

# Highlighting Legend

Chapter

Definition

Propositions / Theorems / Proofs

Procedures

Remarks are made in red pencil, same with questions

# Lecture 1 - Jan 8<sup>th</sup> 2024

## Chapter 1 Iteration and Orbits

**Defn** Let  $f: A \rightarrow \mathbb{R}$  such that  $A \subseteq \mathbb{R}$  and  $f(A) \subseteq A$  [ i.e  $f: A \rightarrow A$  ]. For  $a \in A$ , we may iterate the function at  $a$ :

$$\begin{aligned}x_1 &= a \\x_2 &= f(a) \\x_3 &= f(f(a)) = f^2(a) \\x_4 &= f(f(f(a))) = f^3(a) \\&\vdots\end{aligned}$$

all will exist in  $A$ .

We call  $(x_n)_{n=1}^{\infty} (= (x_n))$  we call that sequence the orbit of  $a$  under  $f$ .

Ex.  $f(x) = x^4 + 2x^2 - 2$ ,  $a = -1$

$$\begin{aligned}x_1 &= a, x_2 = f(a), x_3 = f^2(a) \dots \\-1, 1, 1, \dots &\rightarrow \text{eventually constant / periodic}\end{aligned}$$

Ex  $f(x) = -x^2 - x + 1$ ,  $a = 0$

$$\begin{aligned}x_1, x_2, x_3, x_4, x_5, x_6 \dots \\0, 1, -1, 1, -1, 1 \dots &\rightarrow \text{eventually periodic in period 2}\end{aligned}$$

Ex.  $f(x) = x^2 - 3x + 1$ ,  $a = 1$

$$\begin{aligned}x_1, x_2, x_3, x_4, \dots \\1, -1, 3, 19 \rightarrow \infty \quad [\text{he will talk about convergence / divergence later on}]\end{aligned}$$

Ex.  $f(x) = x^2 + 2x$ ,  $a = -0.5$

My Q: Does it ever equal 1, is that a distinction?

$$\begin{aligned}-0.5, -0.75, -0.9375, -0.9961, \dots &\rightarrow \text{Converges to } -1\end{aligned}$$

Ex.  $f(x) = x^3 - 3x$ ,  $a = 0.75$

$$\begin{aligned}0.75, -1.828, -0.625, 1.631, -0.552, \dots &\rightarrow \text{This is chaotic behaviour. * For P-MATH insights: an interval around zero is dense *}\end{aligned}$$

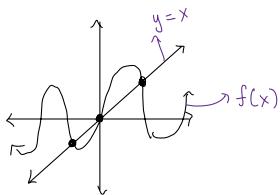
**Defn:**  $f: A \rightarrow \mathbb{R}$ ,  $f(A) \subseteq A$ , We say  $a \in A$  is a fixed point in  $f$  iff  $f(a) = a$ . In this case, the orbit of  $a$  is  $a, a, a, \dots$  which is constant.

Ex. Find all fixed points of  $f(x) = x^2 + x - 4$

When does  $x^2 + x - 4 = x$ ?

$$\Leftrightarrow x^2 - 4 = 0 \Leftrightarrow x = \pm 2 \quad \text{Will often give low degree polynomial}$$

Ex. How many fixed points does  $f(x)$  as pictured below have?



Geometrically, a fixed point occurs when  $f(x)$  intersects  $y=x$

Ex. Prove that  $f(x) = x^4 - 3x + 1$  has a fixed point.

Solve for  $x^4 - 3x + 1 = x$

$$\Rightarrow x^4 - 4x + 1 = 0 \quad \text{Intermediate Value theorem}$$

Since  $g(x) = x^4 - 4x + 1$  is continuous,  $g(0) = 1 > 0$  and  $g(1) = -2 < 0$ . By IVT  $\exists x \in (0, 1)$  s.t.  $g(x) = 0 \Leftrightarrow f(x) = x$ .

Defn  $f: A \rightarrow \mathbb{R}$ ,  $f(A) \subseteq A$

① We say  $a \in A$  is a **periodic point** for  $f$  if its orbit is periodic. I.E.  $\exists n \in \mathbb{N} \quad f^n(a) = a$ . The least such  $n$  is called the period of  $a$  and/or the orbit.

② Eventually periodic  $\exists n < m, f^n(a) = f^m(a)$

## Lecture 2 - Jan 10<sup>th</sup> 2024

### Def'n [Doubling Function]

$D: [0,1] \rightarrow [0,1]$  where  $D(x) = \text{fractional part of } 2x$  aka  $2x \bmod 1$

Ex)  $D(0.4) = 0.8$

$$D(0.6) = 0.2$$

$$D(0.8) = 0.6$$

$$D(0.5) = 0$$

It is an important function as it also provides a rich source of periodic orbits

Ex)  $D, a = \frac{1}{5}$

orbit:  $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \dots$  This has period 4. This leads into some cute number theory with GCDs

Ex)  $D, a = \frac{1}{20}$

orbit  $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \dots$  This is eventually periodic with period 4.

The Doubling function will come up later in a more meaningful way

Every day functions may not exhibit periodic orbits

Q: Given  $f$  and  $a$ , does  $f^n(a)$  tend towards some limit  $L$ ? This does happen surprisingly often

The language of this course vs Elementary Real Analysis

Notation:

If  $(x_n)_{n=1}^{\infty}$  is a sequence of real numbers we write  $(x_n) \subseteq \mathbb{R}$  → a small abuse of notation but reasonable

Def'n  $(x_n) \subseteq \mathbb{R}, x \in \mathbb{R}$  we say  $(x_n)$  converges to  $x$  iff for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  s.t.  $|x_n - x| < \epsilon$  for all  $n \geq N$

We write  $x_n \rightarrow x$  or  $\lim x_n = x$

↓  
N depends \*

Ex) Claim:  $\frac{1}{n} \rightarrow 0$ , Let  $\epsilon > 0$  be given.

Note:  $\frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$

Consider  $N = \frac{2}{\epsilon}$  For  $n \geq N$  we have  $\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \epsilon$  not necessarily a Natural number. Can always use ceiling function  $\lceil \cdot \rceil$  to bring this to a natural number  
↓ bring this to  $\frac{2}{\epsilon}$  so that we ensure strictly less than epsilon.

Ex) Claim  $\frac{2n}{n+3} \rightarrow 2$

Let  $\epsilon > 0$  be given. Let us choose  $N \in \mathbb{N}$  s.t.

$$\frac{1}{N} < \frac{\epsilon}{6}$$

For  $n \geq N$

$$\begin{aligned} & \left| \frac{2n}{n+3} - 2 \right| \\ &= \left| \frac{2n}{n+3} - \frac{2n+6}{n+3} \right| \\ &= \left| \frac{-6}{n+3} \right| = \frac{6}{n+3} < \frac{6}{n} \xrightarrow{\text{if you divide by less of something you get something bigger}} \\ & \frac{6}{n} \leq \frac{6}{N} = 6\left(\frac{1}{N}\right) < 6\left(\frac{\epsilon}{6}\right) = \epsilon \end{aligned}$$

Defn  $(x_n) \subseteq \mathbb{R}$  we say  $(x_n)$  is bounded if  $\exists M > 0$  s.t.  $\forall n \in \mathbb{N}, |x_n| \leq M$

Prop  $(x_n) \subseteq \mathbb{R}$  If  $(x_n)$  is convergent, then  $(x_n)$  is bounded.

Ex)  $x_n = (-1)^n$  Shows converse is not true

Proof Suppose  $x_n \rightarrow x$ . then  $\exists N \in \mathbb{N}, n \geq N \Rightarrow |x_n - x| < 1$  picked \epsilon, blake's favourite is 1

For  $n \geq N$ ,

$$|x_n| - |x| \leq |x_n - x| < 1 \Rightarrow |x_n| < 1 + |x| \quad \text{This is only true for } n \geq N$$

this is the reverse triangle inequality, make sure you know how to prove from triangle inequality

So Let  $M = \max \{|x_1|, \dots, |x_{N-1}|, 1 + |x|\}$  ■

Prop Let  $x_n \rightarrow x, y_n \rightarrow y$

Not super important to us,  
value is in working with  
defn of convergence

①  $x_n + y_n \rightarrow x + y$

②  $x_n y_n \rightarrow xy$

Proof:

① Let  $\epsilon > 0$  be given. There exists  $N_1, N_2 \in \mathbb{N}$  s.t.

$$n \geq N_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$$

$$n \geq N_2 \Rightarrow |y_n - y| < \frac{\epsilon}{2}$$

For  $N = \max\{N_1, N_2\}$  and  $n \geq N$

$$|x_n + y_n - (x+y)| = |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y| < \varepsilon$$

(2) Let  $\varepsilon > 0$  be given

Note •

$$|x_n y_n - xy| = |x_n y - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - y| \quad \star$$

Since  $(x_n)$  is bounded  $\exists M > 0, \forall n, |x_n| < M$

Let  $N_1, N_2 \in \mathbb{N}$  s.t.  $n \geq N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2(|y|+1)}$  make sure not dividing by zero

$$n \geq N_2 \Rightarrow |y_n - y| < \frac{\varepsilon}{2M}$$

For  $n \geq N := \max\{N_1, N_2\}$  we have  $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$  by \star ■

## Lec 3 - Jan 12<sup>th</sup> 2024

Def'n we say  $(X_n) \subseteq \mathbb{R}$  is Cauchy if  $\forall \epsilon > 0, \exists N \in \mathbb{N}, n, m \geq N \Rightarrow |X_n - X_m| < \epsilon$

Point of Clarification. The Wednesday class before the assignment is due. We will cover all the material needed for the assignment. Also we can use calculus 2. A lot of this course is in the language of Calculus and Real Analysis

Prop Convergent  $\Rightarrow$  Cauchy

Proof: Let  $\epsilon > 0$  be given and suppose  $(X_n)$  is convergent. Say  $X_n \rightarrow x \in \mathbb{R}$ . There exist  $N \in \mathbb{N} \quad n \geq N$

$$\Rightarrow |X_n - x| < \boxed{\frac{\epsilon}{2}} \quad \text{Then for } n, m \geq N$$

both are  $< \epsilon$  which gives information on what should be in the box

$$|X_n - X_m| = |X_n - x + x - X_m| \leq |X_n - x| + |x - X_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

take  $\epsilon$  divided by # of parts

□

Fact:  $(X_n) \subseteq \mathbb{R}$ ,  $(X_n)$  Cauchy  $\Leftrightarrow$   $(X_n)$  Convergent This is also a part of the Completeness of  $\mathbb{R}$

Big Idea: To prove  $(X_n)$  is Cauchy you do not have to guess the limit!  $\hookrightarrow$  this will be useful for fixed points

Def'n  $f: A \rightarrow \mathbb{R}$ ,  $A \subseteq \mathbb{R}$ ,  $a \in A$ . we say  $f$  is continuous at  $a$  iff  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $|f(x) - f(a)| < \epsilon$  whenever  $x \in A$  and  $|x - a| < \delta$ .

$\rightarrow$  close values in the domain gives close values in function

Fact:  $f$  is continuous at  $a$  iff  $\forall (X_n) \subseteq A$  with  $X_n \rightarrow a$ , we have  $f(X_n) \rightarrow f(a)$   $\hookrightarrow$  function at the terms in the sequence provides link between convergence and continuity.

↳ can prove as an exercise

highly dependent on that  $f$  goes from  $[a, b] \rightarrow [a, b]$

Prop: If  $f: [a, b] \rightarrow [a, b]$  is continuous then  $f(x)$  has a fixed point

Proof: we know  $f(a) \geq a$  and  $f(b) \leq b$ .  $\Leftrightarrow f(a) - a \geq 0$  and  $f(b) - b \leq 0$

By applying the IVT to the continuous function  $g(x) = f(x) - x \quad \exists x \in [a, b]$  such that  $g(x) = 0 \Leftrightarrow f(x) = x$  □

$\hookrightarrow$  a preview of why Calculus is applicable to this course. Epsilon will come back.

Def'n  $f: A \rightarrow \mathbb{R}$ ,  $A \subseteq \mathbb{R}$ . we say that  $f$  is a contraction iff  $\exists C \in [0, 1)$  s.t.  $\forall x, y \in A$ ,  $|f(x) - f(y)| < C|x - y|$   $\hookrightarrow$  is a type of Lipschitz fn  $\rightarrow$  would that mean fn. is below line  $y = x$  graphically?

Prop Contractions are continuous at every point

Proof Let  $\epsilon > 0$  be given Suppose  $|f(x) - f(y)| < C|x - y|$  as before. Fix  $y \in A$ , Consider  $\delta = \frac{\epsilon}{C+1}$   $\hookrightarrow$  to make sure we aren't dividing by 0

and assume  $x \in A$  s.t.  $|x-y| < \delta$ . Then

$$|f(x) - f(y)| \leq C|x-y| < C\delta < \epsilon \quad \square$$

Say function  $a$  problematic, not a contract b/c it drops off at 0

**Defn** We say  $A \subseteq \mathbb{R}$  is **closed** iff whenever  $(x_n) \subseteq A$  with  $x_n \rightarrow x \in \mathbb{R}$  then  $x \in A$ .

Ex.  $[a, b]$  is closed

Ex.  $(0, 1]$  not closed  $\frac{1}{n} \rightarrow 0 \notin (0, 1]$

Theorem [Banach Contraction Mapping Theorem] Suppose  $A \subseteq \mathbb{R}$  is closed and  $f: A \rightarrow A$  is a contraction.

Then there exist a unique fixed point  $a \in A$  for  $f$ . Moreover  $\forall x \in A$   $f^n(x) \rightarrow a$    
 ↗ has acute proof. allegedly.  
 will prove on monday

Ex.  $f: [0, 1] \rightarrow [0, 1]$   $f(x) = \frac{1}{3-x}$

Note:  $\frac{1}{3} \leq f(x) \leq \frac{1}{2}$  ↗ shows it maps back into  $[0, 1]$   
 ↗ also not an onto function

$$f'(x) = \frac{1}{(3-x)^2}, \quad \frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$$

By the MVT,  $\forall x, y \in [0, 1]$   $\exists c \in (0, 1)$  s.t.  $f(x) - f(y) = f'(c)(x-y)$

$$\begin{aligned} \Rightarrow |f(x) - f(y)| &= |f'(c)| |x-y| \\ &\leq \underbrace{\frac{1}{4}}_c |x-y| \end{aligned}$$

∴  $f(x)$  is a contraction.

$$\frac{1}{3-x} = x$$

$$\Leftrightarrow 1 = 3x - x^2$$

$$\Leftrightarrow x^2 - 3x + 1 = 0$$

$$\Leftrightarrow x = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\Leftrightarrow x = \frac{3-\sqrt{5}}{2}$$

$$\therefore \forall x \in [0, 1] \quad f^n(x) \rightarrow \frac{3-\sqrt{5}}{2}$$

Dec 4 - Jan 15<sup>th</sup> 2023

Note: Graphical Analysis tool posted on piazza. Assignment 1 due next week.

Remark: The Banach Contraction Mapping is almost like a black hole

Recall:  $(a_n) \subseteq \mathbb{R}$  we say  $(a_n)$  is **Strongly Cauchy** if  $\exists \epsilon_0 \in [0, \infty)$  s.t.

$$\textcircled{1} \quad \sum_{n=1}^{\infty} \epsilon_n < \infty$$

$$\textcircled{2} \quad \forall n, |a_n - a_{n+1}| < \epsilon_n$$

This is on assignment.

Hint:  $\sum_{n=1}^{\infty} a_n = L, \sum_{k=1}^{\infty} a_k \xrightarrow{n \rightarrow \infty} L$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$  s.t.  $n \geq N \Rightarrow \left| \sum_{k=N+1}^{\infty} a_k \right| < \epsilon$

### Proof of Banach Contraction Mapping

Let  $A \subseteq \mathbb{R}$  be closed and suppose  $\exists c \in [0, 1)$  s.t.  $|f(x) - f(y)| \leq c|x-y|$  for all  $x, y \in A$ . Take a point  $x_0 \in A$

and construct  $x_1 = f(x_0), x_2 = f(x_1), \dots, x_n = f(x_{n-1}) = f^n(x_0)$  this is the orbit.  $\rightarrow$  we aren't including  $x_0$  in the orbit but it doesn't matter

$$\begin{aligned} \text{For } n \in \mathbb{N} \quad |x_{n+1} - x_n| &\stackrel{\text{defn of orbit}}{=} |f(x_n) - f(x_{n-1})| \stackrel{\text{contraction assumption}}{\leq} c|x_n - x_{n-1}| = c|f(x_{n-1}) - f(x_{n-2})| \leq c^2|x_{n-1} - x_{n-2}| \\ &\vdots \\ &\leq c^n|x_1 - x_0| \end{aligned}$$

$c \in [0, 1)$  this is geometric

Since  $\sum_{n=1}^{\infty} c^n|x_1 - x_0|$  is a convergent geometric series, we have that  $(x_n)$  is strongly Cauchy. Hence  $x_n \rightarrow a$  for some  $a \in A$

$\hookrightarrow$  This is because domain is closed

Since  $f$  is continuous,  $\underbrace{f(x_n)}_{x_{n+1}} \rightarrow f(a) \therefore f(a) = a \rightarrow \text{B}(c, x_n \rightarrow a, f(x_n) \rightarrow f(a) \Rightarrow f(a) = a)$

$\rightarrow$  What's significant about this?

Suppose  $a, b \in A$ , s.t.  $f(a) = a$  and  $f(b) = b$ . Then  $|f(a) - f(b)| \leq c|a-b| \Rightarrow |a-b| \leq c|a-b|$ . Since  $c < 1$ ,

$|a-b| = 0$  and so  $a = b$ . This is our strongest fixed point theorem. This is the big analytic results for existence and uniqueness of ODEs

### Chapter 2 Graphical analysis.

To visualize the orbit of  $a$  under  $f$ :

- ① Superimpose  $y = f(x)$ ,  $y = x$

(2) Use a vertical line:  $(a, a) \longrightarrow (a, f(a))$

(3) Use a horizontal line:  $(a, f(a)) \longrightarrow (f(a), f(a))$

(4) Vertical line:  $(f(a), f(a)) \longrightarrow (f(a), f^2(a))$

(5) Use a horizontal line:  $(f(a), f^2(a)) \longrightarrow (f^2(a), f^3(a))$

Etc...

Ex) Using online tool and  $f(x) = x^2 - x + 1$ , fixed points  $x=1$

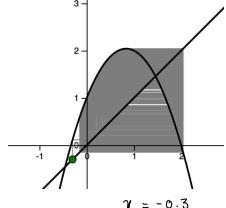
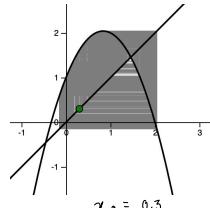
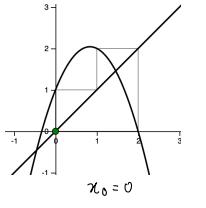
Orbit analysis:

①  $x \in [0, 1]$  we have,  $f^n(x) \rightarrow 1$

② otherwise,  $x \notin [0, 1]$   $f^n(x) \rightarrow \infty$  In assignment, only expecting this sort of an answer.

## Lec 5 - Jan 17<sup>th</sup> 2024

For  $f(x) = \frac{-3}{2}x^2 + \frac{5}{2}x + 1$  at  $x=0$  we have an orbit of  $0, 1, 2$  which is periodic and graphical is a cycle



points near 0, like 0.06 show chaos. The graphical analysis look does show chaos. Chaos also shows density. We can never hit every point in the interval because an orbit is countable but  $[1, 2]$  in  $\mathbb{R}$  is uncountable. It is dense [ex. rational numbers is dense in  $[0, 1] \subset \mathbb{R}$ ]. We will define dense in a bit.

### Chapter 3 - fixed points

Remark. If  $f(x)$  is continuous and  $f^n(a) \rightarrow L$ , then  $f^{n+1}(a) \rightarrow f(L) \rightarrow$  comes from continuity. We also saw this in the B.C.M theorem

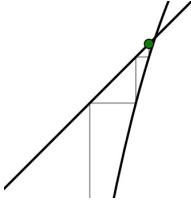
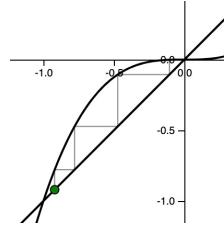
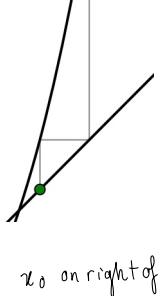
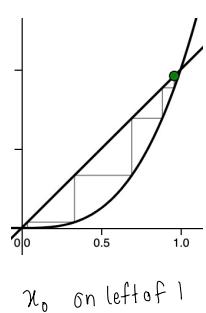
$$\therefore f(L) = L$$

### Motivating Example

$$F(x) = x^3, \text{ fixed points: } 0, \pm 1$$

① for  $x \in (-1, 1)$  we see that  $f^n(x) \rightarrow 0$ . This is an example of an attracting fixed point

②  $x \in (1, \infty)$ ,  $f^n(x) \rightarrow \infty$  and  $x \in (-\infty, -1)$ ,  $f^n(x) \rightarrow -\infty$ . We call  $\pm 1$  repelling fixed points



$x_0$  on left of 1

$x_0$  on right of 1

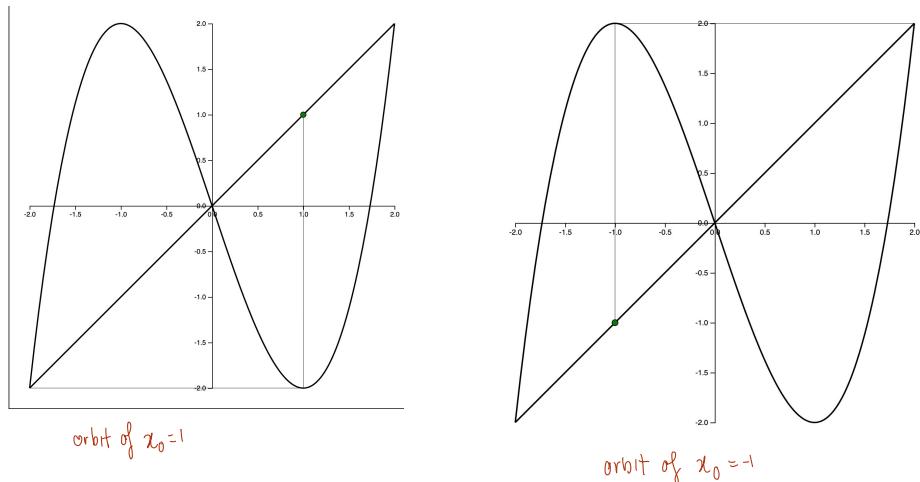
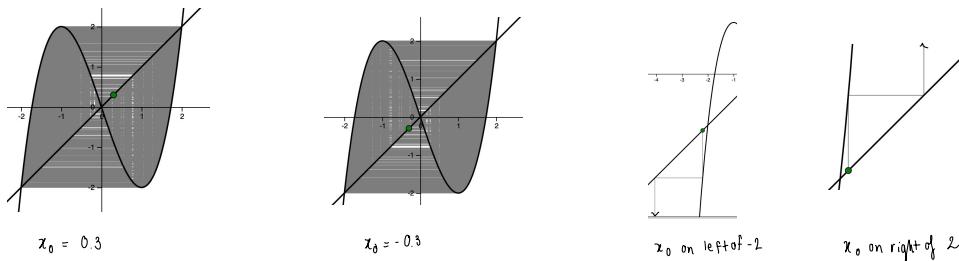
$x_0$  on right of -1

$x_0$  on left of -1

## Motivating Example 2

$$f(x) = x^3 - 3x, \text{ fixed points: } 0, \pm 2$$

- ① 0 is repelling [in a different way than last example b/c right & left to 0 is chaotic]
- ② to the right of 2 the orbits go to infinity, left of -2 orbits go to -infinity. so  $\pm 2$  is also repelling.
- ③ at  $x_0 = 1$ , the orbit is eventually fixed to the fixed point -2; for  $x_0 = -1$ , the orbit is ev



**Defn** let  $a$  be a fixed point of  $f(x)$

- ① if  $|f'(a)| > 1$ , we call  $a$  a **repelling** fixed point.
- ② if  $|f'(a)| < 1$ , we call  $a$  an **attracting** fixed point
- ③ if  $|f'(a)| = 1$ , we call  $a$  a **neutral** fixed point  $\rightarrow$  this can mean a whole bunch of things

**Theorem [Attracting fixed point thm]**

Suppose  $a$  is an attracting fixed of  $f(x)$  then  $\exists$  an open interval  $I$  such that  $a \in I$  and

- ①  $\forall x \in I, \forall n \in \mathbb{N}, f^n(x) \in I$   $\hookrightarrow$  slightly unnecessary because if  $f(x) \in I \Rightarrow f^n(x) \in I$
- ②  $\forall x \in I, f^n(x) \rightarrow a$ .

Defn:  $f: A \rightarrow \mathbb{R}$ ,  $A \subseteq \mathbb{R}$ .

① we say  $a \in A$  is non-isolated iff  $\forall \epsilon > 0$ ,  $\exists b \in A$  with  $b \neq a$  s.t.  $b \in (a-\epsilon, a+\epsilon)$

② Let  $a \in A$  be non-isolated. We say  $\lim_{x \rightarrow a} f(x) = L$  iff  $\forall \epsilon > 0 \exists \delta > 0$ ,  $|f(x) - L| < \epsilon$  whenever  $a \in A$  and  $0 < |x-a| < \delta$

$\rightarrow$  if  $a$  was an isolated, we could choose a  $\delta$  where  $|x-a| < \delta$  is false. Leading to a false hypothesis which leads to an always true answer and hence anything could be the limit.

Ex  $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$  0 is non-isolated

Ex  $\{1\} \cup (2, 3)$  1 is isolated

Proof: [Attracting fixed point thm]  $\rightarrow$  to continue in Lec 6

Assume  $|f'(a)| < 1$ . Then  $\exists c \in \mathbb{R}$  s.t.  $|f'(a)| < c < 1$

$$\therefore \lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x-a|} < c$$

Thus  $\exists \delta > 0$  s.t.  $\frac{|f(x) - f(a)|}{|x-a|} \leq c \quad \forall x \in (a-\delta, a+\delta)$

Hence, for  $x \in (a-\delta, a+\delta)$   $|f(x) - f(a)| \leq c|x-a| \rightarrow$  contracting

# Lec 6<sup>th</sup> - Jan 19<sup>th</sup> 2024

First assignment due 11:59 on Tuesday

## Theorem [Attracting fixed point thm]

Suppose  $a$  is an attracting fixed point of  $f(x)$ . There exists an open interval  $I$  with  $a \in I$  s.t.

- ①  $\forall x \in I, \forall n \in \mathbb{N}, f^n(x) \in I \rightarrow$  saying  $f(x) \in I$  is enough but we are choosing to write this to give a fuller picture
- ②  $\forall x \in I, f^n(x) \rightarrow a$

$a$  is not a boundary pt.  
b/c it is differentiable due to defn  
of attracting

b/c this exists, we can choose a maximal  $\delta$  such that all orbits that go to  $a$  exist in it and no other orbits

Proof:

Say  $|f'(a)| < c < 1$  so that  $\lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x - a|} < c$ . Thus,  $\exists \delta > 0$  s.t.  $\forall x \in I := (a - \delta, a + \delta), |f(x) - f(a)| < c|x - a|$

In particular, for  $x \in I$   $|f(x) - f(a)| = |f(x) - a| \leq c|x - a| \leq c|x - a| < \delta \Rightarrow f(x) \in I$

Continuing, for  $x \in I$   $|f^n(x) - a| \leq c^n|x - a| \leq c^n|x - a| < \delta$  so that  $f^n(x) \in I$ . Finally, for  $x \in I$   $0 \leq |f^n(x) - a| \leq c^n|x - a| \xrightarrow{\text{C} \in (0, 1)} 0$

Note: If the question asks to use a  $\epsilon$ - $\delta$  proof then do so.

$f$  means  $f$  is a contraction

$C \in (0, 1) \Rightarrow C^n \text{ shrinks to } 0$

Can show using  $\epsilon$ - $\delta$  proof if needed

## Theorem [repelling fixed point theorem]

Suppose  $a$  is a repelling fixed point for  $f(x)$ . There exist an open interval  $a \in I$ , s.t.  $\forall x \in I, x \neq a, \exists n \in \mathbb{N}$  s.t.  $f^n(x) \notin I$

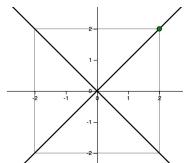
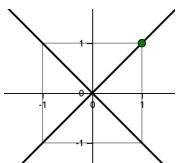
Proof: Say  $|f'(a)| > c > 1$ . then  $\lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x - a|} > c$  and so  $\exists \delta > 0$  s.t. for all  $x \in I := (a - \delta, a + \delta)$   $|f(x) - f(a)| > c|x - a|$

Since  $a$  is a fixed point,  $|f(x) - f(a)| = |f(x) - a|$ . Suppose  $\forall n, f^n(x) \in I$ . As before  $|f^n(x) - a| \geq c^n|x - a| \rightarrow \infty$  This infinity goes well beyond delta!

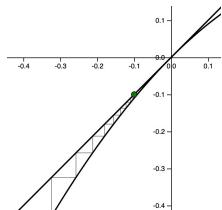
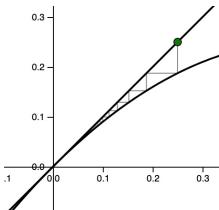
What is a neutral fixed point? A lot.

Investigation: Neutral fixed Points

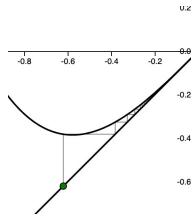
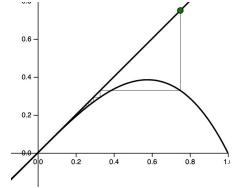
Ex  $f(x) = -x$ ,  $0$  is a fixed point and  $|f'(0)| = 1$ . bounces around



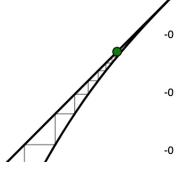
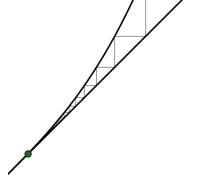
Ex  $f(x) = x - x^2$ ,  $|f'(0)| = 1$  attracting to the right and repelling to the left.



Ex  $f(x) = x - x^3$   $|f'(0)| = 1$ , this is weakly attracting, attracting but too slowly, happens in assymptote situations



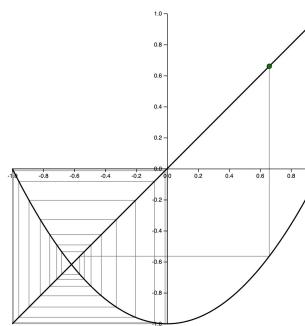
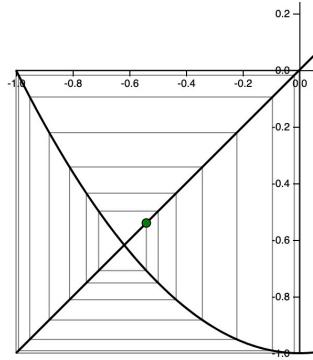
Ex  $f(x) = x + x^3$ ,  $|f'(0)| = 1$ , weakly repelling, repelling too slow



Motivating Example:  $f(x) = x^2 - 1$ ,  $a=0$ .

orbit,  $(0, -1, 0, 1, 0, -1)$ ; 0 is a periodic point of period 2.  $x_0$  near 0 shows orbits getting closer to  $\{0, 1\}$ -cycle.

we will call 0 an attracting periodic point because. 0 is an attracting fixed point of  $f^2(x)$ .



# Lec 7 - Jan 22<sup>nd</sup> 2024

**Defn:** Let  $a$  be a periodic point for  $f(x)$  with period  $n$ . We say  $a$  is an attracting / repelling / neutral per point for  $f(x)$  iff  $a$  is an attracting / repelling / neutral fixed point of  $f^n(x)$ .

for strictly periodic orbits. [no eventually periodic] the points in the periodic orbit are fixed points of  $f^n(x)$ . Also, every point in the orbit will have the same classification as the point tested.  $\rightarrow$  by proposition value of derivative must be the same for any point in the cycle.

**Prop:** Let  $f(x)$  be a function. Then  
 $\hookrightarrow f(x)$  need to be diffable

$$(f^n)'(a) = f'(a) f'(f(a)) f'(f^2(a)) \dots f'(f^{n-1}(a))$$

**Proof:** [Induction]

If  $n=1$ ,  $f'(a) = f'(a)$

Assume the result holds for some  $n$ ,  $n \geq 1$ .

$$\text{then } \frac{d}{dx} f^{n+1}(x) = \frac{d}{dx} f(f^n(x)) = f'(f^n(x)) \cdot (f^n)'(x)$$

$\hookrightarrow$  plugin  $a$

$$\text{Then } (f^{n+1})'(a) = \underbrace{f'(f^n(a)) \cdot f'(a) \cdot f'(f(a)) \dots f'(f^{n-1}(a))}_{\text{inductive hypothesis}}$$

■

$$\text{Ex) } f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1, a=0, \text{ orbit } (0, 1, 2, 0, \dots)$$

$$f'(x) = -3x + \frac{5}{2}$$

$$\begin{aligned} (f^3)'(0) &= f'(0) f'(1) f'(2) \\ &= \left(-\frac{7}{2}\right) \left(\frac{-1}{2}\right) \left(\frac{5}{2}\right) \\ &= \frac{35}{8} > 1 \quad \therefore \text{repelling} \end{aligned}$$

## Chapter 4. Bifurcations

In general Bifurcation theory is the study of how a family of curves can change when a defining parameter is changed

$\hookrightarrow$  very important family

We will consider the quadratic family:  $Q_c(x) = x^2 + c$  Idea: how does  $c$  affect the dynamics of  $Q_c(x)$ ?

to find the fixed points of  $Q_c(x)$ :

$$x^2 + c = x \iff x^2 - x + c = 0$$

$$\iff x = \frac{1 \pm \sqrt{1-4c}}{2}$$

Thus:

- ①  $Q_c(x)$  has 2 fixed points when  $c < \frac{1}{4}$

②  $Q_c(x)$  has 1 fixed point when  $c = \frac{1}{4}$

③  $Q_c(x)$  has no fixed points [really] when  $c > \frac{1}{4}$ .

If  $c > \frac{1}{4}$ ,  $Q_c^n(x)$  must diverge to  $\infty$ .

If  $c = \frac{1}{4}$   $Q_c(x)$  has the unique fixed point of  $p = \frac{1}{2}$ . Since  $Q'_c(x) = 2x$ ,  $Q'_c(\frac{1}{2}) = 1$  so thus fixed point is neutral → attracting to left and repelling to right

If  $c < \frac{1}{4}$ ,  $Q_c(x)$  has 2 fixed points:

$$p_+ = \frac{1 + \sqrt{1-4c}}{2}, \quad p_- = \frac{1 - \sqrt{1-4c}}{2}$$

First,

$$Q'_c(p_+) = 1 + \sqrt{1-4c} > 1 \Rightarrow p_+ \text{ is repelling}$$

Next,

$$-1 < Q'_c(p_-) < 1$$

$$\Leftrightarrow -1 < 1 - \sqrt{1-4c} < 1$$

$$\Leftrightarrow -2 < -\sqrt{1-4c} < 0$$

$$\Leftrightarrow 2 > \sqrt{1-4c} > 0$$

$$\Leftrightarrow -\frac{3}{4} < c < \frac{1}{4} \rightarrow \text{this is the range for attracting fixed point}$$

if  $c < -\frac{3}{4}$ ,  $Q'_c(p_-) < -1$

if  $c = -\frac{3}{4}$ ,  $Q'_c(p_-) = -1$

Theorem: For the family  $Q_c(x) = x^2 + c$ :

① All orbits tend to  $\infty$  if  $c > \frac{1}{4}$

② when  $c = \frac{1}{4}$   $Q_c(x)$  has a unique fixed point,  $\frac{1}{2}$ , and it is neutral

③ if  $c < \frac{1}{4}$   $Q_c(x)$  has 2 fixed points  $p_+$  and  $p_-$ . The point  $p_+$  is repelling

Moreover

a) If  $-\frac{3}{4} < c < \frac{1}{4}$ ,  $p_-$  is attracting

b) if  $c = -\frac{3}{4}$ ,  $p_-$  is neutral

c) if  $c < -\frac{3}{4}$ ,  $p_-$  is repelling

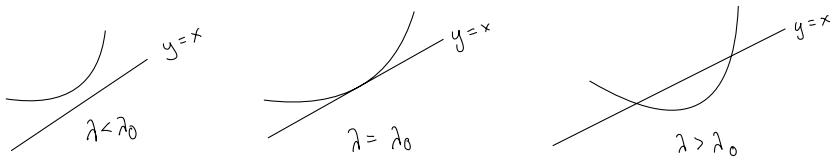
## Lec 8 - Jan 24 2024

**Def'n:** We say a family of Functions  $F_\lambda(x)$  undergoes a **bifurcation** at  $\lambda_0$  if there is a change in fixed point structure at  $\lambda_0$ .

Ex)  $Q_c(x) = x^2 + c$ ,  $\lambda_0 = \frac{1}{4}$

**Def'n:** A family  $F_\lambda(x)$  undergoes a **tangent bifurcation** at  $\lambda_0$  if there is an open interval  $I$  and  $\epsilon > 0$  s.t.

- ① For  $\lambda_0 - \epsilon < \lambda < \lambda_0$ ,  $F_\lambda(x)$  has no fixed points on  $I$
  - ② For  $\lambda = \lambda_0$ ,  $F_\lambda(x)$  has one fixed point and it is neutral
  - ③ For  $\lambda_0 < \lambda < \lambda_0 + \epsilon$ ,  $F_\lambda(x)$  has 2 fixed points in  $I$ , one attracting and the other repelling
- or ①, ②, ③ with  $>$



Ex)  $E_\lambda(x) = e^x + \lambda$ ,  $\lambda_0 = -1$ . This is an example of a tangent bifurcation.

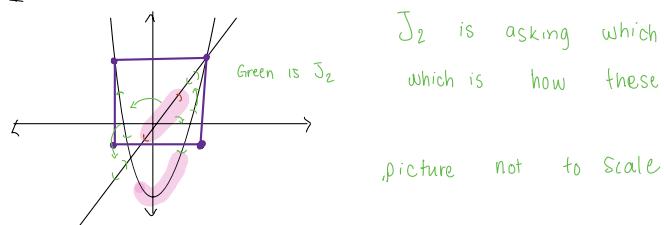
Ex)  $F_\lambda(x) = \lambda x(1-x)$ ,  $\lambda_0 = 1$ . Not tangent bifurcation.

## Chapter 5 Cantor Set

Investigation:  $Q_c(x) = x^2 + c$ ,  $c < -2$ .

$$p_+ = \frac{1 + \sqrt{1 - 4c}}{2} > 2, \quad -p_- < -2$$

Consider  $I = [-p_+, p_+]$  and  $I \times I$



$J_2$  is asking which  $x$  has  $y$  values in  $J_1$ , which is how these intervals are picture not to scale

Let  $J \subseteq I$  be the interval s.t.  $Q_c(x) \notin I$  for all  $x \in J$ ,

For  $x \in J_1$ ,  $Q_c^n(x) \rightarrow \infty$ . Moreover, If  $\exists n$  s.t.  $Q_c^n(x) \in J$ , then  $Q_c^n(x) \rightarrow \infty$

Consider  $\Lambda = \{x \in I : Q_c^n(x) \in I \ \forall n\}$

Big Idea:  $\Lambda$  contains all the points with interesting orbits.

$$J_1 = \{x \in I : Q_c(x) \notin I\}$$

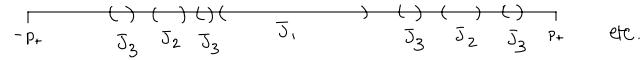
$$J_2 = \{x \in I : Q_c(x) \in J_1\} = \{x \in I : Q_c^2(x) \notin I\}$$

$$J_3, J_4, J_5 \dots$$

what is a cantor set?

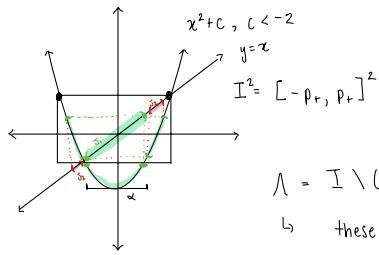
Then,  $\Lambda = I \setminus (J_1 \cup J_2 \cup \dots)$  is a cantor set. Drawing  $\Lambda$  on the  $x$ -axis:

→ Symmetry comes from the cune (parabola)



It is a fractal  
b/c of its self repeating nature!

# Lec 9 - Jan 26<sup>th</sup> 2024



One quarter mark for the course.

$$\Lambda = I \setminus (J_1 \cup J_2 \cup \dots)$$

$$A = \{x \in I : \forall n, Q_c^n(x) \in I\}$$

↳ these are all the interesting orbits

↳ Cantor set

**Construction:** Cantor Middle thirds set

$$C_0 = [0, 1] \rightarrow \text{remove open middle third interval}$$

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

$$\vdots$$

these are nested

$$K = \bigcap_{n=1}^{\infty} C_n \rightarrow \text{Cantor [middle-third] set. A cantor set is taking out a chunk of the interval, infinitely. We can see that.}$$

the end points will always survive.

Proposition:  $A_n \subseteq \mathbb{R}$  closed. Then  $\bigcap_{n=1}^{\infty} A_n$  is closed.

Why?  $(a_n) \subseteq \bigcap A_n, a_n \rightarrow a. \forall n, (a_n) \subseteq A_n \Rightarrow a \in A_n \Rightarrow a \in \bigcap A_n$

Proposition:  $A, B \subseteq \mathbb{R}$  closed. Then  $A \cup B$  is closed.

Why?  $(a_n) \subseteq A \cup B, a_n \rightarrow a. \text{WLOG. } \{n : a_n \in A\} \text{ is infinite. This allows us to construct } (b_n) \subseteq A \text{ s.t. } b_n \rightarrow a. \text{ Since } A \text{ is closed, } a \in A \subseteq A \cup B.$

**Theorem**  $K$  is closed [in fact all Cantor sets are]  $\rightarrow$  combine the 2 propositions above to get this

**Theorem:**  $K$  contains no non-empty open intervals  $\rightarrow$  Hausdorff property of  $\mathbb{R}$ ?

no isolated points  
interior points?

Why?  $I \subseteq K \Rightarrow \forall n, I \subseteq C_n \Rightarrow \forall n \quad l(I) \leq \frac{1}{3^n} \Rightarrow l(I) = 0 \Rightarrow I = \emptyset \text{ which is a contradiction.}$

↳ length of interval.

Let's consider the base -3 expansion of  $x \in [0, 1]$ .  $x = 0.s_1 s_2 s_3 \dots, s_i \in \{0, 1, 2\}$

$$s_1 = 0$$

$$s_1 = 1$$

$$s_1 = 2$$

$$[0, \frac{1}{3}]$$

$$[\frac{2}{3}, 1]$$

$$\begin{array}{l} s_1 = 0 \\ s_2 = 0 \\ s_1 = 1 \\ s_2 = 1 \\ s_1 = 0 \\ s_2 = 2 \end{array}$$

$$\begin{array}{l} s_1 = 0 \\ s_2 = 0 \\ s_1 = 1 \\ s_2 = 2 \\ \dots \\ s_1 = 0 \\ s_2 = 1 \\ s_1 = 2 \end{array}$$

Fact:

$x \in K$  iff  $x$  can be written in base-3 using only 0's and 2's

Ex  $\frac{1}{3} \in K$ .  $\frac{1}{3} = [0.1]_3 = [0.022222\ldots]_3 \rightarrow$  # theory? base  $\alpha$ ?

Theorem:  $K$  is uncountable  $|K| = |\mathbb{R}| \rightarrow$  cardinality  $\rightarrow$  not important to this course, just for interest.

How to remember? 0 means left, 2 means right.

Start Ch 6 on monday.

# Lec 10 - Jan 29<sup>th</sup> 2024

## Chapter 6 - Symbolic Dynamics

Recall:  $Q_c(x) = x^2 + c$ ,  $c < -2$ .

\* Assignment 2 now out, First 4 Q easy, Last 4 NOT

Midterm March 1<sup>st</sup>, Content cut off is already happening

Midterm may have proofs from next part of course (after assignment 2 content)

For Q w classify neutral point use cobwebbing thing

$$p_+ = \frac{1 + \sqrt{1 - 4c}}{2}, \quad I = [-p_-, p_+] \quad J_1 = \{x \in I : Q_c(x) \in I\}$$

$$J_2 = \{x \in I : Q_c(x) \in J_1\}$$

$$J_3 = \{x \in I : Q_c(x) \in J_2\}$$

$$\vdots$$

$\Lambda = I \setminus (\cup J_i) = \{x \in I : \forall n Q_c^n(x) \in I\} \rightarrow$  This is a cantor set. All cantor sets are isomorphic verify iso vs. homeo

Notation:

$$I \setminus J_1 = I_0 \cup I_1$$

$I = [I_0 \setminus \text{removed}] \cup I_1 \rightarrow$  Idea we have a left part and a right part.

Defn: For  $x \in \Lambda$ , we define the **Itinerary** of  $x$  to be the sequence

→ kind telling you direction, dealing with orbit indirectly

$$S(x) = (x_0 x_1 x_2 x_3 \dots), \quad x_i \in \{0, 1\}$$

where  $x_i = 0 \iff Q_c^i(x) \in I_0$

$x_i = 1 \iff Q_c^i(x) \in I_1$

Goal: Understand  $S(x)$  better

Notation:  $\Sigma = \{(x_0 x_1 x_2 \dots) : x_i \in \{0, 1\}\} \rightarrow$  we drop the commas?

→ function, specifically a transform. claims it's easier to work with.

$S: \Lambda \rightarrow \Sigma$

↪ S is a homeomorphism  $\Rightarrow \Lambda$  is same as  $\Sigma$ , what we are doing this week.

Defn: Let  $X$  be a set. We say  $d: X \times X \rightarrow [0, \infty)$  is a **Metric** iff

$$\textcircled{1} \quad d(x, y) = 0 \iff x = y$$

$$\textcircled{2} \quad \forall x, y \in X, \quad d(x, y) = d(y, x)$$

$$\textcircled{3} \quad \forall x, y, z \in X \\ d(x, y) \leq d(x, z) + d(z, y) \rightarrow \text{triangle inequality}$$

We call  $(X, d)$  a metric space

Idea  $d(x, y) =$  distance between  $x$  and  $y$ .

Ex)  $X = \mathbb{R}$ ,  $d(x, y) = |x - y|$

Ex)  $X = \mathbb{R}^n$ ,  $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$

Ex)  $X$  any set,  $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$  discrete metric

These are all Metric Space

Ex)  $A \subseteq \mathbb{R}$ , For  $x, y \in A$ ,  $d(x, y) = |x - y|$  is a metric  $(A, d)$

Ex) [Cantor Space]

$X = \sum$ ,  $d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$

$\downarrow$   
 $x_0 x_1 \dots$      $y_0 y_1 \dots$

ensuring we get a number  
bc it gives us a convergent series

Ex)  $x = (111\dots)$

$y = (1010\dots)$

$d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} \rightarrow$  1's cancel in every even step

How  $\begin{aligned} &= \sum_{i=0}^{\infty} \frac{1}{2^{2i+1}} = \sum_{i=0}^{\infty} \left(\frac{1}{2^{2i}}\right) \frac{1}{2} ? \\ &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^i} = \frac{1}{2} \left(\frac{1}{1 - \frac{1}{4}}\right) = \frac{1}{2} \left(\frac{4}{3}\right) = \frac{4}{6} = \frac{2}{3} \end{aligned}$

### Connections

- dynamical systems on manifolds
- p-adic numbers? Vertasium video

Proposition:  $x, y \in \sum$

① if  $x_i = y_i$  for  $i \leq n$  then  $d(x, y) \leq \frac{1}{2^n}$  First few terms carry a lot of weight

② if  $d(x, y) < \frac{1}{2^n}$  then  $x_i \neq y_i$  for  $i \leq n$

Why?

①  $d(x, y) \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{1}{2^n} \rightarrow$  convergence of geometric series formula?

②  $d(x, y) < \frac{1}{2^n}$

$\exists k \leq n, x_k \neq y_k \Rightarrow |x_k - y_k| = 1$  hence contradiction

Ex)  $x = (000\dots)$

$y = (1000\dots)$

$d(x, y) = \frac{1}{2^0} = 1$  BUT  $x_0 \neq y_0$  This shows that Proposition cannot be iff.

Defn: Shift map.

$\sigma : \sum \rightarrow \sum$

$\sigma : (x_0 x_1 x_2 \dots) = (x_1 x_2 x_3 \dots)$

Remark  $\sigma^k(x_0 x_1 x_2 \dots) = (x_k x_{k+1} x_{k+2} \dots)$

What's next?

Show ①  $\sigma: \Sigma \rightarrow \Sigma$  is continuous  $\curvearrowright$  gives us

②  $\sigma: \Sigma \rightarrow \Sigma$  dynamical System

③  $\varsigma: \Lambda \rightarrow \Sigma$  homeomorphism  $\rightarrow$  means  $\Lambda$  and  $\Sigma$  are "same space"

④  $\sigma: \Sigma \rightarrow \Sigma$  chaotic  $\Rightarrow Q_c: \Lambda \rightarrow \Lambda$

# Lec 11 - Jan 31 2024

Recall:  $Q_c(x) = x^2 + c$ ,  $c < -2$

$\Lambda = \{x \in I : \forall n, Q_n(x) \in I\}$   $\delta : \Lambda \rightarrow \Sigma$ ,  $\delta(x) = (x_0 x_1 x_2 \dots)$  iff  $Q_{n_i}(x) \in I_{x_i}$  either  $x_i$  or  $x_0$

is this extendable  
to other iterations  
in the cantor set.

if  $d'$  and  $d$  where abs then we get  
continuous Defn for  $f : \mathbb{R} \rightarrow \mathbb{R}$

Def'n:  $(X, d)$ ,  $(Y, d')$

We say  $f : X \rightarrow Y$  is continuous at  $y \in Y$  iff  $\forall \epsilon > 0$ ,  $\exists \delta > 0$ ,  $x \in X$ ,  $d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \epsilon$

we say  $f : X \rightarrow Y$  is continuous iff it is continuous at every  $y \in Y$

Proposition:  $\sigma : \Sigma \rightarrow \Sigma$  is continuous

Proof: Fix  $y \in \Sigma$  and let  $\epsilon > 0$  be given. Say  $y = (y_0 y_1 y_2 \dots)$ . Take  $n \in \mathbb{N}$  s.t.  $\frac{1}{2^n} < \epsilon$ .

Consider  $\delta = \frac{1}{2^{n+1}}$ . Let  $x = (x_0 x_1 x_2 \dots) \in \Sigma$  s.t.  $d(x, y) < \delta$ .  $\therefore x_i = y_i$  for  $i = 0, 1, \dots, n+1$

Then,  $\sigma(x) = x_1 x_2 x_3 \dots$ ,  $\sigma(y) = y_1 y_2 y_3 \dots$  and  $x_i = y_i$  for  $i = 1, 2, \dots, n+1$ .

By Previous Prop  $d(\sigma(x), \sigma(y)) \leq \frac{1}{2^n} < \epsilon$

Def'n:  $(X, d)$ ,  $(Y, d')$ . We say  $(x_n)$  converges to  $x$ ,  $x_n \rightarrow x$ , iff  $\forall \epsilon > 0$ ,  $\exists N \in \mathbb{N}$ ,  $n > N \Rightarrow d(x_n, x) < \epsilon$

Proposition  $(X, d)$ ,  $(Y, d')$ ,  $f : X \rightarrow Y$ . Then,  $f$  is continuous iff  $f(x_n) \rightarrow f(x)$  whenever  $x_n \rightarrow x$

Def'n:  $(X, d)$ ,  $(Y, d')$ ,  $f : X \rightarrow Y$ . We call  $f$  a homeomorphism iff

- ①  $f$  is injective (one to one)
- ②  $f$  is Surjective
- ③  $f$  is continuous
- ④  $f^{-1}$  is continuous

Investigation:  $f : X \rightarrow Y$  is a homeomorphism

①  $(x_n) \subseteq X$ ,  $x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x)$

by continuity of  $f^{-1}$

②  $(y_n) \subseteq Y$ ,  $y_n \rightarrow y$ . Say  $y_n = f(x_n)$ ,  $y = f(x) \Rightarrow f(x_n) \rightarrow f(x) \Rightarrow f^{-1}(f(x_n)) \rightarrow f^{-1}(f(x)) \Rightarrow x_n \rightarrow x$

Big Idea:  $f$  is a relabelling of  $X$  to  $Y$ . we think of  $X$  and  $Y$  as the "Same" metric space

Homeomorphic spaces are basically equal.

### Remark

①  $x \in \Lambda, S(x) = (x_0, x_1, \dots)$

we know that  $x_0 \in I_{x_0}, x_1 \in I_{x_1}, Q_c^2(x) \in I_{x_2}, \dots$

$$\therefore S(Q_c(x)) = (x, x_2 \dots) = \sigma(S(x))$$

②  $S(Q_c^n(x)) = \sigma^n(S(x)) \rightarrow$  These 2 are friends and work together

Theorem:  $S: \Lambda \rightarrow \Sigma$  is a homeomorphism.

Aim is to define  
and prove chaos

Theorem: Monotone Convergence theorem

If  $(a_n) \subset \mathbb{R}$  is increasing / decreasing and bounded. then  $(a_n)$  converges.

Theorem: [Nested Intervals Lemma]

If  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$  are closed intervals, then  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$  → open intervals could.

Idea:  $I_k = [a_k, b_k]$

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \dots$$

$(a_n)$  increasing,  $(a_n) \subseteq [a_1, b_1]$

$(b_n)$  decreasing,  $(b_n) \subseteq [a_1, b_1]$

they converge

By MCT,  $a_n \rightarrow a$  and  $b_n \rightarrow b \quad \therefore [a, b] \subseteq \bigcap_{n=1}^{\infty} I_n \quad \rightarrow$  worst case scenario  $a = b$

# Lec 12<sup>th</sup> 2024 - Feb 2<sup>nd</sup> 2024

Thm:  $S: \Lambda \rightarrow \Sigma$  is a homeomorphism. → what's the difference between transforms and homeomorphism

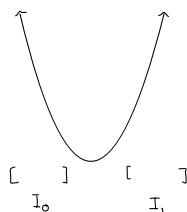
Remark: This is true for  $c < -2$ . We will prove the result for  $c < \frac{-(5+2\sqrt{5})}{4}$ .

A2:  $\forall x \in I \setminus J_1 = I_0 \cup I_1$ ,  $|Q_c'(x)| \geq M > 1$ .  $M$  exists by our A2 Q for the above  $c$  values  
↓ First step in cantor process / itinerary

Proof:

Let's first show it is injective.

Suppose  $x, y \in \Lambda$  with  $S(x) = S(y)$  but  $x \neq y$ . Therefore  $\forall n$ ,  $Q_c^n(x)$  and  $Q_c^n(y)$  live in the same  $I_0$  and  $I_1$ .



By the Mean Value Theorem

$$|Q_c(x) - Q_c(y)| \geq M|x-y|$$

↑ parabola is injective on small intervals

Since  $Q_c$  is injective on  $I_0$  and  $I_1$ , we have that  $Q_c(x) \neq Q_c(y)$ . Thus,

$$|Q_c^2(x) - Q_c^2(y)| \geq M^2|x-y|$$

⋮

$$|Q_c^n(x) - Q_c^n(y)| \geq M^n|x-y|$$

Since  $M > 1$ ,  $M^n|x-y| \rightarrow \infty$ . However,

$$|Q_c^n(x) - Q_c^n(y)| \leq \max \{l(I_0), l(I_1)\} \xrightarrow{\text{length}} \text{length will be the same.}$$

↑ contradiction!

■

Surjective: Let  $y = (y_0, y_1, y_2, \dots) \in \Sigma$ . For  $n \in \mathbb{N}$ , let  $I_{y_0, y_1, \dots, y_n} = \{x \in I : x \in I_{y_0}, Q_c(x) \in I_{y_1}, \dots, Q_c^n(x) \in I_{y_n}\}$ .

It is enough to show that there exist  $x \in \bigcap_{n=1}^{\infty} I_{y_0, y_1, \dots, y_n}$  ( $\Rightarrow S(x) = y$ )

↗ Nested interval lemma  
Can apply if this is an interval

Clearly,  $I_{y_0} \supseteq I_{y_0, y_1} \supseteq I_{y_0, y_1, y_2} \supseteq \dots$

Claim: For all  $n \geq 0$ ,  $I_{y_0, y_1, \dots, y_n}$  is a closed interval. First,  $I_{y_0} \in \{I_0, I_1\}$  is a closed interval. Assume  $I_{y_0, y_1, \dots, y_n}$  is a closed interval for some  $n \geq 0$ .

Note:  $x \in I_{y_0, \dots, y_n, y_{n+1}} \Leftrightarrow x \in I_{y_0}$ ,

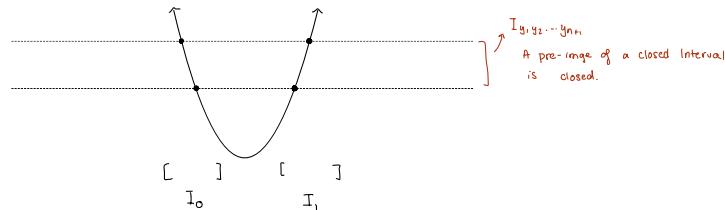
$$f: X \rightarrow Y, \quad B \subseteq Y$$

$Q_c(x) \in I_{y_1}, \quad Q_c(Q_c(x)) \in I_{y_2} \quad Q_c(Q_c^2(x)) \in I_{y_3}, \dots$

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

$\textcircled{*} \Leftrightarrow I_{y_0} \cap Q_c^{-1}(I_{y_1, \dots, y_n}) \rightarrow$  similar to how shift map relates  
means pre-image

By induction,  $I_{y_1, \dots, y_n}$  is closed interval. We have  $I_{y_1, \dots, y_n} \subseteq I_{y_0} \in \{I_0, I_1\}$



$Q_c^{-1}(I_{y_1, \dots, y_n})$  is a union of two disjoint closed intervals, one in  $I_0$ , and one in  $I_1$ . Returning to  $\textcircled{*}$

$$I_{y_0} \cap Q_c^{-1}(I_{y_1, \dots, y_n})$$

is one of these closed intervals. This proves the claim. By the Nested Intervals Lemma,  $\exists x \in \bigcap_{n=1}^{\infty} I_{y_0, \dots, y_n}$ . Hence  $S(x) = y$  and so  $S$  is surjective

### Continuity

Fix  $y \in A$  and say  $S(y) = (y_0, y_1, y_2, \dots)$ . Let  $\epsilon > 0$  be given and choose  $n$  such that  $\frac{1}{2^n} < \epsilon$ . Consider the  $2^{n+1}$  disjoint closed intervals  $I_{y_0, \dots, y_n}$ . Pick  $\delta > 0$  such that  $(y - \delta, y + \delta)$  only overlaps with  $I_{y_0, \dots, y_n}$ . Note:  $y \in I_{y_0, \dots, y_n}$

means first n terms agree

For  $x \in A$  with  $|x - y| < \delta$ ,  $x \in I_{y_0, \dots, y_n}$  and so

$$d(S(x), S(y)) \leq \frac{1}{2^n} < \epsilon$$

### Continuity of $S^{-1}$

Similar

■

# Lec 13 - Feb 5<sup>th</sup> 2024

week 5. 6 weeks before reading week. 6 weeks after reading week. Nothing Due this week. Midterm March 1. Midterm will be in a different room.

## Chapter 7 - Chaos

Def'n:  $(X, d)$ , we say  $A \subseteq X$  is dense iff  $\forall x \in X, \forall \varepsilon > 0, \exists a \in A$  s.t.  $d(a, x) < \varepsilon$

Ex.  $\mathbb{Q}$  is dense in  $\mathbb{R}$ .  $\rightarrow$  easiest way to do so is by Decimal expansion and Cut-off at a certain # of decimal points

Ex.  $\mathbb{Z}$  is not dense in  $\mathbb{R}$ .

Ex.  $A = \{x \in \Sigma : \exists N, \forall i > N, x_i = 0\} \rightarrow$  Sequence of eventually all zeros. This is dense in  $\Sigma$

Proof: Let  $x \in \Sigma$  and let  $\varepsilon > 0$  be given. Take  $n \in \mathbb{N}$  s.t.  $\frac{1}{2^n} < \varepsilon$ . Consider  $y = (x_0 x_1 \dots x_n 00 \dots) \in A$ . Then  $d(x, y) < \frac{1}{2^n} < \varepsilon$  ■

$\rightarrow$  SUPER IMPORTANT TO DO

Ex.  $A = \{x \in \Sigma : x \text{ periodic}\}$  is dense in  $\Sigma$  hint  $y = (x_0 x_1 \dots x_n x_0 x_1 \dots x_n \dots)$  where  $x \in \Sigma$  and  $x \approx y$  agree up to the first  $n$ -terms

Remark  $A = \{x \in \Sigma : x \text{ periodic}\}$  is exactly the periodic points for  $\sigma: \Sigma \rightarrow \Sigma$  ↪ Shift map.

Prop: There exists  $z \in \Sigma$  s.t.  $\{\sigma^k(z) : k \in \mathbb{N} \cup \{0\}\}$  is dense in  $\Sigma$ .  $\rightarrow$  There could be multiple  $z$ . but we only need 1.

$\rightarrow$  writing out all combinations of 01 blocks for all possible lengths. This ensures any  $x \in \Sigma$  will be close

Proof: Take  $z = (01\underline{0001}0\underline{11}00\underline{01}01\underline{11}\dots)$

Take  $x \in \Sigma$  and  $\varepsilon > 0$ . Let  $\frac{1}{2^n} < \varepsilon$ .  $\rightarrow$   $k$  allows you to shift to the right block. The bigger the  $n$ , the more chopping. For some  $k$ ,  $\sigma^k(z)$  and  $x$  agree on the first  $n$  terms. Thus  $d(\sigma^k(z), x) \leq \frac{1}{2^n} < \varepsilon$  ■

Def'n: A **Dynamical System** is a metric space  $(X, d)$  together with a continuous function  $f: X \rightarrow X$ . This allows you to do what we have done on any metric space

Notation:  $f: X \rightarrow X$  dynamical system.

Ex)  $\sigma: \Sigma \rightarrow \Sigma$  is a dynamical system.

$\rightarrow$  gross?

Def'n: Let  $f: X \rightarrow X$  be a dynamical system. We say  $f: X \rightarrow X$  is transitive iff  $\forall x, y \in X, \forall \varepsilon > 0, \exists z \in X, n, m \in \mathbb{N} \cup \{0\}$

s.t.  $d(x, f^n(z)) < \varepsilon$  and  $d(y, f^m(z)) < \varepsilon$  No matter which 2 points you start with, there is an orbit of a point that is close to the original 2 points.

Proposition:  $f: \Sigma \rightarrow \Sigma$  is transitive

Def'n: Take  $z \in \Sigma$  s.t.  $\{\sigma^k(z) : k \in \mathbb{N} \cup \{0\}\}$  is dense in  $\Sigma$ . For all  $\varepsilon > 0, x, y \in \Sigma, \exists n, m \in \mathbb{N} \cup \{0\}$  s.t.  $d(x, \sigma^n(z)) < \varepsilon$  and  $d(y, \sigma^m(z)) < \varepsilon$  ■  
 $\hookrightarrow$  having a dense set makes transitivity easy.

It doesn't need to be dense in order for the function to be transitive.

$\rightarrow$  B doesn't depend on  $x \approx y$ .

Def'n: Let  $f: X \rightarrow X$  be a Dynamical System. We say  $f: X \rightarrow X$  depends sensitively on initial conditions (is sensitive) iff  $\exists \beta > 0, \forall \varepsilon > 0, \forall x \in X, \exists y \in X, \exists k \in \mathbb{N}$  such that

$$\textcircled{1} \quad d(x, y) < \epsilon$$

$$\textcircled{2} \quad d(f^k(x), f^k(y)) \geq \beta \quad \rightarrow \text{this is similar to stable/unstable}$$

Prop:  $\sigma: \Sigma \rightarrow \Sigma$  is sensitive.

Proof: Take  $\boxed{\beta = 1}$ . Let  $\epsilon > 0$  be given and let  $x \in \Sigma$ . Say  $\frac{1}{2^n} < \epsilon$  and suppose  $y \in \Sigma$  is s.t.  $0 < d(x, y) < \frac{1}{2^n}$ . Thus,  $\exists k \geq n$ , s.t.  $x_k \neq y_k \quad \therefore \quad d(\sigma^k(x), \sigma^k(y)) \geq \frac{|x_k - y_k|}{2^k} = 1 = \beta$  (either 0 or 1  $\Rightarrow |x_k - y_k| = 1$ ).

# Lec 14 - Feb 7<sup>th</sup> 2024

Def'n: A dynamical system  $f: X \rightarrow X$  is chaotic iff

- (1) The periodic points for  $f$  are dense in  $X$ .
- (2) transitive
- (3) Sensitive

Theorem:  $\sigma: \Sigma \rightarrow \Sigma$  is chaotic  $\rightarrow$  we have already proved this above

$\rightarrow$  Does chaos imply repelling? Prof must think about it.

Proposition:  $(X, d)$ ,  $(Y, d')$  and  $f: X \rightarrow Y$  continuous and surjective. If  $A \subseteq X$  is dense in  $X$  then  $f(A)$  is dense in  $Y$ .

$\rightarrow$  can do b/c of surjective

$\rightarrow$  continuous defn

Proof: Let  $y \in Y$  and say  $y = f(x)$ . Let  $\epsilon > 0$  be given. Since  $f$  is continuous at  $x$ ,  $\exists \delta > 0$ ,  $d(x, z) < \delta \Rightarrow d'(f(x), f(z)) < \epsilon$

$\rightarrow$  density defn

Since  $A$  is dense in  $X$ , we may find  $a \in A$  such that  $d(a, z) < \delta \Rightarrow d'(f(a), f(z)) < \epsilon$   
 $\Rightarrow d'(f(a), y) < \epsilon$

■

Theorem:  $c < \frac{-(5 + 2\sqrt{5})}{4}$  then  $Q_c: \Lambda \rightarrow \Lambda$  is chaotic.

Proof: IDEA: Relate this theorem to the above theorem that  $\sigma: \Sigma \rightarrow \Sigma$  is chaotic

## (1) Periodic Points

$\rightarrow$  b/c  $S$  is one-to-one.

Note that  $Q_c^n(x) = x$  for  $x$  that is periodic  $\Leftrightarrow S(Q_c^n(x)) = S(x)$   
 $\Leftrightarrow \sigma^n(S(x)) = S(x)$

By Using the previous proposition with  $S^{-1}: \Sigma \rightarrow \Lambda$ , the periodic point for  $Q_c$  are dense in  $\Lambda$

## (2) Transitive

Take  $z \in \Sigma$  such that  $\{\sigma^k(z): k \in \mathbb{N} \cup \{0\}\}$  is dense. Again,  $\{S^{-1}(\sigma^k(z)): k \in \mathbb{N} \cup \{0\}\}$  is dense in  $\Lambda$

Note: Say  $S(x) = z$ .  $S(Q_c^k(x)) = \sigma^k(S(x)) \Leftrightarrow Q_c^k(x) = S^{-1}(\sigma^k(S(x)))$ . Thus  $\{Q_c^k(x): k \in \mathbb{N} \cup \{0\}\}$  is dense in  $\Lambda$ .

as before,  $Q_c: \Lambda \rightarrow \Lambda$  is transitive.

## (3) Sensitivity

Recall that  $\Lambda \subseteq I \setminus J = I_0 \cup I_1$ . Let  $\beta > 0$  be less than the distance between  $I_0$  and  $I_1$ . For  $x, y \in \Lambda$  with  $x \neq y$ ,  $S(x) \neq S(y)$

$\Rightarrow \exists k$  such that  $k^{\text{th}}$  term of  $S(x) \neq k^{\text{th}}$  term of  $S(y)$ . Hence  $|Q_c^k(x) - Q_c^k(y)| > \beta$

This is slightly stronger than our defn of sensitivity  
■

End of Midterm  
Material

Lec 15 - Feb 9<sup>th</sup> 2024

## Chapter 8 Sarkovskii's Theorem

### Theorem: Period Three

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be continuous. If  $f$  has a point with period 3, then  $f$  has a point with  $n$  for all  $n \in \mathbb{N}$ .

**Proposition 1:**  $I \subseteq J$  are closed intervals.  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous. If  $f(I) \supseteq J$ , then  $f(x)$  has a fixed point in  $I$ .

Why?  $[a, b] = I \subseteq J = [c, d]$

$$c = f(p), d = f(q); p, q \in I$$

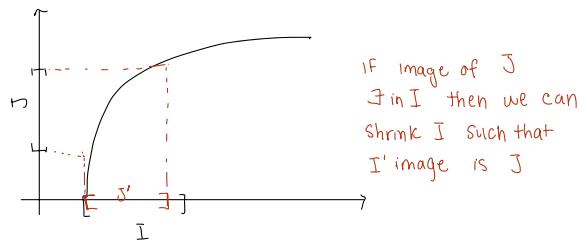
$g(x) = f(x) - x$  is also continuous.

$$g(p) = f(p) - p = c - p \leq 0$$

$$g(q) = f(q) - q = d - q \geq 0$$

By IVT,  $\Rightarrow \exists z, g(z) = f(z) - z = 0 \Rightarrow f(z) = z \therefore \exists$  a fixed point. ■

**Proposition 2:**  $I, J$  be closed intervals.  $f: \mathbb{R} \rightarrow \mathbb{R}$  be continuous.  $f(I) \supseteq J$  then,  $\exists$  closed interval  $I' \subseteq I$  such that  $f(I') = J$



Proof: [Period 3]

Let  $a \in \mathbb{R}$  be a period 3 point for  $f(x)$ . Say  $f(a) = b, f(b) = c, f(c) = a$ . WLOG, suppose  $a < b < c$ .

Case 1  $a < b < c$ .  $\rightarrow$  Case  $a < c < b$  similar

Let  $I = [a, b]$  and  $J = [b, c]$ . Observe  $f(a) = b$  and  $f(b) = c \xrightarrow{\text{IVT}} J \subseteq f(I)$  and  $f(b) = c, f(c) = a \xrightarrow{\text{IVT}} I \cup J \subseteq f(J)$

$\xrightarrow{\text{prop 2}}$

Since  $J \subseteq f(J) \exists$  closed interval  $A_1 \subseteq J$  such that  $f(A_1) = J$ . Again,  $A_1 \subseteq J = f(A_1)$ , and so there exists a closed interval  $A_2 \subseteq A_1$ ,

such that  $f(A_2) = A_1$ .

Fix  $n > 3$ . Repeating the above process we can find  $A_{n-2} \subseteq A_{n-3} \subseteq \dots \subseteq A_2 \subseteq A_1 \subseteq J$ , such that  $f(A_i) = A_{i-1}$ .

$\downarrow$   
in  $J$

in I

Now  $f(I) \supseteq J \supseteq A_{n-2} \Rightarrow \exists A_{n-1} \subseteq I$  closed interval such that  $f(A_{n-1}) = A_{n-2}$

Moreover,  $f(J) \supseteq I \supseteq A_{n-1} \Rightarrow \exists A_n \subseteq J$  closed interval such that  $f(A_n) = A_{n-1}$ . We have  $f^n(A_n) = J$  and  $A_n \subseteq J$ . By prop 1,

$\exists x_0 \in A_n$  such that  $f^n(x_0) = x_0$

Note:  $x_0 \in A_n$ ,  $f(x_0) \in A_{n-1} \subseteq I$  and  $f^i(x_0) \in J$  for  $i = 2, 3, \dots, n$ . Suppose  $f^i(x_0) = x_0$  for some  $i < n$ .

only overlap in both intervals

Then, 
$$\begin{array}{ccc} f(x_0) & = & f^{i+1}(x_0) = b \\ \downarrow & & \downarrow \\ \in I & & \in J \end{array}$$

$\Rightarrow f(x_0) = b$ ,  $f^2(x_0) = c$ ,  $f^3(x_0) = a \Rightarrow a \in J$  but that's not true. at I. Hence  $x_0$  has period n.

Further,  $f(J) \supseteq J$  and so  $f$  has a fixed point (Prop 1) in  $J$ .

Finally,  $f(I) \supseteq J \Rightarrow J = f(I')$

$$f(J) \supseteq I' \Rightarrow f(J) = I'$$

by prop 1

$\Rightarrow f^2(J') = f(I') = J \supseteq J' \Rightarrow \exists x_0 \in J' \text{ s.t. } f^2(x_0) = x_0$ . If  $f(x_0) = x_0$ ,  $x_0 \in J'$  and  $f(x_0) \in I' \Rightarrow x_0 = b$  but  $f(b) \neq b$  b/c  $f(b) = c$ . Thus a contradiction. Hence  $x_0$  has period 2.

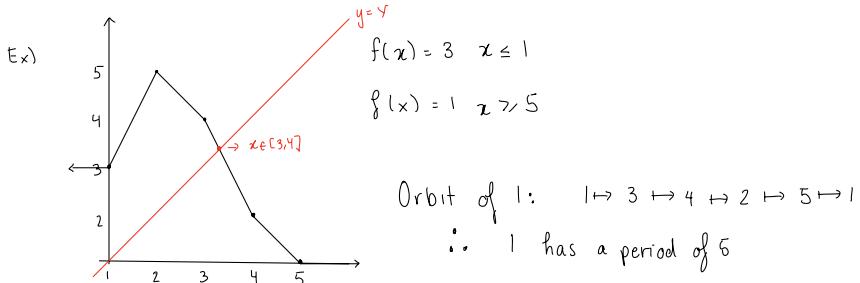
HOMEWORK:  $a < c < b$

## Lec 16 - Feb 12<sup>th</sup> 2023

Theorem: Period Three

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be continuous. If  $f$  has a point with period 3, then  $f$  has point with  $n$  for all  $n \in \mathbb{N}$ .

Proof is basically repeating applications of the Intermediate Value Theorem. This is important because you can create 2 intervals with the 3 numbers



Claim:  $f$  has no point with period 3. Suppose  $f$  has a point with period 3. Note  $1 \leq x \leq 5$

Since we are assuming  $x$  is a 3-periodic point

$$\begin{aligned} ① \quad x \in [1, 2] \quad & x \in [1, 2] \cap f^3([1, 2]) \\ &= [1, 2] \cap [2, 5] \\ &\Rightarrow x = 2 \quad \text{but contradiction b/c 2 has period 5.} \end{aligned}$$

$$\begin{aligned} ② \quad x \in [2, 3] \Rightarrow x \in [2, 3] \cap f^3([2, 3]) \\ &= [2, 3] \cap [3, 5] \Rightarrow x = 3 \quad \text{but again contradiction} \end{aligned}$$

$$\begin{aligned} ③ \quad x \in [4, 5] \Rightarrow x \in [4, 5] \cap f^3([4, 5]) \\ &= [4, 5] \cap [1, 4] \\ &\Rightarrow x = 4 \quad \text{but contradiction} \end{aligned}$$

$$\begin{aligned} ④ \quad x \in [3, 4] \Rightarrow x \in [3, 4] \cap f^3([3, 4]) \\ &= [3, 4] \cap [1, 5] \end{aligned}$$

$f: [3, 4] \rightarrow [2, 4]$  strictly decreasing

$f: [2, 4] \rightarrow [2, 5]$  strictly decreasing

$f: [2, 5] \rightarrow [1, 5]$  strictly decreasing.

$\therefore f^3(x)$  is strictly decreasing.  $\rightarrow$  Since strictly decreasing  $f^3(x)$  only crosses the line  $y = x$  once

$\therefore f^3(x)$  has a unique fixed point in  $[3, 4]$  but it is just the fixed point of  $f$  in  $[3, 4]$

$\therefore f$  has no point with period three.

Sometimes a function has period  $n$  but not period  $k$ .

$$\text{Ex) } f(x) = \begin{cases} 1 & x < -1 \\ -x & -1 \leq x \leq 1 \\ -1 & x > 1 \end{cases}$$

For  $x < -1$  not periodic  
 $-1 \leq x \leq 1, x \neq 0$  period 2  
 $x = 0$  period 1  
 $x > 1$  not periodic

### Sarkovskii Ordering

$$3 < 5 < 7 < 9 < 11 \dots$$

↓

$$2 \cdot 3 < 2 \cdot 5 < 2 \cdot 7 < 2 \cdot 9 < 2 \cdot 11$$

↓

$$2^2 \cdot 3 < 2^2 \cdot 5 < 2^2 \cdot 7 < 2^2 \cdot 9 < 2^2 \cdot 11$$

↓  
⋮  
↓

$$\dots 2^n < 2^{n-1} < \dots < 2^2 < 2 < 1$$

This has created a total ordering of the natural numbers where 3 is the smallest, 2 is the second largest and 1 is the largest

Ex  $2 \cdot 13 < 2^2 \cdot 5$

$$2^{10} \cdot 3 < 2^5 \quad \forall n \in \mathbb{N}, n \leq 1.$$

### Theorem: Sarkovski

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be continuous. Suppose  $n, m \in \mathbb{N}$  with  $n \prec m$  (in Sarkovski ordering). If  $f$  has a point with period  $n$ , then it has a point with period  $m$ .

Lee 17 - Feb 14 2024

"I'll switch up the polynomial, trig / exponential on the mid term..."

## Chapter 9 - Fractals

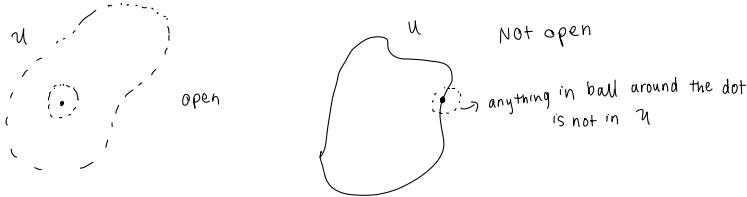
### Defn / Notation

$$\textcircled{1} \quad \vec{x} \in \mathbb{R}^n, \quad \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

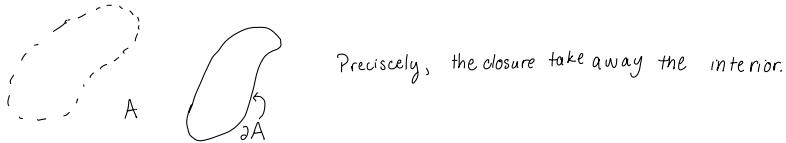
$$\textcircled{2} \quad d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| \quad \text{is a metric on } \mathbb{R}^n$$

\textcircled{3} \quad \vec{x} \in \mathbb{R}^n, \quad \varepsilon > 0. \quad \text{The open ball of radius epsilon centered at } \vec{x}: \quad B\_\varepsilon(\vec{x}) = \{\vec{y} \in \mathbb{R}^n : \|\vec{x} - \vec{y}\| < \varepsilon\}

\textcircled{4} \quad \text{We say } U \subseteq \mathbb{R}^n \text{ is open if and only if } \forall \vec{x} \in U, \exists \varepsilon > 0, \quad B\_\varepsilon(\vec{x}) \subseteq U



\textcircled{5} \quad A \subseteq \mathbb{R}^n, \quad \partial(A) = \text{boundary of } A.



Def'n We say  $S \subseteq \mathbb{R}^n$  has topological Dimension 0 iff  $\forall \vec{x} \in S, \exists$  arbitrarily small open  $\vec{x} \in U$  s.t.

$$\partial(U) \cap S \neq \emptyset$$

Ex) call this  $x$    
 → this has dimension 0  $\dim_t X = 0$   $\left. \begin{array}{l} \text{isolated points} \\ \text{non-isolated points} \end{array} \right\}$

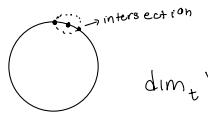
Ex)  $X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$   $\dim_t X = 0$

Because of the Hausdorff property of  $\mathbb{R}$ , doesn't this imply any Hausdorff  $\dim_t(X) = 0$ ?

Def'n:  $S \subseteq \mathbb{R}^n$ , we say  $S$  has topological dimension  $k \in \mathbb{N}$  iff  $\forall \vec{x} \in S$  there exists arbitrarily small  $\vec{x} \in U$  s.t.  $\partial(U) \cap S$  has topological dimension  $k-1$ , where  $k$  is minimal with this property.

Ex)  $X$   $\dim_t X = 1 \Rightarrow \dim_t(X) = 1$

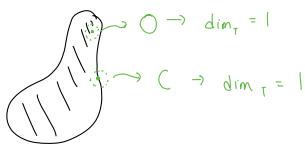
Ex



$$\dim_t X = 1$$

$\rightarrow \dim_t = 1$  from previous example.

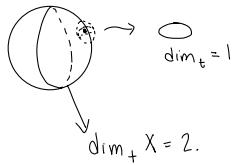
Ex)



$$\dim_t X = 2$$

This is a recursive definition

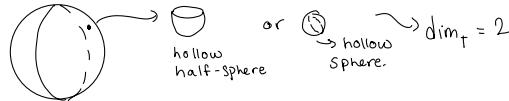
Ex)  $X$  is a non-filled sphere



even though 3-D, Topological dim is 2

$$\dim_t X = 2.$$

Ex)  $X$  is a filled-sphere



$$\dim_t X = 3.$$

$\Rightarrow$  filled sphere is bigger than a hollow sphere

**Defn:**  $S \subseteq \mathbb{R}^n$ . we say  $S$  is **Self-Similar** if  $S$  may be divided into  $k$  congruent subsets each of which may be magnified by a fixed  $M$  to yield  $S$  itself

We then define the **fractal dimension** of  $S$  by

$$\dim_f S = \frac{\ln(k)}{\ln(M)}$$

**Defn:** A **fractal** is a self-similar  $S \subseteq \mathbb{R}^n$  s.t.  $\dim_f S > \dim_t S$ .

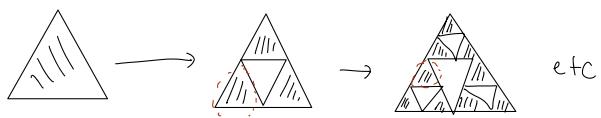
Ex



$$\dim_t X = 1$$

$$\dim_f X = \frac{\ln(n)}{\ln(n)} \xrightarrow{n \text{ partition}} n \text{ is the multiplier to get back the line}$$

Ex) X be Sierpinski Triangle



$$\dim_t X = 1$$

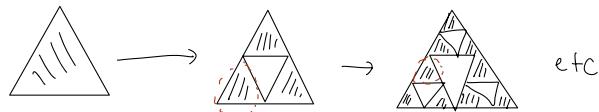
$$\dim_f X = \frac{\ln(3)}{\ln(2)} = \frac{\ln(9)}{\ln(4)}$$

ln helps so that no matter what iteration, it gives same #  
 $\approx 1.585\dots$

Distance between fractal dimension and Topological Dimension gives a measure of how crazy the fractal is. Sierpinski Triangle pretty tame.

## Lec 18 - Feb 16<sup>th</sup> 2024

Ex)  $X$  be Sierpinski Triangle

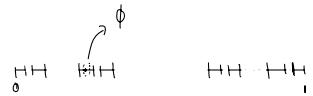


$$\dim_t X = 1$$

$$\dim_f X = \frac{\ln(3)}{\ln(2)} = \log_2 3$$

→ change of basis formula

Ex) Cantor Set.  $K$



For every point in the cantor set, there are arbitrarily small amounts of points around that point. hence  $\dim_t K = 0$ . Minimally comes into play here. b/c you could find a point near a boundary

$$\dim_f K = \frac{\ln(2)}{\ln(3)} = \log_3(2) = 0.63$$

→ magnification.

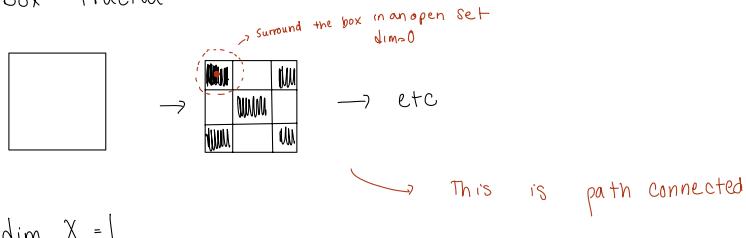
Ex) Koch Curve



This is a fractal. The Koch Snowflake is not totally a fractal according to our definition.

$$\dim_t X = 1, \quad \dim_f X = \frac{\ln(4)}{\ln(3)} = \log_3 4$$

Ex) Box Fractal

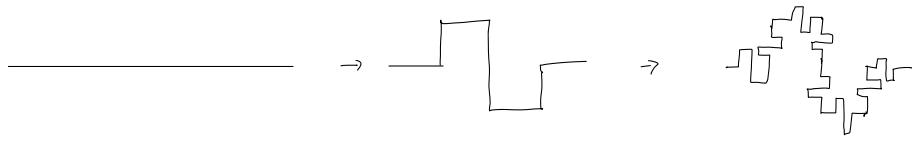


$$\dim_t X = 1$$

$$\dim_f X = \frac{\ln(5)}{\ln(3)} = \log_3 5$$

→ This is path connected

Ex) Minkowski's Sausage



$$\dim_t X = 1, \quad \dim_f X = \frac{\ln(8)}{\ln(4)} = \log_4(8) = \frac{3}{2}$$

The Chaos game: [sodar.org/interactive/activities/The Chaos Game.](http://sodar.org/interactive/activities/The%20Chaos%20Game)

- ↳ Can create fractals
- ↳ fractals can come from iterated function system

Google CGR DNA.

# Lec 19 - Feb 26<sup>th</sup> 2024

Recall: Chaos Game.

① Start with the vertices of an equilateral triangle



② Pick  $p \in \mathbb{R}^2$

③ Randomly Select a vertex.  $v_i$

④ Replace  $p$  with the mid point  $(p, v_i)$

⑤ iterate.

Question: where does the orbit of  $p$  end up? Answer: Sierpinski triangle.

Goal: Formalize this.

Iterated Function Systems

Fix  $p_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$  and fix  $0 < \beta < 1$  (contraction factor)

Consider  $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \beta \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$  ie  $F(p) = \beta(p - p_0) + p_0$

Note:

①  $F(p_0) = p_0 \rightarrow$  fixed point

②  $\|F(p) - F(p_0)\| = \|\beta(p - p_0)\| = \beta\|p - p_0\|$

③  $\|F^n(p) - p_0\| = \beta^n\|p - p_0\| \rightarrow 0 \therefore F^n(p) \rightarrow p_0$

Def'n:  $0 < \beta < 1$ ,  $p_1, p_2, \dots, p_n \in \mathbb{R}^2$

$F_i(p) = \beta(p - p_i) + p_i$ . We call  $\{F_1, F_2, \dots, F_n\}$  an iterated function System. Fix  $p_0 \in \mathbb{R}^2$ . Randomly select  $F_i$ . Let  $q_1 = F_i(p_0)$  and continuing...

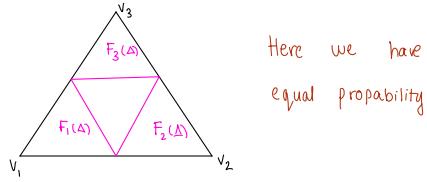
(iterated function system)

The set of points in which the orbit  $q_1, q_2, q_3, \dots$  must eventually end up in is called the attractor for the IFS

Ex) Chaos Game  $P_1 = V_1$ ,  $P_2 = V_2$ ,  $P_3 = V_3$ ,  $\beta = \frac{1}{2}$

$$F_i(p) = \frac{1}{2}(p - p_i) + p_i = \frac{1}{2}(p + v_i) = \text{mid point}(p, v_i)$$

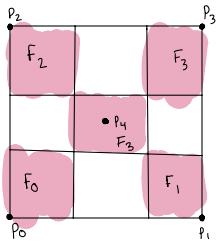
$\{F_1, F_2, F_3\}$  IFS and the attractor = Sierpinski Triangle.



Here we have  
equal probability

"The orbits are dense in  
the attractor set"

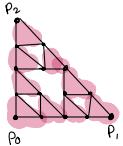
Ex)  $p_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $p_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $p_4 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$  and  $\beta = \frac{1}{3}$



Box fractal = Attractor.

Think where one iteration of functions go to

Ex  $p_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$   $p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$   $p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$   $\beta = \frac{1}{2}$



Attractor = Right Sierpinski triangle.

## Lec 20 - Feb 29<sup>th</sup> 2028

Ex  $P_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ,  $P_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\beta = \frac{1}{3}$

The IFS is:

$$F_0(\vec{x}) = \frac{1}{3} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x \\ \frac{1}{3}y \end{bmatrix}$$

$$F_1(\vec{x}) = \frac{1}{3} \begin{bmatrix} x-1 \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x + \frac{2}{3} \\ \frac{1}{3}y \end{bmatrix}$$

$q_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$ . Say the orbit of  $\{F_0, F_1\}$  is  $q_0, q_1, q_2, \dots, q_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$  with random selections  $s_1, s_2, s_3, \dots, s_i \in \{0, 1\}$

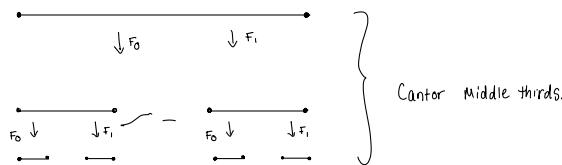
Note:  $y_n = \frac{1}{3^n} y_0 \rightarrow 0$

$$\begin{aligned} x_1 &= \frac{1}{3} x_0 + \frac{2s_1}{3} \\ x_2 &= \frac{1}{3^2} x_0 + \frac{2s_1}{3^2} + \frac{2s_2}{3} \\ x_3 &= \frac{1}{3^3} x_0 + \frac{2s_1}{3^3} + \frac{2s_2}{3^2} + \frac{2s_3}{3} \\ &\vdots \\ x_n &= \frac{1}{3^n} x_0 + \frac{2s_1}{3^n} + \frac{2s_2}{3^{n-1}} + \frac{2s_3}{3^{n-2}} + \dots + \frac{2s_n}{3} \end{aligned}$$

$\therefore x_n$  gets arbitrarily close to points of the form  $\sum_{i=1}^{\infty} \frac{t_i}{3^i}$  where  $t_i \in \{0, 2\}$ .

Therefore the attractor of the IFS is  $\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \text{Cantor Set} \right\}$

graphically:



### (Generalized) Iterated Function Systems

Def'n: An affine transformation is a transformation  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $F(\vec{x}) = A(\vec{x}) + \vec{b}$  where  $A \in M_n(\mathbb{R})$ ,  $\vec{b} \in \mathbb{R}^n$ .  
The idea is that it's basically a linear transformation b/c it doesn't need to pass through the origin.

Moreover, we call  $F$  a linear contraction if  $\exists 0 < \lambda < 1$  such that  $\|F(\vec{x}) - F(\vec{y})\| < \lambda \|\vec{x} - \vec{y}\|$ . There is a connection to the eigenvalues.

Ex  $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  and  $0 < \theta < 1$

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad F(\vec{x}) = \beta A \vec{x} + \vec{b}$$

$\rightarrow$  Scales by  $\beta$

$\rightarrow$  rotates counter-clockwise by  $\theta$

$\rightarrow$  translates by  $\vec{b}$

} only 2 out of the 3 rigid motions  
3<sup>rd</sup> in reflection

Def'n : we say  $A \subseteq \mathbb{R}^n$  is compact iff  $A$  is closed and bounded.

Notation:  $K_n = \{A \subseteq \mathbb{R}^n : A \text{ compact}\}$

Def'n: Let  $F_1, F_2, \dots, F_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be linear contractions. We call  $F: K_n \rightarrow K_n$   $F(A) = F_1(A) \cup F_2(A) \cup \dots \cup F_k(A)$  an iterated function system

$\downarrow$   
not well defined in this defn but it is compact & Bounded on both sides

We will:

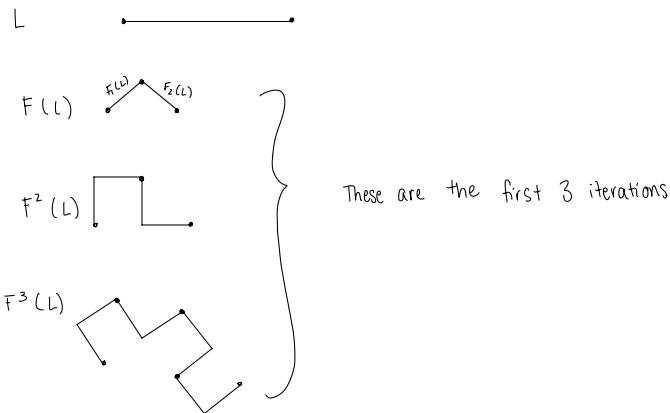
- ① Equip  $K_n$  with a metric  $\rightarrow$  v. similar to Banach Contraction
- ② Show  $F$  has a unique fixed point  $A^*$  and  $\forall A \in K_n, F^n(A) \rightarrow A^*$
- ③ We call  $A^*$  the attractor of  $F$ .

$$\text{Ex) } F_1(\vec{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \vec{x} \quad \rightarrow \text{this rotates by } \frac{\pi}{4}$$

$\downarrow$  Squashes

$$F_2(\vec{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \rightarrow \text{translates 1 step}$$

$L$  = line segment from  $(0, 0)$  to  $(1, 0)$



Dragon Curve: Video by mathematical visual proofs

## Lec 21 - Mar 4<sup>th</sup> 2024

2 speakers lined up for end of semester, trying to get as many speakers as he can  
assignment due Mar 12<sup>th</sup>. May have class on April 8<sup>th</sup>

Recall:

$$F_1, F_2, \dots, F_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

each  $F_i(\vec{x}) = A_i \vec{x} + \vec{b}_i$  and  $\|F_i(\vec{x}) - F_i(\vec{y})\| \leq \lambda \|\vec{x} - \vec{y}\|$  where  $\lambda \in (0, 1)$  each  $F_i$  will have their own  $\lambda_i$  but here  $\lambda = \max_i (\lambda_i)$ . These are Linear Contractions.

Consider  $K_n = \{A \in \mathbb{R}^n : A \text{ compact}\}$  Remember compact = closed and bounded.

Fact:  $\forall A \in K_n \Rightarrow F_i(A) \in K_n$  means we can iterate.

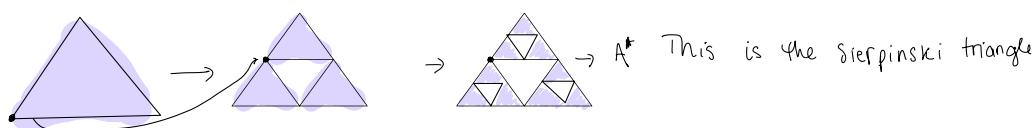
To do: Equip  $K_n$  with a metric. We will consider  $F: K_n \rightarrow K_n$ ,  $F(A) = F_1(A) \cup F_2(A) \cup \dots \cup F_k(A)$   $F$  gives you all of the images at once

To do: We show that  $F$  has a unique fixed point  $A^* \in K_n$  and  $\forall A \in K_n, F(A) \rightarrow A^*$  Like Banach contraction theorem, but we need metric

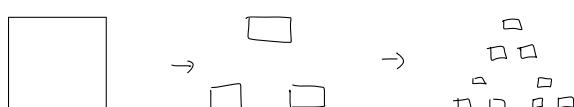
We call  $F$  an (generalized) iterated function system. We call  $A^*$  its attractor.

Ex)  $F_1 = \frac{1}{2} \vec{x}$   
 $F_2 = \frac{1}{2} \vec{x} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$   
 $F_3 = \frac{1}{2} \vec{x} + \begin{bmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{bmatrix}$

Let  $A$  be the filled triangle with vertices  $(0, 0), (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2})$



OR.



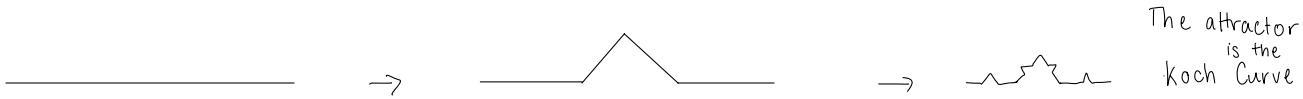
$$\begin{array}{c} - \\ - \end{array} \rightarrow \begin{array}{c} - \\ - \\ - \end{array}$$

No matter what compact set you start out with, you converge to same thing

Ex)  $F_1(\vec{x}) = \frac{1}{3} \vec{x}$   
 $F_2(\vec{x}) = \frac{1}{3} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \vec{x} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} \quad \theta = \frac{\pi}{3}$   
 $F_3(\vec{x}) = \frac{1}{3} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \vec{x} + \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{bmatrix} \quad \text{special } \theta \text{ for rotation matrix} \quad \theta = -\frac{\pi}{3}$

$$F_1(\vec{x}) = \frac{1}{3}\vec{x} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$$

and  $L =$  line segment from  $(0,0)$  to  $(1,0)$



Ex)  $A = [0,1] \times [0,1]$  the filled square

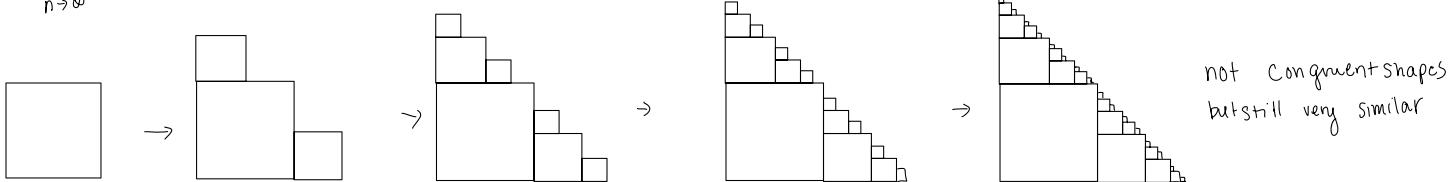
$$F_1(\vec{x}) = \vec{x}$$

$$F_2(\vec{x}) = \frac{1}{2}\vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F_3(\vec{x}) = \frac{1}{2}\vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Remark: Not quite an IFS b/c  $F_1$  is not a contraction.  $\Rightarrow$  no longer always converge to an attractor. Will depend on IC.

i.e.  $\lim_{n \rightarrow \infty} f^n(A)$  will depend on  $A$ . But turns out for us limit does exist



Defn: [Hausdorff Metric]

$\forall \vec{v}, \vec{w} \in \mathbb{R}^n$ ,  $A, B \in \mathbb{R}^n$

$$d(\vec{v}, B) = \min \{ \|\vec{v} - \vec{b}\| : \vec{b} \in B \} \quad \text{impf is a min. Extreme Value theorem:}$$

$$d(A, B) = \max \{ d(a, B) : a \in A \} \rightarrow \text{The biggest shortest distance}$$

$$D(A, B) = \max \{ d(A, B), d(B, A) \}$$

Lec 22 · Mar 6<sup>th</sup> 2024

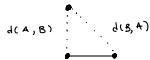
Recall: The Hausdorff metric ( $D$ ), formulation is in last lecture

↳ The way little  $d$  is not symmetric that the motivation for  $D$  b/c the max in it solves the Symmetry issue

Fact:  $D$  is a metric on  $\mathbb{K}_n$

$$\text{Ex) } A = \{(1,1)\}, B = \{(x,0) : 0 \leq x \leq 1\}$$

graph



$$d(A, B) = 1$$

$$d(B, 0) = \sqrt{2} \quad (a^2 + b^2 = c^2)$$

Q Is class: Would min work? Probably not b/c triangle inequality probably fails

The reason we have a max in  $d(A, B)$  is b/c we want to only have 2 sets be the same is that if they are the same set

$$\text{Finally, } D(A, B) = \max \{1, \sqrt{2}\} = \sqrt{2}$$

Lemma 1: Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear contraction such that  $\|f(\vec{x}) - f(\vec{y})\| \leq \lambda \|\vec{x} - \vec{y}\|$ ,  $\lambda \in (0, 1)$

For  $A, B \subseteq \mathbb{K}_n$  then  $D(f(A), f(B)) \leq \lambda D(A, B)$

$$\begin{aligned} \text{Proof: First, } d(f(a), f(b)) &= \min_{b \in B} \|f(a) - f(b)\| \\ &\leq \min_{b \in B} \lambda \|a - b\| = \lambda \min_{b \in B} \|a - b\| \\ &= \lambda d(a, b) \end{aligned}$$

$$\begin{aligned} \text{and so } d(f(A), f(B)) &= \max_{a \in A} d(f(a), f(B)) \\ &\leq \lambda \max_{a \in A} d(a, B) = \lambda d(A, B) \\ &\leq \lambda D(A, B) \end{aligned}$$

comes from  
defn & properties of min & max

$$\therefore d(f(A), f(B)) \leq \lambda D(A, B)$$

Similarly  $d(f(B), f(A)) \leq \lambda D(A, B) \rightarrow$  convince yourself of this

$$\therefore D(f(A), f(B)) \leq \lambda D(A, B)$$

■

Lemma 2: For  $A_1, A_2, B_1, B_2 \subseteq \mathbb{K}_n$ ;  $D(A_1 \cup A_2, B_1 \cup B_2) \leq \max \{D(A_1, B_1), D(A_2, B_2)\}$

$$\begin{aligned} \text{Proof: First, } d(A_1 \cup A_2, B_1 \cup B_2) &= \max_{a \in A_1 \cup A_2} d(a, B_1 \cup B_2) \\ &= \max \left\{ \max_{a \in A_1} d(a, B_1 \cup B_2), \max_{a \in A_2} d(a, B_1 \cup B_2) \right\} \quad \text{ensure biggest distance} \\ &\leq \max \left\{ \max_{a \in A_1} d(a, B_1), \max_{a \in A_2} d(a, B_2) \right\} \quad \text{By the min in the definition of } d(\cdot, B) \\ &= \max \{d(A_1, B_1), d(A_2, B_2)\} \\ &\leq \max \{D(A_1, B_1), D(A_2, B_2)\} \end{aligned}$$

$$\therefore d(A_1 \cup A_2, B_1 \cup B_2) \leq \max \{d(A_1, B_1), d(A_2, B_2)\}$$

$$\text{Similarly } d(B_1 \cup B_2, A_1 \cup A_2) \leq \max \{d(A_1, B_1), d(A_2, B_2)\}$$

$$\therefore D(A_1 \cup A_2, B_1 \cup B_2) \leq \max \{d(A_1, B_1), d(A_2, B_2)\}$$

**Lemma 3:** Let  $F_1, \dots, F_k$  be linear contractions with contraction factor  $\lambda \in (0, 1)$

Consider  $F: K_n \rightarrow K_n$ ,  $F(A) = F_1(A) \cup F_2(A) \cup \dots \cup F_k(A)$ . Then  $D(F(A), F(B)) < \lambda D(A, B)$

**Proof:** We have,  $D(F(A), F(B)) \leq \max_{i=1, \dots, k} D(F_i(A) \cup F_i(B))$  By the inductive generalization of Lemma 2  
 $\leq \max_{i=1, \dots, k} \lambda D(A, B)$  By Lemma 1  
 $= \lambda D(A, B)$  b/c no more i-dependency. This is technically a corollary

Q from class: Can be infinitely many? probs not b/c we may lose compactness.

**Defn:**  $(X, d)$  metric space.

① We say  $(x_n) \subseteq X$  is Cauchy iff  $\forall \epsilon > 0, \exists N \in \mathbb{N}, n, m \geq N \Rightarrow d(x_n, x_m) < \epsilon$ .

② We say  $X$  is complete is complete iff every Cauchy Sequence  $(x_n) \subseteq X$  converges to some  $x \in X$ .

Fact:  $(K_n, D)$  is complete! yay for us

↳ useful for us

# Lec 23 - Mar 8 2024

Guest speakers! right now 4 but hoping to get 5. This will take last 2 weeks of guest speaker. Speaker content will be on final. 50% of final MC and the content may show up there.

Correction:  $K_n = \{ A \subseteq \mathbb{R}^n : A \text{ compact}, A \neq \emptyset \}$

Recall:  $F_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$   $1 \leq i \leq k$  are linear contractions

$$\|F_i(\vec{x}) - F_i(\vec{y})\| \leq \lambda \|\vec{x} - \vec{y}\|$$

$F : K_n \rightarrow K_n$

B/c the function

$F(A) = F_1(A) \cup \dots \cup F_k(A)$  then

contracts so does the

$$D(F(A), F(B)) \leq \lambda D(A, B)$$

distance

## Theorem

Let  $F_1, \dots, F_k$  be linear contraction with Contraction factor  $\lambda \in (0, 1)$ . Let  $F : K_n \rightarrow K_n$  be the corresponding IFS. Then:

- ①  $F$  has a unique fixed point  $A^*$
- ② For all  $A^* \in K_n$ ,  $F^m(A) \rightarrow A^*$ .

We call  $A^*$  the attractor of  $F$ .

Can there be IFS that are not contracts such that there are multiple fixed points? like a Kalidoscope  $\rightarrow$  one of our example of Square Stair case is an example of this

## Proof:

Fix  $A \in K_n$ . Consider the Orbit  $F^m(A)$ , then,

$$\begin{aligned} D(F^{m+r}(A), F^m(A)) &= D(F^m(F(A)), F^m(A)) \\ &\leq \lambda^m D(F(A), A) \end{aligned}$$

call this  $\epsilon_m$

Note:  $\sum \epsilon_m < \infty$ ,  $|\lambda| < 1$ . Therefore  $(F^m(A)) \subseteq K_n$  is strongly Cauchy.  $\rightarrow$  technically we didn't define it in general space. but just change R defn. from  $\|\cdot\|$  to  $D$ .

$\Rightarrow (F^m(A))$  is cauchy  $\Rightarrow F^m(A) \rightarrow A^* \in K_n$  Since  $K_n$  is complete

Since  $F$  is continuous  $F^{m+1}(A) \rightarrow F(A^*)$  Hence,  $F(A^*) = A^*$

Suppose  $A^*, B^*$  are fixed points for  $F$ . then

$$\begin{aligned} D(A^*, B^*) &= D(F(A^*), F(B^*)) \leq \lambda D(A^*, B^*) \quad \lambda \in (0, 1) \\ \Rightarrow D(A^*, B^*) &= 0 \Rightarrow A^* = B^* \end{aligned}$$

## Chapter 10 - Complex functions

assume polar form. and complex algebra

these are modulus not absolute value

Defn  $f(z)$ ,  $\mathbb{C}$ -valued,  $z \in \mathbb{C}$

?

① For  $z_0 \in \mathbb{C}$ , we say  $\lim_{z \rightarrow z_0} f(z) = L \in \mathbb{C}$  iff  $\forall \varepsilon > 0$ ,  $\exists \delta > 0$ ,  $0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$

② We define the derivative of  $f(z)$  at  $z_0$  by  $f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ , provided the limit exists.

Defn: Let  $a \in \mathbb{C}$  be a fixed point of  $f(z)$ .

- ① We say  $a$  is attracting  $\Leftrightarrow |f'(a)| < 1$
- ② We say  $a$  is repelling  $\Leftrightarrow |f'(a)| > 1$
- ③ We say  $a$  is neutral  $\Leftrightarrow |f'(a)| = 1$

Remark We obtain the  $\mathbb{C}$ -analogue of the attracting / repelling fixed point theorems.

$$\text{Ex) } f(z) = z^2 + z + 1$$

$$\text{Fixed points: } z^2 + z + 1 = z \Leftrightarrow z^2 + 1 = 0 \Leftrightarrow z = \pm i$$

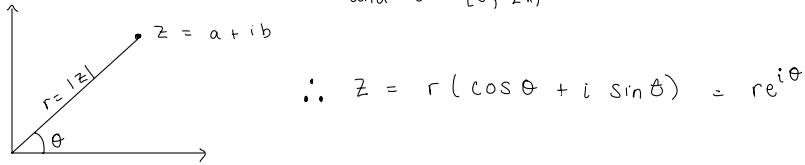
$$f'(z) = 2z + 1$$

$$|f'(i)| = |2i + 1| = \sqrt{4+1} = \sqrt{5} > 1$$

$$|f'(-i)| = |-2i + 1| = \sqrt{5} > 1$$

$\Rightarrow$  repelling

Recall



$$\text{Fact: } e^{i\theta} \cdot e^{i\phi} = e^{i(\theta + \phi)}$$

$$(re^{i\theta})^n = r^n e^{in\theta}$$

$$\text{Recall: } \left(e^{\frac{2\pi i}{n}}\right)^n = e^{2\pi i} = 1 \quad \text{This gets of periodic points}$$

$n^{\text{th}}$  roots of unity.

## Lec 24 - Mar 11<sup>th</sup> 2023

Assignment 2 due tomorrow.  
hint:  
2: not chaotic, focus on transitivity

$$\text{Ex) } z = e^{2\pi i/3} \\ = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \\ = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$$

$$f(w) = w^2 \quad \text{very easy in this form} \\ f(z) = (e^{2\pi i/3})^2 = e^{4\pi i/3} = -\frac{1}{2} - i \frac{\sqrt{3}}{2} \\ f^2(z) = e^{8\pi i/3} = e^{2\pi i/3} = z \quad \text{reducing by } 2\pi?$$

period 2 derivative formula from before works in  $\mathbb{C}$

$$|(f^2)'(z)| = |f'(z) \cdot f'(f(z))| \\ = |(-1 + i\sqrt{3}) \cdot (-1 - i\sqrt{3})| \\ = 2 \cdot 2 = 4 > 1$$

$\therefore z$  is a repelling periodic point for  $f(w)$

## Chapter 11 Julia Sets

Notation: For  $c \in \mathbb{C}$ ,  $Q_c(z) = z^2 + c$

orbit is bounded.

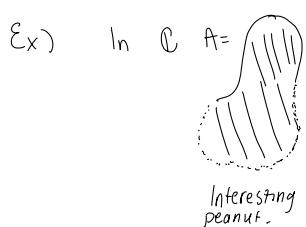
Def'n: Let  $c \in \mathbb{C}$ . Then the filled Julia set for  $c$  is  $K_c = \{z \in \mathbb{C} : (Q_c^n(z)) \text{ is bounded}\}$   
 $= \{z \in \mathbb{C} : \exists M > 0, \forall n \in \mathbb{N}, |Q_c^n(z)| \leq M\}$

Complex analogue of  $\Lambda$

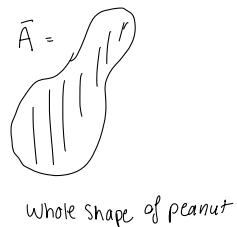
Remark: This is the complex analogue of  $\Lambda$  for  $Q_c(z) = z^2 + c$  where  $c \in \mathbb{R}$  and  $c < -2$ .

Def'n  $(X, d)$ ,  $A \subseteq X$

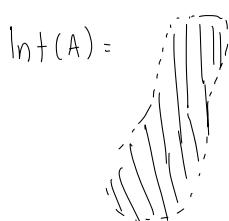
- ① the closure of  $A$  is  $\bar{A} = \{x \in X, \exists (a_n) \subseteq A \text{ s.t. } a_n \rightarrow x\}$  includes point on boundary which may not be in the set.
- ② the interior of  $A$  is  $\text{int}(A) = \{x \in A : \exists \epsilon > 0, B_\epsilon(x) \subseteq A\}$
- ③ the boundary of  $A$  is  $\partial(A) = \bar{A} \setminus \text{int}(A)$



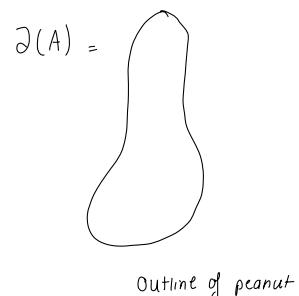
Interesting peanut.



Whole shape of peanut



open filled peanut



Outline of peanut

Remark:  $A$  is closed iff  $A = \bar{A}$

Assignment 4: Will show  $K_c$  is closed  $\Rightarrow$  means  $J_c$  wont contain any new points from the filled Julia set

Def'n: Cf. C. The Julia Set for  $c$  is  $J_c = \partial(K_c)$

Remark:  $J_c = \overline{K_c} \setminus \text{Int}(K_c) = K_c \setminus \text{Int}(K_c)$

Ex)  $c=0$ ,  $Q_0(z) = z^2$

$$z = r e^{i\theta} \rightarrow \text{modulus of } z = \cos^2 + \sin^2 = 1$$

$$|Q_0(z)| = |r^2 e^{2i\theta}| = r^2$$

$$|Q_0^2(z)| = |r^4 e^{4i\theta}| = r^4$$

$$|Q_0^3(z)| = |r^8 e^{8i\theta}| = r^8$$

$\vdots$

$$|Q_0^n(z)| = r^{2^n}$$

if  $r \in [0, 1] \Rightarrow Q_0^n(z)$  bounded

$$K_0 = \{z \in \mathbb{C} : |z| \leq 1\}.$$

$$J_0 = \{z \in \mathbb{C} : |z| = 1\}. \rightarrow \text{unit circle}$$

Ex)  $c = -2$

$$Q_{-2}(z) = z^2 - 2$$

$$\textcircled{1} \quad R = \{z \in \mathbb{C} : |z| > 1\}$$

$$H: R \rightarrow \mathbb{C}$$

$$H(z) = z + \frac{1}{z} \quad \left. \right\} \text{not clear at first.}$$

$H$  is a translation function, aim to bridge information b/w  $Q_0$  and  $Q_{-2}$

\textcircled{2}

$$\begin{aligned} \text{Claim } H \text{ is Injective: } H(z) = H(w) &\Rightarrow z + \frac{1}{z} = w + \frac{1}{w} \\ &\Rightarrow zw = z^2 + 1 - \frac{z}{w} = w^2 + 1 - \frac{w}{z} \\ &\Rightarrow w^2 - z^2 = \frac{w}{z} - \frac{z}{w} = \frac{w^2 - z^2}{zw} \quad \rightarrow \text{only if } zw = 1 \text{ or } w^2 = z^2 \end{aligned}$$

Since  $w, z \in R$ , we have  $w^2 - z^2 = 0 \Rightarrow w = \pm z$ .

Since  $H(w) = H(z) \Rightarrow w = z$

But  $zw \neq 1$   
b/c  $|z|$  and  $|w| > 1$

\textcircled{3}  $H: R \rightarrow \mathbb{C} \setminus [-2, 2]$  is surjective

$$H(z) = w \Leftrightarrow z + \frac{1}{z} = w$$

$$\Rightarrow z^2 - wz + 1 = 0$$

$$\Rightarrow z_{\pm} = \frac{1}{2}(w \pm \sqrt{w^2 - 4})$$

Note:  $z_+ z_- = 1$

if  $z_+ \in R$  or  $z_- \in R$        $H(z_+) = w$     or     $H(z_-) = w$ .    Otherwise     $|z_{\pm}| = 1$ .

Say  $z_{\pm} = e^{i\theta}$

$$H(z_{\pm}) = H(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos \theta \in [-2, 2]$$

## Lec 25 - Mar 13<sup>th</sup> 2024

Recall:  $K_c = \{z \in \mathbb{C} : Q_c(z) \text{ bounded}\}$

$$J_c = \partial(K_c)$$

$$K_0 = \{z : |z| \leq 1\}$$

$$J_0 = \{z : |z|=1\}$$

Ex:  $c = -2$   $Q_{-2}(z) = z^2 - 2$

$$\textcircled{1} \quad R = \{z : |z| > 1\} \quad H: R \rightarrow \mathbb{C} \quad H(z) = z + \frac{1}{z}$$

\textcircled{2}  $H$  injective

\textcircled{3}  $H: R \rightarrow \mathbb{C} \setminus [-2, 2]$  surjective

$$\textcircled{4} \quad H(Q_0(z)) = H(z^2) = z^2 + \frac{1}{z^2}$$

$$z^2 + \frac{1}{z^2} + 2(\cancel{\frac{z}{z}})^2 - 2 \quad \downarrow$$

$$Q_{-2}(H(z)) = (z + \frac{1}{z})^2 - 2 = z^2 + \frac{1}{z^2}$$

$$H(Q_0^n(z)) = Q_{-2}^n(H(z))$$

said to be conjugate but we want define it.

Compare:  $H$  is playing the same role

$$S(Q_c(x)) = \sigma^n(S(x))$$

$$\textcircled{5} \quad |z_n| \rightarrow \infty$$

$$|H(z_n)| = |z_n + \frac{1}{z_n}| \geq |z_n| - |\frac{1}{z_n}| \rightarrow \infty \quad \text{reverse triangle inequality}$$

→ How does a complex # go to infinity?

$$\Rightarrow |H(z_n)| \rightarrow \infty$$

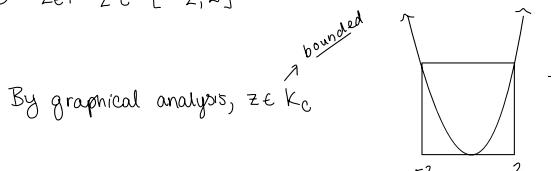
$$\textcircled{6} \quad z \in \mathbb{C} \setminus [-2, 2] \quad z = H(w), \quad w \in R$$

$$|Q_{-2}^n(z)| = |Q_{-2}^n(H(w))| = |H(Q_0^n(w))| \rightarrow \infty$$

$\downarrow \infty \rightarrow$  b/c  $w \in R, z > 1 \Rightarrow z \notin K_0 \Rightarrow Q_0^n(z) \rightarrow \infty$

$\therefore z \notin K_{-2}$

$$\textcircled{7} \quad \text{Let } z \in [-2, 2]$$



$$\text{Therefore } K_{-2} = [2, 2]$$

$$J_{-2} = [-2, 2] \rightarrow \text{this is the boundary in the complex plane}$$

These are the only 2 Julia sets that aren't fractals

[Escape criteria]

Proposition If  $|z| \geq |c| > 2$  then  $|Q_c^n(z)| \rightarrow \infty$ . In particular,  $z \notin K_c$

Proof: we have  $|Q_c^n(z)| = |z^2 + c| \geq |z|^2 - |c| \xrightarrow{\text{Reverse triangle}} \text{from proposition}$

$$\geq |z|^2 - |z|$$

$$= |z|(|z| - 1)$$

Say  $|z| > 2 + \lambda$ ,  $\lambda > 0$ .  $\Rightarrow |z| - 1 > 1 + \lambda$

Hence  $|Q_c^n(z)| \geq |z|(1 + \lambda)$

$\Rightarrow |Q_c^n(z)| \geq |z|(1 + \lambda)^n \rightarrow$  how come  $|z|$  doesn't get raised to the  $n$

and  $(1 + \lambda)^n \rightarrow \infty$  hence  $|Q_c^n(z)| \rightarrow \infty$

Corollary:  $|c| > 2$  Then  $|Q_c^n(0)| \rightarrow \infty$  so that  $0 \notin K_c$

Why?

$z = Q_c(0) = c$

$|z| = |c| > 2$

$\Rightarrow |Q_c^n(z)| \rightarrow \infty$  By previous proposition

$\Rightarrow |Q_c^n(0)| \rightarrow \infty$

Corollary:  $M = \max \{ |c|, 2 \}$  some complex #  
 If  $|z| > M$  then  $|Q_c^n(z)| \rightarrow \infty$ . Thus,  $K_c \subseteq \{ z : |z| \leq M \}$  hence bounded sets

Why?

$|Q_c^n(z)| \geq (1 + \lambda)^n |z| \rightarrow \infty$

Cannot use escape criterion but use exact same proof b/c the assumption  $|z| \geq |c| > 2$  in the Escape Criterion isn't met.

Hint we haven't showed  $K_c$  is closed but knowing  $K_c \subseteq \{ z : |z| \leq M \}$  helps in showing that

Corollary: if  $\exists K$  s.t  $|Q_c^K(z)| > \max \{ |c|, 2 \}$  then  $|Q_c^\infty(z)| \rightarrow \infty$  thus  $z \notin K_c$

Algorithm:

- ① choose a large  $N \in \mathbb{N}$
- ② if  $|Q_c^i(z)| > \max \{ |c|, 2 \}$  for  $i \leq N$  colour  $z$  white
- ③ if  $|Q_c^i(z)| \leq \max \{ |c|, 2 \}$  for all  $i \leq N$ , colour  $z$  black

The black shaded region approximates  $K_c$

## Lec 26<sup>th</sup> - Mar 15<sup>th</sup> 2024

Recall: If  $\exists k$  s.t.  $|Q_c^k(z)| > \max\{|c|, 2\}$  then  $|Q_c^n(z)| \rightarrow \infty$ . In particular,  $z \notin K_c$ .

Ex) Is  $i \in K_{2+i}$

$$Q(z) = z^2 + 2i \quad c = 2+i \quad |c| = \sqrt{5}$$

$$\text{Max} = \max\{\sqrt{5}, 2\} = \sqrt{5}$$

$$i \mapsto 1+i \mapsto 2+3i$$

image under  $Q(z)$   $|1+i| = \sqrt{2}$

$$|2+3i| = \sqrt{4+9} = \sqrt{13} > \sqrt{5}$$

$\therefore$  No!  $i \notin K_{2+i}$

$K_c$  tends to be sparse so it was highly likely  $i \notin K_{2+i}$

Remark:  $n \in \mathbb{Z}$ ,  $n \neq 0$

$$\begin{aligned} \int_0^{2\pi} e^{int} dt &= \int_0^{2\pi} (\cos(nt) + i \sin(nt)) dt \\ &\stackrel{\text{by defn}}{=} \int_0^{2\pi} \cos(nt) dt + i \int_0^{2\pi} \sin(nt) dt \\ &= \left[ \frac{1}{n} \sin(nt) \right]_0^{2\pi} + i \left[ \frac{1}{n} \cos(nt) \right]_0^{2\pi} \\ &= 0 \quad \rightarrow \text{it being zero is useful and important to Fourier analysis} \end{aligned}$$

if  $n=0$ ,

$$\int_0^{2\pi} e^{int} dt = \int_0^{2\pi} 1 dt = 2\pi.$$

Prop [Cauchy's Estimate] not most general

Let  $P(z) = \sum_{n=0}^d a_n z^n$  be a polynomial. s.t.  $|P(z)| \leq M$  for all  $|z - z_0| \leq r$  then

$$|P'(z_0)| \leq \frac{M}{r}$$

Proof

① assume  $z_0 = 0$

Assume  $|P(z)| \leq M$  for  $|z| \leq r$

$$\begin{aligned} \text{Then, } \frac{1}{2\pi} \int_0^{2\pi} \frac{P(re^{it})}{re^{it}} dt &\quad \checkmark \div \text{ by } re^{it} \text{ gives } -1 \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^d a_n r^{n-1} e^{i(n-1)t} dt \end{aligned}$$

When  $n-1=0$  the integral is zero

$$= a_1$$

$$= P'(0)$$

from integral properties of absolute value



$$\text{and so } |P'(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{P(re^{it})}{P(e^{it})} \right| dt$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{r} dt = \frac{M}{r}$$

Case 2 General Case.

Assume  $|P(z)| < M$  for  $|z - z_0| \leq r$

Let  $w = z - z_0$  so that  $|P(w + z_0)| \leq M$  for all  $|w| \leq r$ .

Libenz notation.

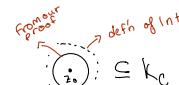
$$\text{By Case 1, } \left| \frac{d}{dw} \Big|_{w=0} P(w + z_0) \right| \leq \frac{M}{r}$$

$$\Rightarrow \left| \frac{d}{dz} \Big|_{z=z_0} P(z) \right| \leq \frac{M}{r}$$

Theorem: If  $z_0$  is a repelling periodic point for  $Q_c(z)$ , then  $z_0 \in J_c$ .

Proof: Assume  $z_0$  is a repelling periodic point with period  $n$ . Suppose  $z_0 \notin J_c$ . Since  $z_0$  is periodic,  $z_0 \in K_c$ .

thus  $z_0 \in \text{int}(K_c)$  and so  $\exists r > 0$  s.t.  $z \in K_c$  for all  $|z - z_0| \leq r$ .



Using escape criterion (contra positive)

For all  $z$  s.t.  $|z - z_0| \leq r$  and all  $k \in \mathbb{N}$ ,  $\left| (Q_c^n)^k(z) \right| \leq M$ , where  $M = \max\{|c|, 2\}$

By the Cauchy estimate.

$$\left| (Q_c^n)^k(z_0) \right| \leq \frac{M}{r} \quad \text{for all } k \in \mathbb{N}.$$

Haven't used repelling yet.

$$\text{Say } \left| (Q_c^n)'(z_0) \right| = \lambda > 1$$

Blake called it

acute proof. it

may be on the final

$$\text{Finally, } \left| (Q_c^n)'(z_0) \right| = \left| (Q_c^n)'(z_0) \right| \cdot \left| (Q_c^n)'(Q_c(z_0)) \right| \cdots \cdot \left| (Q_c^n)'(Q_c^{k-1}(z_0)) \right|$$

$$= \lambda^k \rightarrow \infty \quad \text{which contradicts the Cauchy estimate}$$

Fact:  $K_c \subseteq \mathbb{C}$  closed. If  $z \in \text{int}(K_c)$  then  $Q_c(z) \in \text{int}(Q_c(K_c))$

Proposition: If  $Q_c(z) \in J_c$ , then  $z \in J_c$  the Julia set of the preimage.

Why?

$$Q_c(z) \in J_c \Rightarrow Q_c(z) \in K_c \Rightarrow z \in K_c$$

$$Q_c(z) \notin \text{int}(K_c) \stackrel{\text{fact}}{\Rightarrow} z \notin \text{int}(K_c) \quad \text{note } Q_c(K_c) = K_c \Rightarrow z \in J_c$$

## Lec 27 - Mar 18<sup>th</sup> 2024

Friday next week is holiday. Wednesday next week will start Guest speaker.

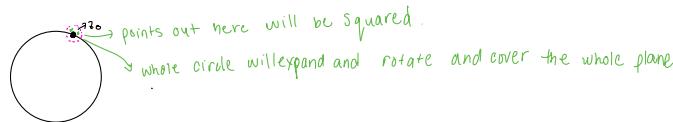
Recall: if  $Q_c(z) \in J_c \Rightarrow z \in J_c$

functions explode  
they cover the complex numbers

Def'n: We say  $Q_c$  is supersensitive at  $z_0 \in \mathbb{C}$  iff whenever  $U \subseteq \mathbb{C}$  is open such that  $z_0 \in U$  then  $\mathbb{C} = \bigcup_{n=0}^{\infty} Q_c^n(U)$

Fact:  $\forall z_0 \in J_c, Q_c$  is supersensitive at  $z_0$

Ex.  $J_0$  is the unit circle



Big Idea:

- ① pick  $z \in \mathbb{C}$  and  $z_0 \in J_c$
- ② For  $\epsilon > 0$ ,  $U = B_\epsilon(z_0)$
- ③ by Supersensitivity,  $\exists k, \exists w \in U$  s.t.  $Q_c^k(w) = z$
- ④ By looking at the backwards orbit of  $z$ . We can find a very close  $w \in U$  to  $J_c$ . Why bc w in U  $\Rightarrow$  close to Julia set.

More precisely:

- ① choose  $z \in \mathbb{C}$
- ② Compute 10 000 terms in the backwards orbit, randomly selecting a pre-image at each step.
- ③ Plot all but the first 100 points from the backwards orbit. → b/c consider sufficient. for numerics we aren't talking about

Path Connected, for  $c \in$  Mandelbrot set, the Julia set is path connected.

Different  $c \in$  Mandelbrot set will have different periodic points

The Mandelbrot set is a dictionary for the Julia sets

### Chapter 12 - The Mandelbrot Set

Def'n: for  $(X, d)$

- ① Let  $a, b \in X$ . A path from  $a$  to  $b$  is a continuous function  $\gamma: [0, 1] \rightarrow X$  s.t.  $\gamma(0) = a$  and  $\gamma(1) = b$



- ② We say  $A \subseteq X$  is path connected iff  $\forall a, b \in A \quad \exists$  a path  $\gamma: [0, 1] \rightarrow X$  from  $a$  to  $b$  s.t.  $\gamma([0, 1]) \subseteq A$





③ The maximal path connected subsets of  $A$  are called the path connected components of  $A$ .



3 path connected components

④ We say  $A$  is totally disconnected if the path connected components of  $A$  are exactly  $\{a\}_{a \in A}$ .

Ex. Cantor Set.

not necessarily isolated

## Lec 28 - Mar 20<sup>th</sup> 2024

Recall:  $A \subseteq \mathbb{C}$

(1)  $A$  is path connected iff  $\forall a, b \in A \exists$  a continuous  $\gamma: [0, 1] \rightarrow A$ ,  $\gamma(0) = a$ ,  $\gamma(1) = b$   $\gamma([0, 1]) \subseteq A$

(2)  $A$  is totally disconnected iff  $\forall a \neq b$  there is no path in  $A$  from  $a$  to  $b$

Ex.  $A = \{z : |z| \leq 1\}$



Ex:  $A:$



Ex.  $k \subseteq \mathbb{R}$  Cantor set is totally disconnected. ★ remember that the Cantor set is not isolated.

Theorem: For  $Q_c(z) = z^2 + c$  either:

(1)  $|Q_c^n(0)|$  is bounded ( $0 \in K_c$ ) in which case  $K_c$  is path connected

OR

(2)  $|Q_c^n(0)| \rightarrow \infty$ , [ $0 \notin K_c$ ], in which case  $K_c$  is totally disconnected.

Def'n: The Mandelbrot set  $M = \{c \in \mathbb{C} : |Q_c^n(0)| \text{ is bounded}\}$   
 $= \{c \in \mathbb{C} : |Q_c^n(0)| \neq \infty\}$   
 $= \{c \in \mathbb{C} : 0 \in K_c\}$   
 $= \{c \in \mathbb{C} : K_c \text{ path connected}\}$

Recall: If  $|c| > 2$ ,  $|Q_c^n(0)| \rightarrow \infty \Rightarrow M \subseteq \{z : |z| \leq 2\}$

Ex.  $0, -2 \in M$

$c = 0$   $c = -2$  — both are path connected and contain zero

Ex)  $c = 2$   $Q_c(z) = z^2 + 2$

$$0 \mapsto 2 \mapsto 6 \mapsto 38$$

$$|Q_c^n(0)| \rightarrow \infty \quad 2 \notin M$$

Ex)  $c = i$

$$Q_c(z) = z^2 + i$$

$$0 \mapsto i \mapsto -1 + i \mapsto -i \mapsto -1 + i \mapsto i \dots \text{ Eventually periodic}$$

$$\therefore |Q_c(\omega)| \text{ bounded} \Rightarrow i\epsilon \mathcal{M}$$

Goal: Describe the shape of  $\mathcal{M}$ .



## Lec 2a - Mar 22<sup>nd</sup> 2024

Assignment due next next tuesday. Guest speakers start Wednesday next week.  
no class april 8<sup>th</sup>. Exam breakdown Monday, Theorem list on Monday

Recall:  $\mathcal{M} = \{c \in \mathbb{C} : Q_c^n(0) \text{ is bounded}\}$   
 $= \{c \in \mathbb{C} : 0 \in K_c\}$   
 $= \{c \in \mathbb{C} : |Q_c^n(0)| \rightarrow \infty\}$   
 $= \{c \in \mathbb{C} : K_c \text{ is path connected}\}$

Recall:  $c \notin \mathcal{M}$   $K_c$  is totally disconnected

Remark:  $z \in \mathbb{C}, Q_c = z^2 + c$

$M = \max\{|c|, 2\}$ . by the escape criterion

$$\textcircled{1} \quad \forall n, |Q_c^n(z)| \leq M$$

Something subtle, not all limits that are unbounded go to infinity like  $0, 1, 0, 2, 0, 3 \dots$

$$\textcircled{2} \quad |Q_c^n(z)| \rightarrow \infty$$

Goal: Describe the general shape of  $\mathcal{M}$ :

Investigation:

\textcircled{1} When does  $Q_c(z)$  have an attracting fixed point?

Suppose  $z \in \mathbb{C}$  is an attracting fixed point for  $Q_c$ .  $\exists r > 0, x \in B_r(z) \Rightarrow Q_c^n(x) \rightarrow z$

Complex  
Analog of attracting fixed pt.  
therem

$$\Rightarrow B_r(z) \subseteq K_c$$

$$\Rightarrow K_c \text{ path connected} \Rightarrow c \in \mathcal{M}$$

$$\text{Now, } z^2 + c = z, |2z| < 1$$

$$\Rightarrow c = z - z^2, |z| < 1/2$$

Let's explore the boundary: The most extreme value is when  $|z| = 1/2$

$$z = \frac{1}{2}e^{i\theta}, c = z - z^2$$

$$c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta} \quad \text{where } \theta \in \mathbb{R} \quad \text{famous shape, Cardioid}$$

$\curvearrowright$  dotted bc boundary  
isn't included in attracting fixed point



$$\subseteq \mathcal{M}$$

② When does  $Q_c$  have an attracting 2-cycle?

If  $Q_c$  has an attracting periodic point  $z$  with period 2, we can similarly justify that  $c \in \mathcal{M}$ .

We know:  $(z^2 + c)^2 + c = z$

$\downarrow$   
Simplified, try for yourself

$\Rightarrow z^4 + 2cz^2 - z + c^2 + c = 0$  the fixed points of  $Q_c$  are also roots of this polynomial.

Let  $p_1$  &  $p_2$  be the fixed points of  $Q_c$

$\therefore (z - p_1)(z - p_2) = z^2 - z + c$  is a factor of the previous polynomial.

Thus,  $z$  is a root of  $\frac{z^4 + 2cz^2 - z + c^2 + c}{z^2 - z + c} \rightarrow$  this will isolate the roots that are 2-cycles

$$= z^2 + z + c + 1 \rightarrow \text{polynomial division try for yourself.}$$

Let  $z_1$  and  $z_2$  be the roots of  $z^2 + z + c + 1$ . i.e.  $z_1, z_2$  are the period 2 points for  $Q_c$ . For these to be attracting  $|Q_c'(z)| < 1$ .

$$\begin{aligned} &\Rightarrow |Q_c'(z_1) \cdot Q_c'(z_2)| < 1 \\ &\Rightarrow |2(z_1) \cdot 2(z_2)| < 1 \\ &\Rightarrow 4|z_1 z_2| < 1 \\ &\Rightarrow |z_1 z_2| < \frac{1}{4} \end{aligned}$$

$$\Rightarrow |c+1| < \frac{1}{4} \rightarrow \text{Circle of radius } < \frac{1}{4} \text{ starting at } -1.$$



If you do this & n then you get mandelbrot set. There are algorithms for the Mandelbrot set to visualize it. We don't touch that.  
by escape criterion

Proposition:  $\mathcal{M} = \{c \in \mathbb{C} : |c| \leq 2\} \cap \{c : |c^2 + c| \leq 2\} \cap \{c : |(c^2 + c)^2 + c| \leq 2\} \cap \dots$

HOMEWORK!

↳ should be algorithm for Computer

each of these are a preimage of a closed sets from  $-2 \rightarrow 2$

Proposition:  $\mathcal{M}$  is closed.



# Lec 30 - march 25<sup>th</sup> 2024

last traditional lecture. Guest speakers start Wednesday

## Chapter 13- Polynomial Julia Sets

Theorem: [Polynomial Escape Criterion]

Let  $P(z) = a_n z^n + \dots + a_1 z + a_0$  be a complex polynomial  $[\mathbb{C} \rightarrow \mathbb{C}]$ . with  $a_n \neq 0$  and  $n \geq 2$ . There exists  $R > 0$  depending only on  $n$  and  $a_i$  s.t.  $|P^k(z)| \rightarrow \infty$  for all  $|z| \geq R$

Proof:

Fix  $\lambda > 1$  and let  $C = \sum_{i=0}^{n-1} |a_i|$ . Consider  $R = \max \left\{ 1, \frac{2C}{|a_n|}, \left( \frac{2\lambda}{|a_n|} \right)^{\frac{1}{n-1}} \right\}$ . Assume  $|z| \geq R$ . Then,

$\nearrow$  non leading coefficients       $\swarrow$  used in proof       $\nearrow$  why these 3 things must ask over piazza

$\nearrow$  Reverse triangle inequality

$$|P(z)| \geq |a_n z^n| - |a_{n-1} z^{n-1}| - \dots - |a_1 z| - |a_0|$$

$\nearrow$  Regular triangle inequality

$$\geq |a_n z^n| - (|a_{n-1}| \cdot |z|^{n-1} + \dots + |a_1| \cdot |z| + |a_0|)$$

$\nearrow$  C from before

$$\geq |a_n z^n| - [C |z|^{n-1}] \quad \text{blc } |z| \text{ must be bigger than } R.$$

$$= |z^n| \left( |a_n| - \frac{C}{|z|} \right)$$

$$\geq |z^n| \left( |a_n| - \frac{|a_n|}{2} \right)$$

$$= |z|^n \cdot \frac{1}{2} |a_n|$$

$$= |z| \cdot \frac{1}{2} |a_n| \cdot |z|^{n-1}$$

$$\begin{aligned} \text{Note: } |z|^{n-1} &\geq \frac{2\lambda}{|a_n|} \\ &\geq |z| \cdot \frac{1}{2} |a_n| \frac{2\lambda}{|a_n|} \\ &= |z| \cdot \lambda \end{aligned}$$

Thus  $|P^k(z)| \geq \lambda^k |z| \rightarrow \infty$

Remark: Either

- ①  $\forall k, |P^k(z)| \leq R$
- ②  $|P^k(z)| \rightarrow \infty$

Def'n: Let  $P(z)$  be a complex polynomial with degree  $\geq 2$ .

- (1) the filled Julia set of  $P(z)$ :  $K_P = \{z \in \mathbb{C} : P^k(z) \text{ is bounded}\}$
- (2) The Julia set of  $P(z)$  is  $J_P = \partial(K_P)$

$$\text{Ex } p(z) = z^3 + z - 0.15$$

$$\text{Ex } z^3 + z - 0.2$$

$$\text{Ex } z^3 + z - 0.25$$

$$\text{Ex } z^4 + z^3 + z - 0.25$$

Multibrot Sets. Look up.

are all Julia Sets Symmetric?