Highlighting Legend

Chapter

Definition

Propositions / Theorems / Proofs

Procedures

Remarks are made in red pencil, same with questions

Lecture 1 – Jan 8th 2024

Chapter 1 Iteration and Orbits

allows for indefinite iteration

Defn Let $f: A \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$ [I.E $f: A \to A$]. For $a \in A$, we may iterate the function at a:

$$x_1 = a$$
$$x_2 = f(a)$$
$$x_3 = f(f(a)) = f^2(a)$$
$$x_4 = f(f(f(a))) = f^3(a)$$
$$\vdots$$

all will exist in A.

We call $(x_n)_{n=1}^{\infty} = (= (x_n))$ we call that sequence **the orbit of a under f.**

Ex. $f(x) = x^4 + 2x^2 - 2$, $a = -1$

$x_1 = a,\ x_2 = f(a),\ x_3 = f^2(a) \ldots$

$-1, 1, 1, \ldots \to$ eventually constant / periodic

Ex $f(x)\ -x^2 - x + 1$, $a = 0$

$x_1, x_2, x_3, x_4, x_5, x_6 \ldots$

$0, 1, -1, 1, -1, 1 \ldots \to$ eventually perodic w period 2

Ex. $f(x) = x^2 - 3x + 1$, $a = 1$

$x_1, x_2, x_3, x_4, \ldots$

$1, -1, 3, 19 \to \infty$ goes to infinity [he will talk about convergence / divergence later on]

Ex $f(x) = x^2 + 2x$, $a = -0.5$

x_1, x_2, x_3, x_4

$-0.5, -0.75, -0.9375, -0.9961$, converges to -1

My Q: Does it ever equal 1, is that a distinction?

Ex. $f(x) = x^3 - 3x$, $a = 0.75$

$x_1, x_2, x_3, x_4, x_5, \ldots$

$0.75, -1.828, -0.625, 1.631, -0.552, \ldots$ This is chaotic behaviour.

* For PMATH insights: an Interval around zero is dense *

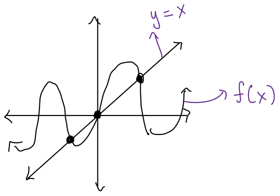
Defn: $f: A \to \mathbb{R}$, $f(A) \subseteq A$, We say $a \in A$ is a fixed point in f iff $f(a) = a$. In this case, the orbit of a is $a, a, a \ldots$ which is constant.

Ex. Find all fixed points of $f(x) = x^2 + x - 4$

When does $x^2 + x - 4 = x$?

$\Leftrightarrow$ $x^2 - 4 = 0$ $\Leftrightarrow$ $x = \pm 2$ Will often give low degree polynomial

Ex. How many fixed points does $f(x)$ as pictured below have?



$$\boxed{\text{Geometrically, a fixed point occurs when } f(x) \text{ intersects } y = x}$$

Ex. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

Solve for $x^4 - 3x + 1 = x$

$\Rightarrow x^4 - 4x + 1 = 0$

Intermediate Value theorem

Since $g(x)$ is continuous $g(0) = 1 > 0$ and $g(1) = -2 < 0$. By IVT $\exists\, x \in (0,1)$ s.t. $g(x) = 0 \Leftrightarrow f(x) = x$.

Defn $f: A \to \mathbb{R}$, $f(A) \subseteq A$

① We say $a \in A$ is a periodic point for f if its orbit is periodic. I.E $\exists n \in \mathbb{N}$ $f^n(a) = a$. The least such n is called the period of a and/or the orbit.

② Eventually periodic $\exists n < m$, $f^n(a) = f^m(a)$

Lecture 2 – Jan 10th 2024

Def'n [Doubling Function]

$D: [0,1) \to [0,1)$ where $D(x) =$ fractional part of $2x$ aka $2x \bmod 1$

Ex) $D(0.4) = 0.8$
$D(0.6) = 0.2$
$D(0.8) = 0.6$
$D(0.5) = 0$

It is an important function as it also provides a rich source of periodic orbits

Ex) D, $a = \frac{1}{5}$
Orbit: $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \ldots$ This has period 4. This leads into some cute number theory with GCDs

Ex) D, $a = \frac{1}{20}$
Orbit $\frac{1}{20}, \frac{1}{10}, \frac{1}{5} \ldots$ This is eventually periodic with period 4.

The Doubling function will come up later in a more meaningful way

Every day functions may not exhibit periodic orbits

Q: Given f and a, does $f^n(a)$ tend towards some limit L? This does happen suprisingly often

The language of this course vs Elementary Real Analysis

Notation:

If $(x_n)_{n=1}^{\infty}$ is a sequence of real number we write $(X_n) \subseteq \mathbb{R}$ a small abuse of notation but reasonable

Def'n $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$ we say (x_n) **converges** to x iff for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ for all $n \geq N$

* N depends on ε *

We write $x_n \longrightarrow x$ or $\lim x_n = x$

Ex) Claim: $\frac{1}{n} \to 0$, Let $\varepsilon > 0$ be given.

Note: $\frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$

not necessarily a Natural number. Can always use ceiling function $\lceil\ \rceil$ to bring this to a natural number

Consider $N = \frac{2}{\varepsilon}$ For $n \geq N$ we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$

bring this to $2/\varepsilon$ so that we ensure strictly less than Epsilon.

Ex) Claim $\frac{2n}{n+3} \to 2$

Let $\varepsilon > 0$ be given. Let us choose $N \in \mathbb{N}$ s.t.

$$\boxed{\frac{1}{N} < \frac{\varepsilon}{6}}$$

For $n \geq N$ $\left| \frac{2n}{n+3} - 2 \right|$

$$= \left| \frac{2n}{n+3} - \frac{2n+6}{n+3} \right|$$

$$= \left| \frac{-6}{n+3} \right| = \frac{6}{n+3} < \frac{6}{n}$$ if you divide by less of something you get something bigger

$$\frac{6}{n} \leq \frac{6}{N} = 6\left(\frac{1}{N}\right) < 6\left(\frac{\varepsilon}{6}\right) = \varepsilon$$

Defn $(x_n) \subseteq \mathbb{R}$ we say (x_n) is **bounded** if $\exists M > 0$ s.t. $\forall n \in \mathbb{N}, |x_n| \leq M$

Prop $(x_n) \subseteq \mathbb{R}$ If (x_n) is convergent, then (x_n) is bounded.

Ex) $x_n = (-1)^n$ Shows converse is not true

picked ε, blake's favourite is 1

Proof Suppose $x_n \to x$. then $\exists N \in \mathbb{N}$, $n \geq N \Rightarrow |x_n - x| < 1$

For $n \geq N$,

this is the reverse triangle inequality, make sure you know how to prove from triangle inequality

$$|x_n| - |x| \leq |x_n - x| < 1 \Rightarrow |x_n| < 1 + |x|$$ } This is only true for $n \geq N$

So Let $M = \max\{|x_1|, \dots, |x_{N-1}|, 1 + |x|\}$ ∎

Prop Let $x_n \to x$, $y_n \to y$

Not super important to us, value is in working with defn of convergence

① $x_n + y_n \to x + y$

② $x_n y_n \to xy$

Proof:

① Let $\varepsilon > 0$ be given. There exists $N_1, N_2 \in \mathbb{N}$ s.t.

$$n \geq N_1 \Rightarrow |x_n - x| < \boxed{\frac{\varepsilon}{2}}$$

$$n \geq N_2 \Rightarrow |y_n - y| < \boxed{\frac{\varepsilon}{2}}$$

For $N = \max\{N_1, N_2\}$ and $n \geq N$

$$|x_n + y_n - (x+y)| = |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y| < \varepsilon$$

(2) Let $\varepsilon > 0$ be given

Note:

$$|x_n y_n - xy| = |x_n y - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - y| \quad ✡$$

Since (x_n) is bounded $\exists M > 0, \forall n, |x_n| < M$

Let $N_1, N_2 \in \mathbb{N}$ s.t. $n \geq N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2(|y|+1)}$ make sure not dividing by zero

$n \geq N_2 \Rightarrow |y_n - y| < \frac{\varepsilon}{2M}$

For $n \geq N := \max\{N_1, N_2\}$ we have $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by ✡ ∎

Lec 3 - Jan 12th 2024

Def'n we say $(X_n) \subseteq \mathbb{R}$ is **Cauchy** if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $n, m \geq N \Rightarrow |x_n - x_m| < \varepsilon$

Point of Clarification. The Wednesday class before the assigment is due we will cover all the material needed for the assignment. Also we can use calculus 2. Alot of this course is in the language of Calculus and Real Analysis

Prop Convergent $\Rightarrow$ Cauchy

Proof: Let $\varepsilon > 0$ be given and suppose (X_n) is convergent. Say $x_n \to x \in \mathbb{R}$. There exist $N \in \mathbb{N}$ $n \geq N$

$\Rightarrow |x_n - x| < \boxed{\frac{\varepsilon}{2}}$ Then for $n, m \geq N$

both are $< \varepsilon$ which gives information on what should be in the box

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \qquad \square$$

take ε divided by # of parts

Fact: $(X_n) \subseteq \mathbb{R}$, (X_n) Cauchy $\Leftrightarrow$ (X_n) Convergent This is also apart of the Completeness of $\mathbb{R}$

Big Idea: To prove (X_n) is Cauchy you do not have to guess the limit! → This will be useful for fixed points

Def'n $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, $a \in A$. we say f is **continuous** at a iff $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - f(a)| < \varepsilon$ whenever $x \in A$ and $|x - a| < \delta$.

→ close values in the domain gives close values in function

Fact: f is continuous at a iff $\forall (x_n) \subseteq A$ (sequence) with $x_n \to a$, we have $f(x_n) \to f(a)$ (function at the terms in the sequence) Provides link between convergence and continuity.

↳ can prove as an excersize

Prop: If $f: [a,b] \to [a,b]$ is continuous then $f(x)$ has a fixed points

→ highly dependent on that f goes from $[a,b] \to [a,b]$

Proof: we know $f(a) \geq a$ and $f(b) \leq b$. $\Leftrightarrow$ $f(a) - a \geq 0$ and $f(b) - b \leq 0$

By appling the IVT to the continuous function $g(x) = f(x) - x$ $\exists x \in [a,b]$ such that $g(x) = 0 \Leftrightarrow f(x) = x$ $\square$

↳ a preview of why Calculus is applicable to this course. Epsilons will come back.

Def'n $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$. we say that f is a **contraction** iff $\exists C \in [0,1)$ s.t. $\forall x, y \in A$, $|f(x) - f(y)| \leq C|x - y|$

is a type of Lipshitz fcn → would that mean fcn. is below line $x = y$ graphically?

Prop Contractions are continuous at every point

Proof Let $\varepsilon > 0$ be given Suppose $|f(x) - f(y)| \leq C|x - y|$ as before. Fix $y \in A$, Consider $\delta = \boxed{\frac{\varepsilon}{C+1}}$ → to make sure we arent dividing by 0 and Assume $x \in A$ s.t. $|x-y| < \delta$. Then

$$|f(x) - f(x)| \leq C|x-y| < C\delta < \varepsilon \qquad \square$$

Sqrt function a problematic, not a contract b/c it drops off a to 0

Defn We say $A \subseteq \mathbb{R}$ is **close** iff whenever $(x_n) \subseteq A$ with $x_n \to x \in \mathbb{R}$ then $x \in A$.

Ex. $[a,b]$ is closed

Ex $(0,1]$ not closed $\frac{1}{n} \to 0 \notin (0,1]$

Thereom [Banach Contraction Mapping theorem] Suppose $A \subseteq \mathbb{R}$ is closed and $f: A \to A$ is a contraction. Then there exist a unique fixed point $a \in A$ for f. Moreover $\forall x \in A$ $f^n(x) \longrightarrow a$

↳ has a cute proof. allegedy. will prove on monday

Ex $f: [0,1] \to [0,1]$ $f(x) = \frac{1}{3-x}$

Note: $\frac{1}{3} \leq f(x) \leq \frac{1}{2}$ → shows it maps back into $[0,1]$

↳ also <u>not</u> an onto function

$$f'(x) = \frac{1}{(3-x)^2}, \qquad \frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$$

By the MVT, $\forall x, y \in [0,1]$ $\exists c \in (0,1)$ s.t. $f(x) - f(y) = f'(c)(x-y)$

$$\Rightarrow |f(x) - f(y)| = |f'(c)|\,|x-y|$$
$$\leq \underbrace{\tfrac{1}{4}}_{C}|x-y|$$

$\therefore$ $f(x)$ is a contraction.

$$\frac{1}{3-x} = x$$

$$\Leftrightarrow \quad 1 = 3x - x^2$$

$$\Leftrightarrow \quad x^2 - 3x + 1 = 0$$

$$\Leftrightarrow \quad x = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\Leftrightarrow \quad x = \frac{3-\sqrt{5}}{2}$$

$\therefore$ $\forall x \in [0,1]$ $f^n(x) \to \frac{3-\sqrt{5}}{2}$

Lec 4 - Jan 15th 2023

Note: Graphical Analysis tool posted on piazza. Assignment 1 due next week.

Remark: The Banach Contraction Mapping is almost like a black hole

Recall: $(a_n) \subseteq \mathbb{R}$ we say (a_n) is **Strongly Cauchy** if $\exists$ $(\varepsilon_n) \subseteq [0, \infty)$ s.t.

① $\sum_{n=1}^{\infty} \varepsilon_n < \infty$

② $\forall n, \ |a_n - a_{n+1}| < \varepsilon_n$

This is on assigment.

Hint: $\sum_{n=1}^{\infty} a_n = L$, $\underbrace{\sum_{k=1}^{n} a_k}_{S_n} \xrightarrow{n\to\infty} L$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $n \geq N \Rightarrow \underbrace{|S_n - L|}_{\left|\sum_{k=n+1}^{\infty} a_k\right|} < \varepsilon$ → the fact you can make up a tail end, a series of them, and adding up is arbitrally small

Proof of Banach Contraction Mapping

Let $A \subseteq \mathbb{R}$ be closed and suppose $\exists C \in [0,1)$ s.t. $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in A$. Take a point $x_0 \in A$

and construct $x_1 = f(x_0)$, $x_2 = f(x_1), \ldots, x_n = f(x_{n-1}) = f^n(x_0)$ } this is the orbit. → we arent including x_0 in the orbit but it doesnt matter

For $n \in \mathbb{N}$ $|x_{n+1} - x_n| \overset{\text{defn of orbit}}{=} |f(x_n) - f(x_{n-1})| \overset{\text{contraction assumption}}{\leq} c|x_n - x_{n-1}| = c|f(x_{n-1}) - f(x_{n-2})| \leq c^2|x_{n-1} - x_{n-2}|$

$\vdots$

$\leq c^n |x_1 - x_0|$, $c \in [0,1)$ → this is geometric

Since $\sum_{n=1}^{\infty} c^n |x_1 - x_0|$ is a convergent geometric series, we have that (x_n) is strongly Cauchy. Hence $x_n \to a$ for some $a \in A$

↳ This is because domain is closed

Since f is continuous, $\underbrace{f(x_n)}_{x_{n+1}} \to f(a)$. $\therefore f(a) = a$ → B/c $x_n \to a$, $f(x_n) \to f(a) \Rightarrow f(a) = a$

→ whats significant about this?

Suppose $a, b \in A$, s.t $f(a) = a$ and $f(b) = b$. Then $|f(a) - f(b)| \leq c|a - b| \Rightarrow |a - b| \leq c|a - b|$. Since $c < 1$,

$|a - b| = 0$ and so $a = b$. This is our strongest fixed point theorem. This is the big analytic results for existence and uniqueness of ODEs

Chapter 2 Graphical Analysis.

to Visualize the orbit of a under f:

① Super impose $y = f(x)$, $y = x$

② Use a vertical line: (a, a) — $(a, f(a))$

③ Use a horizontal line: $(a, f(a))$ — $(f(a), f(a))$

④ Vertical line: $(f(a), f(a))$ — $(f(a), f^2(a))$

⑤ Use a horizontal line: $(f(a), f^2(a))$ — $(f^2(a), f^2(a))$

Etc...

Ex) Using online tool and $f(x) = x^2 - x + 1$, fixed points $x = 1$

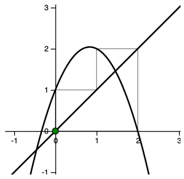
Orbit Analysis:

① $x \in [0,1]$ we have, $f^n(x) \to 1$

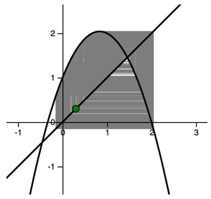
② otherwise, $x \notin [0,1]$ $f^n(x) \to \infty$ In assignment, only expecting this sort of an answer.

Lec 5 - Jan 17th 2024

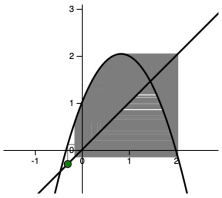
For $f(x) = \frac{-3}{2}x^2 + 5/2\, x + 1$ at $x = 0$ we have an orbit of 0, 1, 2 which is periodic and graphical is a cycle



$x_0 = 0$



$x_0 = 0.3$



$x_0 = -0.3$

points near 0, like 0.06 show chaos. The graphical Analysis look does show chas
Chaos also shows density. We can never hit ever point in the interval ble an orbit is countable but [1, 2] in $\mathbb{R}$ is uncountable. It is dense [ex. rational numbers is dense in $[0,1] \in \mathbb{R}$]
We will define dense in a bit.

Chapter 3 - fixed points

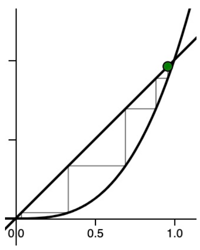
Remark. If f(x) is continuous and $f^n(a) \to L$, then $f^{n+1}(a) \to F(L)$ → comes from continuity. We also saw this in the BCM theorem
$\therefore f(L) = L$

Motivating Example

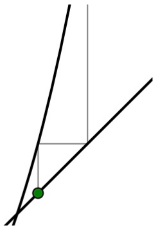
$F(x) = x^3$, fixed points : $0, \pm 1$

① for $x \in (-1, 1)$ we see that $f^n(x) \to 0$. This is an example of an attracting fixed point

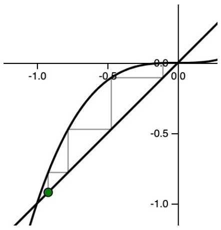
② $x \in (1, \infty)$, $f^n(x) \to \infty$. and $x \in (\infty, -1)$, $f^n(x) \to -\infty$. We call ± 1 repelling fixed points



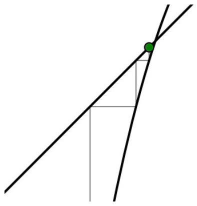
x_0 on left of 1



x_0 on right of 1



x_0 on right of -1



x_0 on left of -1

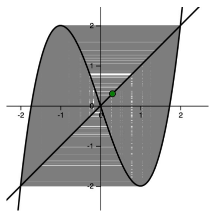
Motivating Example 2

$f(x) = x^3 - 3x$, fixed points : $0, \pm 2$

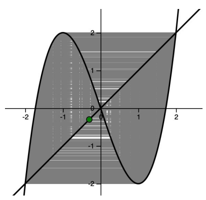
① 0 is repelling [in a different way then last example b/c right & left to 0 is chaotic

② to the right of 2 the orbits go to infinity, left of −2 orbits go to − infinity. so ±2 is also repelling.

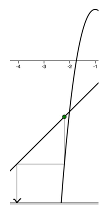
③ at $x_0 = 1$, the orbit is eventually fixed to the fixed point −2; for $x_0 = -1$, the orbit is ev



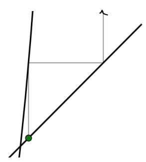
$x_0 = 0.3$



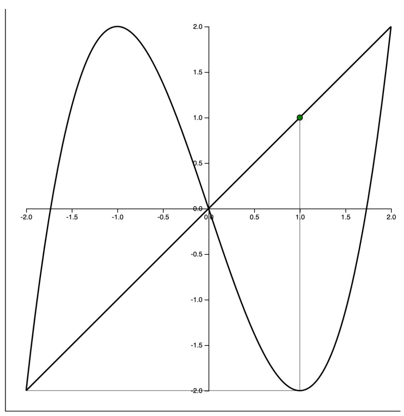
$x_0 = -0.3$



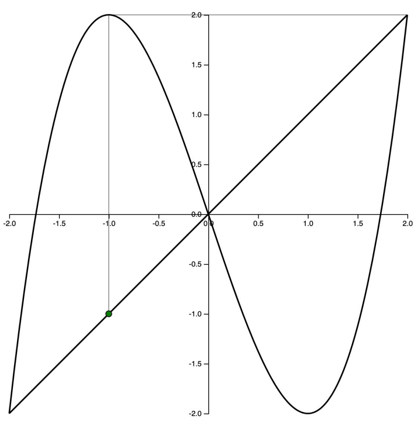
x_0 on left of −2



x_0 on right of 2



orbit of $x_0 = 1$



orbit of $x_0 = -1$

Defn Let $\underline{a}$ be a fixed point of $f(x)$

① if $|f'(\underline{a})| > 1$, we call $\underline{a}$ a **repelling** fixed point.

② if $|f'(\underline{a})| < 1$, we call $\underline{a}$ an **attracting** fixed point

③ if $|f'(a)| = 1$, we call $\underline{a}$ a **neutral** fixed point → this can mean a whole bunch of things

Theorem [Attracting fixed point thm]

Suppose a is an attracting fixed of $f(x)$ then $\exists$ an open interval I such that $a \in I$ and

① $\forall x \in I$, $\forall n \in \mathbb{N}$, $f^n(x) \in I$ → slightly unecessary because if $f(x) \in I \Rightarrow f^n(x) \in I$

② $\forall x \in I$, $f^n(x) \to a$.

Defn: $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$.

① We say $a \in A$ is **non-isolated** iff $\forall \varepsilon > 0$, $\exists b \in A$ with $b \neq a$ s.t $b \in (a-\varepsilon, a+\varepsilon)$

② Let $a \in A$ be non-isolated. We say $\lim_{x \to a} f(x) = L$ iff $\forall \varepsilon > 0$ $\exists \delta > 0$, $|f(x) - L| < \varepsilon$ whenever $a \in A$ and $0 < |x-a| < \delta$

→ if a was an isolated. we could choose a δ where $|x-a| < \delta$ is false. Leading to a false hypothesis which leads to an always true awnser and hence anything could be the limit.

Ex $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\}$ 0 is non isolated

Ex $\{1\} \cup (2,3)$ 1 is isolated

Proof: [Attracting fixed point thm] → to continue in Lec 6

Assume $|f'(a)| < 1$. Then $\exists c \in \mathbb{R}$ s.t. $|f'(a)| < c < 1$

$$\therefore \lim_{x \to a} \frac{|f(x) - f(a)|}{x-a} < c$$

Thus $\exists \delta > 0$ s.t. $\frac{|f(x) - f(a)|}{|x-a|} \leq c$ $\forall x \in (a-\delta, a+\delta)$

Hence, for $x \in (a-\delta, a+\delta)$ $|f(x) - f(a)| \leq c|x-a|$ → contracting

Lec 6th - Jan 19th 2024

First assignment due 11:59 on Tuesday

Theorem [Attracting fixed point thm]

b/c this exists, we can choose a maximal δ such that all orbits that go to a exist in it and no other orbits

Suppose a is an attracting fixed point of $f(x)$. There exists an open interval I with $a \in I$ s.t.

① $\forall x \in I$, $\forall n \in \mathbb{N}$, $f^n(x) \in I$ → Saying $f(x) \in I$ is enough but we are choosing to write this to give a fuller picture

② $\forall x \in I$, $f^n(x) \to a$

a is not a boundary pnt. b/c is differentiable due to defn of attracting

Proof:

we have defined the I that was stated in the thm

means f is a contraction

Say $|f'(a)| < c < 1$ so that $\lim\limits_{x \to a} \frac{|f(x) - f(a)|}{|x-a|} < c$. Thus, $\exists \delta > 0$ s.t. $\forall x \in I := (a-\delta, a+\delta)$, $|f(x) - f(a)| \leq c|x-a|$

In particular, for $x \in I$ $|f(x) - f(a)| \underset{\text{b/c a is a fixed point}}{=} |f(x) - a| \leq \underset{c \in (0,1)}{c} |x-a| \leq |x-a| \underset{\text{b/c } x \in I}{<} \delta \Rightarrow f(x) \in I$

Continuing, for $x \in I$ $|f^n(x) - a| \underset{\text{due to n-interations of contracting}}{\leq} c^n|x-a| \underset{\text{b/c of the case where } x=a}{\leq} |x-a| < \delta$ So that $f^n(x) \in I$. Finally, for $x \in I$ $0 \leq |f^n(x) - a| \leq \underset{c \in (0,1) \Rightarrow c^n \text{ shrinks} \to 0}{c^n} |x-a| \longrightarrow 0$ ■

Can show using ε-δ proof if needed

Note: If the question asks to use a ε-δ proof then do so.

Theorem [repelling fixed point theorem]

Suppose a is a repelling fixed point for $f(x)$. There exist an open interval $a \in I$, s.t. $\forall x \in I$, $x \neq a$, $\exists n \in \mathbb{N}$ s.t. $f^n(x) \notin I$

a repelling point need not shoot of to inifinte, it may be chaotic where for some n, $f^n(x) \notin I$

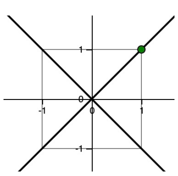
Proof: Say $|f'(a)| > c > 1$. then $\lim\limits_{x \to a} \frac{|f(x) - f(a)|}{|x-a|} > c$ and so $\exists \delta > 0$ s.t. for all $x \in I := (a-\delta, a+\delta)$ $|f(x) - f(a)| \geq c|x-a|$

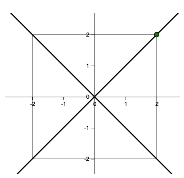
Since a is a fixed point, $|f(x) - f(a)| = |f(x) - a|$. Suppose $\underset{\text{for contradiction}}{\forall n}$, $f^n(x) \in I$. As before $|f^n(x) - a| \geq c^n|x-a| \longrightarrow \infty$ This infinity goes well beyond delta! ■

What is a neutral fixed point? A Lot.

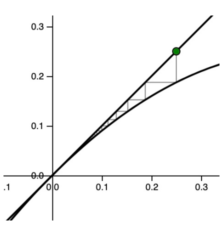
Investigation: Neutral fixed Points

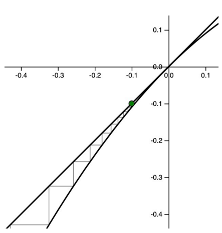
Ex $f(x) = -x$, 0 is a fixed point an $|f'(0)| = 1$. bounces around



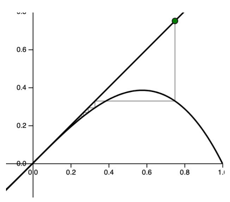


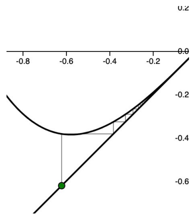
Ex $f(x) = x - x^2$, $|f'(0)| = 1$ attracting to the right and repelling to the left.



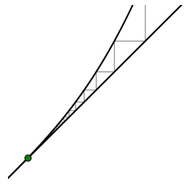


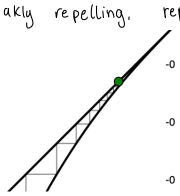
Ex $f(x) = x - x^3$ $|f'(0)| = 1$, this is weakly attracting, attracting but too slowly. happens in assymtope situations





Ex $f(x) = x + x^3$, $|f'(0)| = 1$, weakly repelling. repelling too slow

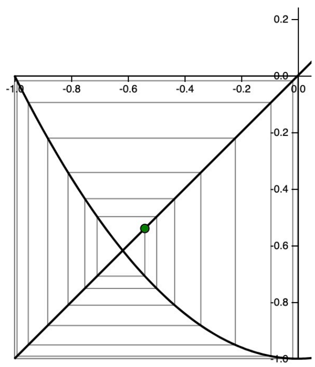


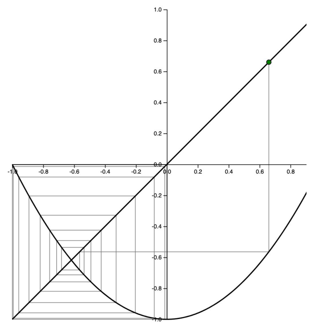


Motivating Example: $f(x) = x^2 - 1$, $a = 0$.

orbit, $(0, -1, 0, -1, 0, -1)$; 0 is a periodic point of period 2. x_0 near 0 shows orbits getting closer to $(0,1)$-cycle.

we will call 0 an attracting periodic point because. 0 is an attracting fixed point of $f^2(x)$.





Lec 7 - Jan 22nd 2024

Defn: Let a be a periodic point for $f(x)$ with period n. We say a is an attracting / repelling / neutral per point for $f(x)$ iff a is an attracting / repelling / neutral fixed point of $f^n(x)$.

for strictly periodic orbits. [no eventually periodic] the points in the periodic orbit are fixed points of $f^n(x)$. Also, every point in the orbit will have the same classification as the point tested. → by proposition value of derivative must be the same for any point in the cycle.

Prop: Let $f(x)$ be a function. Then
↳ $f(x)$ need to be diff'able

$$(f^n)'(a) = f'(a)\, f'(f(a))\, f'(f^2(a)) \dots f'(f^{n-1}(a))$$

Proof: [Induction]

if $n=1$, $f'(a) = f'(a)$

Assume the result holds for some n, $n \geq 1$.

then $\frac{d}{dx} f^{n+1}(x) = \frac{d}{dx} f(f^n(x)) = f'(f^n(x)) \cdot (f^n)'(x)$

plug in a

Then $(f^{n+1})'(a) = \underbrace{f'(f^n(a)) \cdot f'(a) \cdot f'(f(a)) \dots f'(f^{n-1}(a))}_{\text{inductive hypothesis}}$ ■

Ex) $f(x) = \frac{-3}{2}x^2 + \frac{5}{2}x + 1$, $a=0$, orbit $(0, 1, 2, 0, \dots)$

$f'(x) = -3x + 5/2$

$$\begin{aligned}(f^3)'(0) &= f'(0)\, f'(1)\, f'(2) \\ &= \left(\tfrac{-7}{2}\right)\left(\tfrac{-1}{2}\right)\left(\tfrac{5}{2}\right) \\ &= \tfrac{35}{8} > 1 \quad \therefore \text{ repelling}\end{aligned}$$

Chapter 4. Bifurcations

In general Bifurcation theory is the study of how a family of curves can change when a defining parameter is change

We will consider the quadratic family: $Q_c(x) = x^2 + c$ (very important family) Idea: how does c affect the dynamics of $Q_c(x)$?

to find the fixed points of $Q_c(x)$:

$$x^2 + c = x \iff x^2 - x + c = 0$$

$$\iff x = \frac{1 \pm \sqrt{1-4c}}{2}$$

Thus:

① $Q_c(x)$ has 2 fixed points when $c < \frac{1}{4}$

② $Q_c(x)$ has 1 fixed point when $c = \frac{1}{4}$

③ $Q_c(x)$ has no fixed points [real] when $c > \frac{1}{4}$.

If $c > \frac{1}{4}$, $Q_c^n(x)$ must diverge to ∞.

If $c = \frac{1}{4}$ $Q_c(x)$ has the unique fixed point of $p = \frac{1}{2}$. Since $Q_c'(x) = 2x$, $Q_c'(\frac{1}{2}) = 1$ so this fixed point is neutral → attracting to left and repelling to right

If $c < \frac{1}{4}$, $Q_c(x)$ has 2 fixed points:

$$p_+ = \frac{1 + \sqrt{1-4c}}{2}, \quad p_- = \frac{1 - \sqrt{1-4c}}{2}$$

First,

$$Q_c'(p_+) = 1 + \sqrt{1-4c} > 1 \Rightarrow p_+ \text{ is repelling}$$

Next,

$$-1 < Q_c'(p_-) < 1$$

$$\Leftrightarrow -1 < 1 - \sqrt{1-4c} < 1$$

$$\Leftrightarrow -2 < -\sqrt{1-4c} < 0$$

$$\Leftrightarrow 2 > \sqrt{1-4c} > 0$$

$$\Leftrightarrow -\frac{3}{4} < c < \frac{1}{4}$$ → this is the range for attracting fixed point

if $c < \frac{-3}{4}$, $Q_c'(p_-) < -1$

if $c = \frac{-3}{4}$, $Q_c'(p_-) = -1$

Theorem: For the family $Q_c(x) = x^2 + c$:

① All orbits tend to ∞ if $c > \frac{1}{4}$

② when $c = \frac{1}{4}$ $Q_c(x)$ has a unique fixed point, $\frac{1}{2}$, and it is neutral

③ if $c < \frac{1}{4}$ $Q_c(x)$ has 2 fixed points p_+ and p_-. The point p_+ is repelling

Moreover

a) If $-\frac{3}{4} < c < \frac{1}{4}$, p_- is attracting

b) if $c = \frac{-3}{4}$, p_- is neutral

c) if $c < \frac{-3}{4}$, p_- is repelling

Lec 8 - Jan 24 2024

Def'n: We say a family of Functions $F_\lambda(x)$ undergoes a **bifurcation** at λ_0 if there is a change in fixed point structure at λ_0

Ex) $Q_c(x) = x^2 + c$, $\lambda_0 = \frac{1}{4}$

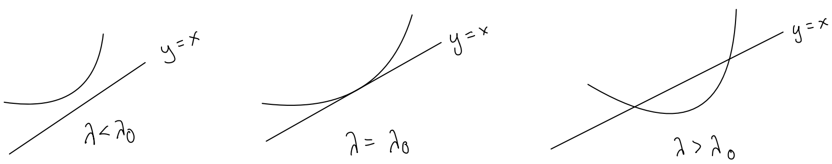
Def'n: A family $F_\lambda(x)$ undergoes a **tangent bifurcation** at λ_0 if there is an open interval I and $\varepsilon > 0$ s.t

① For $\lambda_0 - \varepsilon < \lambda < \lambda_0$, $F_\lambda(x)$ has no fixed points on I

② For $\lambda = \lambda_0$, $F_\lambda(x)$ has one fixed point and it is neutral

③ For $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, $F_\lambda(x)$ has 2 fixed points in I, one attracting and the other repelling

or ①, ②, ③ with $>$.



Ex) $E_\lambda(x) = e^x + \lambda$, $\lambda_0 = -1$. This is an example of a tangent bifurcation.

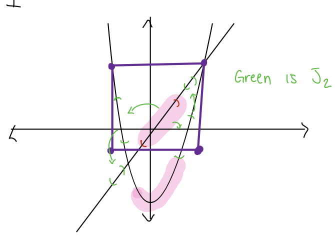
Ex) $F_\lambda(x) = \lambda x(1-x)$, $\lambda_0 = 1$. Not tangent bifurcation.

Chapter 5 Cantor Set

Investigation: $Q_c(x) = x^2 + c$, $c < -2$.

$$p_+ = \frac{1 + \sqrt{1-4c}}{2} > 2, \quad -p_+ < -2$$

Consider $I = [-p_+, p_+]$ and $I \times I$



J_2 is asking which x has y values in J_1 which is how these intervals are

picture not to scale

Let $J_1 \subseteq I$ be the Interval s.t. $Q_c(x) \notin I$ for all $x \in J_1$

For $x \in J_1$, $Q_c^n(x) \longrightarrow \infty$. Moreover, If $\exists n$ s.t. $Q_c^n(x) \in J_1$ then $Q_c^n(x) \to \infty$

Consider $\Lambda = \{x \in I : Q_c^n(x) \in I \ \forall n\}$

Big Idea: Λ contains all the points with interesting orbits.

$J_1 = \{ x \in I : Q_c(x) \notin I \}$

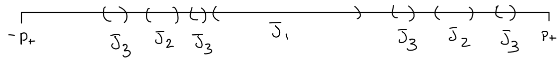
$J_2 = \{ x \in I : Q_c(x) \in J_1 \} = \{ x \in I : Q_c^2(x) \notin I \}$

$J_3, J_4, J_5 \ldots$

what is a cantor set?

Then, $\Lambda = I \setminus (J_1 \cup J_2 \cup \ldots)$ is a cantor set. Drawing Λ on the x-axis:

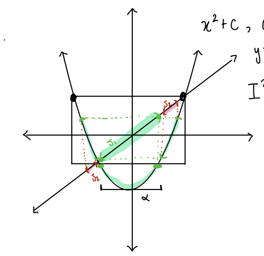
Symmetry comes from the curve (parabola)



etc.

It is a fractal b/c of its self repeating nature!

Lec 9 - Jan 26th 2023



$x^2+c,\ c<-2$

$y=x$

$I^2 = [-p_r, p_+]^2$

One quarter mark for the course.

$\Lambda = I \setminus (J_1 \cup J_2 \cup \dots)$ $\qquad A = \{x \in I : \forall n,\ Q_c^n(x) \in I\}$

↳ these are all the interesting orbits

↳ cantor set

Construction: Cantor Middle thirds set

$C_0 = [0,1]$ → remove open middle third Interval

$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

$C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$

$\vdots$

these are nested

$K = \bigcap_{n=1}^{\infty} C_n$ → Cantor [middle - third] set. A cantor set is taking out a chunk of the interval, infinitely. We can see that.

the end points will always survive.

Proposition: $A_n \subseteq \mathbb{R}$ closed. Then $\bigcap_{n=1}^{\infty} A_n$ is closed.

Why? $(a_k) \subseteq \bigcap A_n$, $a_k \longrightarrow a$. $\forall n$, $(a_k) \subseteq A_n \Rightarrow a \in A_n \Rightarrow a \in \bigcap A_n$

Proposition: $A, B \subseteq \mathbb{R}$ closed. Then $A \cup B$ is closed.

Why? $(a_n) \subseteq A \cup B$, $a_n \to a$. WLOG. $\{n : a_n \in A\}$ is infinite. This allows us to construct $(b_n) \subseteq A$ s.t. $b_n \to a$. Since A is closed, $a \in A \subseteq A \cup B$.

Theorem K is closed [in fact all Cantor Sets are] → combine the 2 propositions above to get this

Theorem: K contains no non-empty open intervals → Hausdorff property of $\mathbb{R}$?

no isolated points

interior points?

Why? $I \subseteq K \Rightarrow \forall n,\ I \subseteq C_n \Rightarrow \forall n\ \ \ell(I) \le \frac{1}{3^n} \Rightarrow \ell = (I) = 0 \Rightarrow I = \emptyset$ which is a Contradiction.

↳ length of interval.

$s_1 \cdot \frac{1}{3}$ $\quad s_2 \cdot \frac{1}{9}$ $\quad s_3 \cdot \frac{1}{27}$

Lets consider the base -3 expansion of $x \in [0,1]$. $x = 0.s_1 s_2 s_3 \dots,$ $\quad s_i \in [0,1,2]$

$s_1 = 0$ $\qquad s_1 = 1$ $\qquad s_1 = 2$

$[0, \frac{1}{3}]$ $\qquad\qquad [\frac{2}{3}, 1]$

$s_1 = 0, s_2 = 0$ $\quad s_1 = 0, s_2 = 1$ $\quad s_1 = 0, s_2 = 2$ $\quad$ etc..

$[0, \frac{1}{9}]$ $\quad [\frac{2}{9}, \frac{1}{3}]$ $\qquad [\frac{2}{3}, \frac{7}{9}]$ $\quad [\frac{8}{9}, 1]$

Fact:

$x \in K$ iff x can be written in base-3 using only 0's and 2's

Ex $\frac{1}{3} \in K$. $\frac{1}{3} = [0.1]_3 = [0.022222\ldots]_3$ → # theory? base α?

Theorem: K is uncountable $|K| = |\mathbb{R}|$ → cardinality → not important to this course, just for interest.

How to remember? 0 means left, 2 means right.

Start Ch 6 on monday.