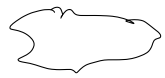
Creating a Sundial by Joaco Prandi - Mar 27th 2024

Speaker: Joaco Prandi, PhD Candidate, Department of Pure Mathematics, University of Waterloo

Title: Creating a Sundial

Abstract: In this talk, we will try to understand how to make a sundial. Not just any sundial, but a sundial that requires too much math and probably will never be built. At least it will give the time in a digital format.

2 New dimension. 2nd Most helpful. Take a set like  draw a grid over it and count the amount of boxes

Call it $N_\delta(K) \approx \delta^s$

Box counting dimension $\dim_B(K) \lim_{\delta \to 0} \dfrac{\log(N_\delta(K))}{-\log(\delta)}$ → lim doesn't always exist

$\overline{\dim}_B(K) = \lim_{\delta \to 0} \sup \dfrac{\log(N_\delta(K))}{-\log(\delta)}$

$\underline{\dim}_B(K) = \lim_{\delta \to 0} \inf \dfrac{\log(N_\delta(K))}{-\log(\delta)}$

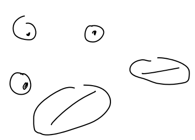
these always exist

$N_\delta(K)$ can be replace to: Maximal Number of balls of radius δ centered in K with them being disjonit

or Minimal # of balls of radius δ centered at K covering the set.

What happens if the boxes are different sizes? We need a measure b/c counting is sufficent.

Define $\mathcal{H}_\delta^s(K) = \inf \left\{ \sum_{k=0}^{\infty} |U_i|^s : \bigcup_{i=0}^{\infty} U_i \supset K \text{ and } |U_i| \leq \delta \right\}$

Ex  smallest is measure

as δ gets smaller ________________?

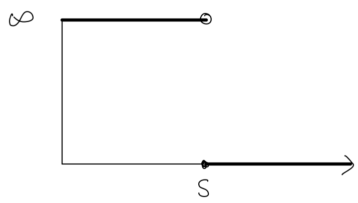
$\mathcal{H}^s(k) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(k)$ for most sets its either 0 or infinite

In $\mathbb{R}^d$ and $n \leq d$ then $\mathcal{H}^n$ is measuring n-dimensionally.

$\mathcal{H}^1$ = length, $\mathcal{H}^2$ = area, $\mathcal{H}^3$ = Volume. So on.

$\mathcal{H}^s(k)$ can think of as a function of S. If $S < t$, then if $\mathcal{H}^T(k)$ is $(\mathcal{H}^T(k) > 0)$ positive then $\mathcal{H}^S(k) > \infty$

If $\mathcal{H}^S(k) < \infty$ (finitie) then then $\mathcal{H}^T(k) = 0$



The Hausdorff dimension is $\dim_{\mathcal{H}} = \sup\{s : \mathcal{H}^s(k) \text{ is infinite}\}$
$= \inf\{s : \mathcal{H}^s(k) \text{ is zero}\}$.

Facts: If C is the ternary Cantor Set. then $\dim_B(C) = \dim_{\mathcal{H}}(C) = \frac{\log(2)}{\log(3)}$

$F = \{\frac{1}{n}\}_{n=0}^{\infty}$ $\quad \dim_B(F) = \frac{1}{2}$, $\quad \dim_B(F) = \dim_B(\overline{F})$ Closure of Set

$\dim_{\mathcal{H}}(F) = 0$, $\quad \dim_{\mathcal{H}}(F) \neq \dim_{\mathcal{H}}(\overline{F})$
in general

Fact: If $\dim_{\mathcal{H}}(F) < 1$ then the set is totally disconnected.

Fact the function f is Lipschitz with ratio c if $\mathcal{H}^s(f(k)) \leq c^s \mathcal{H}^s(k)$

like a contraction
but the contraction
constant doesnt have
to be between 0 & 1

$$\Rightarrow \quad \dim_H(f(K)) \le \dim_H(K)$$

Projections: (orthogonal projections)

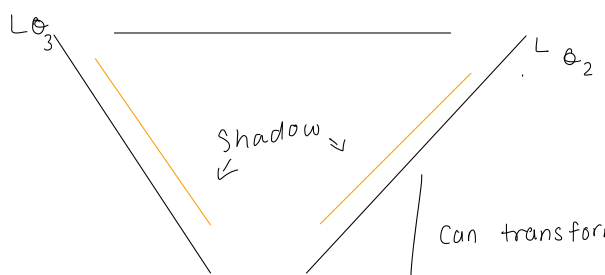
Let L_θ be the the line through the origin with angle $\theta \in [0,\pi]$ and $\text{Proj}_\theta(F)$ is the orthogonal projection to L_θ. In general the are Lipschitz.

Theorem: For almost all θ

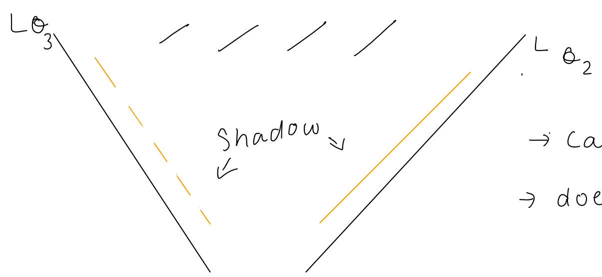
↗ stronger than pervious.

a) If $\dim_H(f) \le 1$ then $\dim_H(\text{proj}_\theta(f)) = \dim_H(f)$

b) If $\dim_H(f) > 1$ then $\dim_H(\text{Proj}_\theta(f)) = 1$ and $\mathcal{H}^1(\text{Prog}_\theta(f)) > 0$



↓ Can transform by cutting and rotating



→ Calle Iterated venetian (blind) process

→ doesnt work $\forall$ lines, but can for some angles.

Theorem Let $G_\theta \subset L_\theta$, $\theta \in [0,\pi)$ be a collection of sets s.t. $\bigcup_\theta G_\theta$ is measureable in 2 dimensions. then there exists a set $F \subseteq \mathbb{R}^2$ s.t. $G_0 \subset \text{Proj}_\theta(F)$ and

$$\mathcal{H}^1(\text{Proj}_\theta(F) \setminus G_0) = 0 \quad \text{for almost all } \theta.$$

This is in $\mathbb{R}^2$

In theorey wecan use this to make a Sundial.

Youtube video for Sundial

http://mate.dm.uba.ar/~umolter/materias/referencias/2.pdf

This talk was based on chapter 2, 3 and 6 of Fractal Geometry Mathematical Foundations and Applications by Kenneth Falconer (link about)

Sundial video: https://www.youtube.com/watch?v=78I-A7ikXYU

Logistics

Q3 is ugly piazza is the way

and Course perception survey out

Guest Speaker : Paul Fieguth - April 1st 2024

The role of bifurcation in Continuous-time and discrete time Dynamical Systems

Book:

Complex System: many elements, interacting, non linear.

Problem: all uni courses focus on linear, Gaussian and small.
all major world issues are non linear, non Gausian, and large.
↳ black swan events

his aim is to teach ppl humility when approaching world issues

Bifurcations in ~~non~~ Linear Dynamics

Continuous : $\dot{z}(t) = f(z(t), \theta)$ parameter

very facisnating connections between the two.

Discreete : $z_{n+1} = \bar{f}(z_n, \theta)$

Basic motivation for Discrete time is as time-discretization of continuous time.

Cont $\dot{z}(t) = \lim_{\delta \to 0} \dfrac{z(t+\delta) - z(t)}{\delta}$

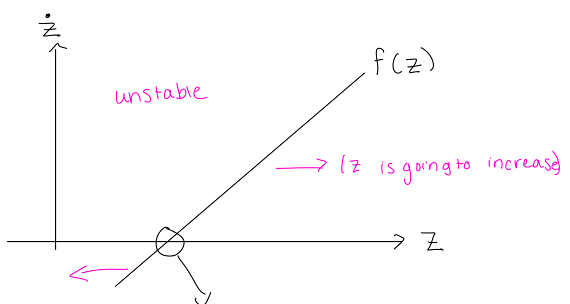
$\therefore \quad \dot{z} = f(z)$ and $z(t+\delta) = z(t) + \delta \cdot f(z)$

↓

Discrete time known as forward Euler

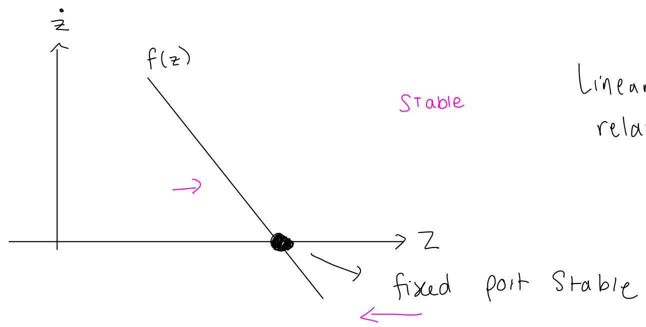
There are other C.T to D.T possibilities (later)

System Diagram



unstable fixed point

convention is unfilled if unstable



Linear b/c relation of $\delta(z)$ is linear

convention is filled when Stable

Aside: D.T. uses $y = x$ to find fixed points

Linear System: Key Attributes.

1. Super position: $x_1 \to \boxed{} \to y_1 \qquad x_2 \to \boxed{} \to y_2$

then $\alpha x_1 + \beta y_1 \to \boxed{} \to \alpha y_1 + \beta y_2$

Ex Streching Spring obeys super position

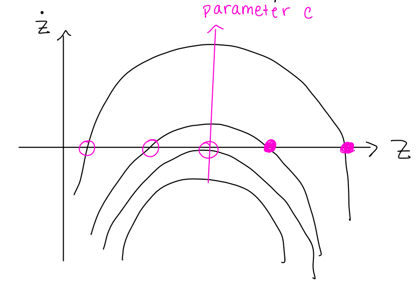
2. Sine wave $A \sin(\omega t + \phi) \to \boxed{} \to \beta \sin(\omega t + \varphi)$

<u>Cannot</u> change frequency.

∴ Constant input cannot lead to oscillating output.

every electronic using a battery is non linear, b/c constant input but it buzzes, flashes

Defn: A bifurcation is a discontinuous change in attribute or behaviour in response to a continuous change in parameter.

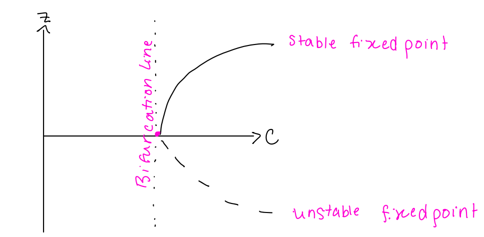


$\dot{z} = -(z-5)^2 + C$

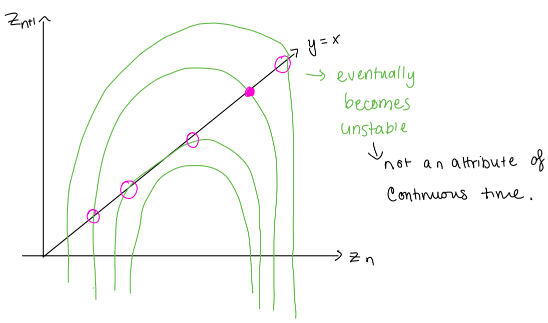
Double root is Called degeneracy.
realistictly never happen.

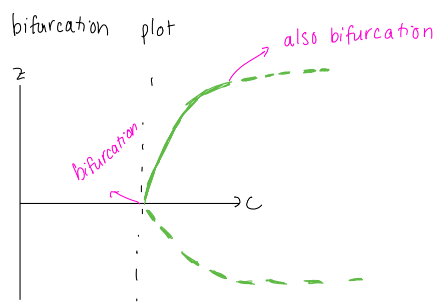
An example of bifurcation 0 fixed to 2 fixed.

Bifurcation Plot



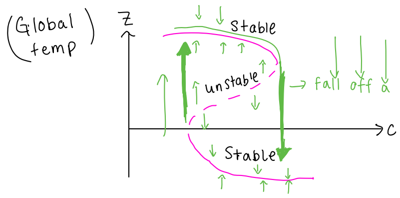
Also have bifurcations in D.T



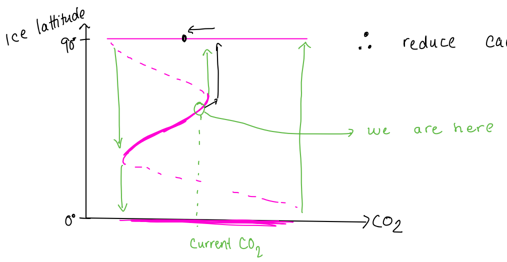


Two major bifurcations:

① Double - fold:



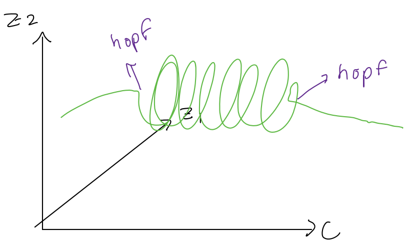
Bistable system another example light switch, temperature. of fridge. nearly all thermal.



∴ reduce carbon rn. b/c if we go past our current stable point we would need to reduce alot before getting to our stable nice state

② Hopf bifurcation:

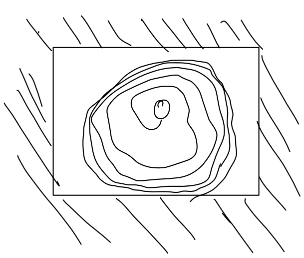
Transition between cycling and not cycling.



Why is cycling common?

Ex Flag pole, guiser, Sailboat mass. pooring water on live plants

b/c bouneded unstable system.

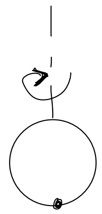


Lineeant cross

→ limit cycle.

Bifurcations that matter to us:

- Stick - slip. (Brakes locking, rude and unpleasent noises)
- covid : fizzling out vs. pandemic
- Bead on a hoop



- Toys: slap braclet, jumping disc, Candle boats
- Ecology, : lake ecology
- Humain brain, epilectic seizures,
- Chocolate production

Fixed point propreties in ~~to~~pological Dynamics.

By Andy Zucker April 3rd 2024

Let G be a group. Often countable & discrete or some times topological)
(Integers over "countable & discrete"; Reals w addition over "topological")

most he will talk about are this.

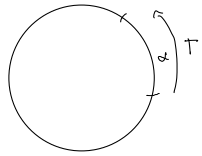
Defn : A G-flow is a compact Hausdorff space X equipped with a continuous action

$$a : G \times X \to X$$

Satisfying ① $a(1_G, x) = x \ (\forall x \in X)$ identity
② $a(g, a(h,x)) = a(gh, x)$ $\longrightarrow g(hx) = (gh)x$

notation : typically a is understood. just write $g \cdot x$, or gx for $a(g,x)$

Ex 1) $G = \mathbb{Z}$, X is unit circle. I'll write $T : X \to X$ for the generating homeomorphism which here will be irrational rotation



Since α is irrational, this action is called free, i.e, $\forall x \in X$, $\forall n \in \mathbb{Z} \setminus \{0\}$ $T^n(x) \neq x$
(an irrational multiple of 2π ; no fixed points)

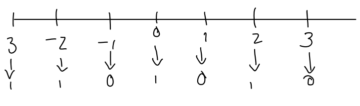
Ex, $G = \mathbb{Z}$, $X = 2^{\mathbb{Z}}$ $\to$ one way of talking about Cantor space (biinfite sequences of 0's and 1)

Again let $T : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ be the generating homeomorphism, (generator) we take

$$T(x)(n) = x(n-1) \quad \text{called the shift.}$$

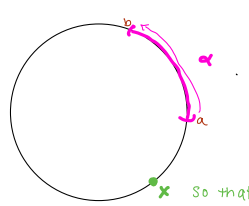
This $\mathbb{Z}$-flow is some time called Bernoulli shift. This action is not free. Consider $x \equiv 0$. This is a fixed point (so is 1). This is call essentially free. I.e. big open dense set that will move (orbits in X)

This is a function in X.



$\rightarrow$ Like shif map in class but ..... more?

However, there is a closed, non-$\emptyset$, T-invariant subspace of $2^{\mathbb{Z}}$ which is free. We can do this by looking at irrational rotation



so that $T^n(x)$ is never a or b

If $T^n(x)$ is in blue window = 1
If not = 0

This gives biinfinte string of 0,1s
Then under shift, it's <u>free</u>

But, Some Spaces we <u>Cant</u> find free actions
$\hookrightarrow$ primes?

For now, fix $X =$ Cantor space

Def'n: $\text{Clop}(X)$ = clopen subsets of X (both open and closed)

Defn: A probability measure on X is a map $\mu: \text{Clop}(X) \to [0,1]$ with $\mu(X) =$ and μ is finitely additive. [nice metric]

Regular baril? Algebras??? ↑ this good enough,

finitely additive: $\forall A, B \in \text{Clop}(X)$ if $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$ gives notation of size

Ex: View X as $2^{\mathbb{Z}}$, if $A \in \text{Clop}(X)$ is defined by $\{x : x(n) = i\}$ for some fixed n, $i<2$.

then can set $\mu(A) = 1/2$. Generate the rest independently. Imagine flipping a fair coin and 1 side is 1 other 0

Ex: Fix $x \in X$. The Dirac Delta at x is the measure $\mu(A) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$

very dependent on X.

in real # $\emptyset$ is clopen
Cantor space is cool b/c a lot of clopens

That was probability, Now throw group action in the mix

If G is a countable group and G acts on X the G acts on $P(X)$, the space of probability measures on X:

$$(g \cdot \mu)(A) = \mu(g^{-1} A)$$

using action on Cantor space, to get preimage and use μ to measure, introduces new measure of mass

What space? topology on $P(X)$ via $\mu_n \to \mu$ iff $\forall A \in \text{Clop}(X)$ $\mu_n(A) \to \mu(A)$

is called "weak* - topology" on $P(X)$.

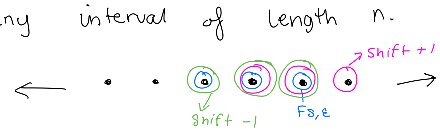
Fact: $P(X)$ is also compact Who found this? sombody infamous

Defn: Countable group G is amenable if for any finite $S \subseteq G$ and $\varepsilon > 0$. $\exists F_{S,\varepsilon} := F$ Such that

generators of group

$$\frac{|\underbrace{SF}_{\text{enlarge}} \setminus \underbrace{F}_{\text{throwaway starters}}|}{|F|} < \varepsilon$$

Ex $\mathbb{Z}$ is amenable. It is enough to consider $S = \{\pm 1\}$. Given $\varepsilon > 0$, find $n \in \mathbb{N}$. with $\frac{2}{n} < \varepsilon$. Let $F_{S,\varepsilon}$ be any interval of length n.



$\Rightarrow |SF \setminus F| = 2$ and $\frac{|SF \setminus F|}{n} = \frac{2}{n} < \varepsilon$

F is called a Følner set

Theorem: (Følner verify) A Countable Group G is amenable iff whenever G acts on counter space (X) The induced action on $P(x)$ has a fixed points
measures

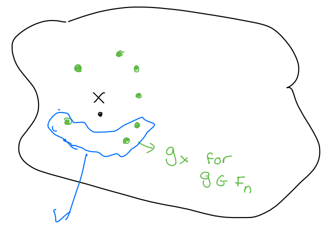
Proof: Assume G is amenable. Write $G = \bigcup_n S_n$ union. with $S_1 \subseteq S_2 \subseteq \dots$ finite. Let $F_n \subseteq G$ be $(S_n, \frac{1}{n})$ - følner. Given an action G on X which is Cantor. [use compactness]:

① pick $x \in X$ arbitrary

② For every $n \in \mathbb{N}$, $\mu(n) = \sum_{g \in F_n} \frac{1}{|F_n|} \delta_g$

taking avg of δ-measure

X =



Clopen, what proportion is in clopen, thats the measure

Idea: B/c $\langle F_n : n \in \mathbb{N} \rangle$ are more & more Følner we have $\mu_n(A) - (g \cdot \mu_n)(A) \to 0$ for every $A \in \mathrm{Clop}(X)$ and every $g \in G$.

- Eventually g is in S_n and $\frac{1}{n}$ is v. small.
- Pass to a convergent subsequence of μ_n [Compactness of $P(x)$] limit will be invariant