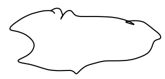
Creating a Sundial by Joaco Prandi - Mar 27th 2024

Speaker: Joaco Prandi, PhD Candidate, Department of Pure Mathematics, University of Waterloo

Title: Creating a Sundial

Abstract: In this talk, we will try to understand how to make a sundial. Not just any sundial, but a sundial that requires too much math and probably will never be built. At least it will give the time in a digital format.

2 New dimension. 2^{nd} most helpful. Take a set like  draw a grid over it and count the amount of boxes

Call it $N_\delta(K) \approx \delta^s$

Box counting dimension $\dim_B(K) \lim_{\delta \to 0} \dfrac{\log(N_\delta(K))}{-\log(\delta)}$ → lim doesn't always exist

$$\overline{\dim}_B(K) = \lim_{\delta \to 0} \sup \frac{\log(N_\delta(K))}{-\log(\delta)}$$

$$\underline{\dim}_B(K) = \lim_{\delta \to 0} \inf \frac{\log(N_\delta(K))}{-\log(\delta)}$$

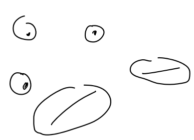
these always exist

$N_\delta(K)$ can be replace to: Maximal Number of balls of radius δ centered in K with them being disjonit

Or Minimal # of balls of radius δ centered at K covering the set.

What happens if the boxes are different sizes? We need a measure b/c counting is sufficent.

Define $\mathcal{H}^s_\delta(K) = \inf \left\{ \sum_{k=0}^{\infty} |U_i|^s : \bigcup_{i=0}^{\infty} U_i \supset K \text{ and } |U_i| \le \delta \right\}$

Ex  smallest is measure

as δ gets smaller __________________?

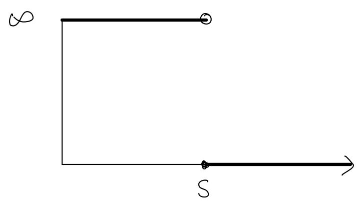
$\mathcal{H}^s(k) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(k)$ for most sets its either 0 or infinite

In $\mathbb{R}^d$ and $n \le d$ then $\mathcal{H}^n$ is measuring n-dimensionally.

$\mathcal{H}^1$ = length, $\mathcal{H}^2$ = area, $\mathcal{H}^3$ = Volume. So on.

$\mathcal{H}^s(k)$ can think of as a function of S. If $s < t$; then if $\mathcal{H}^T(k)$ is positive ($\mathcal{H}^T(k) > 0$) then $\mathcal{H}^s(k) > \infty$

If $\mathcal{H}^s(k) < \infty$ (finitie) then then $\mathcal{H}^T(k) = 0$



The Hausdorff dimension is $\dim_H = \sup\{s : \mathcal{H}^s(k) \text{ is infinite}\}$
$= \inf\{s : \mathcal{H}^s(k) \text{ is zero}\}$.

Facts: If C is the ternary Cantor Set. then $\dim_B(C) = \dim_H(C) = \frac{\log(2)}{\log(3)}$

$F = \{\frac{1}{n}\}_{n=0}^{\infty}$ $\quad \dim_B(F) = \frac{1}{2}$, $\quad \dim_B(F) = \dim_B(\bar{F})$ closure of Set

$\dim_H(F) = 0$, $\quad \dim_H(F) \neq \dim_H(\bar{F})$ in general

Fact: If $\dim_H(F) < 1$ then the set is totally disconnected.

Fact the function f is Lipschitz with ratio c if $\mathcal{H}^s(f(k)) \le c^s \mathcal{H}^s(k)$

like a contraction but the contraction constant doesnt have to be between 0 & 1

$$\Rightarrow \quad \dim_H(f(K)) \leq \dim_H(K)$$

<u>Projections:</u> (orthogonal projections)

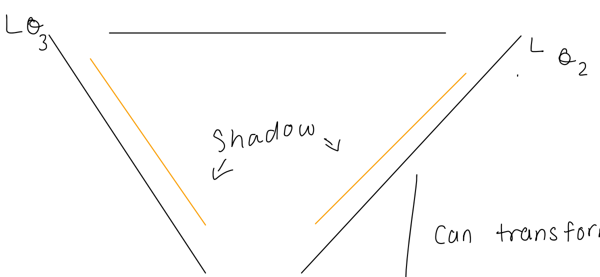
Let L_θ be the the line through the origin with angle $\theta \in [0,\pi]$ and $\text{Proj}_\theta(F)$ is the orthogonal projection to L_θ. In general the are Lipschitz.

Theorem: For almost all θ

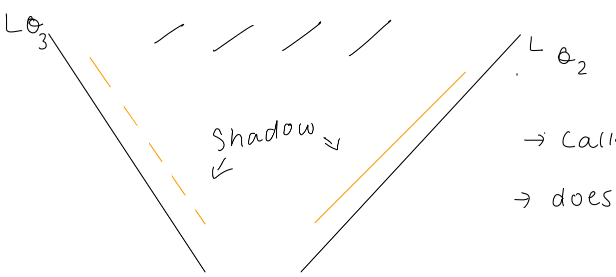
↗ stronger than pervious.

a) If $\dim_H(f) \leq 1$ then $\dim_H(\text{proj}_\theta(f)) = \dim_H(f)$

b) If $\dim_H(f) > 1$ then $\dim_H(\text{Proj}_\theta(f)) = 1$ and $\mathcal{H}^1(\text{Prog}_\theta(f)) > 0$



Can transform by cutting and rotating



→ Calle Iterated venetian (blind) process

→ doesnt work ∀ lines, but can for some angles.

Theorem Let $G_\theta \subset L_\theta$, $\theta \in [0,\pi)$ be a collection of sets s.t. $\bigcup_\theta G_\theta$ is measureable in 2 dimensions. then there exists a set $F \subseteq \mathbb{R}^2$ s.t. $G_0 \subset \text{Proj}_\theta(F)$ and $\mathcal{H}^1(\text{Proj}_\theta(F) \setminus G_0) = 0$ for almost all θ.

This is in $\mathbb{R}^2$

In theorey wecan use this to make a Sundial.

Youtube video for Sundial

Logistics

Q3 is ugly piazza is the way

and Course perception survey out

Guest Speaker: Paul Fieguth - April 1st 2024

The role of bifurcation in Continuous-time and discrete time Dynamical Systems

Book:

Complex System: many elements, interacting, non linear.

Problem: all uni courses focus on linear, Gaussian and small.
all major world issues are non linear, non Gausian, and large.
↳ black swan events

his aim is to teach ppl humility when approaching world issues

Bifurcations in ~~non~~ Linear Dynamics

Continuous : $\dot{z}(t) = f(z(t), \theta)$ — parameter

very facisnating connections between the two.

Discreete : $z_{n+1} = \bar{f}(z_n, \theta)$

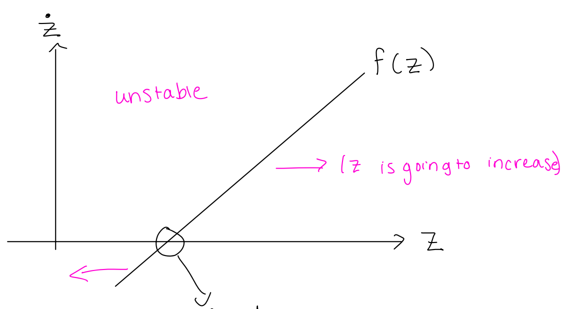
Basic motivation for Discrete time is as time-discretization of continuous time.

Cont $\dot{z}(t) = \lim_{\delta \to 0} \dfrac{z(t+\delta) - z(t)}{\delta}$

$\therefore \quad \dot{z} = f(z)$ and $z(t+\delta) = z(t) + \delta \cdot f(z)$
↓
Discrete time known as Forward Euler

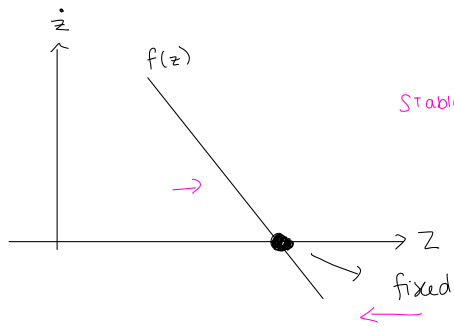
There are other C.T to D.T possibilities (later)

System Diagram



un stable fixed point

convention is un filled if unstable



fixed point Stable

convention is filled when Stable

Linear b/c relation of $\delta(z)$ is linear

Aside: D.T. uses $y = x$ to find fixed points

Linear System: Key Attributes.

① Super position : $x_1 \to \square \to y_1$ $\quad x_2 \to \square \to y_2$

then $\alpha x_1 + \beta y_1 \to \square \to \alpha y_1 + \beta y_2$

Ex Streching Spring obeys super position

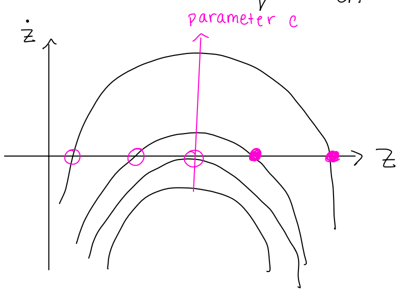
② Sine wave $A \sin(\omega t + \phi) \to \square \to \beta \sin(\omega t + \varphi)$

<u>Cannot</u> change frequency.

∴ Constant input cannot lead to oscillating out put.

every electronic using a battery is non linear, b/c constant input but it buzzes, flashes

Defn: A bifurcation is a dis continuous change in attribute or behaviour in response to a continuous change in para meter.



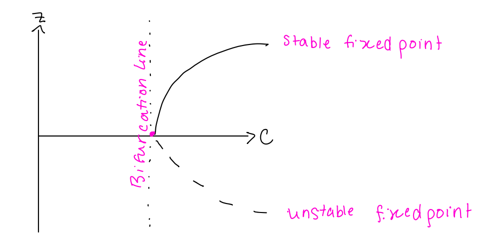
$$\dot{z} = -(z-5)^2 + c$$

Double root is called degeneracy.

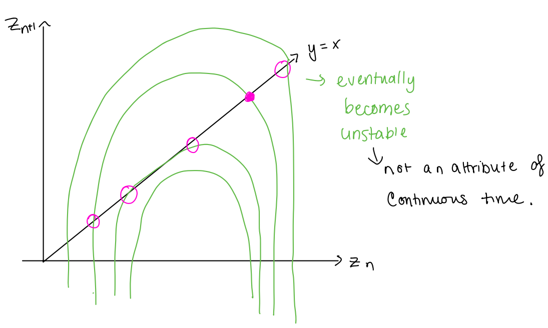
realisticly never happen.

An example of bifurcation 0 fixed to 2 fixed.

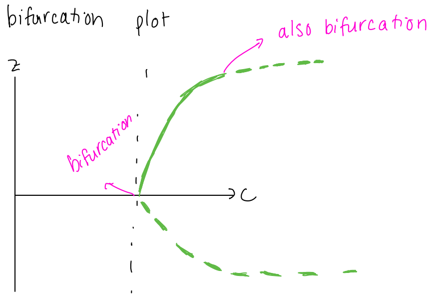
Bifurcation Plot



Also have bifurcations in D.T

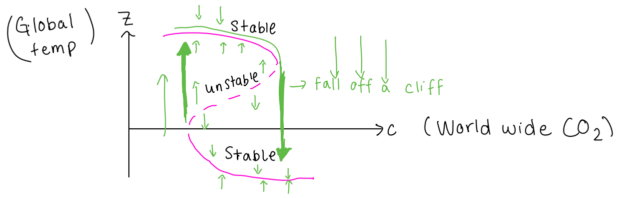


bifurcation plot

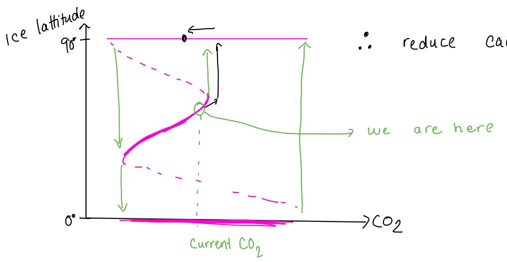


Two major bifurcations:

① Double - fold:



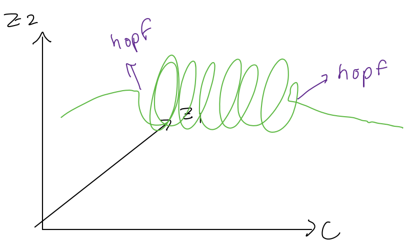
Bistable system another example light switch, temperature of fridge. nearly all thermal.



∴ reduce carbon rn. b/c if we go past our current stable point we would need to reduce alot before getting to our stable nice state

② Hopf bifurcation:

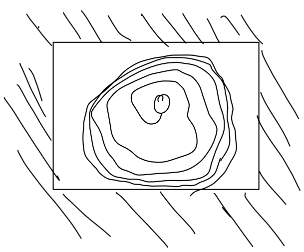
Transition between cycling and not cycling.



Why is cycling common?

Ex Flag pole, guiser, Sailboat mass. poring water on live plants

b/c bouneded unstable system.

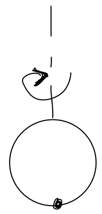


Lineeant cross

→ limit cycle.

Bifurcations that matter to us:

- Stick - Slip. (Brakes locking, rude and unpleasent noises)
- covid : fizzling out vs. pandemic
- Bead on a hoop



- Toys: slap braclet, jumping disc, Candle boats
- Ecology, : lake ecology
- Humain brain, epilectic seizures,
- Chocolate production