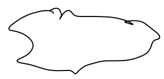
Creating a Sundial by Joaco Prandi - Mar 27^{th} 2024

Speaker: Joaco Prandi, PhD Candidate, Department of Pure Mathematics, University of Waterloo

Title: Creating a Sundial

Abstract: In this talk, we will try to understand how to make a sundial. Not just any sundial, but a sundial that requires too much math and probably will never be built. At least it will give the time in a digital format.

2 New dimension. 2^{nd} Most helpful. Take a set like  draw a grid over it and count the amount of boxes

Call it $N_\delta(K) \approx \delta^s$

Box counting dimension $\dim_B(k) \lim_{\delta \to 0} \dfrac{\log(N_\delta(k))}{-\log(\delta)}$ → lim doesnt always exist

$$\overline{\dim}_B(K) = \lim_{\delta \to 0} \sup \frac{\log(N_\delta(k))}{-\log(\delta)}$$

$$\underline{\dim}_B(k) = \lim_{\delta \to 0} \inf \frac{\log(N_\delta(k))}{-\log(\delta)}$$

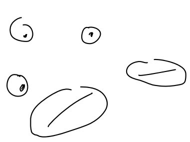
these always exist

$N_\delta(K)$ can be replace to: Maximal Number of balls of radius δ centered in K with them being disjonit

Or Minimal # of balls of radius δ centered at K covering the set.

What happens if the boxes are different sizes? We need a measure b/c counting is sufficent.

Define $\mathcal{H}^s_\delta(k) = \inf \left\{ \sum_{k=0}^{\infty} |U_i|^s : \bigcup_{i=0}^{\infty} U_i \supset k \text{ and } |U_i| \leq \delta \right\}$

Ex  smallest is measure

as δ gets smaller ______________?

$\mathcal{H}^s(k) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(k)$ for most sets its either 0 or infinite

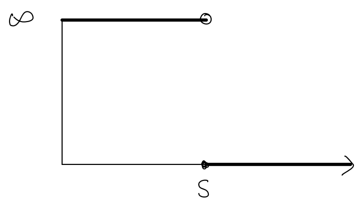
In R^d and $n \leq d$ -then H^n is measuring n-dimensionally.

H^1 = length, H^2 = area, H^3 = Volume. So on.

$(H^T(k) > 0)$

$H^S(k)$ can think of as a function of S. If $S < t$, then if $\mathcal{H}^T(k)$ is positive then $H^S(k) > \infty$

If $H^S(k) < \infty$ (finitie) then then $\mathcal{H}^T(k) = 0$



The Hausdorff dimension is $\dim_H = \sup\{s : \mathcal{H}^S(k) \text{ is infinite}\}$
$= \inf\{s : H^S(k) \text{ is zero}\}$.

Facts: If C is the ternary Cantor Set. then $\dim_B(C) = \dim_H(C) = \frac{\log(2)}{\log(3)}$

$F = \{\frac{1}{n}\}_{n=0}^{\infty}$ $\quad \dim_B(F) = \frac{1}{2}$, $\quad \dim_B(F) = \dim_B(\bar{F})$ ← closure of Set

$\dim_H(F) = 0$, $\quad \dim_H(F) \neq \dim_H(\bar{F})$
in general

Fact: If $\dim_H(F) < 1$ then the set is totally disconnected.

Fact the function f is Lipschitz with ratio c if $\mathcal{H}^S(f(k)) \leq c^S \mathcal{H}^S(k)$

like a contraction but the contraction constant doesnt have to be between 0 & 1

$$\Rightarrow \quad \dim_H(f(K)) \leq \dim_H(K)$$

<u>Projections:</u> (orthogonal projections)

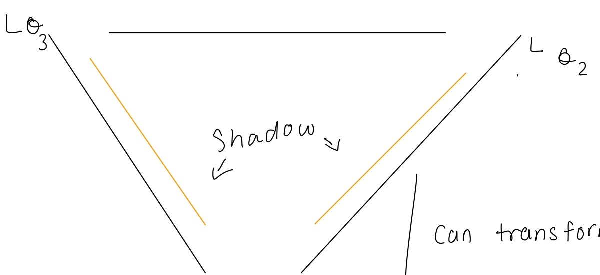
Let L_θ be the the line through the origin with angle $\theta \in [0,\pi]$ and $\text{Proj}_\theta(F)$ is the orthogonal projection to L_θ. In general the are Lipschitz.

Theorem: For almost all θ

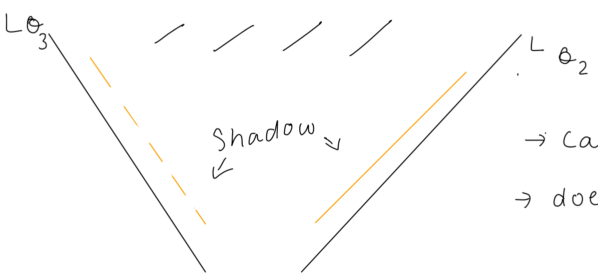
↗ stronger than perrious.

a) If $\dim_H(F) \leq 1$ then $\dim_H(\text{proj}_\theta(F)) = \dim_H(F)$

b) If $\dim_H(F) > 1$ then $\dim_H(\text{Proj}_\theta(F)) = 1$ and $\mathcal{H}^1(\text{Proj}_\theta(F)) > 0$



Can transform by cutting and rotating



→ Calle Iterated venetian (blind) process

→ doesnt work $\forall$ lines, but can for some angles.

Theorem Let $G_\theta \subset L_\theta$, $\theta \in [0,\pi)$ be a collection of sets s.t. $\bigcup_\theta G_\theta$ is measureable in 2 dimensions. then there exists a set $F \subseteq \mathbb{R}^2$ s.t. $G_\theta \subset \text{Proj}_\theta(F)$ and $\mathcal{H}^1(\text{Proj}_\theta(F) \setminus G_\theta) = 0$ for almost all θ.

This is in $\mathbb{R}^2$

In theorey wecan use this to make a Sundial.

Youtube video for Sundial

http://mate.dm.uba.ar/~umolter/materias/referencias/2.pdf

This talk was based on chapter 2, 3 and 6 of Fractal Geometry Mathematical Foundations and Applications by Kenneth Falconer (link about)

Sundial video: https://www.youtube.com/watch?v=78I-A7ikXYU

Logistics

Q3 is ugly piazza is the way

and Course perception survey out

Guest Speaker: Paul Fieguth - April 1st 2024

The role of bifurcation in Continuous-time and discrete time Dynamical Systems

Book:

Complex System: many elements, interacting, non linear.

Problem: all uni courses focus on linear, Gaussian and small.
all major world issues are non linear, non Gausian, and large.
↳ black swan events

his aim is to teach ppl humility when approaching world issues

Bifurcations in ~~non~~ Linear Dynamics

Continuous: $\dot{z}(t) = f(z(t), \theta)$ ← parameter

very facisnating connections between the two.

Discreete: $z_{n+1} = \bar{f}(z_n, \theta)$

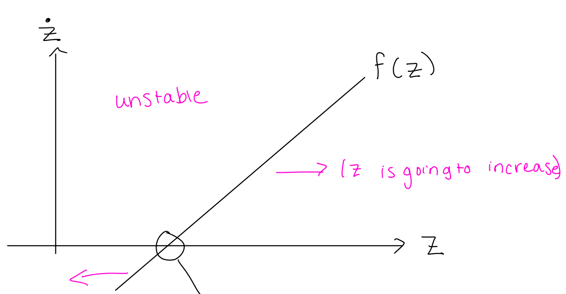
Basic motivation for Discrete time is as time-discretization of continuous time.

Cont $\dot{z}(t) = \lim_{\delta \to 0} \frac{z(t+\delta) - z(t)}{\delta}$

$\therefore \quad \dot{z} = f(z)$ and $z(t+\delta) = z(t) + \delta \cdot f(z)$
↓
Discrete time known as Forward Euler

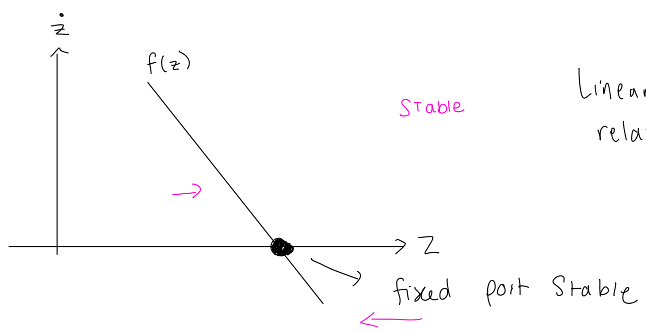
There are other C.T to D.T possibilities (later)

System Diagram



unstable fixed point

convention is unfilled if unstable



Linear b/c relation of $f(z)$ is linear

convention is filled when Stable

Aside: D.T. uses $y = x$ to find fixed points

Linear System: Key Attributes.

① Super position : $x_1 \to \square \to y_1$ $\quad x_2 \to \square \to y_2$

then $\alpha x_1 + \beta y_1 \to \square \to \alpha y_1 + \beta y_2$

Ex Streching Spring obeys super position

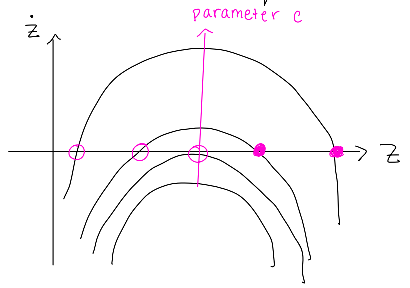
② Sine wave $A\sin(\omega t + \phi) \to \square \to \beta \sin(\omega t + \varphi)$

Cannot change frequency.

∴ Constant input cannot lead to oscillating out put.

every electronic using a battery is non linear, b/c constant input but it buzzes, flashes

Defn: A bifurcation is a discontinuous change in attribute or behaviour in response to a continuous change in para meter.

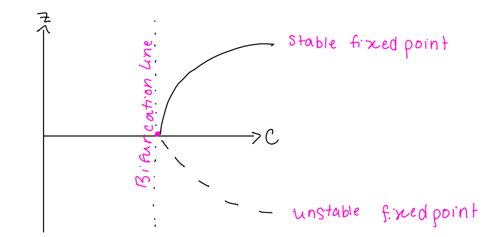


$\dot{z} = -(z-5)^2 + C$

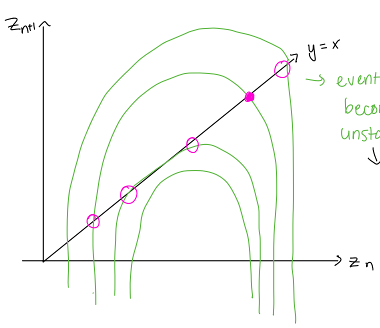
Double root is Called degeneracy.
realistictly never happen.

An example of bifurcation 0 fixed to 2 fixed.

Bifurcation Plot

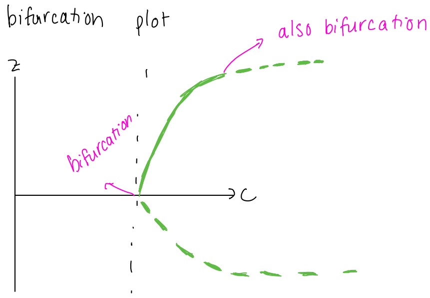


Also have bifurcations in D.T



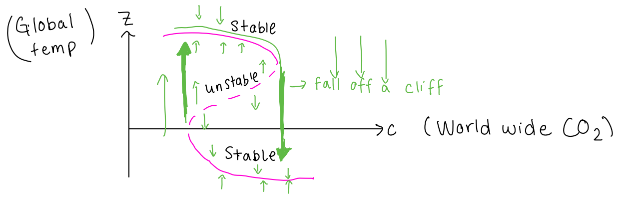
→ eventually becomes unstable
↓ not an attribute of Continuous time.

bifurcation plot

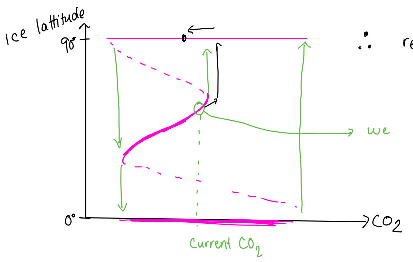


Two major bifurcations:

① Double - fold:



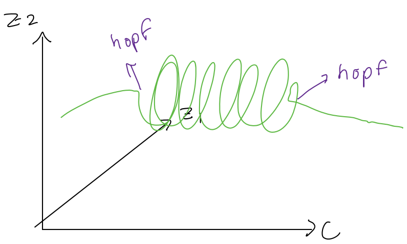
Bistable system another example light switch, temperature. of fridge. nearly all thermal.



∴ reduce carbon m. b/c if we go past our current stable point we would need to reduce alot before getting to our stable nice state

② Hopf bifurcation:

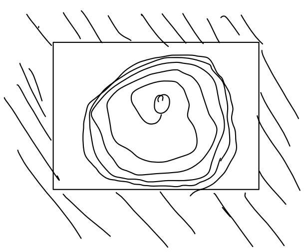
Transition between cycling and not cycling.



Why is cycling Common?

Ex Flag pole, guiser, Sailboat mass. poring water on live plants

b/c bouneded unstable system.

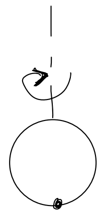


Lineeant cross

→ limit cycle.

Bifurcations that matter to us:

- Stick - slip. (Brakes locking, rude and unpleasent noises)
- covid : fizzling out vs. pandemic
- Bead on a hoop



- Toys: slap braclet, jumping disc, Candle boats
- Ecology, : lake ecology
- Humain brain, epilectic seizures,
- Chocolate production