Highlighting Legend

Chapter

Definition

Propositions / Theorems / Proofs

Procedures

Remarks are made in red pencil, same with questions

Lecture 1 – Jan 8th 2024

Chapter 1 Iteration and Orbits

allows for indefinite iteration

Defn Let $f: A \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$ [I.E $f: A \to A$]. For $a \in A$, we may iterate the function at a:

$$x_1 = a$$
$$x_2 = f(a)$$
$$x_3 = f(f(a)) = f^2(a)$$
$$x_4 = f(f(f(a))) = f^3(a)$$
$$\vdots$$

all will exist in A.

We call $(x_n)_{n=1}^{\infty} = (= (x_n))$ we call that sequence the orbit of a under f.

Ex. $f(x) = x^4 + 2x^2 - 2$, $a = -1$

$x_1 = a$, $x_2 = f(a)$, $x_3 = f^2(a)$ ...
$-1, 1, 1, \ldots$ → eventually constant / periodic

Ex $f(x)$ $-x^2 - x + 1$, $a = 0$

$x_1, x_2, x_3, x_4, x_5, x_6 \ldots$
$0, 1, -1, 1, -1, 1 \ldots$ → eventually perodic w period 2

Ex. $f(x) = x^2 - 3x + 1$, $a = 1$

$x_1, x_2, x_3, x_4, \ldots$
$1, -1, 3, 19 \to \infty$ goes to infinity [he will talk about convergence / divergence later on]

Ex $f(x) = x^2 + 2x$, $a = -0.5$

x_1, x_2, x_3, x_4
$-0.5, -0.75, -0.9375, -0.9961$, converges to -1

My Q: Does it ever equal 1, is that a distinction?

Ex. $f(x) = x^3 - 3x$, $a = 0.75$

$x_1, x_2, x_3, x_4, x_5, \ldots$
$0.75, -1.828, -0.625, 1.631, -0.552, \ldots$ This is chaotic behaviour.

* For PMATH insights: an Interval around zero is dense *

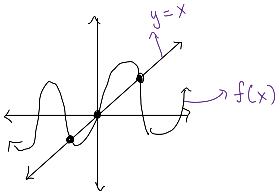
Defn: $f: A \to \mathbb{R}$, $f(A) \subseteq A$, We say $a \in A$ is a fixed point in f iff $f(a) = a$. In this case, the orbit of a is $a, a, a \ldots$ which is constant.

Ex. Find all fixed points of $f(x) = x^2 + x - 4$

When does $x^2 + x - 4 = x$?

$\Leftrightarrow$ $x^2 - 4 = 0$ $\Leftrightarrow$ $x = \pm 2$ Will often give low degree polynomial

Ex. How many fixed points does $f(x)$ as pictured below have?



Geometrically, a fixed point occurs when $f(x)$ intersects $y = x$

Ex. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

Solve for $x^4 - 3x + 1 = x$

$\Rightarrow x^4 - 4x + 1 = 0$

Intermediate Value theorem

Since $g(x)$ is continuous $g(0) = 1 > 0$ and $g(1) = -2 < 0$. By IVT $\exists\, x \in (0,1)$ s.t. $g(x) = 0 \Leftrightarrow f(x) = x$.

Defn $f: A \to \mathbb{R}$, $f(A) \subseteq A$

① We say $a \in A$ is a periodic point for f if its orbit is periodic. I.E $\exists n \in \mathbb{N}$ $f^n(a) = a$. The least such n is called the period of a and/or the orbit.

② Eventually periodic $\exists n < m$, $f^n(a) = f^m(a)$

Lecture 2 – Jan 10th 2024

Def'n [Doubling Function]

$D: [0,1) \to [0,1)$ where $D(x) =$ fractional part of $2x$ aka $2x \bmod 1$

Ex) $D(0.4) = 0.8$
$D(0.6) = 0.2$
$D(0.8) = 0.6$
$D(0.5) = 0$

It is an important function as it also provides a rich source of periodic orbits

Ex) D, $a = \frac{1}{5}$
Orbit: $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \ldots$ This has period 4. This leads into some cute number theory with GCDs

Ex) D, $a = \frac{1}{20}$
Orbit $\frac{1}{20}, \frac{1}{10}, \frac{1}{5} \ldots$ This is eventually periodic with period 4.

$\boxed{\text{The Doubling function will come up later in a more meaningful way}}$

Every day functions may not exhibit periodic orbits

Q: Given f and a, does $f^n(a)$ tend towards some limit L? This does happen suprisingly often

$\boxed{\text{The language of this course vs Elementary Real Analysis}}$

Notation:

If $(x_n)_{n=1}^{\infty}$ is a sequence of real number we write $(X_n) \subseteq \mathbb{R}$ a small abuse of notation but reasonable

Def'n $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$ we say (x_n) **converges** to x iff for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ for all $n \geq N$

* N depends on ε *

We write $x_n \to x$ or $\lim x_n = x$

Ex) Claim: $\frac{1}{n} \to 0$, Let $\varepsilon > 0$ be given.

Note: $\frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$

not necessarily a Natural number. Can always use ceiling function $\lceil \ \rceil$ to bring this to a natural number

Consider $N = \frac{2}{\varepsilon}$ For $n \geq N$ we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$

bring this to $2/\varepsilon$ so that we ensure strictly less than epsilon.

Ex) Claim $\frac{2n}{n+3} \to 2$

Let $\varepsilon > 0$ be given. Let us choose $N \in \mathbb{N}$ s.t.

$$\boxed{\frac{1}{N} < \frac{\varepsilon}{6}}$$

For $n \geq N$ $\left| \frac{2n}{n+3} - 2 \right|$

$$= \left| \frac{2n}{n+3} - \frac{2n+6}{n+3} \right|$$

$$= \left| \frac{-6}{n+3} \right| = \frac{6}{n+3} < \frac{6}{n}$$

if you divide by less of something you get something bigger

$$\frac{6}{n} \leq \frac{6}{N} = 6\left(\frac{1}{N}\right) < 6\left(\frac{\varepsilon}{6}\right) = \varepsilon$$

Defn $(x_n) \subseteq \mathbb{R}$ we say (x_n) is **bounded** if $\exists M > 0$ s.t. $\forall n \in \mathbb{N}, |x_n| \leq M$

Prop $(x_n) \subseteq \mathbb{R}$ If (x_n) is convergent, then (x_n) is bounded.

Ex) $x_n = (-1)^n$ shows converse is not true

picked ε, blake's favourite is 1

Proof Suppose $x_n \to x$. then $\exists N \in \mathbb{N}$, $n \geq N \Rightarrow |x_n - x| < 1$

For $n \geq N$,

this is the reverse triangle inequality, make sure you know how to prove from triangle inequality

$$|x_n| - |x| \leq |x_n - x| < 1 \Rightarrow |x_n| < 1 + |x| \Big\}$$ This is only true for $n \geq N$

So let $M = \max\{|x_1|, \ldots, |x_{n-1}|, 1 + |x|\}$ ∎

Prop Let $x_n \to x$, $y_n \to y$

Not super important to us, value is in working with defn of convergence

① $x_n + y_n \to x + y$

② $x_n y_n \to xy$

Proof:

① Let $\varepsilon > 0$ be given. There exists $N_1, N_2 \in \mathbb{N}$ s.t.

$$n \geq N_1 \Rightarrow |x_n - x| < \boxed{\frac{\varepsilon}{2}}$$

$$n \geq N_2 \Rightarrow |y_n - y| < \boxed{\frac{\varepsilon}{2}}$$

For $N = \max\{N_1, N_2\}$ and $n \geq N$

$$\left| x_n + y_n - (x+y) \right| = \left| x_n - x + y_n - y \right| \leq \left| x_n - x \right| + \left| y_n - y \right| < \varepsilon$$

(2) Let $\varepsilon > 0$ be given

Note:

$$\left| x_n y_n - xy \right| = \left| x_n y - x_n y + x_n y - xy \right| \leq \left| x_n \right| \cdot \left| y_n - y \right| + \left| y \right| \cdot \left| x_n - y \right| \quad ✡$$

Since (x_n) is bounded $\exists M > 0, \forall n, \left| x_n \right| < M$

Let $N_1, N_2 \in \mathbb{N}$ s.t. $n \geq N_1 \Rightarrow \left| x_n - x \right| < \dfrac{\varepsilon}{2(|y|+1)}$ → make sure not dividing by zero

$n \geq N_2 \Rightarrow \left| y_n - y \right| < \dfrac{\varepsilon}{2M}$

For $n \geq N := \max\{N_1, N_2\}$ we have $\left| x_n y_n - xy \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by ✡ ∎

Lec 3 - Jan 12th 2024

Def'n we say $(X_n) \subseteq \mathbb{R}$ is **Cauchy** if $\forall \varepsilon > 0, \exists N \in \mathbb{N}, n, m \geq N \Rightarrow |X_n - X_m| < \varepsilon$

Point of Clarification. The Wednesday class before the assigment is due We will cover all the material needed for the assignment. Also we can use calculus 2. Alot of this course is in the language of Calculus and Real Analysis

Prop Convergent $\Rightarrow$ Cauchy

Proof: Let $\varepsilon > 0$ be given and suppose (X_n) is convergent. Say $x_n \to x \in \mathbb{R}$. There exist $N \in \mathbb{N}$ $n \geq N$

$\Rightarrow |x_n - x| < \boxed{\frac{\varepsilon}{2}}$ Then for $n, m \geq N$

bothare $< \varepsilon$ which gives information on what should be in the box

$$|x_n - x_m| = |X_n - X + X - X_m| \leq |X_n - X| + |X - X_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \qquad \square$$

take ε divided by # of parts

Fact: $(X_n) \subseteq \mathbb{R}$, (X_n) Cauchy $\Leftrightarrow$ (X_n) Convergent This is also apart of the Completeness of $\mathbb{R}$

Big Idea: To prove (X_n) is Cauchy you do not have to guess the limit! This will be useful for fixed points

Def'n $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, $a \in A$. we say f is **Continuous** at a iff $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|F(x) - F(a)| < \varepsilon$ whenever $x \in A$ and $|x - a| < \delta$.

→ close values in the domain gives close values in function

Fact: f is continuous at a iff $\forall (x_n) \subseteq A$ (sequence) with $x_n \to a$, we have $f(x_n) \to f(a)$ (function at the terms in the sequence) Provides link between convergence and Continuity.

↳ can prove as an excersize

highly dependent on that f goes from $[a,b] \to [a,b]$

Prop: If $f: [a,b] \to [a,b]$ is Continuous then $f(x)$ has a fixed points

Proof: we know $f(a) \geq a$ and $f(b) \leq b$. $\Leftrightarrow$ $f(a) - a \geq 0$ and $f(b) - b \leq 0$

By appling the IVT to the continuous function $g(x) = f(x) - x$ $\exists x \in [a,b]$ such that $g(x) = 0 \Leftrightarrow f(x) = x$ $\square$

↳ a preview of why Calculus is applicable to this course. Epsilons will come back.

is a typeof Lipshitz fcn → would that mean fcn. is below line $x = y$ graphically?

Def'n $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$. We say that f is a **contraction** iff $\exists C \in [0,1)$ s.t. $\forall x, y \in A$, $|f(x) - f(y)| \leq C|x - y|$

Prop Contractions are Continuous at every point

Proof Let $\varepsilon > 0$ be given Suppose $|f(x) - f(y)| \leq C|x - y|$ as before. Fix $y \in A$, Consider $\delta = \boxed{\frac{\varepsilon}{C+1}}$ to make sure we arent dividing by 0 and Assume $x \in A$ s.t. $|x-y| < \delta$. Then

$$|f(x) - f(x)| \leq C|x-y| < C\delta < \varepsilon \quad \square$$

Sqrt function a problematic, not a contract b/c it drops off at 0

Defn We say $A \subseteq \mathbb{R}$ is **close** iff whenever $(X_n) \subseteq A$ with $x_n \to x \in \mathbb{R}$ then $x \in A$.

Ex. $[a,b]$ is closed

Ex $(0,1]$ not closed $\frac{1}{n} \to 0 \notin (0,1]$

Thereom [Banach Contraction Mapping theorem] Suppose $A \subseteq \mathbb{R}$ is closed and $f: A \to A$ is a contraction. Then there exist a unique fixed point $a \in A$ for f. Moreover $\forall x \in A$ $f^n(x) \longrightarrow a$

↪ has a cute proof. allegedy. will prove on monday

Ex $f: [0,1] \to [0,1]$ $f(x) = \frac{1}{3-x}$

Note: $\frac{1}{3} \leq f(x) \leq \frac{1}{2}$ → shows it maps back into $[0,1]$

↪ also not an onto function

$$f'(x) = \frac{1}{(3-x)^2}, \qquad \frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$$

By the MVT, $\forall x,y \in [0,1]$ $\exists c \in (0,1)$ s.t. $f(x) - f(y) = f'(c)(x-y)$

$$\Rightarrow |f(x) - f(y)| = |f'(c)|\,|x-y|$$
$$\leq \underbrace{\tfrac{1}{4}}_{C}|x-y|$$

$\therefore$ $f(x)$ is a contraction.

$$\frac{1}{3-x} = x$$

$$\Leftrightarrow \quad 1 = 3x - x^2$$

$$\Leftrightarrow \quad x^2 - 3x + 1 = 0$$

$$\Leftrightarrow \quad x = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\Leftrightarrow \quad x = \frac{3-\sqrt{5}}{2}$$

$\therefore$ $\forall x \in [0,1]$ $f^n(x) \to \frac{3-\sqrt{5}}{2}$

Lec 4 - Jan 15th 2023

Note: Graphical Analysis tool posted on piazza. Assignment 1 due next week.

Remark: The Banach Contraction Mapping is almost like a black hole

Recall: $(a_n) \subseteq \mathbb{R}$ we say (a_n) is **strongly Cauchy** if $\exists\ (\varepsilon_n) \subseteq [0,\infty)$ s.t.

1. $\sum_{n=1}^{\infty} \varepsilon_n < \infty$
2. $\forall n,\ |a_n - a_{n+1}| < \varepsilon_n$

This is on assigment.

Hint: $\sum_{n=1}^{\infty} a_n = L$, $\underbrace{\sum_{k=1}^{n} a_k}_{S_n} \xrightarrow{n\to\infty} L$

$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ s.t $n \geq N \Rightarrow \underbrace{|S_n - L|}_{\left|\sum_{k=n+1}^{\infty} a_k\right|} < \varepsilon$ → the fact you can make up a tail end, a series of them, and adding up is arbitrally small

Proof of Banach Contraction Mapping

Let $A \subseteq \mathbb{R}$ be closed and suppose $\exists C \in [0,1)$ s.t. $|f(x) - f(y)| \leq c|x-y|$ for all $x, y \in A$. Take a point $x_0 \in A$

and construct $x_1 = f(x_0)$, $x_2 = f(x_1), \dots, x_n = f(x_{n-1}) = f^n(x_0)$ } this is the orbit. → we arent including x_0 in the orbit but it doesnt matter

For $n \in \mathbb{N}$ $|x_{n+1} - x_n| \overset{\text{defn of orbit}}{=} |f(x_n) - f(x_{n-1})| \overset{\text{contraction assumption}}{\leq} c|x_n - x_{n-1}| = c|f(x_{n-1}) - f(x_{n-2})| \leq c^2|x_{n-1} - x_{n-2}|$

$\vdots$

$\leq c^n |x_1 - x_0|$ → $c \in [0,1)$ → this is geometric

Since $\sum_{n=1}^{\infty} c^n |x_1 - x_0|$ is a convergent geometric series, we have that (x_n) is strongly Cauchy. Hence $x_n \to a$ for some $a \in A$ → This is because domain is closed

Since f is continuous, $\underbrace{f(x_n)}_{x_{n+1}} \to f(a)$. $\therefore f(a) = a$ → B/c $x_n \to a$, $f(x_n) \to f(a) \Rightarrow f(a) = a$ → whats significant about this?

Suppose $a, b \in A$, s.t $f(a) = a$ and $f(b) = b$. Then $|f(a) - f(b)| \leq c|a-b| \Rightarrow |a-b| \leq c|a-b|$. Since $c<1$,

$|a-b| = 0$ and so $a = b$. This is our strongest fixed point theorem. This is the big analytic results for existence and uniqueness of ODEs

Chapter 2 Graphical Analysis

to Visualize the orbit of a under f:

1. Super impose $y = f(x)$, $y = x$

② Use a vertical line: (a, a) — $(a, f(a))$

③ Use a horizontal line: $(a, f(a))$ — $(f(a), f(a))$

④ Vertical line: $(f(a), f(a))$ — $(f(a), f^2(a))$

⑤ Use a horizontal line: $(f(a), f^2(a))$ — $(f^2(a), f^2(a))$

Etc...

Ex) Using online tool and $f(x) = x^2 - x + 1$, fixed points $x = 1$

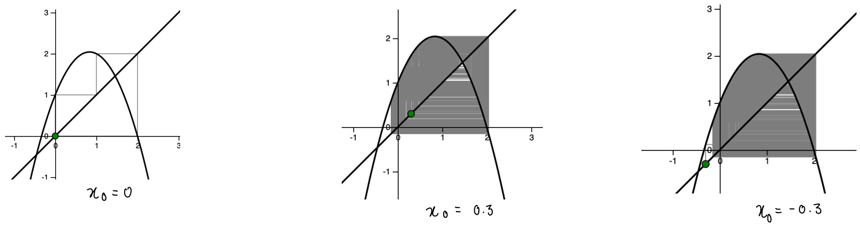
Orbit Analysis:

① $x \in [0,1]$ we have, $f^n(x) \to 1$

② otherwise, $x \notin [0,1]$ $f^n(x) \to \infty$ In assignment, only expecting this sort of an answer.

Lec 5 - Jan 17th 2024

For $f(x) = \frac{-3}{2}x^2 + 5/2\, x + 1$ at $x = 0$ we have an orbit of $0, 1, 2$ which is periodic and graphical is a cycle



points near 0, like 0.06 show chaos. The graphical Analysis look does show chas
Chaos also shows density. We can never hit ever point in the interval ble an orbit is countable but $[1, 2]$ in $\mathbb{R}$ is uncountable. It is dense [ex. rational numbers is dense in $[0, 1] \in \mathbb{R}$]
We will define dense in a bit.

Chapter 3 - fixed points

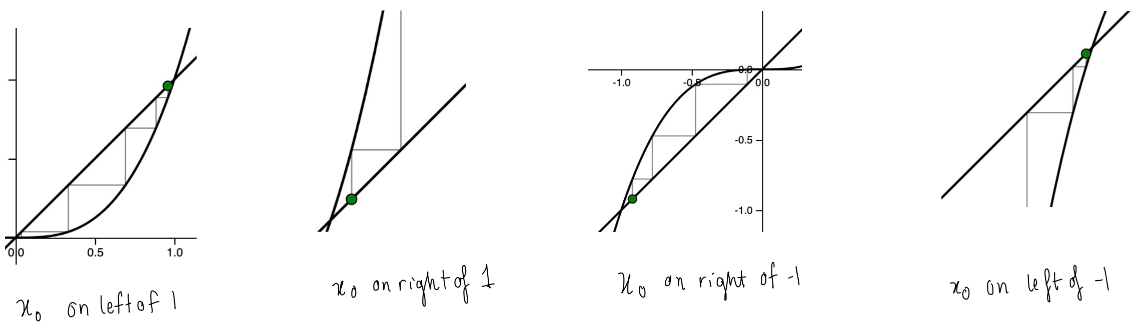
Remark. If $f(x)$ is continuous and $f^n(a) \to L$, then $f^{n+1}(a) \to F(L)$ → comes from continuity. We also saw this in the BCM theorem
$\therefore f(L) = L$

Motivating Example

$F(x) = x^3$, fixed points : $0, \pm 1$

① for $x \in (-1, 1)$ we see that $f^n(x) \to 0$. This is an example of an attracting fixed point

② $x \in (1, \infty)$, $f^n(x) \to \infty$. and $x \in (\infty, -1)$, $f^n(x) \to -\infty$. We call ± 1 repelling fixed points



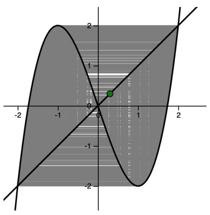
Motivating Example 2

$f(x) = x^3 - 3x$, fixed points : $0, \pm 2$

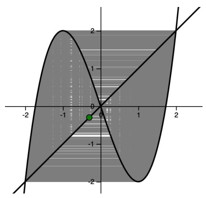
① 0 is repelling [in a different way then last example b/c right & left to 0 is chaotic

② to the right of 2 the orbits go to infinity, left of -2 orbits go to $-$infinity. So ± 2 is also repelling.

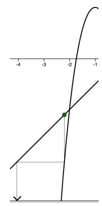
③ at $x_0 = 1$, the orbit is eventually fixed to the fixed point -2; for $x_0 = -1$, the orbit is ev



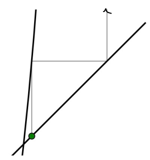
$x_0 = 0.3$



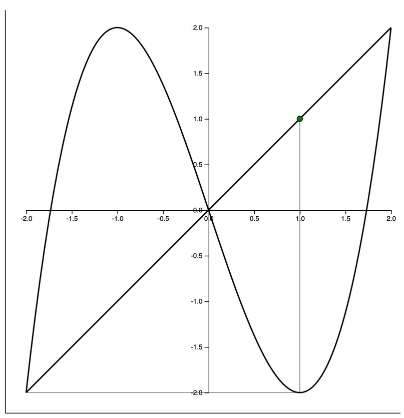
$x_0 = -0.3$



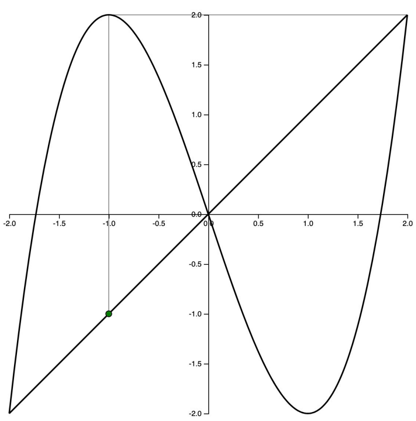
x_0 on left of -2



x_0 on right of 2



orbit of $x_0 = 1$



orbit of $x_0 = -1$

Defn Let $\underline{a}$ be a fixed point of $f(x)$

① if $|f'(\underline{a})| > 1$, we call $\underline{a}$ a repelling fixed point.

② if $|f'(\underline{a})| < 1$, we call $\underline{a}$ an attracting fixed point

③ if $|f'(\underline{a})| = 1$, we call $\underline{a}$ a neutral fixed point $\rightarrow$ this can mean a whole bunch of things

Theorem [Attracting fixed point thm]

Suppose a is an attracting fixed of $f(x)$ then $\exists$ an open interval I such that $a \in I$ and

① $\forall x \in I$, $\forall n \in \mathbb{N}$, $f^n(x) \in I$ $\rightarrow$ slightly unecessary because if $f(x) \in I \Rightarrow f^n(x) \in I$

② $\forall x \in I$, $f^n(x) \rightarrow a$.

Defn: $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$.

① We say $a \in A$ is **non-isolated** iff $\forall \varepsilon > 0$, $\exists b \in A$ with $b \neq a$ s.t $b \in (a-\varepsilon, a+\varepsilon)$

② Let $a \in A$ be non-isolated. We say $\boxed{\lim_{x \to a} f(x) = L}$ iff $\forall \varepsilon > 0$ $\exists \delta > 0$, $|f(x) - L| < \varepsilon$ whenever $a \in A$ and $0 < |x-a| < \delta$

→ if a was an isolated. we could choose a δ where $|x-a| < \delta$ is false. Leading to a false hypothesis which leads to an always true awnser and hence anything could be the limit.

Ex $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\}$ 0 is non isolated

Ex $\{1\} \cup (2,3)$ 1 is isolated

Proof: [Attracting fixed point thm] → to continue in Lec 6

Assume $|f'(a)| < 1$. Then $\exists c \in \mathbb{R}$ s.t. $|f'(a)| < c < 1$

$$\therefore \lim_{x \to a} \frac{|f(x) - f(a)|}{x-a} < c$$

Thus $\exists \delta > 0$ s.t. $\dfrac{|f(x) - f(a)|}{|x-a|} \leq c$ $\forall x \in (a-\delta, a+\delta)$

Hence, for $x \in (a-\delta, a+\delta)$ $|f(x) - f(a)| \leq c|x-a|$ → contracting

Lec 6th - Jan 19th 2024

First assignment due 11:59 on Tuesday

Theorem [Attracting fixed point thm]

b/c this exists, we can choose a maximal δ such that all orbits that go to a exist in it and no other orbits

Suppose a is an attracting fixed point of $f(x)$. There exists an open interval I with $a \in I$ s.t.

① $\forall x \in I$, $\forall n \in \mathbb{N}$, $f^n(x) \in I$ → Saying $f(x) \in I$ is enough but we are choosing to write this to give a fuller picture

② $\forall x \in I$, $f^n(x) \to a$

a is not a boundary pnt. b/c is differentiable due to defn of attracting

Proof: Say $|f'(a)| < c < 1$ so that $\lim_{x \to a} \frac{|f(x) - f(a)|}{|x-a|} < C$. Thus, $\exists \delta > 0$ s.t $\forall x \in I := (a-\delta, a+\delta)$, $|f(x) - f(a)| \leq C|x-a|$

(we have defined the I that was stated in the thm) (means f is a contraction)

In particular, for $x \in I$ $|f(x) - f(a)| = |f(x) - a| \leq C|x-a| \leq |x-a| < \delta \Rightarrow f(x) \in I$

(b/c a is a fixed point) ($C \in (0,1)$) (b/c $x \in I$)

Continuing, for $x \in I$ $|f^n(x) - a| \leq c^n|x-a| \leq |x-a| < \delta$ So that $f^n(x) \in I$. Finally, for $x \in I$ $0 \leq |f^n(x) - a| \leq C^n|x-a| \longrightarrow 0$ ■

(due to n-interations of contracting) ($\leq$ b/c of the case where $x = a$) ($C \in (0,1) \Rightarrow C^n$ shrinks $\to 0$) (Can show using ε-δ proof if needed)

Note: If the question asks to use a ε-δ proof then do so.

Theorem [repelling fixed point theorem]

Suppose a is a repelling fixed point for $f(x)$. There exist an open interval $a \in I$, s.t. $\forall x \in I$, $x \neq a$, $\exists n \in \mathbb{N}$ s.t. $f^n(x) \notin I$

a repelling point need not shoot of to inifinte, it may be chaotic where for some n, $f^n(x) \notin I$

Proof: Say $|f'(a)| > c > 1$. then $\lim_{x \to a} \frac{|f(x) - f(a)|}{|x-a|} > C$ and so $\exists \delta > 0$ s.t. for all $x \in I := (a-\delta, a+\delta)$ $|f(x) - f(a)| \geq C|x-a|$

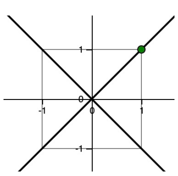
Since a is a fixed point, $|f(x) - f(a)| = |f(x) - a|$. Suppose $\forall n$, $f^n(x) \in I$. As before $|f^n(x) - a| \geq c^n|x-a| \longrightarrow \infty$ This infinity goes well beyond delta! ■

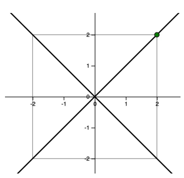
(for contradiction)

What is a neutral fixed point? A Lot.

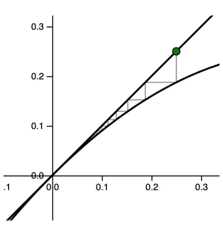
Investigation: Neutral fixed Points

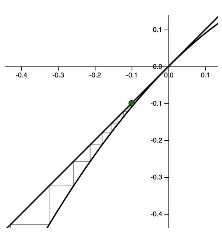
Ex $f(x) = -x$, 0 is a fixed point an $|f'(0)| = 1$. bounces around



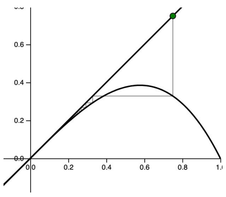


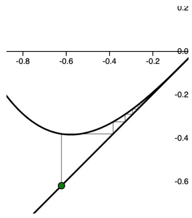
Ex $f(x) = x - x^2$, $|f'(0)| = 1$ attracting to the right and repelling to the left.



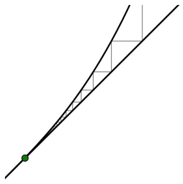


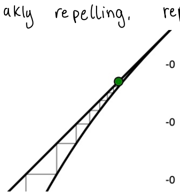
Ex $f(x) = x - x^3$ $|f'(0)| = 1$, this is weakly attracting, attracting but too slowly. happens in assymtope situations





Ex $f(x) = x + x^3$, $|f'(0)| = 1$, weakly repelling. repelling too slow

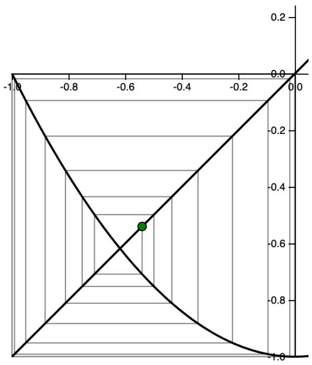


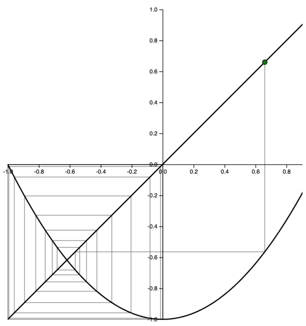


Motivating Example: $f(x) = x^2 - 1$, $a = 0$.

orbit, $(0, -1, 0, -1, 0, -1)$; 0 is a periodic point of period 2. x_0 near 0 shows orbits getting closer to $(0,1)$-cycle.

we will call 0 an attracting periodic point because. 0 is an attracting fixed point of $f^2(x)$.





Lec 7 - Jan 22nd 2024

Defn: Let a be a periodic point for $f(x)$ with period n. We say a is an attracting / repelling / neutral per point for $f(x)$ iff a is an attracting / repelling / neutral fixed point of $f^n(x)$.

for strictly periodic orbits. [no eventually periodic] the points in the periodic orbit are fixed points of $f^n(x)$. Also, every point in the orbit will have the same classification as the point tested. → by proposition value of derivative must be the same for any point in the cycle.

Prop: Let $f(x)$ be a function. Then
↳ $f(x)$ need to be diff'able

$$(f^n)'(a) = f'(a)\, f'(f(a))\, f'(f^2(a)) \dots f'(f^{n-1}(a))$$

Proof: [Induction]

if $n=1$, $f'(a) = f'(a)$

Assume the result holds for some n, $n \geq 1$.

then $\frac{d}{dx} f^{n+1}(x) = \frac{d}{dx} f(f^n(x)) = f'(f^n(x)) \cdot (f^n)'(x)$

↱ plug in a

Then $(f^{n+1})'(a) = \underbrace{f'(f^n(a)) \cdot f'(a) \cdot f'(f(a)) \dots f'(f^{n-1}(a))}_{\text{inductive hypothesis}}$ ■

Ex) $f(x) = \frac{-3}{2}x^2 + \frac{5}{2}x + 1$, $a=0$, orbit $(0, 1, 2, 0, \dots)$

$f'(x) = -3x + 5/2$

$$\begin{aligned}(f^3)'(0) &= f'(0)\, f'(1)\, f'(2) \\ &= \left(-\tfrac{7}{2}\right)\left(\tfrac{-1}{2}\right)\left(\tfrac{5}{2}\right) \\ &= \tfrac{35}{8} > 1 \quad \therefore \text{ repelling}\end{aligned}$$

Chapter 4. Bifurcations

In general Bifurcation theory is the study of how a family of curves can change when a defining parameter is change

↱ very important family

We will consider the quadratic family: $Q_c(x) = x^2 + c$ Idea: how does c affect the dynamics of $Q_c(x)$?

to find the fixed points of $Q_c(x)$:

$$x^2 + c = x \iff x^2 - x + c = 0$$

$$\iff x = \frac{1 \pm \sqrt{1-4c}}{2}$$

Thus:

① $Q_c(x)$ has 2 fixed points when $c < \frac{1}{4}$

② $Q_c(x)$ has 1 fixed point when $c = \frac{1}{4}$

③ $Q_c(x)$ has no fixed points [real] when $c > 1/4$.

If $c > \frac{1}{4}$, $Q_c^n(x)$ must diverge to ∞.

If $c = \frac{1}{4}$ $Q_c(x)$ has the unique fixed point of $p = \frac{1}{2}$. Since $Q_c'(x) = 2x$, $Q_c'(\frac{1}{2}) = 1$ so this fixed point is neutral → attracting to left and repelling to right

If $c < \frac{1}{4}$, $Q_c(x)$ has 2 fixed points:

$$p_+ = \frac{1 + \sqrt{1-4c}}{2}, \quad p_- = \frac{1 - \sqrt{1-4c}}{2}$$

First,

$Q_c'(p_+) = 1 + \sqrt{1-4c} > 1 \Rightarrow p_+$ is repelling

Next,

$$-1 < Q_c'(p_-) < 1$$

$$\Leftrightarrow -1 < 1 - \sqrt{1-4c} < 1$$

$$\Leftrightarrow -2 < -\sqrt{1-4c} < 0$$

$$\Leftrightarrow 2 > \sqrt{1-4c} > 0$$

$$\Leftrightarrow -\frac{3}{4} < c < \frac{1}{4}$$ → this is the range for attracting fixed point

if $c < -\frac{3}{4}$, $Q_c'(p_-) < -1$

if $c = -\frac{3}{4}$, $Q_c'(p_-) = -1$

Theorem: For the family $Q_c(x) = x^2 + c$:

① All orbits tend to ∞ if $c > \frac{1}{4}$

② when $c = 1/4$ $Q_c(x)$ has a unique fixed point, $\frac{1}{2}$, and it is neutral

③ if $c < \frac{1}{4}$ $Q_c(x)$ has 2 fixed points p_+ and p_-. The point p_+ is repelling

More over

a) If $-\frac{3}{4} < c < \frac{1}{4}$, p_- is attracting

b) if $c = -\frac{3}{4}$, p_- is neutral

c) if $c < -\frac{3}{4}$, p_- is repelling

Lec 8 - Jan 24 2024

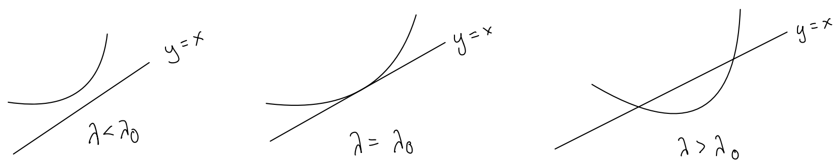
Def'n: We say a family of functions $F_\lambda(x)$ undergoes a **bifurcation** at λ_0 if there is a change in fixed point structure at λ_0

Ex) $Q_c(x) = x^2 + c$, $\lambda_0 = \frac{1}{4}$

Def'n: A family $F_\lambda(x)$ undergoes a **tangent bifurcation** at λ_0 if there is an open interval I and $\varepsilon > 0$ s.t.

1. For $\lambda_0 - \varepsilon < \lambda < \lambda_0$, $F_\lambda(x)$ has no fixed points on I
2. For $\lambda = \lambda_0$, $F_\lambda(x)$ has one fixed point and it is neutral
3. For $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, $F_\lambda(x)$ has 2 fixed points in I, one attracting and the other repelling

or ①, ②, ③ with $>$.



Ex) $E_\lambda(x) = e^x + \lambda$, $\lambda_0 = -1$. This is an example of a tangent bifurcation.

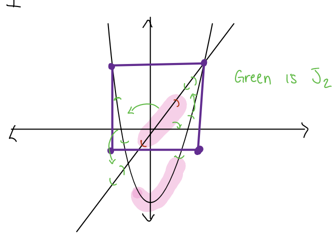
Ex) $F_\lambda(x) = \lambda x(1-x)$, $\lambda_0 = 1$. Not tangent bifurcation.

Chapter 5 Cantor Set

Investigation: $Q_c(x) = x^2 + c$, $c < -2$.

$p_+ = \dfrac{1 + \sqrt{1-4c}}{2} > 2$, $-p_+ < -2$

Consider $I = [-p_+, p_+]$ and $I \times I$



J_2 is asking which x has y values in J_1 which is how these intervals are

picture not to scale

Let $J_1 \subseteq I$ be the interval s.t. $Q_c(x) \notin I$ for all $x \in J_1$

For $x \in J_1$, $Q_c^n(x) \longrightarrow \infty$. Moreover, If $\exists n$ s.t. $Q_c^n(x) \in J_1$ then $Q_c^n(x) \longrightarrow \infty$

Consider $\Lambda = \{x \in I : Q_c^n(x) \in I \ \forall n\}$

Big Idea: Λ contains all the points with interesting orbits.

$J_1 = \{ x \in I : Q_c(x) \notin I \}$

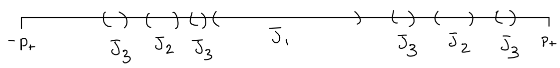
$J_2 = \{ x \in I : Q_c(x) \in J_1 \} = \{ x \in I : Q_c^2(x) \notin I \}$

$J_3, J_4, J_5 \ldots$

what is a cantor set?

Then, $\Lambda = I \setminus (J_1 \cup J_2 \cup \ldots)$ is a cantor set. Drawing Λ on the x-axis:

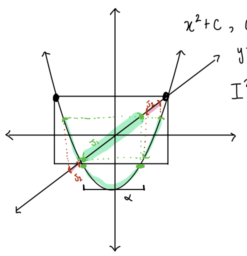
symmetry comes from the curve (parabola)



etc.

It is a fractal b/c of its self repeating nature!

Lec 9 - Jan 26th 2024



$x^2+c,\ c<-2$

$y=x$

$I^2 = [-p_r, p_r]^2$

One quarter mark for the course.

$\Lambda = I \setminus (J_1 \cup J_2 \cup \dots)$ $\qquad$ $A = \{x \in I : \forall n,\ Q_c^n(x) \in I\}$

↳ these are all the interesting orbits

↳ cantor set

Construction: Cantor Middle thirds set

$C_0 = [0,1]$ → remove open middle third Interval

$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9}, 1]$

$\vdots$

these are nested

$K = \bigcap_{n=1}^{\infty} C_n$ → Cantor [middle - third] set. A cantor set is taking out a chunk of the interval, infinitely. We can see that the end points will always survive.

Proposition: $A_n \subseteq \mathbb{R}$ closed. Then $\bigcap_{n=1}^{\infty} A_n$ is closed.

Why? $(a_k) \subseteq \bigcap A_n$, $a_k \longrightarrow a$. $\forall n$, $(a_k) \subseteq A_n \Rightarrow a \in A_n \Rightarrow a \in \bigcap A_n$

Proposition: $A, B \subseteq \mathbb{R}$ closed. Then $A \cup B$ is closed.

Why? $(a_n) \subseteq A \cup B$, $a_n \to a$. WLOG. $\{n : a_n \in A\}$ is infinite. This allows us to construct $(b_n) \subseteq A$ s.t. $b_n \to a$. Since A is closed, $a \in A \subseteq A \cup B$.

Theorem K is closed [in fact all Cantor Sets are] → combine the 2 propositions above to get this

Theorem: K contains no non-empty open intervals → Hausdorff property of $\mathbb{R}$?

no isolated points

interior points?

Why? $I \subseteq K \Rightarrow \forall n,\ I \subseteq C_n \Rightarrow \forall n\ \ \ell(I) \leq \frac{1}{3^n} \Rightarrow \ell = (I) = 0 \Rightarrow I = \emptyset$ which is a Contradiction.

↳ length of interval.

Lets consider the base -3 expansion of $x \in [0,1]$. $x = 0.s_1 s_2 s_3 \dots$, $s_i \in [0,1,2]$

$s_1 \cdot \frac{1}{3}$ $\quad$ $s_2 \cdot \frac{1}{9}$ $\quad$ $s_3 \cdot \frac{1}{27}$

$s_1 = 0$ $\qquad$ $s_1 = 1$ $\qquad$ $s_1 = 2$

$[0, \frac{1}{3}]$ $\qquad\qquad$ $[\frac{2}{3}, 1]$

$s_1 = 0,\ s_2 = 0$ $\quad$ $s_1 = 0,\ s_2 = 1$ $\quad$ $s_1 = 0,\ s_2 = 2$ $\quad$ etc..

$[0, \frac{1}{9}]$ $\quad$ $[\frac{2}{9}, \frac{1}{3}]$ $\qquad$ $[\frac{2}{3}, \frac{7}{9}]$ $\quad$ $[\frac{8}{9}, 1]$

Fact:

$x \in K$ iff x can be written in base-3 using only 0's and 2's

Ex $\frac{1}{3} \in K$. $\frac{1}{3} = [0.1]_3 = [0.022222\ldots]_3$ → # theory? base α?

Theorem: K is uncountable $|K| = |\mathbb{R}|$ → cardinality → not important to this course, just for interest.

How to remember? 0 means left, 2 means right.

Start Ch 6 on monday.

Lec 10 - Jan 29th 2024

* Assignment 2 now out, First 4 Q easy, Last 4 NOT
Midterm march 1^{st}, Content cut off is already happening
Midterm may have proofs from next part of course (after assignment 2 content)
For Q w̄ classify neutral point use cobwebb thing

Chapter 6 - Symbolic Dynamics

Recall: $Q_c(x) = x^2 + c$, $c < -2$

$p_+ = \frac{1 + \sqrt{1-4c}}{2}$, $I = [-p_+, p_+]$

$J_1 = \{x \in I : Q_c(x) \in I\}$
$J_2 = \{x \in I : Q_c(x) \in J_1\}$
$J_3 = \{x \in I : Q_c(x) \in J_2\}$
$\vdots$

$\Lambda = I \setminus (\cup J_i) = \{x \in I : \forall n \; Q_c^n(x) \in I\}$ → This is a cantor set. all cantor sets are isomophic to each other
verify iso vs. homeo

Notation:
$I \setminus J_1 = I_0 \cup I_1$

$I = [\, I_0 \,]$ (removed) $[\, I_1 \,]$ → Idea we have a left part and a right part.

Defn: For $x \in \Lambda$, we define the Itinerary of x to be the sequence

→ kind telling you direction. dealing with orbit indirectly

$$S(x) = (x_0 x_1 x_2 x_3 \dots), \quad x_i \in \{0,1\}$$

where $x_i = 0 \iff Q_c^i(x) \in I_0$
$x_i = 1 \iff Q_c^i(x) \in I_1$

Goal: Understand $S(x)$ better

Notation: $\Sigma = \{(x_0 x_1 x_2 \dots) : x_i \in \{0,1\}\}$ → we drop the commas?

→ function, specifically a transform. Claims its easier to work with.

$S : \Lambda \to \Sigma$

↳ S is a homemorphism ⇒ Λ is same as Σ, what we are doing this week.

Defn: Let X be a set. We say $d : X \times X \to [0, \infty)$ is a Metric iff

1. $d(x,y) = 0$ iff $x = y$
2. $\forall x, y \in X$, $d(x,y) = d(y,x)$
3. $\forall x, y, z \in X$

$$d(x,y) \leq d(x,z) + d(z,y)$$ → triangle inequality

We call (X, d) a metric space

Idea $d(x,y)$ = distance between x and y.

Ex) $X = \mathbb{R}$, $d(x,y) = |x-y|$

Ex) $X = \mathbb{R}^n$, $d(x,y) = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$

Ex) X any set, $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ discrete metric

These are all Metric Space

Ex) $A \subseteq \mathbb{R}$, For $x, y \in A$, $d(x,y) = |x-y|$ is a metric (A, d)

Ex) [Cantor Space]

$X = \Sigma$, $d(\underset{x_0 x_1 \dots}{x}, \underset{y_0 y_1 \dots}{y}) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$ → ensuring we get a number b/c it gives us a convergent series

Ex) $x = (1\ 1\ 1 \dots)$

$y = (1\ 0\ 1\ 0 \dots)$

$d(x,y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$ → 1s cancel in every even step

$= \sum_{i=0}^{\infty} \frac{1}{2^{2i+1}} = \sum_{i=0}^{\infty} \left(\frac{1}{2^{2i}}\right) \frac{1}{2}$?

How

$= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^i} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{1}{2}\left(\frac{4}{3}\right) = \frac{4}{6} = \frac{2}{3}$

Connections

- dynamical systems on manifolds
- p-adic numbers? Vertasium video

Proposition: $x, y \in \Sigma$

① if $x_i = y_i$ for $i \leq n$ then $d(x,y) \leq \frac{1}{2^n}$

First few terms carry alot of weight

② if $d(x,y) < \frac{1}{2^n}$ then $x_i = y_i$ for $i \leq n$

Why?

① $d(x,y) \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2^{n+1}}}{1-\frac{1}{2}} = \frac{1}{2^n}$ → Convergence of geometric series formula. ?

② $d(x,y) < \frac{1}{2^n}$

$\Rightarrow |x_k - y_k| = 1$

$\exists\, k \leq n$, $x_k \neq y_k$. $d(x,y) \geq \frac{1}{2^k} \geq \frac{1}{2^n}$ hence contradiction

Ex) $x = (000 \dots)$

$y = (1000 \dots)$

$d(x,y) = \frac{1}{2^0} = 1$ BUT $x_0 \neq y_0$ This shows that Proposition cannot be iff.

Defn: Shift map.

$\sigma : \Sigma \to \Sigma$

$\sigma : (x_0 x_1 x_2 \dots) = (x_1 x_2 x_3 \dots)$

Remark $\sigma^k(x_0 x_1 x_2 \dots) = (x_k\, x_{k+1}\, x_{k+2} \dots)$

Whats Next?

Show ① $\sigma : \Sigma \to \Sigma$ is continuous

gives us

② $\sigma : \Sigma \to \Sigma$ dynamical System

③ $S : \Lambda \to \Sigma$ homeomorphism $\to$ means Λ and Σ are "Same space"

④ $\sigma : \Sigma \to \Sigma$ Chaotic $\Rightarrow$ $Q_c : \Lambda \to \Lambda$

Lec 11 - Jan 31 2024

Recall: $Q_c(x) = x^2 + c$, $c < -2$

$\Lambda = \{x \in I : \forall n, Q_c^n(x) \in I\}$ $\quad S: \Lambda \to \Sigma_2$, $S(x) = (x_0 x_1 x_2 \ldots)$ iff $Q_c^i(x) \in I_{x_i}$

sequence space

itinerary function

either x_1 or x_0

is this extendable to other iterations in the cantor set.

Def'n: (X,d), (Y,d')

We say $f: X \to Y$ is continuous at $y \in X$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$, $x \in X$, $d(x,y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$

if d' and d where abs then we get continuous Defn for a $f: \mathbb{R} \to \mathbb{R}$

We say $f: X \to Y$ is continuous iff it is continuous at every $y \in X$

Proposition: $\sigma: \Sigma \to \Sigma$ is continuous

Proof: Fix $y \in \Sigma_2$ and let $\varepsilon > 0$ be given. Say $y = (y_0 y_1 y_2 \ldots)$. Take $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \varepsilon$.

Consider $\boxed{\delta = \frac{1}{2^{n+1}}}$. Let $x = (x_0 x_1 x_2 \ldots) \in \Sigma$ s.t. $d(x,y) < \delta$. $\therefore x_i = y_i$ for $i = 0, 1, \ldots, n+1)$

Then, $\sigma(x) = x_1 x_2 x_3 \ldots$, $\sigma(y) = y_1 y_2 y_3 \ldots$ and $x_i = y_i$ for $i = 1, 2, \ldots, n+1$.

By Previous Prop $d(\sigma(x), \sigma(y)) \leq \frac{1}{2^n} < \varepsilon$ $\quad \bullet$

Def'n: (X,d), $(x_n) \subseteq X$, $x \in X$. We say (x_n) converges to x, $x_n \to x$, iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $n \geq N$ $\Rightarrow d(x_n, x) < \varepsilon$

Proposition (X,d), (Y,d'), $f: X \to Y$. Then, f is continuous iff $f(x_n) \to f(x)$ whenever $x_n \to x$

Def'n: (X,d), (Y,d'), $f: X \to Y$, We call f a homeomorphism iff

1. f is injective (one to one)
2. f is surjective
3. f is continuou
4. f^{-1} is continuous

Investigation: $f: X \to Y$ is a homeomorphism

1. $(x_n) \subseteq X$, $x_n \to x$ $\Rightarrow$ $f(x_n) \to f(x)$

by continuity of f^{-1}

2. $(y_n) \subseteq Y$, $y_n \to y$. Say $y_n = f(x_n)$, $y = f(x)$ $\Rightarrow$ $f(x_n) \to f(x)$ $\Rightarrow$ $f^{-1}(f(x_n)) \to f^{-1}(f(x))$ $\Rightarrow$ $x_n \to x$

Big Idea: f is a rellabelling of X to Y. We think of X and Y as the "Same" metric space

Homeomorphic spaces are basically equal.

Remark

① $x \in \Lambda$, $S(x) = (x_0, x_1, \dots)$

we know that $x_0 \in I_{x_0}$, $x_1 \in I_{x_1}$, $Q_c^2(x) \in I_{x_2}, \dots$

$\therefore$ $S(Q_c(x)) = (x_1\ x_2 \dots) = \sigma(S(x))$

② $S(Q_c^n(x)) = \sigma^n(S(x))$ → These 2 are friends and work together

Theorem: $S: \Lambda \to \Sigma$ is a homeomorphism.

Aim is to define and prove chaos

Theorem: Monotone Convergence theorem

If $(a_n) \subseteq \mathbb{R}$ is increasing / decreasing and bounded. then (a_n) converges.

Theorem: [Nested Intervals Lemma]

If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ are closed Intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ → open intervals could.

Idea: $I_k = [a_k, b_k]$

$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \dots$

(a_n) increasing, $(a_n) \subseteq [a_1, b_1]$

(b_n) decreasing, $(b_n) \subseteq [a_1, b_1]$

they converge

By MCT, $a_n \to a$ and $b_n \to b$ $\therefore$ $[a, b] \subseteq \bigcap_{n=1}^{\infty} I_n$ → worst case scenerio $a = b$

Lec 12th 2024 - Feb 2nd 2024

Thm: $S: \Lambda \to \Sigma_2$ is a homeomorphism. → whats the difference between transforms and homeomorphism

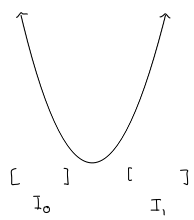
Remark: This is true for $c < -2$. We will prove the result for $c < \frac{-(5+2\sqrt{5})}{4}$.

A2: $\forall x \in I \setminus J_1 = I_0 \cup I_1$ $|Q_c'(x)| \geq \mu > 1$. μ exists by our A2 Q for the above c values

First step in cantor process / itenerary

Proof:
Lets first show it is injective

Suppose $x, y \in \Lambda$ with $S(x) = S(y)$ but $x \neq y$. Therefore $\forall n$, $Q_c^n(x)$ and $Q_c^n(y)$ live in the same I_0 and I_1.



By the Mean Value Theorem

$$|Q_c(x) - Q_c(y)| \geq \mu |x-y|$$

parabola is injective on small? intervals

Since Q_c is injective on I_0 and I_1, we have that $Q_c(x) \neq Q_c(y)$. Thus,

$$|Q_c^2(x) - Q_c^2(y)| \geq \mu^2 |x-y|$$
$$\vdots$$
$$|Q_c^n(x) - Q_c^n(y)| \geq \mu^n |x-y|$$

Since $\mu > 1$, $\mu^n |x-y| \to \infty$. However,

$$|Q_c^n(x) - Q_c^n(y)| \leq \max\{\ell(I_0), \ell(I_1)\}$$ → length will be the same.

length

Contradiction!

■

<u>Surjective:</u> Let $y = (y_0 y_1 y_2 \ldots) \in \Sigma$. For $n \in \mathbb{N}$, Let $I_{y_0 y_1 \cdots y_n} = \{x \in I : x \in I_{y_0}, Q_c(x) \in I_{y_1}, \ldots, Q_c^n(x) \in I_{y_n}\}$.

It is enough to show that there exist $x \in \bigcap_{n=1}^{\infty} I_{y_0 y_1 \cdots y_n}$ $(\Rightarrow S(x) = y)$

Nested interval lemma can apply if this is an interval

Clearly, $I_{y_0} \supseteq I_{y_0 y_1} \supseteq I_{y_0 y_1 y_2} \supseteq \ldots$

<u>Claim:</u> For all $n \geq 0$ $I_{y_0 y_1 y_2 \cdots y_n}$ is a closed interval. First, $I_{y_0} \in \{I_0, I_1\}$ is a closed interval. Assume $I_{y_0 y_1 \cdots y_n}$ is a closed interval for some $n \geq 0$.

Note: $x \in I_{y_0 y_1 \cdots y_n y_{n+1}} \iff x \in I_{y_0}$,

$f: X \to Y$, $B \subseteq Y$

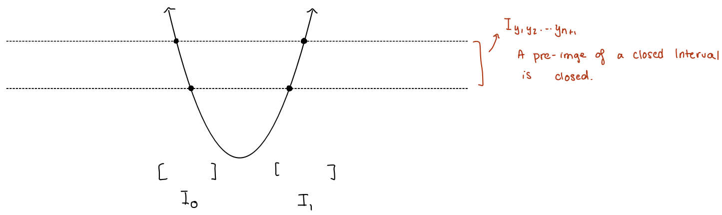
$f^{-1}(B) = \{x \in X : f(x) \in B\}$

$Q_c(x) \in I_{y_1}$, $Q_c(Q_c(x)) \in I_{y_2}$, $Q_c(Q_c^2(x)) \in I_{y_3}, \ldots$

means pre-image

$(*) \iff I_{y_0} \cap Q_c^{-1}(I_{y_1 y_2 \cdots y_n})$ → similar to how shift map relates

By Induction, $I_{y_1 y_2 \cdots y_{n+1}}$ is closed Interval. We have $I_{y_1 y_2 \cdots y_n} \subseteq I_{y_1} \in \{I_0, I_1\}$



$Q_c^{-1}(I_{y_1 \cdots y_{n+1}})$ is a union of two disjoint closed Intervals, one in I_0, and one in I_1. Returning to $(*)$

$$I_{y_0} \cap Q_c^{-1}(I_{y_1 y_2 \cdots y_{n+1}})$$

Is one of these closed Intervals. This proves the claim. By the Nested Intervals Lemma, $\exists\, x \in \bigcap_{n=1}^{\infty} I_{y_0 \cdots y_n}$. Hence, $S(x) = y$. and so S is Sujective

Continuity

Fix $y \in \Lambda$ and say $S(y) = (y_0 y_1 y_2 \ldots)$. Let $\varepsilon > 0$ be given and choose n such that $\frac{1}{2^n} < \varepsilon$. Consider the 2^{n+1}

disjoint, closed Intervals $I_{t_0 t_1 \cdots t_n}$. Pick $\delta > 0$ such that $(y - \delta, y + \delta)$ only overlaps with $I_{y_0 y_1 \cdots y_n}$ Note: $y \in I_{y_0 y_1 \cdots y_n}$

means first n terms agree

For $x \in \Lambda$ with $|x - y| < \delta$, $x \in I_{y_0 y_1 \cdots y_n}$ and so

$$d(S(x), S(y)) \le \frac{1}{2^n} < \varepsilon$$

Continuity of S^{-1}

Similar

■

Lec 13 - Feb 5^{th} 2024

week 5. 6 weeks before Reading week. 6 weeks after Reading week. Nothing Due this week. Midterm march 1. Midterm will be in a different room
no questions after class this week.

Chapter 7 - Chaos

Def'n: (X,d), we say $A \subseteq X$ is **dense** iff $\forall x \in X$, $\forall \varepsilon > 0$, $\exists a \in A$ s.t. $d(a,x) < \varepsilon$

Ex. $\mathbb{Q}$ is dense in $\mathbb{R}$. $\rightarrow$ easiest way to do so is by Decimal expansion and Cut-off at a certain # of decimal points

Ex. $\mathbb{Z}$ is not dense in $\mathbb{R}$.

Ex. $A = \{x \in \Sigma : \exists N, \forall i > N, x_i = 0\}$ $\rightarrow$ Sequence of eventually all zeros. This is Dense in Σ

Proof: Let $x \in \Sigma$ and let $\varepsilon > 0$ be given. Take $n \in \mathbb{N}$ s.t. $\frac{1}{2^n} < \varepsilon$. Consider $y = (x_0 x_1 \dots x_n 0 0 \dots) \in A$. Then $d(x,y) \leq \frac{1}{2^n} < \varepsilon$ ■

$\rightarrow$ SUPER IMPORTANT TO DO

Ex. $A = \{x \in \Sigma : x \text{ periodic}\}$ is dense in Σ hint $y = (x_0 x_1 \dots x_n x_0 x_1 \dots x_n \dots)$ where $x \in \Sigma$ and x & y agree up to the first n - terms

Remark $A = \{x \in \Sigma : x \text{ periodic}\}$ is exactly the periodic points for $\sigma : \Sigma \rightarrow \Sigma$ $\rightarrow$ Shift map.

Prop: There exists $z \in \Sigma$ s.t. $\{\sigma^k(z) : k \in \mathbb{N} \cup \{0\}\}$ is dense in Σ. $\rightarrow$ There could be multiple z. but we only need 1.

$\rightarrow$ writing out all combinations of 01 blocks for all possible lengths. This ensures any $x \in \Sigma$ will be close

Proof: Take $z = (0\,1\,\underline{00}\,\underline{01}\,\underline{10}\,\underline{11}\,\underline{000}\,\underline{001}\,\underline{010}\,\underline{011}, \dots)$

$\rightarrow$ k allows you to shift to the right block. The bigger the n, the more chopping

Take $x \in \Sigma$ and $\varepsilon > 0$. Let $\frac{1}{2^n} < \varepsilon$. For some k, $\sigma^k(z)$ and x agree on the first n terms. Thus $d(\sigma^k(z), x) \leq \frac{1}{2^n} < \varepsilon$ ■

Def'n: A **Dynamical System** is a metric space (X,d) together with a continuous function $f : X \rightarrow X$. This allows you to do what we have done on any metric space

Notation: $f : X \rightarrow X$ dynamical System.

Ex) $\sigma : \Sigma \rightarrow \Sigma$ is a dynamical System.

$\rightarrow$ gross?

Def'n: Let $f : X \rightarrow X$ be a dynamical System. We say $f : X \rightarrow X$ is **transitive** iff $\forall x, y \in X$, $\forall \varepsilon > 0$, $\exists z \in X$, $n, m \in \mathbb{N} \cup \{0\}$ s.t. $d(x, f^n(z)) < \varepsilon$ and $d(y, f^m(z)) < \varepsilon$ No matter which 2 points you start with, There is an orbit of a point that is close to the original 2 points.

Proposition: $\sigma : \Sigma \rightarrow \Sigma$ is transitive

Proof: take $z \in \Sigma$ s.t. $\{\sigma^k(z) : k \in \mathbb{N} \cup \{0\}\}$ is dense in Σ. For all $\varepsilon > 0$, $x, y \in \Sigma$, $\exists n, m \in \mathbb{N} \cup \{0\}$ s.t. $d(x, \sigma^k(z)) < \varepsilon$ and $d(y, \sigma^m(z)) < \varepsilon$ ■
$\hookrightarrow$ having a dense set makes transitivity easy.

It doesnt need to be dense in order for the function to be transitive.

$\rightarrow$ worst one today

$\rightarrow$ β doesnt Depend on x & y.

Def'n: Let $f : X \rightarrow X$ be a Dynamical System. we say $f : X \rightarrow X$ depends sensitively on initial conditions (**is sensitive**) iff $\exists \beta > 0$, $\forall \varepsilon > 0$, $\forall x \in X$ $\exists y \in X$, $\exists k \in \mathbb{N}$ Such that

① $d(x, y) < \varepsilon$

② $d(f^k(x), f^k(y)) \geq \beta$ → this is similar to stable/unstable

Prop: $\sigma : \Sigma \to \Sigma$ is sensitive.

Proof: Take $\boxed{\beta = 1}$. Let $\varepsilon > 0$ be given and let $x \in \Sigma$. Say $\frac{1}{2^n} < \varepsilon$ and suppose $y \in \Sigma$ is s.t. $0 < d(x, y) < \frac{1}{2^n}$. Thus, $\exists\, k \geq n$, s.t.

→ either 0 or 1 ⇒ $|x_k - y_k| = 1$.

$x_k \neq y_k$ $\therefore$ $d(\sigma^k(x), \sigma^k(y)) \geq \frac{|x_k - y_k|}{2^0} = 1 = \beta$

Lec 14 - Feb 7th 2024

Def'n: A dynamical system $f: X \to X$ is chaotic iff
1. The periodic points for f are dense in X.
2. transitive
3. Sensitive

Theorem: $\sigma: \Sigma \to \Sigma$ is chaotic → we have already proved this above

→ Does chaos imply repelling? Prof must think about it.

Proposition: (X,d), (Y,d') and $f: X \to Y$ continuous and surjective. ↗ allows us to not "kill" A when f is applied. If $A \subseteq X$ is dense in X then $f(A)$ is dense in Y.

Proof: Let $y \in Y$ and say $y = f(x)$. ↗ can do b/c of surjective. Let $\varepsilon > 0$ be given. Since f is continuous at x, $\exists\, \delta > 0$, ↗ continuous defn $d(z,x) < \delta \Rightarrow d'(f(z), f(x)) < \varepsilon$

Since A is dense in X, we may find $a \in A$ such that $d(a,x) < \delta$ ↗ density defn $\Rightarrow d'(f(a), f(x)) < \varepsilon$

$\Rightarrow d'(f(a), y) < \varepsilon$ ■

Theorem: $c < \dfrac{-(5+2\sqrt{5})}{4}$ then $Q_c: \Lambda \to \Lambda$ is chaotic.

Proof: IDEA: Relate this theorem to the above thm that $\sigma: \Sigma \to \Sigma$ is chaotic

1. **Periodic Points**

Note that $Q_c^n(x) = x$ for x that is periodic $\Leftrightarrow S(Q_c^n(x)) = S(x)$ ↗ b/c S is one-to one.

$\Leftrightarrow \sigma^n(S(x)) = S(x)$

By using the previous proposition with $S^{-1}: \Sigma \to \Lambda$, the periodic point for Q_c are dense in Λ

2. **Transitive**

Take $z \in \Sigma$ such that $\{\sigma^k(z) : k \in \mathbb{N} \cup \{0\}\}$ is dense. Again, $\{S^{-1}(\sigma^k(z)) : k \in \mathbb{N} \cup \{0\}\}$ is dense in Λ

Note: Say $S(x) = z$. $S(Q_c^k(x)) = \sigma^k(S(x)) \Leftrightarrow Q_c^k(x) = S^{-1}(\sigma^k(S(x)))$. Thus $\{Q_c^k(x) : k \in \mathbb{N} \cup \{0\}\}$ is dense in Λ. as before, $Q_c: \Lambda \to \Lambda$ is transitive.

3. **Sensitivity**

Recall that $\Lambda \subseteq I \setminus J_1 = I_0 \cup I_1$. Let $\beta > 0$ be less than the distance between I_0 and I_1. For $x, y \in \Lambda$ with $x \neq y$, $S(x) \neq S(y)$

$\Rightarrow \exists k$ such that k^{th} term of $S(x) \neq k^{th}$ term of $S(y)$. Hence $|Q_c^k(x) - Q_c^k(y)| > \beta$ ■ This is slightly stronger than our defn of Sensitivity

---

End of Midterm Material

Lec 15 - Feb 9th 2024

Chapter 8 Sarkovskii's Thereom

Thereom: Period Three

Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. If f has a point with period 3, then f has point with n for all $n \in \mathbb{N}$.

Proposition 1: $I \subseteq J$ are closed Intervals. $f: \mathbb{R} \to \mathbb{R}$ is continuous. If $f(I) \supseteq J$, then $f(x)$ has a fixed point in I.

Why? $[a, b] = I \subseteq J = [c, d]$

$c = f(p)$, $d = f(q)$; $p, q \in I$

$g(x) = f(x) - x$ is also continuous.

left end pt of big interval

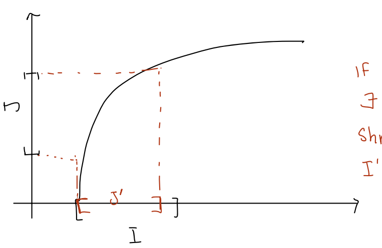
$g(p) = f(p) - p = c - p \leq 0$

right end pt of big Interval

$g(q) = f(q) - q = d - q \geq 0$

By IVT, $\Rightarrow \exists z, g(z) = f(z) - z = 0 \Rightarrow f(z) = z$ $\therefore \exists$ a fixed point. ■

Proposition 2: I, J be closed Intervals. $f: \mathbb{R} \to \mathbb{R}$ be continuous. $f(I) \supseteq J$ then, $\exists$ closed Interval $I' \subseteq I$ such that $f(I') = J$



If image of J $\exists$ in I then we can shrink I such that I' image is J

Proof: [Period 3]

Let $a \in \mathbb{R}$ be a period 3 point for $f(x)$. Say $f(a) = b$, $f(b) = c$, $f(c) = a$. WLOG, suppose $a < b, c$.

Case 1 $a < b < c$. Case $a < c < b$ similar

Let $I = [a, b]$ and $J = [b, c]$. Observe $f(a) = b$ and $f(b) = c \overset{IVT}{\Rightarrow} J \subseteq f(I)$ and $f(b) = c$, $f(c) = a \overset{IVT}{\Rightarrow} I \cup J \subseteq f(J)$

Since $J \subseteq f(J)$ $\exists$ closed Interval $A_1 \subseteq J$ such that $f(A_1) = J$. Again, $A_1 \subseteq J = f(A_1)$, and so there exists a closed interval $A_2 \subseteq A_1$ such that $f(A_2) = A_1$. (prop 2)

Fix $n > 3$. Repeating the above process we can find $A_{n-2} \subseteq A_{n-3} \subseteq \ldots \subseteq A_2 \subseteq A_1 \subseteq J$. such that $f(A_i) = A_{i-1}$.

in J

in I

Now $f(I) \supseteq J \supseteq A_{n-2} \Rightarrow \exists\, A_{n-1} \subseteq I$ closed interval such that $f(A_{n-1}) = A_{n-2}$

Moreover, $f(J) \supseteq I \supseteq A_{n-1} \Rightarrow \exists\, A_n \subseteq J$ closed interval such that $f(A_n) = A_{n-1}$. We have $f^n(A_n) = J$ and $A_n \subseteq J$. By prop1,

must show its the period.

$\exists\, x_0 \in A_n$ such that $f^n(x_0) = x_0$

Note: $x_0 \in A_n$, $f(x_0) \in A_{n-1} \subseteq I$. and $f^i(x_0) \in J$ for $i = 2, 3, \dots n$. Suppose $f^i(x_0) = x_0$ for some $i < n$.

, only overlap. in both Intervals

Then, $\underset{\in I}{f(x_0)} = \underset{\in J}{f^{i+1}(x_0)} = b$

$\Rightarrow f(x_0) = b$, $f^2(x_0) = c$, $f^3(x_0) = a \Rightarrow a \in J$ but that's not true. $a \in I$. Hence x_0 has period n.

Further, $f(J) \supseteq J$ and so f has a fixed point (Prop 1) in J.

Finally, $f(I) \supseteq J \Rightarrow J = f(I')$

$f(J) \supseteq I' \Rightarrow f(J') = I'$

by prop 1

$\Rightarrow f^2(J') = f(I') = J \supseteq J' \Rightarrow \exists\, x_0 \in J'$ s.t. $f^2(x_0) = x_0$. If $f(x_0) = x_0$. $x_0 \in J'$ and $f(x_0) \in I' \Rightarrow x_0 = b$

but $f(b) \neq b$ b/c $f(b) = c$. Thus a contradition. Hence x_0 has period 2

$\boxed{\text{HOMEWORK: } a < c < b}$

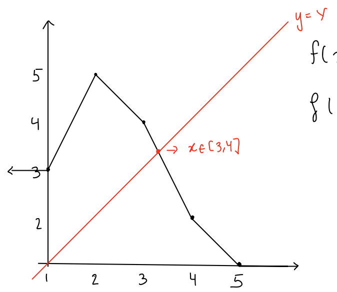
Lec 16 - Feb 12th 2023

Thereom: Period Three

Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. If f has a point with period 3, then f has point with n for all $n \in \mathbb{N}$.

Proof is basically repeating applications of the Intermediate Value Theorem. 3 is important because you can create 2 intervals with the 3 numbers

Ex)



$f(x) = 3 \quad x \leq 1$

$f(x) = 1 \quad x \geq 5$

Orbit of 1: $1 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 5 \mapsto 1$

$\therefore$ 1 has a period of 5

Claim: f has no point with period 3. Suppose f has a point with period 3. Note $1 \leq x \leq 5$

→ Since we are assuming x is a 3-periodic point

① $x \in [1,2]$. $x \in [1,2] \cap f^3([1,2])$

$= [1,2] \cap [2,5]$

$\Rightarrow x = 2$ but contradiction b/c 2 has period 5.

② $x \in [2,3] \Rightarrow x \in [2,3] \cap f^3([2,3])$

$= [2,3] \cap [3,5] \Rightarrow x = 3$ but again contradiction

③ $x \in [4,5] \Rightarrow x \in [4,5] \cap f^3([4,5])$

$= [4,5] \cap [1,4]$

$\Rightarrow x = 4$ but contradiction

④ $x \in [3,4] \Rightarrow x \in [3,4] \cap f^3([3,4])$

$= [3,4] \cap [1,5]$

$f: [3,4] \to [2,4]$ Stricly decreasing

$f: [2,4] \to [2,5]$ stricly decreasing

$f: [2,5] \to [1,5]$ Stricly decreasing.

$\therefore f^3(x)$ is stricty decreasing. → Since Strictly decreasing $f^3(x)$ only crosses the line $y = x$ once

$\therefore f^3(x)$ has a unique fixed point in $[3,4]$ but it is just the fixed point of f in $[3,4]$

$\therefore f$ has no point with period three.

Sometimes a function has period n but not period k.

Ex) $f(x) = \begin{cases} 1 & x < -1 \\ -x & -1 \le x \le 1 \\ -1 & x > 1 \end{cases}$

For $x < -1$ not periodic

$-1 \le x \le 1$, $x \neq 0$ period 2

$x = 0$ period 1

$x > 1$ not periodic

Sarkovskii Ordering

$3 \prec 5 \prec 7 \prec 9 \prec 11 \ldots$

$\prec$

$2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec 2\cdot 9 \prec 2\cdot 11$

$\prec$

$2^2\cdot 3 \prec 2^2\cdot 5 \prec 2^2\cdot 7 \prec 2^2\cdot 9 \prec 2^2\cdot 11$

$\vdots$

$\prec$

$\ldots 2^n \prec 2^{n-1} \prec \cdots \prec 2^2 \prec 2 \prec 1$

This has created a total ordering of the natural numbers where 3 is the smallest, 2 is the second largest and 1 is the largest

Ex $2\cdot 13 \prec 2^2\cdot 5$

$2^{10}\cdot 3 \prec 2^5$ $\forall n \in \mathbb{N}$, $n \preccurlyeq 1$.

Theorem: Sarkovski

Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $n, m \in \mathbb{N}$ with $n \prec m$ (in Sarkovski ordering). If f has a point with period n, then it has a point with period m

Lec 17 – Feb 14 2024

"I'll switch up the polynomial, trig / expenontial on the Midterm..."

Chapter 9 - Fractals

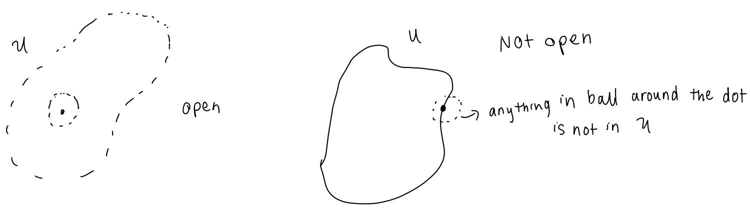
Def n / Notation

① $x \in \mathbb{R}^n$, $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

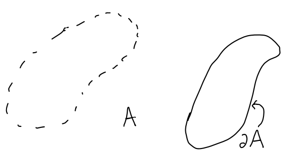
② $d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$ is a metric on $\mathbb{R}^n$

③ $\vec{x} \in \mathbb{R}^n$, $\varepsilon > 0$. The open ball of radius epsilon centered at $\vec{x}$: $B_\varepsilon(\vec{x}) = \{\vec{y} \in \mathbb{R}^n : \|x-y\| < \varepsilon\}$

④ We say $U \subseteq \mathbb{R}^n$ is open if and only if $\forall \vec{x} \in U$, $\exists\, \varepsilon > 0$, $B_\varepsilon(\vec{x}) \subseteq U$



⑤ $A \subseteq \mathbb{R}^n$, $\partial(A) =$ boundary of A.

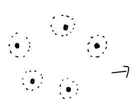


Precisely, the closure take away the interior.

Def'n We say $S \subseteq \mathbb{R}^n$ has topological Dimesion 0 iff $\forall \vec{x} \in S$, $\exists$ arbitrarly small open $\vec{x} \in U$ s.t.

$$\partial(U) \cap S \neq \emptyset$$

Ex) call this x



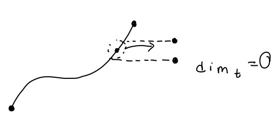
this has dimension 0 $\quad \dim_t X = 0$ } isolated points

Ex) $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ $\quad \dim_t X = 0$ } non-isolated points

Because of the Hausdorff proprety of $\mathbb{R}$, doesn't this imply any Hausdorff $\dim_T(X) = 0$? ~~no~~ LOL

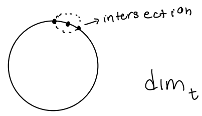
Def'n: $S \subseteq \mathbb{R}^n$, we say S has topological dimension $k \in \mathbb{N}$ iff $\forall \vec{x} \in S$ there exists arbitrarily small $\vec{x} \in U$ s.t $\partial(U) \cap S$ has topological dimension $k-1$, where k is minimal with this property.

Ex) X



$\Rightarrow \quad \dim_t(X) = 1$

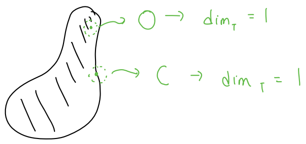
Ex



$\dim_t X = 1$

Ex)

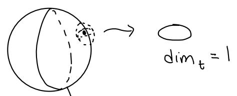
$\nearrow \dim_T = 1$ from previous example.



$\dim_T X = 2$

This is a recursive definition

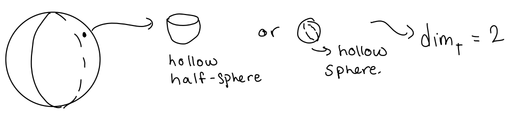
Ex) X is a non-fille sphere



$\dim_t X = 2.$

even though 3-D, Topological dim is 2

Ex) X is a filled-sphere



$\dim_T X = 3.$

$\Rightarrow$ filled sphere is bigger than a hollow sphere

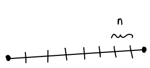
Defn: $S \subseteq \mathbb{R}^n$. we say S is **Self-Similar** if S may be divided into k Congruent Subsets each of which may be magnified by a fixed M to yield S itself

We then define the **fractal dimension** of S by

$$\boxed{\dim_f S = \frac{\ln(k)}{\ln(M)}}$$

Defn: A **fractal** is a self-similar $S \subseteq \mathbb{R}^n$ s.t. $\dim_f S > \dim_t S.$

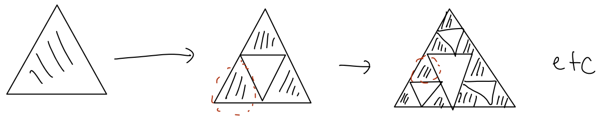
Ex



$\dim_t X = 1$

$\dim_f X = \dfrac{\ln(n)}{\ln(n)}$ → n partition → n is the multiplier to get back the line

Ex) X be Sierpinski Triangle



$\dim_t X = 1$

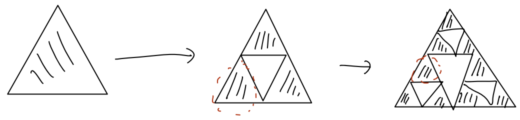
$$\dim_f X = \frac{\ln(3)}{\ln(2)} = \frac{\ln(9)}{\ln(4)} \approx 1.585\ldots$$

ln helps so that no matter what iteration, it gives same #

Distance between fractal dimesion and Topological Dimesion gives a measure of how crazy the fractal is.
Sierpinski Triangle pretty tame.

Lec 18 - Feb 16th 2024

Ex) X be Sierpinski Triangle



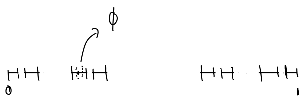
etc

$\dim_t X = 1$

$$\dim_f X = \frac{\ln(3)}{\ln(2)} = \log_2 3$$

→ change of basis formula

Ex) Cantor Set. K



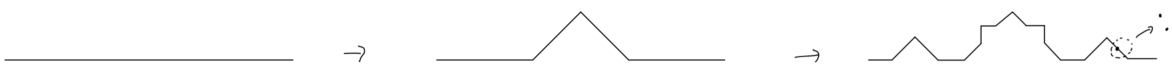
For every point in the cantor set, there are arbitraly small amounts of points around that point. Hence $\dim_T K = 0$. Minimally comes into play here. b/c you could find a point near a boundary

→ pieces

$$\dim_f K = \frac{\ln(2)}{\ln(3)} = \log_3(2) = 0.63$$

→ magnification.

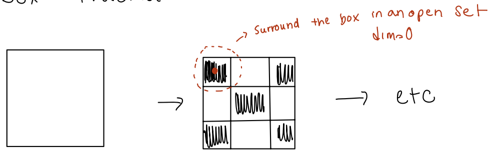
Ex) Koch Curve



This is a fractal. The Koch Snowflake is not totally a fractal according to our definition.

$\dim_t X = 1$, $\dim_f X = \frac{\ln(4)}{\ln(3)} = \log_3 4$

Ex) Box Fractal



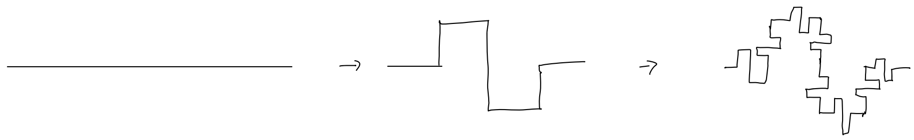
→ etc

→ This is path connected

$\dim_t X = 1$

$$\dim_f X = \frac{\ln(5)}{\ln(3)} = \log_3(5)$$

Ex) Minkowski's Sausage



$\dim_t X = 1$, $\quad \dim_f X = \dfrac{\ln(8)}{\ln(4)} = \log_4(8) = \dfrac{3}{2}$

The Chaos game: Shodor.org/ interactive /activites / The Chaos Game.

↳ Can create fractals

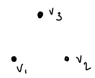
↳ fractals can come from iterated function system

Google CGR DNA.

Lec 19 - Feb 26th 2024

Recall: Chaos Game.

① Start with the vert icies of an equal laterial triangle



② Pick $p \in \mathbb{R}^2$

③ Randomly Select a vert ex. v_i

④ Replace p with the mid point (p, v_i)

⑤ iterate.

Question: where does the orbit of p end up? Awnser: Sierpinski triangle.

Goal: Formalize this.

Iterated Function Systems

Fix $p_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$ and fix $0 < \beta < 1$ (contraction factor)

Consider $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \beta \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ ie $F(p) = \beta(p - p_0) + p_0$

Note :

① $F(p_0) = p_0$ → fixed point

② $\| F(p) - F(p_0) \| = \| \beta(p - p_0) \| = \beta \| p - p_0 \|$

③ $\| F^n(p) - p_0 \| = \beta^n \| p - p_0 \| \to 0$ $\therefore$ $F^n(p) \to p_0$

Def'n : $0 < \beta < 1$, $p_1, p_2, \dots, p_n \in \mathbb{R}^2$

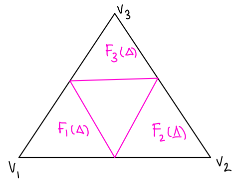
$F_i(p) = \beta(p - p_i) + p_i$. We call $\{F_1, F_2, \dots, F_n\}$ an iterated function System. Fix $p_0 \in \mathbb{R}^2$. Randomly select F_i. Let $q_1 = F_i(q_0)$ and Continuing...

The set of points in which the orbit $q_1, q_2, q_3 \dots$ must eventually end up in, is called the attractor for the IFS (iterated function system)

Ex) Chaos Game $P_1 = V_1$, $P_2 = V_2$, $P_3 = V_3$, $\beta = \frac{1}{2}$

$F_i(P) = \frac{1}{2}(p - p_i) + p_i = \frac{1}{2}(p + v_i) = \text{midpoint}(p, v_i)$

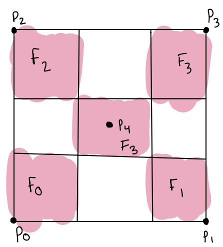
$\{F_1, F_2, F_3\}$ IFS and the attractor = Sierpinski Triangle.



Here we have equal propability

" The orbits are <u>dense</u> in the attractor set"

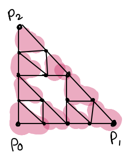
Ex) $p_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $P_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $P_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $P_4 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ and $\beta = \frac{1}{3}$



Box fractal = Attractor.

Think where one iteration of functions go to

Ex $p_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\beta = \frac{1}{2}$



Attractor = Right Sierpinski triangle.

Lec 20 - Feb 29th 2028

Ex $P_0 = \begin{bmatrix}0\\0\end{bmatrix}$, $P_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\beta = \frac{1}{3}$

The IFS is:

$$F_0(\vec{x}) = \frac{1}{3}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\frac{1}{3}x\\\frac{1}{3}y\end{bmatrix}$$

$$F_1(\vec{x}) = \frac{1}{3}\begin{bmatrix}x-1\\y\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{3}x + \frac{2}{3}\\\frac{1}{3}y\end{bmatrix}$$

$q_0 = \begin{bmatrix}x_0\\y_0\end{bmatrix} \in \mathbb{R}^2$. Say the orbit of $\{F_0, F_1\}$ is $q_0, q_1, q_2, \dots$ $q_n = \begin{bmatrix}x_n\\y_n\end{bmatrix}$ with random selections $S_1, S_2, S_3, \dots,$ $S_i \in \{0,1\}$

Note: $y_n = \frac{1}{3^n} y_0 \to 0$

$$x_1 = \frac{1}{3}x_0 + \frac{2S_1}{3}$$

$$x_2 = \frac{1}{3^2}x_0 + \frac{2S_1}{3^2} + \frac{2S_2}{3}$$

$$x_3 = \frac{1}{3^3}x_0 + \frac{2S_1}{3^3} + \frac{2S_2}{3^2} + \frac{2S_3}{3}$$

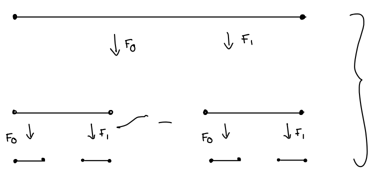
$$\vdots$$

$$x_n = \frac{1}{3^n}x_0 + \frac{2S_1}{3^n} + \frac{2S_2}{3^{n-1}} + \frac{2S_3}{3^{n-2}} + \dots + \frac{2S_n}{3}$$

$\therefore$ x_n gets arbitrarily close to points of the form $\sum_{i=1}^{\infty} \frac{t_i}{3^i}$ where $t_i \in \{0,2\}$.

Therefore the attractor of the IFS is $\left\{ \begin{bmatrix}x\\0\end{bmatrix} : x \in \text{Cantor Set} \right\}$

graphically:



Cantor Middle thirds.

(Generalized) Iterated Function Systems

Def'n: An affine transformation is a transformation $F: \mathbb{R}^n \to \mathbb{R}^n$, $F(\vec{x}) = A(\vec{x}) + \vec{b}$ where $A \in M_n(\mathbb{R})$, $\vec{b} \in \mathbb{R}$

idea is that its basically a linear transformation b/c it doesnt need to pass through the origin

More over, we call F a linear contraction if $\exists\ 0 < \lambda < 1$ such that $\| F(\vec{x}) - F(\vec{y}) \| < \lambda \| \vec{x} - \vec{y} \|$ → There is a connection to the eigenvalues?

counter clockwise ↑ Rotation matrix!!! ☺

Ex $A = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$ and $0 < \beta < 1$

$F: \mathbb{R}^2 \to \mathbb{R}^2$ $F(\vec{x}) = \beta A \vec{x} + \vec{b}$

→ Scales by β

→ rotates counter-clock wise by θ

→ translates by $\vec{b}$

only 2 out of the 3 ridgid motions
3rd in reflection

Def'n : we say $A \subseteq \mathbb{R}^n$ is compact iff A is closed and bounded.

Notation: $K_n = \{ A \subseteq \mathbb{R}^n : A \text{ compact} \}$

Def'n: Let $F_1, F_2, \dots, F_k : \mathbb{R}^n \to \mathbb{R}^n$ be linear contractions. We call $F: K_n \to K_n$ $\quad F(A) = F_1(A) \cup F_2(A) \cup \dots \cup F_k(A)$ an iterated function system

not well defined in this defn but it is compact & Bounded on both sides

We will:

① Equip K_n with a metric

V. similar to Banach Contraction

② Show F has a unique fixed point A^* and $\forall A \in K_n$, $F^n(A) \to A^*$

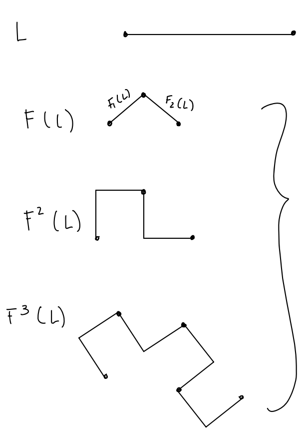
③ We call A^* the attractor of F.

Ex) $F_1(\vec{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \vec{x}$ $\rightarrow$ this rotates by $\frac{\pi}{4}$

Squashes

$F_2(\vec{x}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1/\sqrt{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ translates 1 step

L = line segement from $\binom{0}{0}$ to $\binom{1}{0}$



These are the first 3 iterations

Dragon Curve: Video by mathetical Visual proofs

Lec 21 - Mar 4^{th} 2024

2 speakers lined up for end of semester, trying to get as many speakers as he can

assignment due Mar 12^{th}. May have class on april 8^{th}

Recall:

$F_1, F_2 \dots F_k : \mathbb{R}^n \to \mathbb{R}^n$

each $F_i(\vec{x}) = A_i \vec{x} + \vec{b}_i$ and $\| F_i(\vec{x}) - F_i(\vec{y}) \| \leq \lambda \| x - y \|$ where $\lambda \in (0,1)$ each F_i will have there own λ_i but here $\lambda = \max\limits_{\forall i} (\lambda_i)$. These are Linear Contractions.

Consider $\mathcal{K}_n = \{ A \subseteq \mathbb{R}^n : A \text{ compact} \}$ Remeber compact = closed and bounder.

Fact: $\forall A \in \mathcal{K}_n \Rightarrow F_i(A) \in \mathcal{K}_n$ means we can iterate.

To do: Equip $\mathcal{K}_n$ with a metric. We will consider $F : \mathcal{K}_n \to \mathcal{K}_n$, $F(A) = F_1(A) \cup F_2(A) \cup \dots \cup F_k(A)$ F gives you all of the images at once

To do: We show that F has a unique fixed point $A^* \in \mathcal{K}_n$ and $\forall A \in \mathcal{K}_n$, $F^n(A) \to A^*$ Like banach contraction theorem, but we need metric

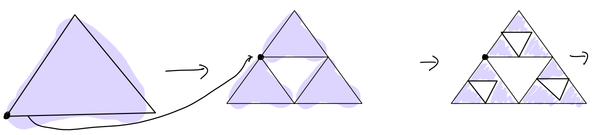
We call F an (generalized) iterated function system. We call A^* its attractor.

Ex) $F_1 = \underbrace{\tfrac{1}{2}}_{\tfrac{1}{2} \cdot \text{Identity matrix}} \vec{x}$

$F_2 = \frac{1}{2}\vec{x} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

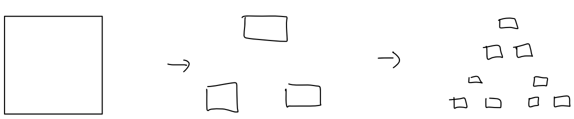
$F_3 = \frac{1}{2}\vec{x} + \begin{bmatrix} 1/4 \\ \sqrt{3}/4 \end{bmatrix}$

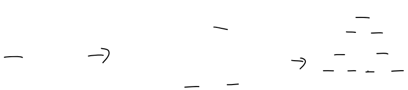
Let A be the filled triangle with vertices $(0,0)$, $(1,0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$



$\to A^*$ This is the Sierpinski triangle

OR.





No matter what compact set you start out with, you converge to same thing

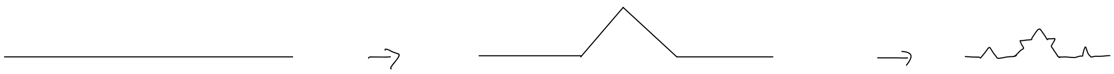
Ex) $F_1(\vec{x}) = \frac{1}{3}\vec{x}$

$F_2(\vec{x}) = \frac{1}{3} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \vec{x} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$ $\theta = \frac{\pi}{3}$

$F_3(\vec{x}) = \frac{1}{3} \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1/2 \\ \sqrt{3}/6 \end{bmatrix}$ special θ for rotation matrix $\theta = -\pi/3$

$F_4(\vec{x}) = \frac{1}{3}\vec{x} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$

and L = line segement from $(0,0)$ to $(1,0)$



The attractor is the Koch Curve

Ex) $A = [0,1] \times [0,1]$ the filled square

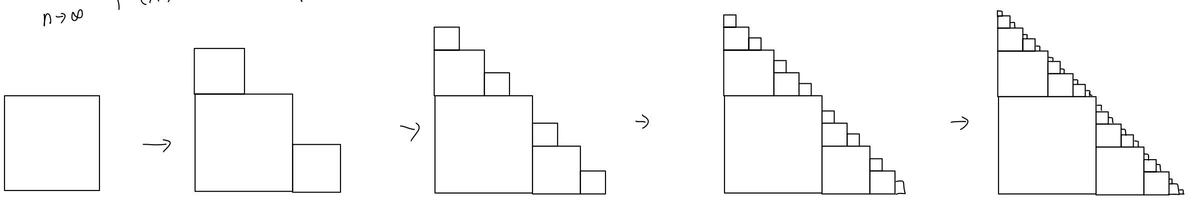
$F_1(\vec{x}) = \vec{x}$

$F_2(\vec{x}) = \frac{1}{2}\vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$F_3(\vec{x}) = \frac{1}{2}\vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Remark: Not quite an IFS b/c F_1 is not a contraction. $\Rightarrow$ no longer always converge to an attractor. Will depend on IC.

i.e. $\lim_{n\to\infty} f^n(A)$ will depend on A. But turns out for us limit does exist



not congruent shapes but still very similar

Defn: [Hausdoff Metric]

$\vec{v} \in \mathbb{R}^n$, $A, B \in \mathcal{R}_n$

$d(\vec{v}, B) = \min\{\|\vec{v} - \vec{b}\| : \vec{b} \in B\}$ impf is a min. Extreme Value theorem.

$d(A, B) = \max\{d(a, B) : a \in A\}$ → The biggest shortest distance

$D(A, B) = \max\{d(A,B), d(B,A)\}$

Lec 22 · Mar 6th 2024

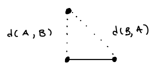
PMATH 367 2/3 point set topology. ↗ quotient spaces 1/3 algebraic topology.
351 co req.

Recall: The Hausdorff metric (D), formulation is in last lecture
↳ The way little d is not symmetric that the motivation for D b/c the max in it solves the symmetry issue

Fact: D is a metric on $\mathcal{K}_n$

Ex) $A = \{(1,1)\}$, $B = \{(x,0) : 0 \le x \le 1\}$

graph



$d(A,B) = 1$

$d(B,A) = \sqrt{2}$ $(a^2 + b^2 = c^2)$

Q is class: Would min work? Probably not b/c triangle inequality probably fails

The reason we have a max in $d(A,B)$ is b/c we want to only have 2 set be the same is that if they <u>are</u> the same set

Finally, $D(A,B) = \max\{1, \sqrt{2}\} = \sqrt{2}$

Lemma 1: Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a Linear contraction such that $\|f(\vec{x}) - f(\vec{y})\| \le \lambda \|\vec{x} - \vec{y}\|$, $\lambda \in (0,1)$
For $A, B \in K_n$ then $D(f(A), f(B)) \le \lambda D(A,B)$

Proof: First, $d(f(a), f(B)) = \min_{b \in B} \|f(a) - f(b)\|$

$$\le \min_{b \in B} \lambda \|a-b\| = \lambda \min_{b \in B} \|a-b\|$$

$$= \lambda\, d(a,B)$$

and so $d(f(A), f(B)) = \max_{a \in A} d(f(a), f(B))$

$$\le \lambda \max_{a \in A} d(a,B) = \lambda\, d(A,B)$$

$$\le \lambda\, D(A,B)$$

comes from defn & propreties of min & max

$\therefore$ $d(f(A), f(B)) \le \lambda D(A,B)$

Similarily $d(f(B), f(A)) \le \lambda D(A,B)$ → convince yourself of this

$\therefore$ $D(f(A), f(B)) \le \lambda D(A,B)$ ∎

Lemma 2: For $A_1, A_2, B_1, B_2 \in \mathcal{K}_n$; $D(A_1 \cup A_2, B_1 \cup B_2) \le \max\{D(A_1,B_1), D(A_2,B_2)\}$

Proof: First, $d(A_1 \cup A_2, B_1 \cup B_2) = \max_{a \in A_1 \cup A_2} d(a, B_1 \cup B_2)$

$$= \max\left\{\max_{a \in A_1} d(a, B_1 \cup B_2), \max_{a \in A_2} d(a, B_1 \cup B_2)\right\}$$

↗ ensure biggest distance

$$\le \max\left\{\max_{a \in A_1} d(a, B_1), \max_{a \in A_2} d(a, B_2)\right\}$$

By the min in the definition of $d(\vec{v}, B)$

$$= \max\{d(A_1,B_1), d(A_2,B_2)\}$$

$$\le \max\{D(A_1,B_1), D(A_2,B_2)\}$$

$\therefore \quad d(A_1 \cup A_2, B_1 \cup B_2) \leq \max \{ D(A_1, B_1), D(A_2, B_2) \}$

Similarily $\quad d(B_1 \cup B_2, A_1 \cup B_2) \leq \max \{ D(A_1, B_2), D(A_2, B_2) \}$

$\therefore \quad D(A_1 \cup A_2, B_1 \cup B_2) \leq \max \{ D(A_1, B_1), D(A_2, B_2) \}$ ■

Lemma 3: Let $F_1, \dots, F_k$ be linear contractions with contraction factor $\lambda \in (0,1)$. Consider $F: K_n \to K_n$, $F(A) = F_1(A) \cup F_2(A) \cup \dots \cup F_k(A)$. Then $D(F(A), F(B)) < \lambda D(A, B)$

Proof: We have, $D(F(A), F(B)) \leq \max\limits_{i=1,\dots,k} D(F_i(A) \cup F_i(B))$ By the inductive generalization of Lemma 2

$\leq \max\limits_{i=1,\dots,k} \lambda D(A, B)$ By Lemma 1

$= \lambda D(A, B)$ b/c no more i-dependency. This is technically a collory ■

Q from class: Can be infinitley many? probs not b/c we may lose compactness.

Defn: (X, d) metric space.

① We say $(x_n) \subseteq X$ is Cauchy ift $\forall \varepsilon > 0, \exists N \in \mathbb{N}, n, m \geq N \Rightarrow d(x_n, x_m) < \varepsilon$.

② We say X is complete is complete iff every Cauchy Sequence $(x_n) \subseteq X$ converges to some $x \in X$.

Fact: (K_n, D) is complete! yay for us

↳ useful for us

Lec 23 - Mar 8 2024

Guest Speakers! right now 4 but hoping to get 5. This will take last 2 weeks of Guest Speaker. Speaker Content will be on final. 50% of Final MC and the content may show up there.

Correction: $K_n = \{ A \subseteq \mathbb{R}^n : A \text{ compact}, A \neq \emptyset \}$

Recall: $F_i = \mathbb{R}^n \to \mathbb{R}^n$ $1 \le i \le k$ are linear contractions

$\| F_i(\vec{x}) - F_i(\vec{y}) \| \le \lambda \| \vec{x} - \vec{y} \|$

$F : K_n \to K_n$

$F(A) = F_1(A) \cup \dots \cup F_k(A)$ then

$D(F(A), F(B)) \le \lambda D(A, B)$

B/c the function contracts so does the distance

Theorem

Let $F_1 \dots F_k$ be linear contraction with contraction factor $\lambda \in (0,1)$. Let $F: K_n \to K_n$ be the corresponding IFS. Then:

① F has a unique fixed point A^*

② For all $A^* \in K_n$, $F^m(A) \to A^*$.

We call A^* the attractor of F.

Can there be IFS that are not contracts such that there are multiple fixed points? like a kalidoscope → one of our example of square stair case is this an example of

Proof:

Fix $A \in K_n$. Consider the Orbit $F^m(A)$, then,

$$D(F^{m+1}(A), F^m(A)) = D(F^m(F(A)), F^m(A))$$
$$\le \underbrace{\lambda^m D(F(A), A)}_{\text{call this } \varepsilon_m}$$

Note: $\sum \varepsilon_m < \infty$, $|\lambda| < 1$. Therefore $(F^m(A)) \le K_n$ is strongly Strongly Cauchy.

technically we didn't define it in general space. but just change R defn. from $\| \ \|$ to D.

$\Rightarrow (F^m(A))$ is cauchy $\Rightarrow F^m(A) \to A^* \in K_n$ since K_n is complete

Since F is continuous $F^{m+1}(A) \to F(A^*)$ Hence, $F(A^*) = A^*$

Suppose A^*, B^* are fixed points for F. then

$$D(A^*, B^*) = D(F(A^*), F(B)) \le \lambda D(A^*, B^*) \quad \lambda \in (0,1)$$
$$\Rightarrow D(A^*, B^*) = 0 \Rightarrow A^* = B^*$$

■

Chapter 10 - Complex functions

assume polar form. and complex algebra

Defn $f(z)$, $\mathbb{C}$ - valued, $z \in \mathbb{C}$

these are modulus not absolute value

① For $z_0 \in \mathbb{C}$, we say $\lim_{z \to z_0} f(z) = L \in \mathbb{C}$ iff $\forall \varepsilon > 0, \exists \delta > 0, 0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$

② We define the derivative of $f(z)$ at z_0 by $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$, provided the limit exists.

Defn: Let $a \in \mathbb{C}$ be a fixed point of $f(z)$.

1. We say a is attracting $\Leftrightarrow |f'(a)| < 1$
1. We say a is repelling $\Leftrightarrow |f'(a)| > 1$
1. We say a is neutral $\Leftrightarrow |f'(a)| = 1$

Remark: We obtain the $\mathbb{C}$ - analouge of the attracting / repelling fixed point theorems.

Ex) $F(z) = z^2 + z + 1$

Fixed points: $z^2 + z + 1 = z \iff z^2 + 1 = 0 \iff z = \pm i$

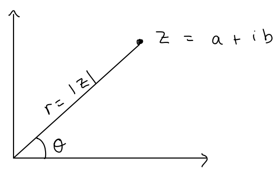
$f'(z) = 2z + 1$

$|f'(i)| = |2i + 1| = \sqrt{4+1} = \sqrt{5} > 1$

$|f'(-i)| = |-2i + 1| = \sqrt{5} > 1$

$\Rightarrow$ repelling

Recall



and $\theta = [0, 2\pi)$

$\therefore z = r(\cos\theta + i \sin\theta) = re^{i\theta}$

Fact: $e^{i\theta} \cdot e^{i\phi} = e^{i(\theta + \phi)}$

$(re^{i\theta})^n = r^n e^{in\theta}$

Recall: $\left(e^{\frac{2\pi i}{n}}\right)^n = e^{2\pi i} = 1$ This gets of periodic points

n^{th} roots of unity.

Lec 24 - Mar 11th 2023

Assigment 2 due to morrow.
hint:
2: not chaotic, focus on transitivity

Ex) $z = e^{2\pi i/3}$

$= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$f(w) = w^2$ → very easy in this form

$f(z) = \left(e^{2\pi i/3}\right)^2 = e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ → from cos & sin

$f^2(z) = e^{8\pi i/3} = e^{2\pi i/3} = z$ → reducing by 2π?

period 2

derivative formula from before works in $\mathbb{C}$

$|(f^2)'(z)| = |f'(z)\cdot f'(f(z))|$

$= |(-1 + i\sqrt{3})|\cdot|(-1 - i\sqrt{3})|$

$= 2\cdot 2 = 4 > 1$

$\therefore$ z is a repelling periodic point for $f(w)$

Chapter 11 Julia Sets

Notation: For $c \in \mathbb{C}$, $Q_c(z) = z^2 + c$

Def'n: Let $c \in \mathbb{C}$. Then the filled Julia Set for c is $K_c = \{z \in \mathbb{C} : (Q_c^n(z)) \text{ is bounded}\}$ → orbit is bounded.

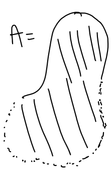
$= \{z \in \mathbb{C} : \exists M > 0, \forall n \in \mathbb{N}, |Q_c^n(z)| \leq M\}$ → complex analouge of Λ

Remark: This is the complex analouge of Λ for $Q_c(x) = x^2 + c$ where $c \in \mathbb{R}$ and $c < -2$.

Def'n (X, d), $A \subseteq X$

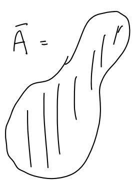
1. the closure of A is $\bar{A} = \{x \in X, \exists (a_n) \subseteq A \text{ s.t. } a_n \to x\}$ → includes point on boundry which may not be in the set.
2. the interior of A is $\text{int}(A) = \{x \in A : \exists \varepsilon > 0, B_\varepsilon(x) \subseteq A\}$
3. the boundary of A is $\partial(A) = \bar{A} \setminus \text{Int}(A)$

Ex) In $\mathbb{C}$ $A =$



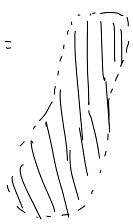
Interesting peanut.

$\bar{A} =$



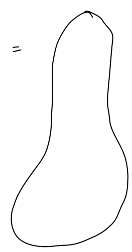
Whole Shape of peanut

$\text{Int}(A) =$



open filled peanut

$\partial(A) =$



Outline of peanut

Remark : A is closed iff $A = \bar{A}$

Assignment 4: Will show K_c is closed → means J_c wont contain any new points from the filled Julia Set

Def'n: $c \in \mathbb{C}$. The Julia Set for c is $J_c = \partial(K_c)$

Remark: $J_c = \partial(K_c) = \overline{K_c} \setminus \text{Int}(K_c) = K_c \setminus \text{Int}(K_c)$

Ex) $c = 0$, $Q_0(z) = z^2$

$z = r e^{i\theta}$ → r is modulus of z
→ $\cos^2 + \sin^2 = 1$

$|Q_0(z)| = |r^2 e^{2i\theta}| = r^2$

$|Q_0^2(z)| = |r^4 e^{4i\theta}| = r^4$

$|Q_0^3(z)| = |r^8 e^{8i\theta}| = r^8$

$\vdots$

$|Q_0^n(z)| = r^{2^n}$

if $r \in [0,1] \Rightarrow Q_0^n(z)$ bounded

$K_0 = \{ z \in \mathbb{C} : |z| \leq 1 \}$.

$J_0 = \{ z \in \mathbb{C} : |z| = 1 \}$. → unit circle

Ex) $c = -2$

$Q_{-2}(z) = z^2 - 2$

① $R = \{ z \in \mathbb{C} : |z| > 1 \}$

$H: R \to \mathbb{C}$

$H(z) = z + \frac{1}{z}$

} not clear at first.
H is a translation function, aim to bridge information b/w Q_0 and Q_{-2}

② Claim H is Injective: $H(z) = H(w) \Rightarrow z + \frac{1}{z} = w + \frac{1}{w}$ ($\times z$) ($\times w$)

$\Rightarrow zw = z^2 + 1 - \frac{z}{w} = w^2 + 1 - \frac{w}{z}$

$\Rightarrow w^2 - z^2 = \frac{w}{z} - \frac{z}{w} = \frac{w^2 - z^2}{zw}$ → only if $zw = 1$ or $w^2 - z^2$

But $zw \neq 1$ b/c $|z|$ and $|w| > 1$

Since $w, z \in R$, we have $w^2 - z^2 = 0 \Rightarrow w = \pm z$.

Since $H(w) = H(z) \Rightarrow w = z$

③ $H: R \to \mathbb{C} \setminus [-2, 2]$ is surjective

$H(z) = w \Leftrightarrow z + \frac{1}{z} = w$

$\Rightarrow z^2 - wz + 1 = 0$

$\Rightarrow z_{\pm} = \frac{1}{2}(w \pm \sqrt{w^2 - 4})$

Note: $z_+ z_- = 1$

if $z_+ \in R$ or $z_- \in R$ $H(z_+) = \omega$ or $H(z_-) = \omega$. Otherwise $|z_\pm| = 1$.

Say $z_\pm = e^{i\theta}$

$H(z\pm) = H(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos\theta \in [-2, 2]$

Lec 25 - Mar 13th 2024

Recall: $K_c = \{z \in \mathbb{C} : Q_c^n(z) \text{ bounded}\}$

$J_c = \partial(K_c)$

$K_0 = \{z : |z| \le 1\}$

$J_0 = \{z : |z| = 1\}$

Ex: $c = -2$ $Q_{-2}(z) = z^2 - 2$

① $R = \{z : |z| > 1\}$ $H : R \to \mathbb{C}$ $H(z) = z + \frac{1}{z}$

② H injective

③ $H : R \to \mathbb{C} \setminus [-2, 2]$ surjective

④ $H(Q_0(z)) = H(z^2) = z^2 + \frac{1}{z^2}$

$z^2 + \frac{1}{z^2} + 2\left(\frac{z}{z}\right) - 2$

$Q_{-2}(H(z)) = (z + \frac{1}{z})^2 - 2 = z^2 + \frac{1}{z}$

$H(Q_0^n(z)) = Q_{-2}^n(H(z))$

said to be conjugate but we won't define it.

H is playing the same role

Compare:

$S(Q_c^n(x)) = \sigma^n(S(x))$

⑤ $|z_n| \to \infty$

$|H(z_n)| = |z_n + \frac{1}{z_n}|$

$\ge |z_n| - |\frac{1}{z_n}| \to \infty$ → Reverse triangle inequality

→ How does a complex # go to infinity?

$\Rightarrow |H(z_n)| \to \infty$

⑥ $z \in \mathbb{C} \setminus [-2, 2]$ $z = H(w)$, $w \in R$

$|Q_{-2}^n(z)| = |Q_{-2}^n(H(w))|$

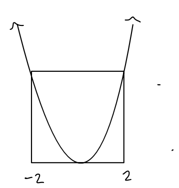
$= |H(Q_0^n(w))| \to \infty$

→ b/c $w \in R$, $z > 1 \Rightarrow z \notin K_0 \Rightarrow Q_0^n(z) \to \infty$

$\therefore z \notin K_{-2}$

⑦ Let $z \in [-2, 2]$

By graphical analysis, $z \in K_c$ bounded



Therefore $K_{-2} = [2, 2]$

$J_{-2} = [-2, 2]$ → this is the boundary in the complex plane

These are the only 2 Julia sets that aren't fractals

[Escape crieteria]

Proposition If $|z| \geq |c| > 2$ then $|Q_c^n(z)| \to \infty$. In particular, $z \notin K_c$

proof: we have $|Q_c(z)| = |z^2 + c| \geq |z|^2 - |c|$ Reverse triangle from proposition

$\geq |z|^2 - |z|$

$= |z|(|z| - 1)$

Say $|z| > 2 + \lambda$, $\lambda > 0$. $\Rightarrow$ $|z| - 1 > 1 + \lambda$

Hence $|Q_c(z)| \geq |z|(1+\lambda)$

$\Rightarrow$ $|Q_c^n(z)| \geq |z|(1+\lambda)^n$ → How come $|z|$ doesn't get raised to the n

and $(1+\lambda)^n \to \infty$ hence $|Q^n_c(z)| \to \infty$

Corollary: $|c| > 2$ Then $|Q_c^n(0)| \to \infty$ so that $0 \notin K_c$

Why?

$z = Q_c(0) = c$

$|z| = |c| > 2$

$\Rightarrow$ $|Q_c^n(z)| \to \infty$ By pervious proposition

$\Rightarrow$ $|Q_c^n(0)| \to \infty$

Corollary: $M = \max\{|c|, 2\}$ some complex #

If $|z| > M$ then $|Q^n_c(z)| \to \infty$. Thus, $K_c \subseteq \{z : |z| \leq M\}$ hence bounded sets

Why?

$|Q_c^n(z)| \geq (1+\lambda^n)|z| \to \infty$

Cannot use escape criterion but use exact same proof b/c the assumption $|z| \geq |c| > 2$ in the Escape Criterion isn't met.

Hint we haven't showed K_c is closed but knowing $K_c \subseteq \{z : |z| \leq M\}$ helps in showing that

Corollary: if $\exists\, k$ s.t $|Q_c^k(z)| > \max\{|c|, 2\}$ then $|Q_c^n(z)| \to \infty$ thus $z \notin K_c$

Algorithm:

1. Choose a large $N \in \mathbb{N}$
2. if $|Q_c^i(z)| > \max\{|c|, 2\}$ for $i \leq N$ colour z white
3. if $|Q_c^i(z)| \leq \max\{|c|, 2\}$ for all $i \leq N$, colour z black

The black shaded region approximates K_c