

Highlighting Legend

Chapter

Definition

Propositions / Theorems / Proofs

Procedures

Remarks are made in red pencil, same with questions

Lecture 1 - Jan 8th 2024

Chapter 1 Iteration and Orbits

Defn Let $f: A \rightarrow \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$ [i.e $f: A \rightarrow A$]. For $a \in A$, we may iterate

allows for indefinite iteration

the function at a :

$$\begin{aligned}x_1 &= a \\x_2 &= f(a) \\x_3 &= f(f(a)) = f^2(a) \\x_4 &= f(f(f(a))) = f^3(a) \\&\vdots\end{aligned}$$

all will exist in A .

We call $(x_n)_{n=1}^{\infty} (= (x_n))$ we call that sequence the orbit of a under f .

Ex. $f(x) = x^4 + 2x^2 - 2$, $a = -1$

$$\begin{aligned}x_1 &= a, x_2 = f(a), x_3 = f^2(a) \dots \\-1, 1, 1, \dots &\rightarrow \text{eventually constant / periodic}\end{aligned}$$

Ex $f(x) = x^2 - x + 1$, $a = 0$

$$\begin{aligned}x_1, x_2, x_3, x_4, x_5, x_6 \dots \\0, 1, -1, 1, -1, 1 \dots &\rightarrow \text{eventually periodic in period 2}\end{aligned}$$

Ex. $f(x) = x^2 - 3x + 1$, $a = 1$

$$\begin{aligned}x_1, x_2, x_3, x_4, \dots \\1, -1, 3, 19 \rightarrow \infty \quad [\text{he will talk about convergence / divergence later on}]\end{aligned}$$

Ex. $f(x) = x^2 + 2x$, $a = -0.5$

My Q: Does it ever equal 1, is that a distinction?

$$\begin{aligned}-0.5, -0.75, -0.9375, -0.9961, \dots &\rightarrow \text{Converges to } -1\end{aligned}$$

Ex. $f(x) = x^3 - 3x$, $a = 0.75$

$$\begin{aligned}0.75, -1.828, -0.625, 1.631, -0.552, \dots &\rightarrow \text{This is chaotic behaviour. * For P-MATH insights: an interval around zero is dense *}\end{aligned}$$

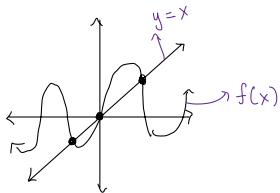
Defn: $f: A \rightarrow \mathbb{R}$, $f(A) \subseteq A$, We say $a \in A$ is a fixed point in f iff $f(a) = a$. In this case, the orbit of a is a, a, a, \dots which is constant.

Ex. Find all fixed points of $f(x) = x^2 + x - 4$

When does $x^2 + x - 4 = x$?

$$\Leftrightarrow x^2 - 4 = 0 \Leftrightarrow x = \pm 2 \quad \text{Will often give low degree polynomial}$$

Ex. How many fixed points does $f(x)$ as pictured below have?



Geometrically, a fixed point occurs when $f(x)$ intersects $y=x$

Ex. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

Solve for $x^4 - 3x + 1 = x$

$$\Rightarrow x^4 - 4x + 1 = 0 \quad \text{Intermediate Value theorem}$$

Since $g(x)$ is continuous $g(0) = 1 > 0$ and $g(1) = -2 < 0$. By IVT $\exists x \in (0, 1)$ s.t. $g(x) = 0 \Leftrightarrow f(x) = x$.

Defn $f: A \rightarrow \mathbb{R}$, $f(A) \subseteq A$

① We say $a \in A$ is a periodic point for f if its orbit is periodic. I.E $\exists n \in \mathbb{N} \quad f^n(a) = a$. The least such n is called the period of a and/or the orbit.

② Eventually periodic $\exists n < m, f^n(a) = f^m(a)$

Lecture 2 - Jan 10th 2024

Def'n [Doubling Function]

$D: [0,1] \rightarrow [0,1]$ where $D(x) = \text{fractional part of } 2x$ aka $2x \bmod 1$

Ex) $D(0.4) = 0.8$

$$D(0.6) = 0.2$$

$$D(0.8) = 0.6$$

$$D(0.5) = 0$$

It is an important function as it also provides a rich source of periodic orbits

Ex) $D, a = \frac{1}{5}$

orbit: $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \dots$ This has period 4. This leads into some cute number theory with GCDs

Ex) $D, a = \frac{1}{20}$

orbit $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \dots$ This is eventually periodic with period 4.

The Doubling function will come up later in a more meaningful way

Every day functions may not exhibit periodic orbits

Q: Given f and a , does $f^n(a)$ tend towards some limit L ? This does happen surprisingly often

The language of this course vs Elementary Real Analysis

Notation:

If $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers we write $(x_n) \subseteq \mathbb{R}$ → a small abuse of notation but reasonable

Def'n $(x_n) \subseteq \mathbb{R}, x \in \mathbb{R}$ we say (x_n) converges to x iff for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|x_n - x| < \epsilon$ for all $n \geq N$

We write $x_n \rightarrow x$ or $\lim x_n = x$

↓
N depends *

Ex) Claim: $\frac{1}{n} \rightarrow 0$, Let $\epsilon > 0$ be given.

Note: $\frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$

Consider $N = \frac{2}{\epsilon}$ For $n \geq N$ we have $\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \epsilon$ not necessarily a Natural number. Can always use ceiling function $\lceil \cdot \rceil$ to bring this to a natural number
↓ bring this to $\frac{2}{\epsilon}$ so that we ensure strictly less than epsilon.

Ex) Claim $\frac{2n}{n+3} \rightarrow 2$

Let $\epsilon > 0$ be given. Let us choose $N \in \mathbb{N}$ s.t.

$$\frac{1}{N} < \frac{\epsilon}{6}$$

For $n \geq N$

$$\begin{aligned} & \left| \frac{2n}{n+3} - 2 \right| \\ &= \left| \frac{2n}{n+3} - \frac{2n+6}{n+3} \right| \\ &= \left| \frac{-6}{n+3} \right| = \frac{6}{n+3} < \frac{6}{n} \xrightarrow{\text{if you divide by less of something you get something bigger}} \\ & \frac{6}{n} \leq \frac{6}{N} = 6\left(\frac{1}{N}\right) < 6\left(\frac{\epsilon}{6}\right) = \epsilon \end{aligned}$$

Defn $(x_n) \subseteq \mathbb{R}$ we say (x_n) is bounded if $\exists M > 0$ s.t. $\forall n \in \mathbb{N}, |x_n| \leq M$

Prop $(x_n) \subseteq \mathbb{R}$ If (x_n) is convergent, then (x_n) is bounded.

Ex) $x_n = (-1)^n$ Shows converse is not true

Proof Suppose $x_n \rightarrow x$. then $\exists N \in \mathbb{N}, n \geq N \Rightarrow |x_n - x| < 1$

For $n \geq N$,

$$|x_n| - |x| \leq |x_n - x| < 1 \Rightarrow |x_n| < 1 + |x| \quad \text{This is only true for } n \geq N$$

↑ this is the reverse triangle inequality, make sure you know how to prove from triangle inequality
picked ϵ , blake's favourite is 1

So Let $M = \max \{ |x_1|, \dots, |x_{N-1}|, 1 + |x| \}$ ■

Prop Let $x_n \rightarrow x, y_n \rightarrow y$

Not super important to us,
value is in working with
defn of convergence

① $x_n + y_n \rightarrow x + y$

② $x_n y_n \rightarrow xy$

Proof:

① Let $\epsilon > 0$ be given. There exists $N_1, N_2 \in \mathbb{N}$ s.t.

$$n \geq N_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$$

$$n \geq N_2 \Rightarrow |y_n - y| < \frac{\epsilon}{2}$$

For $N = \max\{N_1, N_2\}$ and $n \geq N$

$$|x_n + y_n - (x+y)| = |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y| < \varepsilon$$

(2) Let $\varepsilon > 0$ be given

Note •

$$|x_n y_n - xy| = |x_n y - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - y| \quad \star$$

Since (x_n) is bounded $\exists M > 0, \forall n, |x_n| < M$

Let $N_1, N_2 \in \mathbb{N}$ s.t. $n \geq N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2(|y|+1)}$ make sure not dividing by zero

$$n \geq N_2 \Rightarrow |y_n - y| < \frac{\varepsilon}{2M}$$

For $n \geq N := \max\{N_1, N_2\}$ we have $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by \star ■

Lec 3 - Jan 12th 2024

Def'n we say $(X_n) \subseteq \mathbb{R}$ is Cauchy if $\forall \epsilon > 0, \exists N \in \mathbb{N}, n, m \geq N \Rightarrow |X_n - X_m| < \epsilon$

Point of Clarification. The Wednesday class before the assignment is due. We will cover all the material needed for the assignment. Also we can use calculus 2. A lot of this course is in the language of Calculus and Real Analysis

Prop Convergent \Rightarrow Cauchy

Proof: Let $\epsilon > 0$ be given and suppose (X_n) is convergent. Say $X_n \rightarrow x \in \mathbb{R}$. There exist $N \in \mathbb{N} \quad n \geq N$

$$\Rightarrow |X_n - x| < \boxed{\frac{\epsilon}{2}} \quad \text{Then for } n, m \geq N$$

both are $< \epsilon$ which gives information on what should be in the box

$$|X_n - X_m| = |X_n - x + x - X_m| \leq |X_n - x| + |x - X_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

take ϵ divided by # of parts

□

Fact: $(X_n) \subseteq \mathbb{R}, (X_n)$ Cauchy $\Leftrightarrow (X_n)$ Convergent This is also a part of the Completeness of \mathbb{R}

Big Idea: To prove (X_n) is Cauchy you do not have to guess the limit! \hookrightarrow this will be useful for fixed points

Def'n $f: A \rightarrow \mathbb{R}, A \subseteq \mathbb{R}, a \in A$. we say f is continuous at a iff $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$ whenever $x \in A$ and $|x - a| < \delta$.

\rightarrow close values in the domain gives close values in function

Fact: f is continuous at a iff $\forall (X_n) \subseteq A$ with $X_n \rightarrow a$, we have $f(X_n) \rightarrow f(a)$ \hookrightarrow function at the terms in the sequence provides link between convergence and continuity.

\hookleftarrow can prove as an exercise

\hookrightarrow highly dependent on that f goes from $[a, b] \rightarrow [a, b]$

Prop: If $f: [a, b] \rightarrow [a, b]$ is continuous then $f(x)$ has a fixed point

Proof: we know $f(a) \geq a$ and $f(b) \leq b$. $\Leftrightarrow f(a) - a \geq 0$ and $f(b) - b \leq 0$

By applying the IVT to the continuous function $g(x) = f(x) - x \quad \exists x \in [a, b]$ such that $g(x) = 0 \Leftrightarrow f(x) = x$ □

\hookrightarrow a preview of why Calculus is applicable to this course. Epsilon will come back.

Def'n $f: A \rightarrow \mathbb{R}, A \subseteq \mathbb{R}$. we say that f is a contraction iff $\exists C \in [0, 1)$ s.t. $\forall x, y \in A, |f(x) - f(y)| < C|x - y|$

\hookrightarrow is a type of Lipschitz fn \rightarrow would that mean fn. is below line $y = x$ graphically?

Prop Contractions are continuous at every point

Proof Let $\epsilon > 0$ be given Suppose $|f(x) - f(y)| < C|x - y|$ as before. Fix $y \in A$, Consider $\delta = \frac{\epsilon}{C+1}$ \hookrightarrow to make sure we aren't dividing by 0

and assume $x \in A$ s.t. $|x-y| < \delta$. Then

$$|f(x) - f(y)| \leq C|x-y| < C\delta < \epsilon \quad \square$$

Say function f is problematic, not a contraction b/c it drops off at 0

Defn We say $A \subseteq \mathbb{R}$ is closed iff whenever $(x_n) \subseteq A$ with $x_n \rightarrow x \in \mathbb{R}$ then $x \in A$.

Ex. $[a, b]$ is closed

Ex. $(0, 1]$ not closed $\frac{1}{n} \rightarrow 0 \notin (0, 1]$

Theorem [Banach Contraction Mapping Theorem] Suppose $A \subseteq \mathbb{R}$ is closed and $f: A \rightarrow A$ is a contraction.

Then there exist a unique fixed point $a \in A$ for f . Moreover $\forall x \in A$ $f^n(x) \rightarrow a$ has a cute proof. allegedly.
will prove on monday

Ex. $f: [0, 1] \rightarrow [0, 1]$ $f(x) = \frac{1}{3-x}$

Note: $\frac{1}{3} \leq f(x) \leq \frac{1}{2}$ \rightarrow shows it maps back into $[0, 1]$
\hookrightarrow also not an onto function

$$f'(x) = \frac{1}{(3-x)^2}, \quad \frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$$

By the MVT, $\forall x, y \in [0, 1]$ $\exists c \in (0, 1)$ s.t. $f(x) - f(y) = f'(c)(x-y)$

$$\begin{aligned} \Rightarrow |f(x) - f(y)| &= |f'(c)| |x-y| \\ &\leq \underbrace{\frac{1}{4}}_c |x-y| \end{aligned}$$

$\therefore f(x)$ is a contraction.

$$\frac{1}{3-x} = x$$

$$\Leftrightarrow 1 = 3x - x^2$$

$$\Leftrightarrow x^2 - 3x + 1 = 0$$

$$\Leftrightarrow x = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\Leftrightarrow x = \frac{3-\sqrt{5}}{2}$$

$$\therefore \forall x \in [0, 1] \quad f^n(x) \rightarrow \frac{3-\sqrt{5}}{2}$$

Dec 4 - Jan 15th 2023

Note: Graphical Analysis tool posted on piazza. Assignment 1 due next week.

Remark: The Banach Contraction Mapping is almost like a black hole

Recall: $(a_n) \subseteq \mathbb{R}$ we say (a_n) is **Strongly Cauchy** if $\exists \epsilon \in [0, \infty)$ s.t.

$$\textcircled{1} \quad \sum_{n=1}^{\infty} \epsilon_n < \infty$$

$$\textcircled{2} \quad \forall n, |a_n - a_{n+1}| < \epsilon_n$$

This is on assignment.

Hint: $\sum_{n=1}^{\infty} a_n = L, \sum_{k=1}^{\infty} a_k \xrightarrow{n \rightarrow \infty} L$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow \left| \sum_{k=N+1}^{\infty} a_k \right| < \epsilon$

Proof of Banach Contraction Mapping

Let $A \subseteq \mathbb{R}$ be closed and suppose $\exists c \in [0, 1)$ s.t. $|f(x) - f(y)| \leq c|x-y|$ for all $x, y \in A$. Take a point $x_0 \in A$

and construct $x_1 = f(x_0), x_2 = f(x_1), \dots, x_n = f(x_{n-1}) = f^n(x_0)$ this is the orbit. \rightarrow we aren't including x_0 in the orbit but it doesn't matter

$$\begin{aligned} \text{For } n \in \mathbb{N} \quad |x_{n+1} - x_n| &\stackrel{\text{defn of orbit}}{=} |f(x_n) - f(x_{n-1})| \stackrel{\text{contraction assumption}}{\leq} c|x_n - x_{n-1}| = c|f(x_{n-1}) - f(x_{n-2})| \leq c^2|x_{n-1} - x_{n-2}| \\ &\vdots \\ &\leq c^n|x_1 - x_0| \end{aligned}$$

$c \in [0, 1)$ this is geometric

Since $\sum_{n=1}^{\infty} c^n|x_1 - x_0|$ is a convergent geometric series, we have that (x_n) is strongly Cauchy. Hence $x_n \rightarrow a$ for some $a \in A$

\hookrightarrow This is because domain is closed

Since f is continuous, $\underbrace{f(x_n)}_{x_{n+1}} \rightarrow f(a) \therefore f(a) = a \rightarrow \text{B}(c, x_n \rightarrow a, f(x_n) \rightarrow f(a) \Rightarrow f(a) = a)$

\rightarrow What's significant about this?

Suppose $a, b \in A$, s.t. $f(a) = a$ and $f(b) = b$. Then $|f(a) - f(b)| \leq c|a-b| \Rightarrow |a-b| \leq c|a-b|$. Since $c < 1$,

$|a-b| = 0$ and so $a = b$. This is our strongest fixed point theorem. This is the big analytic results for existence and uniqueness of ODEs

Chapter 2 Graphical analysis.

To visualize the orbit of a under f :

- ① Superimpose $y = f(x)$, $y = x$

(2) Use a vertical line: $(a, a) \longrightarrow (a, f(a))$

(3) Use a horizontal line: $(a, f(a)) \longrightarrow (f(a), f(a))$

(4) Vertical line: $(f(a), f(a)) \longrightarrow (f(a), f^2(a))$

(5) Use a horizontal line: $(f(a), f^2(a)) \longrightarrow (f^2(a), f^3(a))$

Etc...

Ex) Using online tool and $f(x) = x^2 - x + 1$, fixed points $x=1$

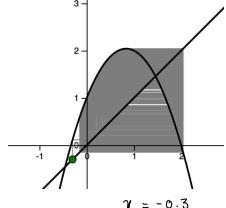
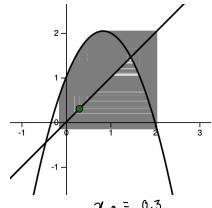
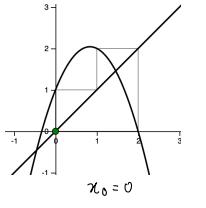
Orbit analysis:

① $x \in [0, 1]$ we have, $f^n(x) \rightarrow 1$

② otherwise, $x \notin [0, 1]$ $f^n(x) \rightarrow \infty$ In assignment, only expecting this sort of an answer.

Lec 5 - Jan 17th 2024

For $f(x) = \frac{-3}{2}x^2 + \frac{5}{2}x + 1$ at $x=0$ we have an orbit of $0, 1, 2$ which is periodic and graphical is a cycle



points near 0, like 0.06 show chaos. The graphical analysis look does show chaos. Chaos also shows density. We can never hit every point in the interval because an orbit is countable but $[1, 2]$ in \mathbb{R} is uncountable. It is dense [ex. rational numbers is dense in $[0, 1] \subset \mathbb{R}$]. We will define dense in a bit.

Chapter 3 - fixed points

Remark. If $f(x)$ is continuous and $f^n(a) \rightarrow L$, then $f^{n+1}(a) \rightarrow f(L) \rightarrow$ comes from continuity. We also saw this in the B.C.M theorem

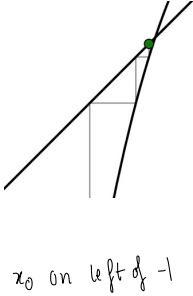
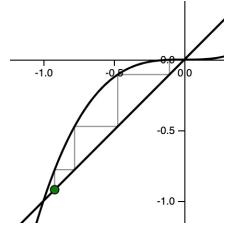
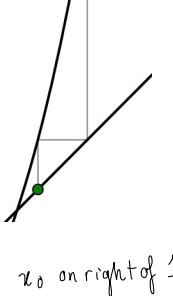
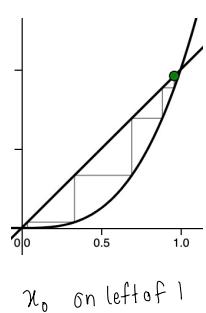
$$\therefore f(L) = L$$

Motivating Example

$$F(x) = x^3, \text{ fixed points: } 0, \pm 1$$

① for $x \in (-1, 1)$ we see that $f^n(x) \rightarrow 0$. This is an example of an attracting fixed point

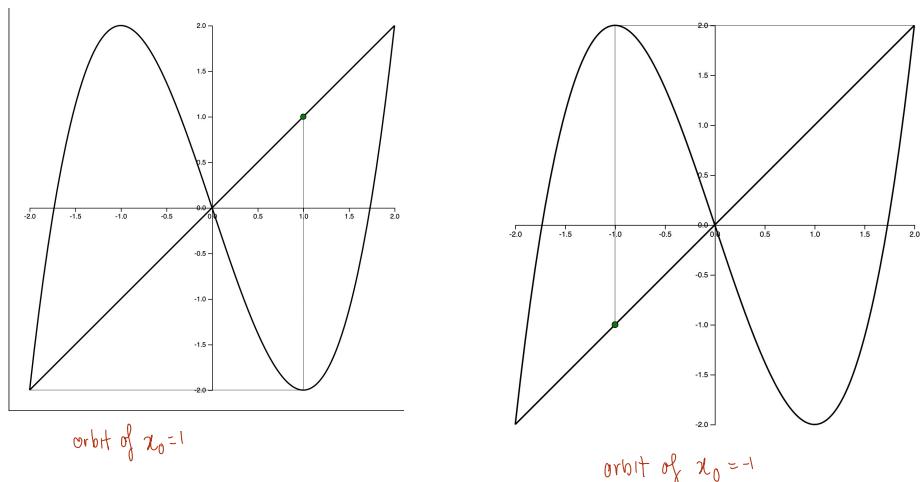
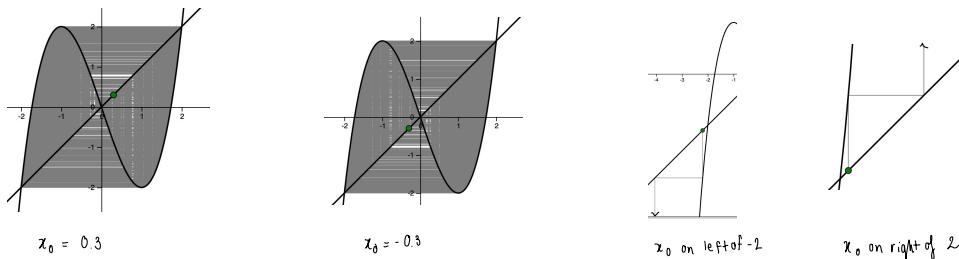
② $x \in (1, \infty)$, $f^n(x) \rightarrow \infty$ and $x \in (-\infty, -1)$, $f^n(x) \rightarrow -\infty$. We call ± 1 repelling fixed points



Motivating Example 2

$$f(x) = x^3 - 3x, \text{ fixed points: } 0, \pm 2$$

- ① 0 is repelling [in a different way than last example b/c right & left to 0 is chaotic]
- ② to the right of 2 the orbits go to infinity, left of -2 orbits go to -infinity. so ± 2 is also repelling.
- ③ at $x_0 = 1$, the orbit is eventually fixed to the fixed point -2; for $x_0 = -1$, the orbit is ev



Defn let \underline{a} be a fixed point of $f(x)$

- ① if $|f'(\underline{a})| > 1$, we call \underline{a} a **repelling** fixed point.
- ② if $|f'(\underline{a})| < 1$, we call \underline{a} an **attracting** fixed point
- ③ if $|f'(\underline{a})| = 1$, we call \underline{a} a **neutral** fixed point \rightarrow this can mean a whole bunch of things

Theorem [Attracting fixed point thm]

Suppose a is an attracting fixed of $f(x)$ then \exists an open interval I such that $a \in I$ and

- ① $\forall x \in I, \forall n \in \mathbb{N}, f^n(x) \in I \rightarrow$ slightly unnecessary because if $f(x) \in I \Rightarrow f^n(x) \in I$
- ② $\forall x \in I, f^n(x) \rightarrow a$.

Defn: $f: A \rightarrow \mathbb{R}$, $A \subseteq \mathbb{R}$.

① we say $a \in A$ is non-isolated iff $\forall \epsilon > 0$, $\exists b \in A$ with $b \neq a$ s.t. $b \in (a-\epsilon, a+\epsilon)$

② Let $a \in A$ be non-isolated. We say $\lim_{x \rightarrow a} f(x) = L$ iff $\forall \epsilon > 0 \exists \delta > 0$, $|f(x) - L| < \epsilon$ whenever $a \in A$ and $0 < |x-a| < \delta$

→ if a was an isolated, we could choose a δ where $|x-a| < \delta$ is false. Leading to a false hypothesis which leads to an always true answer and hence anything could be the limit.

Ex $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ 0 is non-isolated

Ex $\{1\} \cup (2, 3)$ 1 is isolated

Proof: [Attracting fixed point thm] → to continue in Lec 6

Assume $|f'(a)| < 1$. Then $\exists c \in \mathbb{R}$ s.t. $|f'(a)| < c < 1$

$$\therefore \lim_{x \rightarrow a} \frac{|f(x) - f(a)|}{|x-a|} < c$$

Thus $\exists \delta > 0$ s.t. $\frac{|f(x) - f(a)|}{|x-a|} \leq c \quad \forall x \in (a-\delta, a+\delta)$

Hence, for $x \in (a-\delta, a+\delta)$ $|f(x) - f(a)| \leq c|x-a| \rightarrow$ contracting