal{L}_{\bar{\xi}} \tilde{\omega}$$

But the formula for $\mathcal{L}_{\bar{\xi}} \tilde{\omega}$ yields

$$\int_{U(0)} \mathcal{L}_{\bar{\xi}} \tilde{\omega} = \int_{U(0)} \tilde{d}[\tilde{\omega}(\bar{\xi})] + (\tilde{d}(\tilde{\omega}))(\bar{\xi})$$ this is an n+1 form an hence $\to 0$

$$= \int_{U(0)} \tilde{d}[\tilde{\omega}(\bar{\xi})]$$

If we combine our formula's we set

$$\frac{d}{d\epsilon}\int_{u(\epsilon)} \tilde{\omega} = \int_{u(0)} \tilde{\partial}\left[\tilde{\omega}(\bar{\xi})\right] = \int_{\partial u(0)} \tilde{\omega}(\bar{\xi})$$

From above

befor formula with $\mathcal{L}$

Define $\tilde{\alpha} = \tilde{\omega}(\bar{\xi}) = i_{\bar{\xi}}\tilde{\omega}$ and get

$$\boxed{\int_{u(0)} \tilde{\partial}\tilde{\alpha} = \int_{\partial u(0)} \tilde{\alpha}}$$

Stokes theorem

Lec 19 ~ Nov 16th

PHYSICS NEXT WEEK

§ 4.22 Stokes theorem

Recall $\frac{d}{d\epsilon}\int_{U(\epsilon)} \tilde{\omega} = \int_{U(0)} \mathcal{L}_{\vec{\xi}}\, \tilde{\omega}$

Justification:

LHS: We integrate $\tilde{\omega}$ (n-form) over the n-volume $U(\epsilon)$

To obtain the RHS we taylor expand $\tilde{\omega}$ at $U(\epsilon)$ around $U(0)$

Aside: $f(\vec{x}_0 + \epsilon\vec{\xi}) = f(\vec{x}_0) + \epsilon\vec{\xi}\cdot\vec{\nabla} f(\vec{x}_0) + o(\epsilon)$ little o

use this idea to appox $\tilde{\omega}$,

Since the Lie derivative is the directional derivative

$$\tilde{\omega}\big|_{U(\epsilon)} = \tilde{\omega}\big|_{U(0)} + \epsilon\, \mathcal{L}_{\vec{\xi}}\, \tilde{\omega}\big|_{U(0)} + o(\epsilon)$$

If you plug this into the LHS,

$$\frac{d}{d\epsilon}\int_{U(\epsilon)} \tilde{\omega} = \frac{d}{d\epsilon}\int_{U(0)} \tilde{\omega} + \epsilon\, \mathcal{L}_{\vec{\xi}}\, \tilde{\omega} + o(\epsilon)$$

$$= \int_{U(0)} \mathcal{L}_{\vec{\xi}}\, \tilde{\omega} + \text{small stuff}$$

Example: In $\mathbb{E}^2$ consider $\tilde{\alpha} = \alpha_1 \tilde{d}x^1 + \alpha_2 \tilde{d}x^2$ can convert to vetor

we apply $\tilde{d}$ and obtain

$$\tilde{d}\alpha = \tilde{d}(\alpha_1 \tilde{d}x^1 + \alpha_2 \tilde{d}x)$$ is a product rule going on [Next assignment, comes out friday Nov 17)

$$= \alpha_{1,2}\, \tilde{d}x^2 \wedge \tilde{d}x^1 + \alpha_{2,1}\, \tilde{d}x^1 \wedge \tilde{d}x^2$$

$$= \alpha_{2,1} - \alpha_{1,2}\, \tilde{d}x^1 \wedge \tilde{d}x^2$$

Stokes' theorem can be written as

$$\int_U \tilde{d}\tilde{\alpha} = \int_{\partial U} \tilde{\alpha}$$

$$\int_U (\alpha_{2,1} - \alpha_{1,2})\, \tilde{d}x^1 \wedge \tilde{d}x^2 = \int_{\partial U} \alpha_1 \tilde{d}x^1 + \alpha_2 \tilde{d}x^2$$

$(\alpha_1, \alpha_2)\cdot(\tilde{d}x^1, \tilde{d}x^2)$

We can rewrite this in terms of "regular" integrals

$$\int_U \left(\frac{\partial \alpha_2}{\partial x} - \frac{\partial \alpha_1}{\partial y}\right) dx\, dy = \oint (\alpha_1, \alpha_2) \cdot \left(\frac{dx}{d\lambda}, \frac{dy}{d\lambda}\right) d\lambda$$

$$= \oint_{\partial U} \bar{\alpha} \cdot d\bar{x}$$

This is greens theorem, a special case of stokes thm

§ 4.23 Gauss' theorem and the defn of divergence

Recall stokes thm can be written a

$$\int_U \tilde{d}\tilde{\alpha} = \int_{\partial U} \tilde{\alpha} \quad \text{or} \quad \boxed{\int_U \tilde{d}[\tilde{\omega}(\bar{\xi})] = \int_{\partial U} \tilde{\omega}(\bar{\xi})\Big|_{\partial U}} \quad *$$

not really need

Suppose $\tilde{\omega} = \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n$ is it because $\frac{\partial}{\partial x} \wedge \tilde{d}x^1 = 0$

then $\tilde{\omega}(\bar{\xi}) = \xi^1 \tilde{d}x^2 \wedge \dots \wedge \tilde{d}^n - \xi^2 \tilde{d}x^1 \wedge \tilde{d}x^3 \wedge \dots \tilde{d}x^n + \dots$

where does the negative come from?

We compute $\tilde{d}$ of the above and get

$$\tilde{d}[\tilde{\omega}(\bar{\xi})] = \xi^1{}_{,1}\, \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n + \xi^2{}_{,2}\, \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n + \dots + \xi^n{}_{,n}\, \tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n$$

$$\Rightarrow \tilde{d}[\tilde{\omega}(\bar{\xi})] = \xi^i{}_{,i}\, \tilde{\omega}$$

We define the $\tilde{\omega}$-divergence of $\bar{\xi}$ as $\boxed{(\mathrm{div}_{\tilde{\omega}}\, \bar{\xi})\, \tilde{\omega} \equiv \tilde{d}[\tilde{\omega}(\bar{\xi})]}$

If we use components such that ∂U is a surface of constant x^1, then the restriction of $\tilde{\omega}(\bar{\xi})$ to ∂U is

$$\tilde{\omega}(\bar{\xi})\Big|_{\partial U} = \xi^1 \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n$$

or

$$= \tilde{d}x^1(\bar{\xi})\, \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n$$

In general if $\tilde{n}$ is a 1-form normal to the boundary of U (∂U), which means that $\tilde{n}(\bar{\eta}) = 0\ \forall\, \bar{\eta}$ tangent to ∂U

and if $\tilde{\alpha}$ is an $(n-1)$-form with

$$\tilde{\omega} = \tilde{n} \wedge \tilde{\alpha}$$

then

$$\tilde{\omega}(\bar{\xi})\Big|_{\partial U} = \tilde{n}(\bar{\xi})\, \tilde{\alpha}\Big|_{\partial U}$$

Therefore the original form of Stokes thm * becomes

$$\int_U (\mathrm{div}_{\tilde{\omega}} \bar{\xi})\tilde{\omega} = \int_{\partial U} \tilde{n}(\bar{\xi})\tilde{\alpha}$$

with $\tilde{\alpha}$ restricted to ∂U and $\tilde{n} \wedge \tilde{\alpha} = \tilde{\omega}$

In component form, this becomes

$$\boxed{\int_U \xi^i{}_{;i}\, d^n x = \oint_{\partial U} \xi^i n_i\, d^{n-1}x}$$

Gauss' divergence thereom in $\mathbb{R}^n$

§ 4.25 Differential forms and Differential Equations

Consider the DE $\frac{dy}{dx} = f(x,y)$

we often rewrite it as $dy = f(x,y)\,dx$

whats the connection between the two?

If M is a 2D manifold with coordinates (x,y), then we consider the following

$$\tilde{d}y - f(x,y)\tilde{d}x = 0$$

→ this is inspiration

Where f is a function on M.

Suppose $\bar{V}$ is a vector at $P \in M$ with components $(1, f(P))$

$$\text{Consider } \tilde{d}y(\bar{V}) = \tilde{d}y(1, f(P)) = f(P)$$

" $\frac{\partial}{\partial x} + f(P)\frac{\partial}{\partial y}$

$$\tilde{d}x(\bar{V}) = \tilde{d}x(1, f(P)) = 1$$

This implies, $\boxed{(\tilde{d}y - f\tilde{d}x)(\bar{V}) = 0}$ $= \tilde{d}y(\bar{V}) - f\tilde{d}(x)(\bar{V}) = f(P) - f = 0$

Solns to the DEs define a submanifold of M whose tangent annul the 1-form

→ send to 0

Submanifolds that annul the 1-form are solutions to this can be generalized to n-forms with Frobenius thm

Question Given a DE, what are the equivalent form?

example $\frac{dx^2}{dt^2} + \omega_0^2 x =$ ω_0 is constant. Harmonic oscillator

or $\frac{dx}{dt} = \omega_0 y$ and $\frac{dy}{dt} = -\omega_0 x$ System of 1st Order Equations or $\frac{dx}{dt} - \omega_0 y = 0$ and $\frac{dy}{dt} + \omega_0 x = 0$

The 1-forms to consider are:

$$\tilde{\alpha} = \tilde{d}x - \omega_0 y \tilde{d}t$$

$$\tilde{\beta} = \tilde{d}y + \omega_0 x \tilde{d}t$$

Finding Submanifolds that annul these forms is equivalent to Solving DEs.

The manifold is 3D with coordinates $[x, y, z]$ and the Solution is 1D.

§ 4.26 Frobenius' theorem (differential forms version)

The set of forms $\{\tilde{\beta}_i\}$ at $P \in M$ define a Subspace of Vectors, $T_P S \subset T_P M$, each of which annuls $\tilde{\beta}_i$, i.e. For all $\bar{V} \in T_P S$, $\tilde{\beta}_i(\bar{V}) = 0$ $\forall i = 1, \ldots, n$

The set $T_P S$ is called annihilator of $\{\tilde{\beta}_i\}$

The complete ideal consists of all the forms at P whose restriction to $T_P S$ vannishes

Note: if $\tilde{\gamma}$ is a form at P then $\tilde{\gamma} \wedge \tilde{\beta}_i$ is 0 when restricted to $T_P S$ and therefore $\tilde{\gamma} \wedge \tilde{\beta}_i$ is in the complete ideal

A complete ideal has a basis $\{\tilde{\alpha}_j\}$ that generates the ideal ie

the complete ideal of $\{\tilde{\alpha}_j\}$ is the Same as the Complete ideal of $\{\tilde{\beta}_j\}$

All of this extends from vectors to vector fields

Lec 20 - Nov 21st

§ 4.26 Frobenius theorem.

$\{\tilde{\beta}_i\}$ defines a subspace, $T_pS \subset T_pM$, each of which annuls $\tilde{\beta}_i$.

$\forall \bar{V} \in T_pS$ then $\tilde{\beta}_i(\bar{V}) = 0$ $\quad \forall i = 1, \ldots, n$

T_pS is the annihilator of $\{\tilde{\beta}_i\}$.

The complete ideal consists of all the forms whos restriction to T_pS vanishes.

Note: If $\tilde{\gamma}$ is a form then $\tilde{\gamma} \wedge \tilde{\beta}_i$ is 0 when restricted to T_pS $\therefore$ $\tilde{\gamma} \wedge \tilde{\beta}_i$ is in the complete ideal ★

$\{\tilde{\alpha}_j\}$ is closed if each $\tilde{d}\tilde{\alpha}_j$ is in the complete ideal generated by $\{\tilde{\alpha}_j\}$

Aside: A complete ideal has a basis $\{\tilde{\alpha}_j\}$ that generates ideal

Frobenius Theorem:

Suppose $\{\tilde{\alpha}_i, i = 1, \ldots, m\}$ is a linearly independent set of 1-form fields in an open set $U \subset M$, where M is an n-dimensional manifold. The set $\{\tilde{\alpha}_i\}$ is closed iff functions $\{P_{ij}, Q_j \quad i,j = 1 \ldots m\}$ such that

an array w 2 index

a 1- array

$$\tilde{\alpha}_i = \sum_{j=1}^{m} P_{ij}\, \tilde{d}Q_j$$

Idea: In general to solve OEs, we want to find solutions to $\{\tilde{\alpha}_i = 0\}$. The solution to this set of equations by $Q_j =$ constant

This set of Q_j are solution to the equations $\{\tilde{\alpha}_i = 0\}$ and each Q_j defines an m-dimensional submanifold of M. and its tangent vectors annul $\{\tilde{d}Q_j\}$ and also $\{\tilde{\alpha}_j\}$

example: suppose $\tilde{\alpha} = \tilde{d}f$ this satisfieds the above with $P_{11} = 1$ and $f = Q$. f exists iff $\tilde{d}\tilde{\alpha} = 0$

exercise 4.30 $\{\tilde{\alpha}_j, j = 1 \ldots m\}$ is a linearly independent set of 1-forms then any form $\tilde{\gamma}$ is in the complete ideal iff $\tilde{\gamma} \wedge \tilde{\alpha}_1 \wedge \tilde{\alpha}_2 \wedge \cdots \wedge \tilde{\alpha}_m = 0$

§ 5 Applications to Physics

§ 5 A Thermodynamics

§ 5.1 Simple Systems

Consider a one-component fluid where the conservation of energy dictates that

path independent?

$$\boxed{\delta Q = P\,\delta V + dU} \qquad 1^{st} \text{ Law of Thermodynamics}$$

variation (changes?) path dependent?

where U is the internal energy

δQ heat absorbed

$P\delta V$ work done by the fluid

P, V pressure and volume

This law can be written in terms of 1-forms on a 2D manifold with coordinates (V, U) assuming good coordinates

Then $P(\underbrace{V}_{\text{volume}}, \overset{\text{internal energy}}{U})$ is a function on M that is the equation of state

On the RHS, it would make sense to write it as

$$P\underbrace{\tilde{d}V}_{\text{may not be}} + \underbrace{\tilde{d}U}_{\text{exact}}$$

Since $\tilde{d}V$ and $\tilde{d}U$ are one forms, we deduce that the LHS $\tilde{\delta}Q$ is a 1-form as well.

Question is $\tilde{\delta}Q = \tilde{d}Q$, is it an exact one-form? If yes, then $\tilde{d}\tilde{d}Q = 0$ and we deduce

$$0 = \tilde{d}(\tilde{d}Q) = \tilde{d}(P\tilde{d}V + \underbrace{\tilde{d}U}_{0})$$

gradient of function again!

gradient of P

That reduces to

$$0 = \tilde{d}(P\tilde{d}V) = \boxed{\tilde{d}P \wedge \tilde{d}V}$$ Assignment 5.

$$\tilde{d}P \wedge \tilde{d}V + P\underbrace{\tilde{d}\tilde{d}V}_{0}$$

Re look up

we can simplify this,

$$\left(\underbrace{\left(\frac{\partial P}{\partial V}\right)_U \tilde{d}V}_{\tilde{d}V \wedge \tilde{d}V = 0} + \left(\frac{\partial P}{\partial U}\right)_V \tilde{d}U\right) \wedge \tilde{d}V = 0$$

$$\left(\frac{\partial P}{\partial U}\right)_V \tilde{d}U \wedge \tilde{d}V = 0$$

But this can only be true if $\left(\frac{\partial P}{\partial U}\right)_V = 0$ This is typically the case.

In general Q does not exist and we can't write $\tilde{\delta}Q$ as $\tilde{d}Q$

However, since $\tilde{\delta}Q$ is a 1-form but not exact in 2-space $\tilde{d}(\tilde{\delta}Q)$ is a 2-form. This 2-form is in the complete ideal of δQ hence $\tilde{\delta}Q$ is closed.

We can use Frobenius' theorem and deduce that $\exists\ T(V,U)$ and $S(V,U)$ such that

$$\boxed{\tilde{\delta}Q = T\tilde{d}S}$$

This looks like the 2nd Law of Thermal dynamics.

With this choice the first law becomes $\boxed{T\tilde{d}S = P\tilde{d}V + \tilde{d}U}$

§ 5.2 Maxwell and Other mathematical identities

Apply $\tilde{d}$ to the above equation

$$\tilde{d}(T\tilde{d}S) = \tilde{d}(P\tilde{d}V) + \tilde{d}\tilde{d}U \xrightarrow{} 0$$

Assumption 1 presume $T(S,V)$ and $P(S,V)$

partial is proportional to the one form in the gradient calculation.

$$\tilde{d}T \wedge \tilde{d}S = \tilde{d}P \wedge \tilde{d}V$$

$$\left(\frac{\partial T}{\partial V}\right)_S \tilde{d}V \wedge \tilde{d}S = \left(\frac{\partial P}{\partial S}\right)_V \tilde{d}S \wedge \tilde{d}V$$

$$= -\left(\frac{\partial P}{\partial S}\right)_V \tilde{d}V \wedge \tilde{d}S$$

$$\Rightarrow \boxed{\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V}$$ One of Maxwell's indentities.

Assumption 2 $S(T,V)$ and $P(T,V)$

Sub into $\tilde{d}(T\tilde{d}S) = \tilde{d}(P\tilde{d}V)$

$$\tilde{d}T \wedge \tilde{d}S = \left(\frac{\partial P}{\partial T}\right)_V \tilde{d}T \wedge \tilde{d}V$$

$$\left(\frac{\partial S}{\partial V}\right)_T \tilde{d}T \wedge \tilde{d}V = \left(\frac{\partial P}{\partial T}\right)_V \tilde{d}T \wedge \tilde{d}V$$

$$\Rightarrow \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

Assumption : Divide eqn by T and apply $\tilde{d}$ the you can obtain

another Maxwell identity. By dividing (5.2) by T and then taking the exterior derivative we get

$$\frac{1}{T}\tilde{d}P \wedge \tilde{d}V - \frac{P}{T^2}\tilde{d}T \wedge \tilde{d}V - \frac{1}{T^2}\tilde{d}T \wedge \tilde{d}U = 0.$$

By writing $U = U(T, V), P = P(T, V)$, we get

$$\frac{1}{T}\left(\frac{\partial P}{\partial T}\right)_V \tilde{d}T \wedge \tilde{d}V - \frac{P}{T^2}\tilde{d}T \wedge \tilde{d}V - \frac{1}{T^2}\left(\frac{\partial U}{\partial V}\right)_T \tilde{d}T \wedge \tilde{d}V = 0,$$

or

$$T\left(\frac{\partial P}{\partial T}\right)_V - P = \left(\frac{\partial U}{\partial V}\right)_T. \qquad (5.6)$$

$$T\left(\frac{\partial P}{\partial T}\right)_V - P = \left(\frac{\partial U}{\partial V}\right)_T$$ assume $P(T,V)$ and $U(T,V)$

$$\boxed{T\tilde{d}S = P\tilde{d}V + \tilde{d}U}$$

$$\tilde{d}P = \left(\frac{\partial P}{\partial V}\right)_T \tilde{d}V + \left(\frac{\partial P}{\partial T}\right)_V \tilde{d}T$$

Dividing by T

$$\tilde{d}S = \frac{P}{T}\tilde{d}V + \frac{1}{T}\tilde{d}U$$

$$\tilde{d}\tilde{d}S = \tilde{d}\left(\frac{P}{T}\tilde{d}V\right) + \tilde{d}\left(\frac{1}{T}\tilde{d}U\right)$$

$$\frac{1}{T}\overbrace{\left(\left(\frac{\partial U}{\partial V}\right)_T \tilde{d}V + \left(\frac{\partial U}{\partial T}\right)_V \tilde{d}T\right)}^{\tilde{d}U}$$

$$0 = \left(\frac{-1}{T^2}P\tilde{d}T + \frac{1}{T}\tilde{d}(P)\right)\wedge\tilde{d}V + \left(\frac{-1}{T^2}\tilde{d}T\wedge\tilde{d}U + \frac{1}{T}\tilde{d}\tilde{d}U\right) \to 0?$$

$$= \frac{-P}{T^2}\tilde{d}T\wedge\tilde{d}V + \frac{1}{T}\left[\cancel{\left(\frac{\partial P}{\partial V}\right)_T\tilde{d}V}^{\,0} + \left(\frac{\partial P}{\partial T}\right)_V\tilde{d}T\right]\wedge\tilde{d}V - \frac{1}{T^2}\tilde{d}T\wedge\tilde{d}U$$

$$= -\frac{P}{T^2}\tilde{d}T\wedge\tilde{d}V + \frac{1}{T}\left(\frac{\partial P}{\partial T}\right)_V\tilde{d}T\wedge\tilde{d}V - \frac{1}{T^2}\tilde{d}T\wedge\tilde{d}U$$

$$\frac{1}{T^2}\left[\left(\frac{\partial U}{\partial V}\right)_T\tilde{d}V + \left(\frac{\partial U}{\partial T}\right)_V\tilde{d}T\right]$$

$$\frac{1}{T^2}\left[\left(\frac{\partial U}{\partial V}\right)_T\tilde{d}V + \cancel{\left(\frac{\partial U}{\partial T}\right)_V\tilde{d}T}^{\,0}\right]\wedge\tilde{d}T$$

$$-\frac{1}{T^2}\left(\frac{\partial U}{\partial V}\right)_T\tilde{d}T\wedge\tilde{d}V$$

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Phase Space in Mechanics

From section §2.3 c of The Geometry of Physics by Frankel

In classical Mechanics we describe a system using generalized coordinates,

$q^1, \dots, q^n$ } compactly called q

These form an n-dimensial manifold M that we call configuration space.

The Lagrangian is a function of q and $\dot{q}$, where $\dot{q} = \frac{dq}{dt}$ which also has n-coordinates

These 2n coordinates $q, \dot{q}$ completely specify the state

The $\dot{q}$ are generalized velocities and are in T_pM There fore, $(q, \dot{q})$ is in the tangent bundle, TM

The lagrangian, $L(q, \dot{q})$, is a map $L: TM \to \mathbb{R}$.

For Hamiltonian mechanics, we need the generalized momenta

$$\boxed{p_i(q, \dot{q}) \equiv \frac{\partial L}{\partial \dot{q}^i}} \to \text{one form, ie in co-tangent space}$$

↳ sub script means p is in cotangent space

To build the Hamiltonian, we need the Lagrangian and a transformation

$$(q, \dot{q}) \to (q, p)$$

This is not simply changing coordinates

To see this suppose we have a change in generalized coordinates

$$q_u \longrightarrow q_v$$

This can be described as

prime denotes new coordinates

$$q_u = q_v(q_u)$$

$$\dot{q}_v^{i'} = \left(\frac{\partial q_v^{i'}}{\partial q_u^j}\right) \dot{q}_u^j$$

Compare this with $\Lambda^{i'}{}_j = \frac{\partial y^{i'}}{\partial x^j}$ $\quad V^{i'} = \Lambda^{i'}{}_j V^j$ (contravariant) (vectors do this)

The p's transform as follows:

$$p_{i'}^{v} \equiv \frac{\partial L}{\partial \dot q_v^{i'}} = \left(\frac{\partial L}{\partial q_u^{j}} \overbrace{\frac{\partial q_u^{j}}{\partial \dot q_v^{i'}}}^{0} + \frac{\partial L}{\partial \dot q_u^{j}} \frac{\partial \dot q_u^{j}}{\partial \dot q_v^{i'}} \right)$$

$$p_{i'}^{v} = p_j^{u} \left(\frac{\partial \dot q_u^{j}}{\partial \dot q_v^{i'}} \right)$$

This shows this is covariant and must live in the cotangent space

Compare with $\Lambda^{j}_{\ i'} \equiv \dfrac{\partial x^j}{\partial y^{i'}}$

$\dot q$ is in the tangent space (vector)

p is in the cotangent space (one-form)

Hence computing p is not only changing variables but is really a map

$$p : TM \longrightarrow T^*M$$

right hand side could be T_p^*M but if we add q then T^*M

T^*M is the phase space (q,p)

The Hamiltonian is a map H s.t.

$$H : T^*M \longrightarrow \mathbb{R} \qquad H(q,p)$$

The Lagrangian: $L(q,\dot q) = T(q,\dot q) - U(q)$

metric

Where the KE is: $T(q,\dot q) = \frac{1}{2} g_{jk} \dot q^j \dot q^k$

Example: Suppose we have 2 masses in 1D

$M = \mathbb{R}^2$ and $TM = \mathbb{R}^4$

$T = \frac{1}{2} m_1 (\dot q_1)^2 + \frac{1}{2} m_2 (\dot q_2)^2$

need with $g_{ij} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$

Example: If we have a mass in 2D

$T = \frac{1}{2} m (\dot x + \dot y)^2$ [cartesian]

with $g_{ij} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

Involved q

$T = \frac{1}{2} m (\dot r^2 + r^2 \dot \phi^2)$ $\quad g_{ij} = \begin{bmatrix} m & 0 \\ 0 & r^2 \end{bmatrix}$ [polar coordinates]

In general $p_i \equiv \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial T}{\partial \dot{q}^i} = g_{ij}(q)\,\dot{q}^j$

∂T can be used to define a Riemannia metric

$$\langle \dot{q}, \dot{q} \rangle = g_{ij}(q)\,\dot{q}^i \dot{q}^j$$

Kinetic Energy is ½ the length squared of the velocity vector

The generalized momenta p is the covariant version of the generalized velocity.

Example 1: $p_1 = m_1 \dot{q}^1$ and $p_2 = m_2 \dot{q}^2$

In general $p_i = g_{ij}\dot{q}^j$ and $\dot{q}^i = g^{ij} p_j$

§ 5.4 Hamiltonian Vector Fields

Given a Lagrangian, we can obtain the equations of Motion from the Euler Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

Hamiltonian

$$\mathcal{H}(q,p) = p\dot{q} - L$$

Hamiltons' eqns

$$\dot{q}\ \frac{\partial \mathcal{H}}{\partial p} \quad + \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

Phase space is the tangent bundle T^*M, which Includes M and T_p^*M

On T^*M, which is a manifold, we define a 2-form,

$$\boxed{\tilde{\omega} = \tilde{\partial} q \wedge \tilde{\partial} p}$$ area in phase space

can be called M?

Take a curve on T^*M of the form

$$\{ q = f(t),\ p = g(t) \}$$

Which is a solution to Hamilton's equations. The tangent vector to the curve is,

$$\bar{U} = \frac{d}{dt} = \dot{f}\frac{\partial}{\partial q} + \dot{g}\frac{\partial}{\partial p}$$ basis vectors

Theorem: If $\bar{u}$ is a tangent vector to the solution curve then $\mathcal{L}_{\bar{v}}\tilde{\omega} = 0$

Proof: From a formula (4.67)

$$\mathcal{L}_{\bar{v}}\tilde{\omega} = \tilde{d}[\tilde{\omega}(\bar{u})] + \underbrace{(\tilde{d}\tilde{\omega})}_{=0}(\bar{u})$$

The 2nd term is 0 since $\tilde{\omega}$ is a 2-form on a 2 dim Manifold

$$\Rightarrow \mathcal{L}_{\bar{v}}\tilde{\omega} = \tilde{d}[\tilde{\omega}(\bar{u})]$$

$$= \tilde{d}[\tilde{d}q \wedge \tilde{d}p\,(\bar{u})]$$

$$= \tilde{d}[(\tilde{d}q \otimes \tilde{d}p - \tilde{d}p \otimes \tilde{d}q)(\bar{u})]$$

one form

$$= \tilde{d}[(\tilde{d}q(\bar{u})\tilde{d}p - \underbrace{\tilde{d}p(\bar{u})}_{\text{coefficient of 1 form}}\tilde{d}q)]$$

But $\bar{u} = \dot{f}\frac{\partial}{\partial q} + \dot{g}\frac{\partial}{\partial p}$ which yields

$$\tilde{d}q(\bar{u}) = \dot{f} \quad \text{and} \quad \tilde{d}p(\bar{u}) = \dot{g}$$

$$\Rightarrow \mathcal{L}_{\bar{v}}\tilde{\omega} = \tilde{d}[\dot{f}\tilde{d}p - \dot{g}\tilde{d}q]$$

However $\dot{f} = \frac{\partial H}{\partial p}$ and $\dot{g} = -\frac{\partial H}{\partial q}$ from Hamiltons eqn

$$\mathcal{L}_{v}\tilde{\omega} = \tilde{d}\left[\frac{\partial H}{\partial q}\tilde{d}q + \frac{\partial H}{\partial p}\tilde{d}p\right]$$

$$= \tilde{d}[\tilde{d}H] = 0$$

The area in phase space is conserved along solns to Hamiltons equations

A Vector field with $\mathcal{L}_{\bar{v}}\tilde{\omega} = 0$ is a Hamitonian vector field. $\bar{u}$ is tangent to the curves in phase space

The System is conservative (H is constant along Solns)

$$\mathcal{L}_{\bar{v}} H = \frac{dH}{dt} = \dot{q}\frac{\partial H}{\partial q} + \dot{p}\frac{\partial H}{\partial p}$$

$$= \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial H}{\partial p}$$

$$= 0$$

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§ 5.5 canonical transformation

p and q are not unique. P and Q are canonical if $\tilde{d}q \wedge \tilde{d}p = \tilde{d}P \wedge \tilde{d}Q$

This requires

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1$$

example $Q = p$ and $P = -q$

check: $\frac{\partial p}{\partial q}\frac{\partial(-q)}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial(q)}{\partial q} = 1$

§ 5.6 Map between vectors and 1-forms by $\tilde{\omega}$

$\tilde{\omega} = \tilde{d}q \wedge \tilde{d}p$ can be used like the metric tensor to convert vectors to forms and vice versa. Suppose $\bar{V}$ is a vector field on M. then $\tilde{V} = \tilde{\omega}(\bar{V}) = \tilde{d}q \wedge \tilde{d}p\,(\bar{V})$

$$= (\tilde{d}q \otimes \tilde{d}p - \tilde{d}p \otimes \tilde{d}q)\,(\bar{V})$$

$$= \tilde{d}q(\bar{V})\,\tilde{d}p - \tilde{d}p(\bar{V})\,\tilde{d}q$$

If $\bar{V} = V^1 \partial/\partial q + V^2 \partial/\partial p$ then

$$\tilde{V} = V^1 \tilde{d}p - V^2 \tilde{d}q$$

when these become 1-form components doesnt this lower the indices?

$$= -V^2 \tilde{d}q + V^1 \tilde{d}p$$

We can write $(\tilde{V})_i = \omega_{ij} V^j$ and deduce that

$$\omega_{ij} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \omega^{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Using ω^{ij} we can find $\bar{V}$ given $\tilde{V}$

§ 5.7 Poisson Bracket

Say f, g are functions on M and define

$$\bar{X}_f = \overline{df} \quad \text{and} \quad \bar{X}_g = \overline{dg}$$

These are the vector versions of the gradient, From above,

$$\tilde{d}f = \frac{\partial f}{\partial q}\tilde{d}q + \frac{\partial f}{\partial p}\tilde{d}p$$

then

$$\bar{X}_f = \bar{df} = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p}$$

$$\bar{X}_g = \bar{dg} = \frac{\partial g}{\partial p}\frac{\partial}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial}{\partial p}$$

We can then define the Poisson bracket

$$\begin{aligned}\{f,g\} &\equiv \tilde{\omega}(\bar{X}_f, \bar{X}_g) \\ &= \omega_{ij} X_f^i X_g^j \\ &= (X_f)_j X_g^j \\ &= \tilde{d}f(\bar{X}_g) = \langle \tilde{d}f, \bar{X}_g \rangle\end{aligned}$$

To evaluate this we get,

$$\{f,g\} = \left(\frac{\partial f}{\partial q}\tilde{d}q + \frac{\partial f}{\partial p}\tilde{d}p\right)\left(\frac{\partial g}{\partial p}\frac{\partial}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial}{\partial p}\right)$$

$$\boxed{\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}}$$ poisson bracket

Aside $\tilde{d}q\left(\frac{\partial}{\partial q}\right) = 1$

$\tilde{d}q\left(\frac{\partial}{\partial p}\right) = 0$

The above expression is in terms of coordinates. The expression independent of coordinates is

$$\{f,g\} = \tilde{\omega}(\bar{df}, \bar{dg})$$

§ 5.8 Many particle systems: symplectic forms

In 3D with no constraints, and N particles, we have 6N dim'l phase space.

The phase space in general can be said to be 2N where N is the number of generalized coordinates

then

$$\boxed{\tilde{\omega} = \sum_{A=1}^{n} \tilde{d}q^A \wedge \tilde{d}p_A}$$ Symplectic Form

The Phase space is a Symplectic Manifold.

§ 5.9 Linear Dynamical systems: the symplectic inner product and conserval quantities

to begin consider the following hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{A,B=1}^{n} T^{AB} p_A p_B + V_{AB} q^A q^B$$

where we assume T^{AB} and V_{AB} are symmetric. If not, we use the fact the product of the asymetric part and a symmetric function is 0.

For symplicity, assume T^{AB} and V_{AB} are constant.

Hamilton's equations

$$\frac{dP_A}{dt} = -\frac{\partial \mathcal{H}}{\partial q^A} = -\sum_B V_{AB} q^B$$

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p_A} = \sum_B T^{AB} p_B$$

Check: $$\frac{\partial \mathcal{H}}{\partial q^C} = \frac{\partial}{\partial q^C}\left(\frac{1}{2}\sum_{A,B=1}^{n} V_{AB} q^A q^B\right)$$

$$= \frac{1}{2}\sum_{A,B} V_{AB}\,\delta^A_C\, q^B + \frac{1}{2}V\sum_{A,B} V_{AB}\, q^A \delta^B_C$$

$$= \frac{1}{2}\sum_B V_{CB}\, q^B + \frac{1}{2}\sum_A \underbrace{V_{AC}}_{= V_{CA},\ A\to B} q^A$$

$$= \sum_B V_{CB}\, q^B$$

If $\bar{Y}_{(1)}$ is a vector with components $\{q_{(1)}{}^A, P_{(1)A}, A=1\ldots n\}$ and $\bar{Y}_2$ is a vector w components $\{q^A_{(2)}, P_{(2)A}, A=1,\ldots,n\}$ then their sympleetic product is

$$\tilde{\omega}(\bar{Y}_{(1)}, \bar{Y}_{(2)}) = \sum_A q^A_{(1)} P_{(2)A} - q^A_{(2)} P_{(1)A}$$

If $\bar{Y}_{(1)}$ and $\bar{Y}_{(2)}$ are both solutions, then the symplectic inner product is independent of time

$$\frac{d}{dt}\tilde{\omega}(\bar{Y}_{(1)}, \bar{Y}_{(2)}) = \frac{d}{dt}\left[\sum_{A=1}^{n} q^A_{(1)} P_{(2)A} - q^A_{(2)} P_{(1)A}\right]$$

$$= \sum_{A=1}^{n}\left\{ \frac{dq^A_{(1)}}{dt} P_{(2)A} + q^A_{(1)}\frac{dP_{(2)A}}{dt} - \frac{dq^A_{(2)}}{dt} P_{(1)A} - q^A_{(2)}\frac{dP_{(1)A}}{dt}\right\}$$

Using Hamilton's eqns next,

$$\frac{d}{dt}\tilde{\omega}(\bar{Y}_{(1)}, \bar{Y}_{(2)}) = \sum_{A,B}^{n} \{ T^{AB} P_{(1)B} P_{(2)A} - V_{AB}\, q_{(1)}^{A} q_{(2)}^{B} - T^{AB} P_{(2)B} P_{(1)A} + V_{AB}\, q_{(2)}^{B} q_{(1)}^{A} \} = 0$$

If T^{AB} and V_{AB} are independent time then it follow that if $\bar{Y}_{(1)}$ is a soln then so is $\frac{d\bar{Y}_{(1)}}{dt}$

This motivates defining the canonical energy as $E_c(\bar{Y}) = \tilde{\omega}(\frac{d\bar{Y}}{dt}, \bar{Y})$

It can be determined that $E_c(\bar{Y}) = \mathcal{H}$ evaluated at $\bar{Y}$

$$E_c(\bar{Y}) = \frac{1}{2}\tilde{\omega}(\dot{\bar{Y}}, \bar{Y}) = \sum_{A} (\dot{q}_{(1)}^{A} P_{(2)A} - q_{(2)}^{A} \dot{P}_{(1)A})$$

$$= \frac{1}{2}\sum_{A,B} T^{AB} P_{(1)B} P_{(2)A} + V_{AB}\, q^{B}_{(1)} q^{A}_{(2)} = \mathcal{H}$$

So the independence of $\mathcal{H}$ w.r.t yields the conservation of $\mathcal{H}$ or the total Mechanical Energy

Other conserved quantities

In general, T^{AB} and V_{AB} can depend on the coordinates $\{x^i\}$.

If $\exists\, \bar{u}$ such that $\mathcal{L}_{\bar{u}} T^{AB} = 0 = \mathcal{L}_{\bar{u}} V_{AB}$ then there are conserved quantities associated to $\bar{u}$

This can yield expressions for Linear Momentum or angular moment conservation

Noethers theorem?

Exam Content

Lec 23 - Nov 30th

§5.11 Rewritting Maxwell's equations in differential forms
§C electromagnetism

We can non-dimensionalize maxwell's equations in such away that $c = \mu_0 = \epsilon_0 = 1$ to get

A $\quad \vec{\nabla} \times \vec{B} - \frac{\partial E}{\partial t} = 4\pi \vec{J}$ Amperes Law

electric field, magnetic field

B $\quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ Faraday's Law

C $\quad \vec{\nabla} \cdot \vec{B} = 0$

D $\quad \vec{\nabla} \cdot \vec{E} = 4\pi \underbrace{\rho}_{\text{charge}}$

Gaus (C, D)

We will rewrite these using a metric and $\tilde{d}$. The relativistic invariant form requires the Faraday 2-form

$$F_{\mu\nu} = \begin{array}{c|cccc} & t & x & y & \nu = z \\ \mu = t & 0 & -E_x & -E_y & -E_z \\ x & E_x & 0 & B_z & -B_y \\ y & E_y & -B_z & 0 & B_x \\ z & E_z & B_y & -B_x & 0 \end{array} \qquad \mu, \nu = t, x, y, z$$

(μ: rows, ν: colums)

→ How is this a 2-Form

what does this look like?

Then $\tilde{d}F$ is a 3-form on a 4D manifold. Since $\tilde{d}F$ is a 3-form on a 4D Manifold, there are C^4_3 different equations, $C^4_3 = 4$

We can write $F = F_{\mu\nu}\, \tilde{d}\mu \wedge \tilde{d}\nu$, we compute,

$$\tilde{d}F = F_{\mu\nu,\gamma}\, \tilde{d}\gamma \wedge \tilde{d}\mu \wedge \tilde{d}\nu$$

It is observed that $\tilde{d}F = 0$ iff $F_{[\mu\nu,\gamma]} = 0$

① $F_{[xy,z]} = F_{xy,z} + F_{yz,x} + F_{zx,y} = 0$

$B_{z,z} + B_{x,x} \; B_{y,y} = 0$ Div of Magnetic field $= 0$

or $\vec{\nabla} \cdot \vec{B} = 0$ Eqn C

② $F_{[xy,t]} = F_{xy,t} + F_{yt,x} + F_{tx,y} = 0$

$B_{z,t} + E_{y,x} - E_{x,y} = 0$ the z equation of eqn B

③ $F_{[yz,t]} = F_{yz,t} + F_{zt,y} + F_{ty,z} = 0$

$= B_{x,t} + E_{z,y} - E_{y,z} = 0$ the x eqn in B

④ $F_{[xz,t]} = F_{xz,t} + F_{zt,x} + F_{tx,z} = 0$

$-B_{y,t} + E_{z,x} - E_{x,z} = 0$ the y eqn of B

For the other equations we need the special relativistic metric

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \text{Lorentzian metric?}$$

This allows us to find the 2-vector $F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} = g^{\mu\alpha} F_{\alpha\beta}\, g^{\beta\nu}$

Note: $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$

$$F^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

The final four eqn's are $F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu$ where $J^T = \rho$, $J^i = (J)^i$ $i = x, y, z$

We check 4 different equations

① $F^{t\nu}{}_{,\nu} = F^{tx}{}_{,x} + F^{ty}{}_{,y} + F^{tz}{}_{,z} = 4\pi J^t$

$= E_{x,x} + E_{y,y} + E_{z,z} = 4\pi\rho$ eqn D

② $F^{x\nu}{}_{,\nu} = F^{xt}{}_{,t} + F^{xy}{}_{,y} + F^{xz}{}_{,z} = 4\pi J^x$

$= -E_{x,t} + B_{z,y} - B_{y,z} = 4\pi J^x$ This is the x eqn in A

③ $F^{y\nu}{}_{,\nu} = F^{yt}{}_{,t} + F^{yx}{}_{,x} + F^{yz}{}_{,z} = 4\pi J^y$

$= -E_{y,t} - B_{z,x} \quad B_{x,z} = 4\pi J^y$ The y eqn in A

④ $F^{z\nu}{}_{,\nu} = F^{zt}{}_{,t} + F^{zx}{}_{,x} + F^{zy}{}_{,y} = 4\pi J^z$

$= -E_{z,t} + B_{y,x} - B_{x,y} = 4\pi J^z$ The z eqn in A.

Observe that $\tilde{d}F$ is coordinate independent, However $F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu$ is coordinate dependent.

Given a basis of the tangent space $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ then, we can define the volume 4-form as

$$\tilde{\omega} = \tilde{d}t \wedge \tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z$$

We define $* \tilde{F} = \frac{1}{2}\tilde{\omega}(F)$ or $(*\tilde{F})_{\mu\nu} = \frac{1}{2}\omega_{\alpha\beta\mu\nu}F^{\alpha\beta}$ dual or Hodge Star

Next we determine the components of this,

① $(*\tilde{F})_{tx} = \frac{1}{2}\omega_{\alpha\beta tx}F^{\alpha\beta}$

$= \frac{1}{2}\omega_{yztx}F^{yz} + \underbrace{\frac{1}{2}\omega_{zytx}F^{zy}}_{\text{antisymmetric}}$

$yztx \to -ytzx \to ytxz \to -tyxz \to txyz$ $+1$???

$(*\tilde{F})_{tx} = B_x$

② $(*\tilde{F})_{ty} = \frac{1}{2}\omega_{\alpha\beta ty}F^{\alpha\beta} = \frac{1}{2}(\omega_{zxty}F^{zx} + \omega_{xzty}F^{xz})$ $= -F^{zx}$

$zxty \to -xzty \to xtzy \to -txzy \to txyz$ $+1$ coefficent.

$(*\tilde{F})_{ty} = B_y$

③ $(*\tilde{F})_{tz} = B_z$

④ $(*\tilde{F})_{xy} = E_z$

⑤ $(*\tilde{F})_{xz} = -E_y$

⑥ $(*\tilde{F})_{yz} = E_x$

$$(*\tilde{F})_{\mu\gamma} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & F_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & E_x & 0 \end{pmatrix}$$

The exterior derivative of this is

$$\tilde{d}(*\tilde{F}) = (*\tilde{F})_{\mu\nu,\gamma}\, \tilde{d}\gamma \wedge \tilde{d}\mu \wedge \tilde{d}\nu$$

we define $*\tilde{J} = \tilde{\omega}(\bar{J})$ and apply $\tilde{\omega}$ to the eqn $F^{\mu\nu}{}_{,\gamma} = 4\pi J^{\mu}$

$$\tilde{\omega}(F^{\mu\nu}{}_{,\gamma}) = \tilde{\omega}(4\pi J^{\mu})$$

we can write this as

$$\boxed{\tilde{d}(*\tilde{F}) = 4\pi *\tilde{J}}$$

and $$\tilde{d}F = 0$$

§ 5.13 Vector Potential

If $\tilde{d}F = 0$ then F is closed and since it is a 2-form, $\exists$ a 1-form $\tilde{A}$ such that

$$F = \tilde{d}\tilde{A} \quad \text{at least locally}$$

$\tilde{A}$ is the vector potential

Lec 24 Dec 5^{th}

D. Dynamics of a perfect fluid

§ 5.15 Role of Lie derivatives

A perfect fluid (idealized) is one that conserves certain propreties

① Mass

② Entropy

③ Vorticity [will explain] $\nabla \times$ velocity

Today we will express the equations of a fluid using exterior calculus.

§ 5.16 the Comoving time derivative

The conservation of mass (continuity eqn) is:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \bar{V}) = 0$$

things converging, denisty increases

On an assigment we found

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right)(\rho \underbrace{\tilde{\omega}}_{\text{mass}}) = 0$$

This is being written in terms of differential forms

where $\tilde{\omega} = \tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z$

The operator $\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right)$ computes the total rate of change following the flow.

Consider the motion of a fluid parcel. If the change happens over a short time,

$$dt \ll 1$$

makes the following approximation valid

Then the motion is from

$$(x, y, z, t) \to (x + V^x dt,\ y + V^y dt,\ z + V^z dt,\ t + dt)$$

The difference between the two is

$$(V^x, V^y, V^z, 1)\, dt = \bar{U} \quad \text{is a 4-vector}$$

In the formulation, the total rate of change following the flow:

$$\mathcal{L}_{\bar{u}} \bar{W} = \left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{v}}\right) \bar{W}$$ Space time Version.

where $\bar{W}$ is a 4-vector and it would need to be decomposed on the RHS.

§ 5.17 Eqns of Motion

A perfect fluid conserved entropy. If S is the entropy, then the eqn is

1 of 3 $$\boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{v}}\right) S = 0}$$ Thermodynamics

The conservation of Linear momentum (Newton's 2nd Law) can be written as:

$$\underbrace{\frac{\partial}{\partial t} V^i + V^j \frac{\partial}{\partial x^j} V^i}_{\text{total rate of change } F = ma?} + \frac{1}{\rho} \frac{\partial}{\partial x^i} \underset{\text{pressure}}{P} + \frac{\partial}{\partial x^i} \underset{\text{gravity}}{\Phi} = 0$$

P is pressure

Φ is the gravitational potential

V^i is the Velocity

This eqn is a mess as we have both superscripts and subscripts added to each other. Bad!

$\frac{\partial V^i}{\partial x^j}$ is not a $\binom{1}{1}$ tensor. → Partial derivatives don't transform as vectors

→ How does a metric relate to a distance fcn.

Assume we have a metric

This allows us to convert the Vector $\bar{V}$ to the one-form $\tilde{V}$

$$\boxed{\tilde{V} = g_1(\bar{V}, *)}$$ yeilds V_i

↳ star means empty in this case

To rewrite the non-linear term, we need the operator

$$\boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{v}}\right) \tilde{V}}$$

To find out what this term looks like, consider

eqn 3.14 in textbook

$$(\mathcal{L}_{\bar{V}} \tilde{V})_i = V^j \frac{\partial}{\partial x^j} V_i + V_j \frac{\partial}{\partial x^i} V^j$$

Can show using $\vec{\nabla} \cdot V = 0$

$$= V^j \frac{\partial}{\partial x^j} V_i + \frac{1}{2} \frac{\partial}{\partial x^i} (V_j V^j)$$

$$\Rightarrow V^j \frac{\partial}{\partial x^j} V_i = (\mathcal{L}_{\bar{V}} \tilde{V})_i - \frac{\partial}{\partial x^i}\left(\frac{1}{2} V^2\right) \quad \text{if} \quad V^2 = \tilde{V}(\bar{V})$$

Our momentum equation in coordinate independent form becomes

2 of 3 $$\boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right)\tilde{V} + \frac{1}{\rho}\tilde{d}p + \tilde{d}\left(\Phi - \frac{1}{2}V^2\right) = 0}$$

Almost like bernoulli's eqn. ??

§ 5.18 Conservation of Vorticity

The vorticity of a fluid with velocity $\bar{V}$ is

$$\vec{\nabla} \times \vec{V} \quad \text{curl of velocity}$$

we have seen that this can be written as

$$* \tilde{d}\tilde{V} \quad \text{curl} \rightarrow 1 \text{ vector}$$

$$\tilde{d}\tilde{V} \quad \text{curl} \rightarrow 2\text{-form}$$

To get the vorticity equation we apply $\tilde{d}$ to the equation

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right)\tilde{d}\tilde{V} = \frac{1}{\rho^2}\tilde{d}\rho \wedge \tilde{d}p \qquad *$$

Case ① $p = d(\rho)$ then $\tilde{d}\rho \wedge \tilde{d}p = 0$

$$\Rightarrow \boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right)\tilde{d}\tilde{V} = 0}$$

Vorticity is conserved following the flow

Case 2: $p = p(\rho, s)$ Then $\tilde{d}\rho \wedge \tilde{d}P \neq 0$ but

$$\boxed{\tilde{d}S \wedge \tilde{d}\rho \wedge \tilde{d}P = 0}$$ Manifold is 2D (s, ρ)

Apply $\tilde{d}$ to our equation 1 of 3

$$\boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\vec{v}}\right) \tilde{d}S = 0}$$

If we take $\tilde{d}S \wedge$ the vorticity eqn * then

$$\tilde{d}S \wedge \left(\frac{\partial}{\partial t} + \mathcal{L}_{\vec{v}}\right)\tilde{d}\tilde{V} = \frac{1}{\rho^2}\, \overbrace{\tilde{d}S \wedge \tilde{d}\rho \wedge \tilde{d}p}^{0}$$

or

$$\boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\vec{v}}\right) \tilde{d}S \wedge \tilde{d}\tilde{V} = 0}$$ Ertels thereom

$\tilde{d}S \wedge \tilde{d}V$ is a 3-form. By Mass conservation $\rho\tilde{\omega}$ is conserved these are both 3-forms and on a 3D manifold must be Linearlly relates.

$$\boxed{\tilde{d}S \wedge \tilde{d}\tilde{V} = \alpha \rho \tilde{\omega}}$$ α is a function and must exist!

Since $\tilde{d}S \wedge \tilde{d}\tilde{V}$ and $\rho\tilde{\omega}$ is conserved it follows that α is consered.

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\vec{v}}\right)\alpha = 0$$

claim: $\alpha = \frac{1}{\rho}\, \vec{\nabla}S \cdot \vec{\nabla} \times \vec{V}$

proof: take the dual of

$$\tilde{d}S \wedge \tilde{d}V = \alpha \rho \tilde{\omega}$$

$$*\left(\tilde{d}S \wedge \tilde{d}V\right) = *(\alpha\rho\tilde{\omega})$$

Note: $\tilde{d}S \wedge \tilde{d}V = S_{,i}\, \tilde{d}x^i \wedge \epsilon^{ijk} \quad V_{k,j}\, \tilde{d}x^j \wedge \tilde{d}x^k$

$$= \epsilon^{ijk} S_{,i} V_{k,j}\; \tilde{d}x^i \wedge \tilde{d}x^j \wedge \tilde{d}x^k$$

Comparing the coefficent of $\tilde{\omega}$ we get.

$$\alpha = \frac{1}{\rho} \vec{\nabla} S \cdot \vec{\nabla} \times \vec{V}$$ Ertel Potential Vorticity.