

# Highlighting Legend

Chapter

Definition

Propositions / Theorems / Proofs

Procedures

Remarks are made in red pencil, same with questions

# Lecture 1 - Jan 8<sup>th</sup> 2024

## Chapter 1 Iteration and Orbits

**Defn** Let  $f: A \rightarrow \mathbb{R}$  such that  $A \subseteq \mathbb{R}$  and  $f(A) \subseteq A$  [ i.e  $f: A \rightarrow A$  ]. For  $a \in A$ , we may iterate the function at  $a$ :

$$\begin{aligned}x_1 &= a \\x_2 &= f(a) \\x_3 &= f(f(a)) = f^2(a) \\x_4 &= f(f(f(a))) = f^3(a) \\&\vdots\end{aligned}$$

all will exist in  $A$ .

We call  $(x_n)_{n=1}^{\infty} (= (x_n))$  we call that sequence the orbit of  $a$  under  $f$ .

Ex.  $f(x) = x^4 + 2x^2 - 2$ ,  $a = -1$

$$\begin{aligned}x_1 &= a, x_2 = f(a), x_3 = f^2(a) \dots \\-1, 1, 1, \dots &\rightarrow \text{eventually constant / periodic}\end{aligned}$$

Ex  $f(x) = -x^2 - x + 1$ ,  $a = 0$

$$\begin{aligned}x_1, x_2, x_3, x_4, x_5, x_6 \dots \\0, 1, -1, 1, -1, 1 \dots &\rightarrow \text{eventually periodic in period 2}\end{aligned}$$

Ex.  $f(x) = x^2 - 3x + 1$ ,  $a = 1$

$$\begin{aligned}x_1, x_2, x_3, x_4, \dots \\1, -1, 3, 19 \rightarrow \infty \quad [\text{he will talk about convergence / divergence later on}]\end{aligned}$$

Ex.  $f(x) = x^2 + 2x$ ,  $a = -0.5$

My Q: Does it ever equal 1, is that a distinction?

$$\begin{aligned}-0.5, -0.75, -0.9375, -0.9961, \dots \text{ converges to } -1\end{aligned}$$

Ex.  $f(x) = x^3 - 3x$ ,  $a = 0.75$

$$\begin{aligned}0.75, -1.828, -0.625, 1.631, -0.552, \dots\end{aligned}$$

This is chaotic behaviour.

\* For P-MATH insights: an interval around zero is dense \*

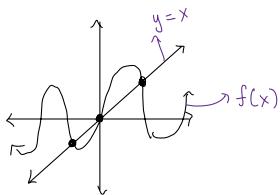
**Defn:**  $f: A \rightarrow \mathbb{R}$ ,  $f(A) \subseteq A$ , We say  $a \in A$  is a fixed point in  $f$  iff  $f(a) = a$ . In this case, the orbit of  $a$  is  $a, a, a, \dots$  which is constant.

Ex. Find all fixed points of  $f(x) = x^2 + x - 4$

When does  $x^2 + x - 4 = x$ ?

$$\Leftrightarrow x^2 - 4 = 0 \Leftrightarrow x = \pm 2 \quad \text{Will often give low degree polynomial}$$

Ex. How many fixed points does  $f(x)$  as pictured below have?



Geometrically, a fixed point occurs when  $f(x)$  intersects  $y=x$

Ex. Prove that  $f(x) = x^4 - 3x + 1$  has a fixed point.

Solve for  $x^4 - 3x + 1 = x$

$$\Rightarrow x^4 - 4x + 1 = 0 \quad \text{Intermediate Value theorem}$$

Since  $g(x)$  is continuous  $g(0) = 1 > 0$  and  $g(1) = -2 < 0$ . By IVT  $\exists x \in (0, 1)$  s.t.  $g(x) = 0 \Leftrightarrow f(x) = x$ .

Defn  $f: A \rightarrow \mathbb{R}$ ,  $f(A) \subseteq A$

① We say  $a \in A$  is a **periodic point** for  $f$  if its orbit is periodic. I.E.  $\exists n \in \mathbb{N} \quad f^n(a) = a$ . The least such  $n$  is called the period of  $a$  and/or the orbit.

② Eventually periodic  $\exists n < m, f^n(a) = f^m(a)$

## Lecture 2 - Jan 10<sup>th</sup> 2024

### Def'n [Doubling Function]

$D: [0,1] \rightarrow [0,1]$  where  $D(x) = \text{fractional part of } 2x$  aka  $2x \bmod 1$

Ex)  $D(0.4) = 0.8$

$$D(0.6) = 0.2$$

$$D(0.8) = 0.6$$

$$D(0.5) = 0$$

It is an important function as it also provides a rich source of periodic orbits

Ex)  $D, a = \frac{1}{5}$

orbit:  $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \dots$  This has period 4. This leads into some cute number theory with GCDs

Ex)  $D, a = \frac{1}{20}$

orbit  $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \dots$  This is eventually periodic with period 4.

The Doubling function will come up later in a more meaningful way

Every day functions may not exhibit periodic orbits

Q: Given  $f$  and  $a$ , does  $f^n(a)$  tend towards some limit  $L$ ? This does happen surprisingly often

The language of this course vs Elementary Real Analysis

Notation:

If  $(x_n)_{n=1}^{\infty}$  is a sequence of real numbers we write  $(x_n) \subseteq \mathbb{R}$  → a small abuse of notation but reasonable

Def'n  $(x_n) \subseteq \mathbb{R}$ ,  $x \in \mathbb{R}$  we say  $(x_n)$  converges to  $x$  iff for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  s.t.  $|x_n - x| < \epsilon$  for all  $n \geq N$

We write  $x_n \rightarrow x$  or  $\lim x_n = x$

↓  
N depends \*

Ex) Claim:  $\frac{1}{n} \rightarrow 0$ , Let  $\epsilon > 0$  be given.

Note:  $\frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$

Consider  $N = \frac{2}{\epsilon}$  For  $n \geq N$  we have  $\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \epsilon$  not necessarily a Natural number. Can always use ceiling function  $\lceil \cdot \rceil$  to bring this to a natural number  
↓ bring this to  $\frac{2}{\epsilon}$  so that we ensure strictly less than epsilon.

Ex) Claim  $\frac{2n}{n+3} \rightarrow 2$

Let  $\epsilon > 0$  be given. Let us choose  $N \in \mathbb{N}$  s.t.

$$\frac{1}{N} < \frac{\epsilon}{6}$$

For  $n \geq N$

$$\begin{aligned} & \left| \frac{2n}{n+3} - 2 \right| \\ &= \left| \frac{2n}{n+3} - \frac{2n+6}{n+3} \right| \\ &= \left| \frac{-6}{n+3} \right| = \frac{6}{n+3} < \frac{6}{n} \xrightarrow{\text{if you divide by less of something you get something bigger}} \\ & \frac{6}{n} \leq \frac{6}{N} = 6\left(\frac{1}{N}\right) < 6\left(\frac{\epsilon}{6}\right) = \epsilon \end{aligned}$$

Defn  $(x_n) \subseteq \mathbb{R}$  we say  $(x_n)$  is bounded if  $\exists M > 0$  s.t.  $\forall n \in \mathbb{N}, |x_n| \leq M$

Prop  $(x_n) \subseteq \mathbb{R}$  If  $(x_n)$  is convergent, then  $(x_n)$  is bounded.

Ex)  $x_n = (-1)^n$  Shows converse is not true

Proof Suppose  $x_n \rightarrow x$ . then  $\exists N \in \mathbb{N}, n \geq N \Rightarrow |x_n - x| < 1$  picked \epsilon, blake's favourite is 1

For  $n \geq N$ ,

$$|x_n| - |x| \leq |x_n - x| < 1 \Rightarrow |x_n| < 1 + |x| \quad \text{This is only true for } n \geq N$$

this is the reverse triangle inequality, make sure you know how to prove from triangle inequality

So Let  $M = \max \{|x_1|, \dots, |x_{N-1}|, 1 + |x|\}$  ■

Prop Let  $x_n \rightarrow x, y_n \rightarrow y$

Not super important to us,  
value is in working with  
defn of convergence

①  $x_n + y_n \rightarrow x + y$

②  $x_n y_n \rightarrow xy$

Proof:

① Let  $\epsilon > 0$  be given. There exists  $N_1, N_2 \in \mathbb{N}$  s.t.

$$n \geq N_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$$

$$n \geq N_2 \Rightarrow |y_n - y| < \frac{\epsilon}{2}$$

For  $N = \max\{N_1, N_2\}$  and  $n \geq N$

$$|x_n + y_n - (x+y)| = |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y| < \varepsilon$$

(2) Let  $\varepsilon > 0$  be given

Note •

$$|x_n y_n - xy| = |x_n y - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - y| \quad \star$$

Since  $(x_n)$  is bounded  $\exists M > 0, \forall n, |x_n| < M$

Let  $N_1, N_2 \in \mathbb{N}$  s.t.  $n \geq N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2(|y|+1)}$  make sure not dividing by zero

$$n \geq N_2 \Rightarrow |y_n - y| < \frac{\varepsilon}{2M}$$

For  $n \geq N := \max\{N_1, N_2\}$  we have  $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$  by \star ■

## Lec 3 - Jan 12<sup>th</sup> 2024

Def'n we say  $(X_n) \subseteq \mathbb{R}$  is Cauchy if  $\forall \epsilon > 0, \exists N \in \mathbb{N}, n, m \geq N \Rightarrow |X_n - X_m| < \epsilon$

Point of Clarification. The Wednesday class before the assignment is due. We will cover all the material needed for the assignment. Also we can use calculus 2. A lot of this course is in the language of Calculus and Real Analysis

Prop Convergent  $\Rightarrow$  Cauchy

Proof: Let  $\epsilon > 0$  be given and suppose  $(X_n)$  is convergent. Say  $X_n \rightarrow x \in \mathbb{R}$ . There exist  $N \in \mathbb{N} \quad n \geq N$

$$\Rightarrow |X_n - x| < \boxed{\frac{\epsilon}{2}} \quad \text{Then for } n, m \geq N$$

both are  $< \epsilon$  which gives information on what should be in the box

$$|X_n - X_m| = |X_n - x + x - X_m| \leq |X_n - x| + |x - X_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

take  $\epsilon$  divided by # of parts

□

Fact:  $(X_n) \subseteq \mathbb{R}$ ,  $(X_n)$  Cauchy  $\Leftrightarrow$   $(X_n)$  Convergent This is also a part of the Completeness of  $\mathbb{R}$

Big Idea: To prove  $(X_n)$  is Cauchy you do not have to guess the limit!  $\hookrightarrow$  this will be useful for fixed points

Def'n  $f: A \rightarrow \mathbb{R}$ ,  $A \subseteq \mathbb{R}$ ,  $a \in A$ . we say  $f$  is continuous at  $a$  iff  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $|f(x) - f(a)| < \epsilon$  whenever  $x \in A$  and  $|x - a| < \delta$ .

$\rightarrow$  close values in the domain gives close values in function

Fact:  $f$  is continuous at  $a$  iff  $\forall (X_n) \subseteq A$  with  $X_n \rightarrow a$ , we have  $f(X_n) \rightarrow f(a)$   $\hookrightarrow$  function at the terms in the sequence provides link between convergence and continuity.

↳ can prove as an exercise

highly dependent on that  $f$  goes from  $[a, b] \rightarrow [a, b]$

Prop: If  $f: [a, b] \rightarrow [a, b]$  is continuous then  $f(x)$  has a fixed point

Proof: we know  $f(a) \geq a$  and  $f(b) \leq b$ .  $\Leftrightarrow f(a) - a \geq 0$  and  $f(b) - b \leq 0$

By applying the IVT to the continuous function  $g(x) = f(x) - x \quad \exists x \in [a, b]$  such that  $g(x) = 0 \Leftrightarrow f(x) = x$  □

$\hookrightarrow$  a preview of why Calculus is applicable to this course. Epsilon will come back.

Def'n  $f: A \rightarrow \mathbb{R}$ ,  $A \subseteq \mathbb{R}$ . we say that  $f$  is a contraction iff  $\exists C \in [0, 1)$  s.t.  $\forall x, y \in A$ ,  $|f(x) - f(y)| < C|x - y|$   $\hookrightarrow$  is a type of Lipschitz fn  $\rightarrow$  would that mean fn. is below line  $y = x$  graphically?

Prop Contractions are continuous at every point

Proof Let  $\epsilon > 0$  be given Suppose  $|f(x) - f(y)| < C|x - y|$  as before. Fix  $y \in A$ , Consider  $\delta = \frac{\epsilon}{C+1}$   $\hookrightarrow$  to make sure we aren't dividing by 0

and assume  $x \in A$  s.t.  $|x-y| < \delta$ . Then

$$|f(x) - f(y)| \leq C|x-y| < C\delta < \epsilon \quad \square$$

Say function  $a$  problematic, not a contract b/c it drops off at 0

**Defn** We say  $A \subseteq \mathbb{R}$  is **closed** iff whenever  $(x_n) \subseteq A$  with  $x_n \rightarrow x \in \mathbb{R}$  then  $x \in A$ .

Ex.  $[a, b]$  is closed

Ex.  $(0, 1]$  not closed  $\frac{1}{n} \rightarrow 0 \notin (0, 1]$

Theorem [Banach Contraction Mapping Theorem] Suppose  $A \subseteq \mathbb{R}$  is closed and  $f: A \rightarrow A$  is a contraction.

Then there exist a unique fixed point  $a \in A$  for  $f$ . Moreover  $\forall x \in A$   $f^n(x) \rightarrow a$    
 ↗ has acute proof. allegedly.  
 will prove on monday

Ex.  $f: [0, 1] \rightarrow [0, 1]$   $f(x) = \frac{1}{3-x}$

Note:  $\frac{1}{3} \leq f(x) \leq \frac{1}{2}$  ↗ shows it maps back into  $[0, 1]$   
 ↗ also not an onto function

$$f'(x) = \frac{1}{(3-x)^2}, \quad \frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$$

By the MVT,  $\forall x, y \in [0, 1]$   $\exists c \in (0, 1)$  s.t.  $f(x) - f(y) = f'(c)(x-y)$

$$\begin{aligned} \Rightarrow |f(x) - f(y)| &= |f'(c)| |x-y| \\ &\leq \underbrace{\frac{1}{4}}_c |x-y| \end{aligned}$$

∴  $f(x)$  is a contraction.

$$\frac{1}{3-x} = x$$

$$\Leftrightarrow 1 = 3x - x^2$$

$$\Leftrightarrow x^2 - 3x + 1 = 0$$

$$\Leftrightarrow x = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\Leftrightarrow x = \frac{3-\sqrt{5}}{2}$$

$$\therefore \forall x \in [0, 1] \quad f^n(x) \rightarrow \frac{3-\sqrt{5}}{2}$$

Dec 4 - Jan 15<sup>th</sup> 2023

Note: Graphical Analysis tool posted on piazza. Assignment 1 due next week.

Remark: The Banach Contraction Mapping is almost like a black hole

Recall:  $(a_n) \subseteq \mathbb{R}$  we say  $(a_n)$  is **Strongly Cauchy** if  $\exists \epsilon_0 \in [0, \infty)$  s.t.

$$\textcircled{1} \quad \sum_{n=1}^{\infty} \epsilon_n < \infty$$

$$\textcircled{2} \quad \forall n, |a_n - a_{n+1}| < \epsilon_n$$

This is on assignment.

Hint:  $\sum_{n=1}^{\infty} a_n = L, \sum_{k=1}^{\infty} a_k \xrightarrow{n \rightarrow \infty} L$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$  s.t.  $n \geq N \Rightarrow \left| \sum_{k=N+1}^{\infty} a_k \right| < \epsilon$

### Proof of Banach Contraction Mapping

Let  $A \subseteq \mathbb{R}$  be closed and suppose  $\exists c \in [0, 1)$  s.t.  $|f(x) - f(y)| \leq c|x-y|$  for all  $x, y \in A$ . Take a point  $x_0 \in A$

and construct  $x_1 = f(x_0), x_2 = f(x_1), \dots, x_n = f(x_{n-1}) = f^n(x_0)$  this is the orbit.  $\rightarrow$  we aren't including  $x_0$  in the orbit but it doesn't matter

$$\begin{aligned} \text{For } n \in \mathbb{N} \quad |x_{n+1} - x_n| &\stackrel{\text{defn of orbit}}{=} |f(x_n) - f(x_{n-1})| \stackrel{\text{contraction assumption}}{\leq} c|x_n - x_{n-1}| = c|f(x_{n-1}) - f(x_{n-2})| \leq c^2|x_{n-1} - x_{n-2}| \\ &\vdots \\ &\leq c^n|x_1 - x_0| \end{aligned}$$

$c \in [0, 1)$  this is geometric

Since  $\sum_{n=1}^{\infty} c^n|x_1 - x_0|$  is a convergent geometric series, we have that  $(x_n)$  is strongly Cauchy. Hence  $x_n \rightarrow a$  for some  $a \in A$

$\hookrightarrow$  This is because domain is closed

Since  $f$  is continuous,  $\underbrace{f(x_n)}_{x_{n+1}} \rightarrow f(a) \therefore f(a) = a \rightarrow \text{B}(c, x_n \rightarrow a, f(x_n) \rightarrow f(a) \Rightarrow f(a) = a)$

$\rightarrow$  What's significant about this?

Suppose  $a, b \in A$ , s.t.  $f(a) = a$  and  $f(b) = b$ . Then  $|f(a) - f(b)| \leq c|a-b| \Rightarrow |a-b| \leq c|a-b|$ . Since  $c < 1$ ,

$|a-b| = 0$  and so  $a = b$ . This is our strongest fixed point theorem. This is the big analytic results for existence and uniqueness of ODEs

### Chapter 2 Graphical analysis.

To visualize the orbit of  $a$  under  $f$ :

- ① Superimpose  $y = f(x)$ ,  $y = x$

(2) Use a vertical line:  $(a, a) \longrightarrow (a, f(a))$

(3) Use a horizontal line:  $(a, f(a)) \longrightarrow (f(a), f(a))$

(4) Vertical line:  $(f(a), f(a)) \longrightarrow (f(a), f^2(a))$

(5) Use a horizontal line:  $(f(a), f^2(a)) \longrightarrow (f^2(a), f^3(a))$

Etc...

Ex) Using online tool and  $f(x) = x^2 - x + 1$ , fixed points  $x=1$

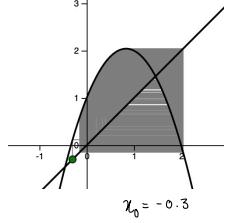
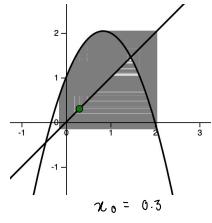
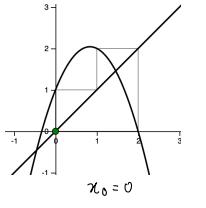
Orbit analysis:

①  $x \in [0, 1]$  we have,  $f^n(x) \rightarrow 1$

② otherwise,  $x \notin [0, 1]$   $f^n(x) \rightarrow \infty$  In assignment, only expecting this sort of an answer.

## Lec 5 - Jan 17<sup>th</sup> 2024

For  $f(x) = \frac{-3}{2}x^2 + \frac{5}{2}x + 1$  at  $x=0$  we have an orbit of  $0, 1, 2$  which is periodic and graphical is a cycle



points near 0, like 0.06 show chaos. The graphical analysis look does show chaos. Chaos also shows density. We can never hit every point in the interval because an orbit is countable but  $[1, 2]$  in  $\mathbb{R}$  is uncountable. It is dense [ex. rational numbers is dense in  $[0, 1] \subset \mathbb{R}$ ]. We will define dense in a bit.

### Chapter 3 - fixed points

Remark. If  $f(x)$  is continuous and  $f^n(a) \rightarrow L$ , then  $f^{n+1}(a) \rightarrow f(L) \rightarrow$  comes from continuity. We also saw this in the B.C.M theorem

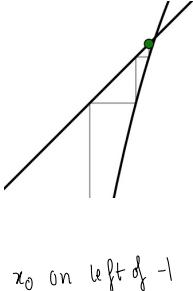
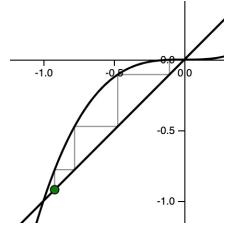
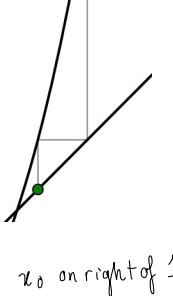
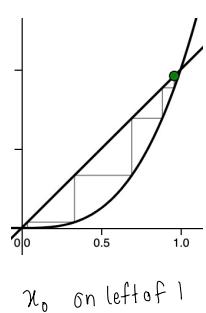
$$\therefore f(L) = L$$

### Motivating Example

$$F(x) = x^3, \text{ fixed points: } 0, \pm 1$$

① for  $x \in (-1, 1)$  we see that  $f^n(x) \rightarrow 0$ . This is an example of an attracting fixed point

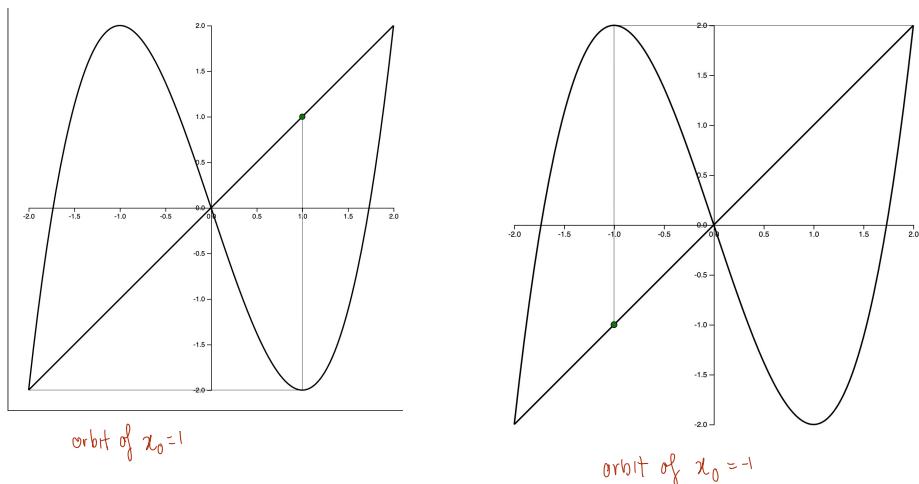
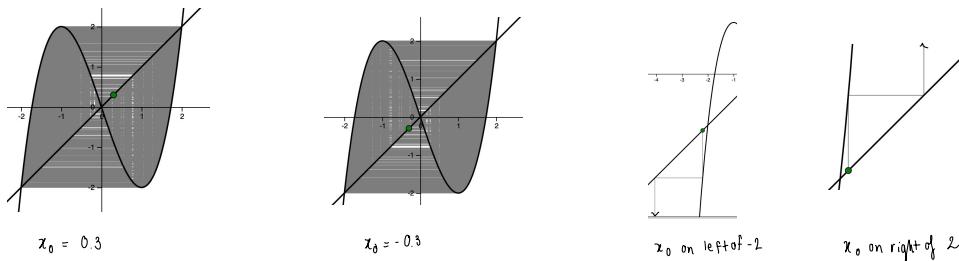
②  $x \in (1, \infty)$ ,  $f^n(x) \rightarrow \infty$ . and  $x \in (-\infty, -1)$ ,  $f^n(x) \rightarrow -\infty$ . we call  $\pm 1$  repelling fixed points



## Motivating Example 2

$$f(x) = x^3 - 3x, \text{ fixed points: } 0, \pm 2$$

- ① 0 is repelling [in a different way than last example b/c right & left to 0 is chaotic]
- ② to the right of 2 the orbits go to infinity, left of -2 orbits go to -infinity. so  $\pm 2$  is also repelling.
- ③ at  $x_0 = 1$ , the orbit is eventually fixed to the fixed point -2; for  $x_0 = -1$ , the orbit is ev



**Defn** let  $a$  be a fixed point of  $f(x)$

- ① if  $|f'(a)| > 1$ , we call  $a$  a **repelling** fixed point.
- ② if  $|f'(a)| < 1$ , we call  $a$  an **attracting** fixed point
- ③ if  $|f'(a)| = 1$ , we call  $a$  a **neutral** fixed point  $\rightarrow$  this can mean a whole bunch of things

**Theorem** [Attracting fixed point thm]

Suppose  $a$  is an attracting fixed of  $f(x)$  then  $\exists$  an open interval  $I$  such that  $a \in I$  and

- ①  $\forall x \in I, \forall n \in \mathbb{N}, f^n(x) \in I$   $\hookrightarrow$  slightly unnecessary because if  $f(x) \in I \Rightarrow f^n(x) \in I$
- ②  $\forall x \in I, f^n(x) \rightarrow a$ .

Defn:  $f: A \rightarrow \mathbb{R}$ ,  $A \subseteq \mathbb{R}$ .

① we say  $a \in A$  is non-isolated iff  $\forall \epsilon > 0$ ,  $\exists b \in A$  with  $b \neq a$  s.t.  $b \in (a-\epsilon, a+\epsilon)$

② Let  $a \in A$  be non-isolated. We say  $\lim_{x \rightarrow a} f(x) = L$  iff  $\forall \epsilon > 0 \exists \delta > 0$ ,  $|f(x) - L| < \epsilon$  whenever  $a \in A$  and  $0 < |x-a| < \delta$

$\rightarrow$  if  $a$  was an isolated, we could choose a  $\delta$  where  $|x-a| < \delta$  is false. Leading to a false hypothesis which leads to an always true answer and hence anything could be the limit.

Ex  $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$  0 is non-isolated

Ex  $\{1\} \cup (2, 3)$  1 is isolated

Proof: [Attracting fixed point thm]  $\rightarrow$  to continue in Lec 6

Assume  $|f'(a)| < 1$ . Then  $\exists c \in \mathbb{R}$  s.t.  $|f'(a)| < c < 1$

$$\therefore \lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x-a|} < c$$

Thus  $\exists \delta > 0$  s.t.  $\frac{|f(x) - f(a)|}{|x-a|} \leq c \quad \forall x \in (a-\delta, a+\delta)$

Hence, for  $x \in (a-\delta, a+\delta)$   $|f(x) - f(a)| \leq c|x-a| \rightarrow$  contracting

# Lec 6<sup>th</sup> - Jan 19<sup>th</sup> 2024

First assignment due 11:59 on Tuesday

## Theorem [Attracting fixed point thm]

Suppose  $a$  is an attracting fixed point of  $f(x)$ . There exists an open interval  $I$  with  $a \in I$  s.t.

- ①  $\forall x \in I, \forall n \in \mathbb{N}, f^n(x) \in I \rightarrow$  saying  $f(x) \in I$  is enough but we are choosing to write this to give a fuller picture
- ②  $\forall x \in I, f^n(x) \rightarrow a$

$a$  is not a boundary pt.  
b/c it is differentiable due to defn  
of attracting

b/c this exists, we can choose a maximal  $\delta$  such that all orbits that go to  $a$  exist in it and no other orbits

Proof:

Say  $|f'(a)| < c < 1$  so that  $\lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x - a|} < c$ . Thus,  $\exists \delta > 0$  s.t.  $\forall x \in I := (a - \delta, a + \delta), |f(x) - f(a)| < c|x - a|$

In particular, for  $x \in I$   $|f(x) - f(a)| = |f(x) - a| \leq c|x - a| \leq c|x - a| < \delta \Rightarrow f(x) \in I$

Continuing, for  $x \in I$   $|f^n(x) - a| \leq c^n|x - a| \leq c^n|x - a| < \delta$  so that  $f^n(x) \in I$ . Finally, for  $x \in I$   $0 \leq |f^n(x) - a| \leq c^n|x - a| \xrightarrow{\text{c} < 1} 0$

Note: If the question asks to use a  $\epsilon$ - $\delta$  proof then do so.

$f$  means  $f$  is a contraction

$c \in (0, 1)$   
 $\Rightarrow c^n \text{ shrinks} \rightarrow 0$

Can show using  $\epsilon$ - $\delta$  proof if needed

## Theorem [repelling fixed point theorem]

Suppose  $a$  is a repelling fixed point for  $f(x)$ . There exist an open interval  $a \in I$ , s.t.  $\forall x \in I, x \neq a, \exists n \in \mathbb{N}$  s.t.  $f^n(x) \notin I$

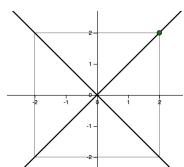
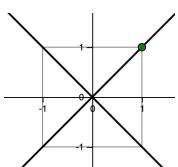
Proof: Say  $|f'(a)| > c > 1$ . then  $\lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x - a|} > c$  and so  $\exists \delta > 0$  s.t. for all  $x \in I := (a - \delta, a + \delta)$   $|f(x) - f(a)| > c|x - a|$

Since  $a$  is a fixed point,  $|f(x) - f(a)| = |f(x) - a|$ . Suppose  $\forall n, f^n(x) \in I$ . As before  $|f^n(x) - a| \geq c^n|x - a| \rightarrow \infty$  This infinity goes well beyond delta!

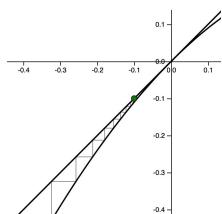
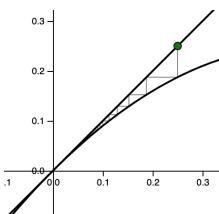
What is a neutral fixed point? A lot.

Investigation: Neutral fixed Points

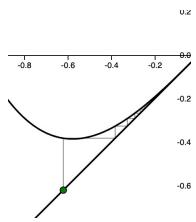
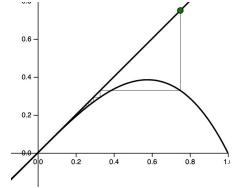
Ex  $f(x) = -x$ ,  $0$  is a fixed point and  $|f'(0)| = 1$ . bounces around



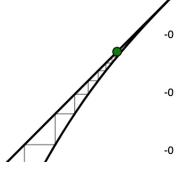
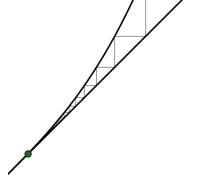
Ex  $f(x) = x - x^2$ ,  $|f'(0)| = 1$  attracting to the right and repelling to the left.



Ex  $f(x) = x - x^3$   $|f'(0)| = 1$ , this is weakly attracting, attracting but too slowly, happens in assymptote situations



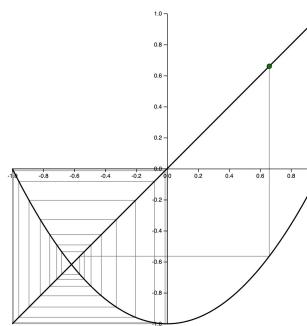
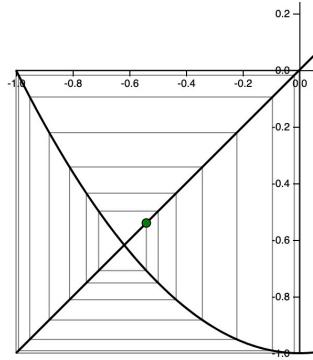
Ex  $f(x) = x + x^3$ ,  $|f'(0)| = 1$ , weakly repelling, repelling too slow



Motivating Example:  $f(x) = x^2 - 1$ ,  $a=0$ .

orbit,  $(0, -1, 0, 1, 0, -1)$ ; 0 is a periodic point of period 2.  $x_0$  near 0 shows orbits getting closer to  $\{0, 1\}$ -cycle.

we will call 0 an attracting periodic point because. 0 is an attracting fixed point of  $f^2(x)$ .



# Lec 7 - Jan 22<sup>nd</sup> 2024

**Defn:** Let  $a$  be a periodic point for  $f(x)$  with period  $n$ . We say  $a$  is an attracting / repelling / neutral per point for  $f(x)$  iff  $a$  is an attracting / repelling / neutral fixed point of  $f^n(x)$ .

for strictly periodic orbits. [no eventually periodic] the points in the periodic orbit are fixed points of  $f^n(x)$ . Also, every point in the orbit will have the same classification as the point tested.  $\rightarrow$  by proposition value of derivative must be the same for any point in the cycle.

**Prop:** Let  $f(x)$  be a function. Then  
 $\hookrightarrow f(x)$  need to be diffable

$$(f^n)'(a) = f'(a) f'(f(a)) f'(f^2(a)) \dots f'(f^{n-1}(a))$$

**Proof:** [Induction]

If  $n=1$ ,  $f'(a) = f'(a)$

Assume the result holds for some  $n$ ,  $n \geq 1$ .

$$\text{then } \frac{d}{dx} f^{n+1}(x) = \frac{d}{dx} f(f^n(x)) = f'(f^n(x)) \cdot (f^n)'(x)$$

$\hookrightarrow$  plugin  $a$

$$\text{Then } (f^{n+1})'(a) = \underbrace{f'(f^n(a)) \cdot f'(a) \cdot f'(f(a)) \dots f'(f^{n-1}(a))}_{\text{inductive hypothesis}}$$

■

$$\text{Ex) } f(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1, a=0, \text{ orbit } (0, 1, 2, 0, \dots)$$

$$f'(x) = -3x + \frac{5}{2}$$

$$\begin{aligned} (f^3)'(0) &= f'(0) f'(1) f'(2) \\ &= \left(-\frac{7}{2}\right) \left(\frac{-1}{2}\right) \left(\frac{5}{2}\right) \\ &= \frac{35}{8} > 1 \quad \therefore \text{repelling} \end{aligned}$$

## Chapter 4. Bifurcations

In general Bifurcation theory is the study of how a family of curves can change when a defining parameter is changed

$\hookrightarrow$  very important family

We will consider the quadratic family:  $Q_c(x) = x^2 + c$  Idea: how does  $c$  affect the dynamics of  $Q_c(x)$ ?

to find the fixed points of  $Q_c(x)$ :

$$x^2 + c = x \iff x^2 - x + c = 0$$

$$\iff x = \frac{1 \pm \sqrt{1-4c}}{2}$$

Thus:

- ①  $Q_c(x)$  has 2 fixed points when  $c < \frac{1}{4}$

②  $Q_c(x)$  has 1 fixed point when  $c = \frac{1}{4}$

③  $Q_c(x)$  has no fixed points [really] when  $c > \frac{1}{4}$ .

If  $c > \frac{1}{4}$ ,  $Q_c^n(x)$  must diverge to  $\infty$ .

If  $c = \frac{1}{4}$   $Q_c(x)$  has the unique fixed point of  $p = \frac{1}{2}$ . Since  $Q'_c(x) = 2x$ ,  $Q'_c(\frac{1}{2}) = 1$  so thus fixed point is neutral → attracting to left and repelling to right

If  $c < \frac{1}{4}$ ,  $Q_c(x)$  has 2 fixed points:

$$p_+ = \frac{1 + \sqrt{1-4c}}{2}, \quad p_- = \frac{1 - \sqrt{1-4c}}{2}$$

First,

$$Q'_c(p_+) = 1 + \sqrt{1-4c} > 1 \Rightarrow p_+ \text{ is repelling}$$

Next,

$$-1 < Q'_c(p_-) < 1$$

$$\Leftrightarrow -1 < 1 - \sqrt{1-4c} < 1$$

$$\Leftrightarrow -2 < -\sqrt{1-4c} < 0$$

$$\Leftrightarrow 2 > \sqrt{1-4c} > 0$$

$$\Leftrightarrow -\frac{3}{4} < c < \frac{1}{4} \rightarrow \text{this is the range for attracting fixed point}$$

if  $c < -\frac{3}{4}$ ,  $Q'_c(p_-) < -1$

if  $c = -\frac{3}{4}$ ,  $Q'_c(p_-) = -1$

Theorem: For the family  $Q_c(x) = x^2 + c$ :

① All orbits tend to  $\infty$  if  $c > \frac{1}{4}$

② when  $c = \frac{1}{4}$   $Q_c(x)$  has a unique fixed point,  $\frac{1}{2}$ , and it is neutral

③ if  $c < \frac{1}{4}$   $Q_c(x)$  has 2 fixed points  $p_+$  and  $p_-$ . The point  $p_+$  is repelling

Moreover

a) If  $-\frac{3}{4} < c < \frac{1}{4}$ ,  $p_-$  is attracting

b) if  $c = -\frac{3}{4}$ ,  $p_-$  is neutral

c) if  $c < -\frac{3}{4}$ ,  $p_-$  is repelling

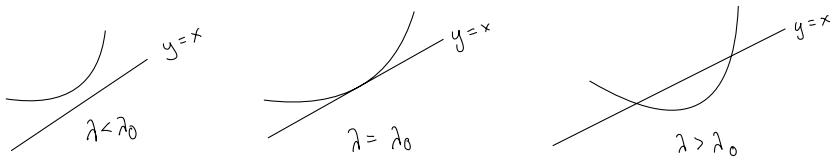
## Lec 8 - Jan 24 2024

**Def'n:** We say a family of Functions  $F_\lambda(x)$  undergoes a **bifurcation** at  $\lambda_0$  if there is a change in fixed point structure at  $\lambda_0$ .

Ex)  $Q_c(x) = x^2 + c$ ,  $\lambda_0 = \frac{1}{4}$

**Def'n:** A family  $F_\lambda(x)$  undergoes a **tangent bifurcation** at  $\lambda_0$  if there is an open interval  $I$  and  $\epsilon > 0$  s.t.

- ① For  $\lambda_0 - \epsilon < \lambda < \lambda_0$ ,  $F_\lambda(x)$  has no fixed points on  $I$
  - ② For  $\lambda = \lambda_0$ ,  $F_\lambda(x)$  has one fixed point and it is neutral
  - ③ For  $\lambda_0 < \lambda < \lambda_0 + \epsilon$ ,  $F_\lambda(x)$  has 2 fixed points in  $I$ , one attracting and the other repelling
- or ①, ②, ③ with  $>$



Ex)  $E_\lambda(x) = e^x + \lambda$ ,  $\lambda_0 = -1$ . This is an example of a tangent bifurcation.

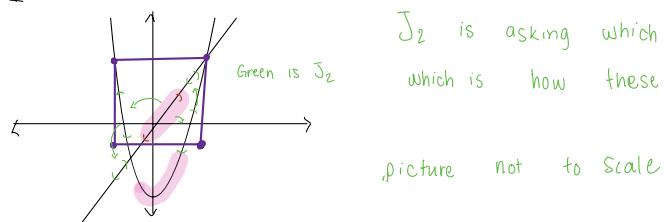
Ex)  $F_\lambda(x) = \lambda x(1-x)$ ,  $\lambda_0 = 1$ . Not tangent bifurcation.

## Chapter 5 Cantor Set

Investigation:  $Q_c(x) = x^2 + c$ ,  $c < -2$ .

$$p_+ = \frac{1 + \sqrt{1 - 4c}}{2} > 2, \quad -p_- < -2$$

Consider  $I = [-p_+, p_+]$  and  $I \times I$



$J_2$  is asking which  $x$  has  $y$  values in  $J_1$ , which is how these intervals are picture not to scale

Let  $J_i \subseteq I$  be the interval s.t.  $Q_c(x) \notin I$  for all  $x \in J_i$ ,

For  $x \in J_1$ ,  $Q_c^n(x) \rightarrow \infty$ . Moreover, If  $\exists n$  s.t.  $Q_c^n(x) \in J_1$ , then  $Q_c^n(x) \rightarrow \infty$

Consider  $\Lambda = \{x \in I : Q_c^n(x) \in I \ \forall n\}$

Big Idea:  $\Lambda$  contains all the points with interesting orbits.

$$J_1 = \{x \in I : Q_c(x) \notin I\}$$

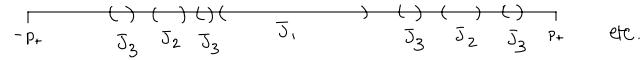
$$J_2 = \{x \in I : Q_c(x) \in J_1\} = \{x \in I : Q_c^2(x) \notin I\}$$

$$J_3, J_4, J_5 \dots$$

what is a cantor set?

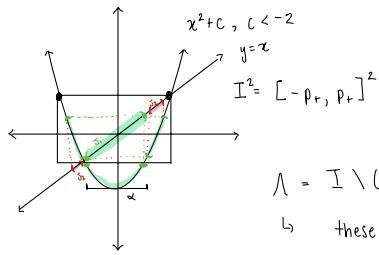
Then,  $\Lambda = I \setminus (J_1 \cup J_2 \cup \dots)$  is a cantor set. Drawing  $\Lambda$  on the  $x$ -axis:

→ Symmetry comes from the cune (parabola)



It is a fractal  
b/c of its self repeating nature!

# Lec 9 - Jan 26<sup>th</sup> 2024



One quarter mark for the course.

$$\Lambda = I \setminus (J_1 \cup J_2 \cup \dots)$$

$$A = \{x \in I : \forall n, Q_c^n(x) \in I\}$$

↳ these are all the interesting orbits

↳ Cantor set

**Construction:** Cantor Middle thirds set

$$C_0 = [0, 1] \rightarrow \text{remove open middle third interval}$$

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

⋮

$$K = \bigcap_{n=1}^{\infty} C_n \rightarrow \text{Cantor [middle-third] set.}$$

these are nested  
↗

A cantor set is taking out a chunk of the interval, infinitely. We can see that.

the end points will always survive.

Proposition:  $A_n \subseteq \mathbb{R}$  closed. Then  $\bigcap_{n=1}^{\infty} A_n$  is closed.

Why?  $(a_n) \subseteq \bigcap A_n, a_n \rightarrow a. \forall n, (a_n) \subseteq A_n \Rightarrow a \in A_n \Rightarrow a \in \bigcap A_n$

Proposition:  $A, B \subseteq \mathbb{R}$  closed. Then  $A \cup B$  is closed.

Why?  $(a_n) \subseteq A \cup B, a_n \rightarrow a. \text{WLOG. } \{n : a_n \in A\}$  is infinite. This allows us to construct  $(b_n) \subseteq A$  s.t.  $b_n \rightarrow a$ . Since  $A$  is closed,  $a \in A \subseteq A \cup B$ .

**Theorem**  $K$  is closed [in fact all Cantor sets are]  $\rightarrow$  combine the 2 propositions above to get this

**Theorem:**  $K$  contains no non-empty open intervals  $\rightarrow$  Hausdorff property of  $\mathbb{R}$ ?

no isolated points  
↗ interior points?  
↙

Why?  $I \subseteq K \Rightarrow \forall n, I \subseteq C_n \Rightarrow \forall n \quad l(I) \leq \frac{1}{3^n} \Rightarrow l(I) = 0 \Rightarrow I = \emptyset$  which is a contradiction.  
↳ length of interval.

Let's consider the base -3 expansion of  $x \in [0, 1]$ .  $x = 0.s_1 s_2 s_3 \dots, s_i \in \{0, 1, 2\}$

$$s_1 = 0$$

$$s_1 = 1$$

$$s_1 = 2$$

$$[0, \frac{1}{3}]$$

$$[\frac{2}{3}, 1]$$

$$\begin{array}{l} s_1 = 0 \\ s_2 = 0 \\ s_2 = 1 \\ s_2 = 2 \end{array}$$

$$\begin{array}{l} s_1 = 0 \\ s_2 = 1 \\ s_2 = 2 \\ \dots \\ s_3 = 0 \\ s_3 = 1 \\ s_3 = 2 \end{array}$$

Fact:

$x \in K$  iff  $x$  can be written in base-3 using only 0's and 2's

Ex  $\frac{1}{3} \in K$ .  $\frac{1}{3} = [0.1]_3 = [0.022222\ldots]_3 \rightarrow$  # theory? base  $\alpha$ ?

Theorem:  $K$  is uncountable  $|K| = |\mathbb{R}| \rightarrow$  cardinality  $\rightarrow$  not important to this course, just for interest.

How to remember? 0 means left, 2 means right.

Start Ch 6 on monday.

# Lec 10 - Jan 29<sup>th</sup> 2024

## Chapter 6 - Symbolic Dynamics

Recall:  $Q_c(x) = x^2 + c$ ,  $c < -2$ .

\* Assignment 2 now out, First 4 Q easy, Last 4 NOT

Midterm March 1<sup>st</sup>, Content cut off is already happening

Midterm may have proofs from next part of course (after assignment 2 content)

For Q w classify neutral point use cobwebbing thing

$$p_+ = \frac{1 + \sqrt{1 - 4c}}{2}, \quad I = [-p_-, p_+] \quad J_1 = \{x \in I : Q_c(x) \in I\}$$

$$J_2 = \{x \in I : Q_c(x) \in J_1\}$$

$$J_3 = \{x \in I : Q_c(x) \in J_2\}$$

$$\vdots$$

$\Lambda = I \setminus (\cup J_i) = \{x \in I : \forall n Q_c^n(x) \in I\} \rightarrow$  This is a Cantor set. All Cantor sets are isomorphic verify iso vs. homeo

Notation:

$$I \setminus J_1 = I_0 \cup I_1$$

$I = [I_0 \setminus \text{removed}] \cup I_1 \rightarrow$  Idea we have a left part and a right part.

Defn: For  $x \in \Lambda$ , we define the **Itinerary** of  $x$  to be the sequence

→ kind telling you direction, dealing with orbit indirectly

$$S(x) = (x_0 x_1 x_2 x_3 \dots), \quad x_i \in \{0, 1\}$$

where  $x_i = 0 \iff Q_c^i(x) \in I_0$

$x_i = 1 \iff Q_c^i(x) \in I_1$

Goal: Understand  $S(x)$  better

Notation:  $\Sigma = \{(x_0 x_1 x_2 \dots) : x_i \in \{0, 1\}\} \rightarrow$  we drop the commas?

→ function, specifically a transform. claims it's easier to work with.

$S: \Lambda \rightarrow \Sigma$

↪ S is a homeomorphism  $\Rightarrow \Lambda$  is same as  $\Sigma$ , what we are doing this week.

Defn: Let  $X$  be a set. We say  $d: X \times X \rightarrow [0, \infty)$  is a **Metric** iff

$$\textcircled{1} \quad d(x, y) = 0 \iff x = y$$

$$\textcircled{2} \quad \forall x, y \in X, \quad d(x, y) = d(y, x)$$

$$\textcircled{3} \quad \forall x, y, z \in X \\ d(x, y) \leq d(x, z) + d(z, y) \rightarrow \text{triangle inequality}$$

We call  $(X, d)$  a metric space

Idea  $d(x, y) =$  distance between  $x$  and  $y$ .

Ex)  $X = \mathbb{R}$ ,  $d(x, y) = |x - y|$

Ex)  $X = \mathbb{R}^n$ ,  $d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$

Ex)  $X$  any set,  $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$  discrete metric

These are all Metric Space

Ex)  $A \subseteq \mathbb{R}$ , For  $x, y \in A$ ,  $d(x, y) = |x - y|$  is a metric  $(A, d)$

Ex) [Cantor Space]

$X = \sum$ ,  $d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}$

ensuring we get a number  
bc it gives us a convergent series

Ex)  $x = (111\dots)$

$y = (1010\dots)$

$d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} \rightarrow$  1's cancel in every even step

How  $\begin{aligned} &= \sum_{i=0}^{\infty} \frac{1}{2^{2i+1}} = \sum_{i=0}^{\infty} \left(\frac{1}{2^{2i}}\right) \frac{1}{2} ? \\ &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^i} = \frac{1}{2} \left(\frac{1}{1 - \frac{1}{4}}\right) = \frac{1}{2} \left(\frac{4}{3}\right) = \frac{4}{6} = \frac{2}{3} \end{aligned}$

### Connections

- dynamical systems on manifolds
- p-adic numbers? Vertasium video

Proposition:  $x, y \in \sum$

① if  $x_i = y_i$  for  $i \leq n$  then  $d(x, y) \leq \frac{1}{2^n}$  First few terms carry a lot of weight

② if  $d(x, y) < \frac{1}{2^n}$  then  $x_i \neq y_i$  for  $i \leq n$

Why?

①  $d(x, y) \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{1}{2^n} \rightarrow$  convergence of geometric series formula?

②  $d(x, y) < \frac{1}{2^n}$

$\exists k \leq n, x_k \neq y_k \Rightarrow |x_k - y_k| = 1$  hence contradiction

Ex)  $x = (000\dots)$

$y = (1000\dots)$

$d(x, y) = \frac{1}{2^0} = 1$  BUT  $x_0 \neq y_0$  This shows that Proposition cannot be iff.

Defn: Shift map.

$\sigma : \sum \rightarrow \sum$

$\sigma : (x_0 x_1 x_2 \dots) = (x_1 x_2 x_3 \dots)$

Remark  $\sigma^k(x_0 x_1 x_2 \dots) = (x_k x_{k+1} x_{k+2} \dots)$

What's next?

Show ①  $\sigma: \Sigma \rightarrow \Sigma$  is continuous  $\curvearrowright$  gives us

②  $\sigma: \Sigma \rightarrow \Sigma$  dynamical System

③  $\varsigma: \Lambda \rightarrow \Sigma$  homeomorphism  $\rightarrow$  means  $\Lambda$  and  $\Sigma$  are "same space"

④  $\sigma: \Sigma \rightarrow \Sigma$  chaotic  $\Rightarrow Q_c: \Lambda \rightarrow \Lambda$