Highlighting Legend

Chapter

Definition

Propositions / Theorems / Proofs

Procedures

Remarks are made in red pencil, same with questions

Lecture 1 – Jan 8th 2024

Chapter 1 Iteration and Orbits

allows for indefinite iteration

Defn Let $f: A \to \mathbb{R}$ such that $A \subseteq \mathbb{R}$ and $f(A) \subseteq A$ [I.E $f: A \to A$]. For $a \in A$, we may iterate the function at a:

$$x_1 = a$$
$$x_2 = f(a)$$
$$x_3 = f(f(a)) = f^2(a)$$
$$x_4 = f(f(f(a))) = f^3(a)$$
$$\vdots$$

all will exist in A.

We call $(x_n)_{n=1}^{\infty} = (= (x_n))$ we call that sequence the orbit of a under f.

Ex. $f(x) = x^4 + 2x^2 - 2$, $a = -1$

$x_1 = a$, $x_2 = f(a)$, $x_3 = f^2(a)$ ...
$-1, 1, 1, \ldots$ → eventually constant / periodic

Ex $f(x)$ $-x^2 - x + 1$, $a = 0$

$x_1, x_2, x_3, x_4, x_5, x_6 \ldots$
$0, 1, -1, 1, -1, 1 \ldots$ → eventually perodic $\bar{w}$ period 2

Ex. $f(x) = x^2 - 3x + 1$, $a = 1$

$x_1, x_2, x_3, x_4, \ldots$
$1, -1, 3, 19 \to \infty$ goes to infinity [he will talk about convergence / divergence later on]

Ex $f(x) = x^2 + 2x$, $a = -0.5$

x_1, x_2, x_3, x_4
$-0.5, -0.75, -0.9375, -0.9961$, converges to -1

My Q: Does it ever equal 1, is that a distinction?

Ex. $f(x) = x^3 - 3x$, $a = 0.75$

$x_1, x_2, x_3, x_4, x_5, \ldots$
$0.75, -1.828, -0.625, 1.631, -0.552, \ldots$ This is chaotic behaviour.

* For PMATH insights: an Interval around zero is <u>dense</u> *

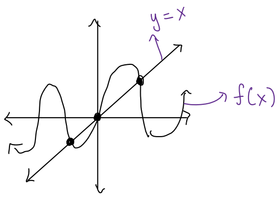
Defn: $f: A \to \mathbb{R}$, $f(A) \subseteq A$, We say $a \in A$ is a fixed point in f iff $f(a) = a$. In this case, the orbit of a is $a, a, a \ldots$ which is constant.

Ex. Find all fixed points of $f(x) = x^2 + x - 4$

When does $x^2 + x - 4 = x$?

$\Leftrightarrow \quad x^2 - 4 = 0 \quad \Leftrightarrow \quad x = \pm 2$ Will often give low degree polynomial

Ex. How many fixed points does $f(x)$ as pictured below have?



$\boxed{\text{Geometrically, a fixed point occurs when } f(x) \text{ intersects } y = x}$

Ex. Prove that $f(x) = x^4 - 3x + 1$ has a fixed point.

Solve for $x^4 - 3x + 1 = x$

$\Rightarrow \quad x^4 - 4x + 1 = 0$

Intermediate Value theorem

Since $g(x)$ is continuous $g(0) = 1 > 0$ and $g(1) = -2 < 0$. By IVT $\exists\, x \in (0,1)$ s.t. $g(x) = 0 \Leftrightarrow f(x) = x$.

Defn $f: A \to \mathbb{R}$, $f(A) \subseteq A$

① We say $a \in A$ is a periodic point for f if its orbit is periodic. I.E $\exists n \in \mathbb{N}$ $f^n(a) = a$. The least such n is called the period of a and/or the orbit.

② Eventually periodic $\exists n < m$, $f^n(a) = f^m(a)$

Lecture 2 — Jan 10^{th} 2024

Def'n [Doubling Function]

$D: [0,1) \to [0,1)$ where $D(x) =$ fractional part of $2x$ aka $2x \bmod 1$

Ex) $D(0.4) = 0.8$
$D(0.6) = 0.2$
$D(0.8) = 0.6$
$D(0.5) = 0$

It is an important function as it also provides a rich source of periodic orbits

Ex) D, $a = \frac{1}{5}$
Orbit: $\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \ldots$ This has period 4. This leads into some cute number theory with GCDs

Ex) D, $a = \frac{1}{20}$
Orbit $\frac{1}{20}, \frac{1}{10}, \frac{1}{5} \ldots$ This is eventually periodic with period 4.

> The Doubling function will come up later in a more meaningful way

Every day functions may not exhibit periodic orbits

Q: Given f and a, does $f^n(a)$ tend towards some limit L? This does happen suprisingly often

> The language of this course vs Elementary Real Analysis

Notation:

If $(x_n)_{n=1}^{\infty}$ is a sequence of real number we write $(X_n) \subseteq \mathbb{R}$ a small abuse of notation but reasonable

Def'n $(x_n) \subseteq \mathbb{R}$, $x \in \mathbb{R}$ we say (x_n) **converges** to x iff for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon$ for all $n \geq N$

* N depends on ε *

We write $x_n \to x$ or $\lim x_n = x$

Ex) Claim: $\frac{1}{n} \to 0$, Let $\varepsilon > 0$ be given.

Note: $\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$

not necessarily a Natural number. Can always use ceiling function $\lceil\,\rceil$ to bring this to a natural number

Consider $N = \frac{2}{\varepsilon}$ For $n \geq N$ we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$

bring this to $2/\varepsilon$ so that we ensure strictly less than Epsilon.

Ex) Claim $\frac{2n}{n+3} \to 2$

Let $\varepsilon > 0$ be given. Let us choose $N \in \mathbb{N}$ s.t.

$$\boxed{\frac{1}{N} < \frac{\varepsilon}{6}}$$

For $n \geq N$ $\left| \frac{2n}{n+3} - 2 \right|$

$$= \left| \frac{2n}{n+3} - \frac{2n+6}{n+3} \right|$$

$$= \left| \frac{-6}{n+3} \right| = \frac{6}{n+3} < \frac{6}{n}$$ if you divide by less of something you get something bigger

$$\frac{6}{n} \leq \frac{6}{N} = 6\left(\frac{1}{N}\right) < 6\left(\frac{\varepsilon}{6}\right) = \varepsilon$$

Defn $(x_n) \subseteq \mathbb{R}$ we say (x_n) is **bounded** if $\exists M > 0$ s.t. $\forall n \in \mathbb{N}, |x_n| \leq M$

Prop $(x_n) \subseteq \mathbb{R}$ If (x_n) is convergent, then (x_n) is bounded.

Ex) $x_n = (-1)^n$ Shows converse is not true

picked ε, blake's favourite is 1

Proof Suppose $x_n \to x$. then $\exists N \in \mathbb{N}$, $n \geq N \Rightarrow |x_n - x| < 1$

For $n \geq N$,

this is the reverse triangle inequality, make sure you know how to prove from triangle inequality

$$|x_n| - |x| \leq |x_n - x| < 1 \Rightarrow |x_n| < 1 + |x| \Big\}$$ This is only true for $n \geq N$

So let $M = \max\{|x_1|, \dots, |x_{n-1}|, 1 + |x|\}$ ■

Prop Let $x_n \to x$, $y_n \to y$

Not super important to us, value is in working with defn of convergence

① $x_n + y_n \to x + y$

② $x_n y_n \to xy$

Proof:

① Let $\varepsilon > 0$ be given. There exists $N_1, N_2 \in \mathbb{N}$ s.t.

$$n \geq N_1 \Rightarrow |x_n - x| < \boxed{\frac{\varepsilon}{2}}$$

$$n \geq N_2 \Rightarrow |y_n - y| < \boxed{\frac{\varepsilon}{2}}$$

For $N = \max\{N_1, N_2\}$ and $n \geq N$

$$|x_n + y_n - (x+y)| = |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y| < \varepsilon$$

(2) Let $\varepsilon > 0$ be given

Note:

$$|x_n y_n - xy| = |x_n y - x_n y + x_n y - xy| \leq |x_n| \cdot |y_n - y| + |y| \cdot |x_n - y| \quad ☆$$

Since (x_n) is bounded $\exists M > 0, \forall n, |x_n| < M$

Let $N_1, N_2 \in \mathbb{N}$ s.t. $n \geq N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2(|y|+1)}$ → make sure not dividing by zero

$n \geq N_2 \Rightarrow |y_n - y| < \frac{\varepsilon}{2M}$

For $n \geq N := \max\{N_1, N_2\}$ we have $|x_n y_n - xy| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by ☆ ∎

Lec 3 - Jan 12th 2024

Def'n we say $(X_n) \subseteq \mathbb{R}$ is **Cauchy** if $\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n, m \geq N \Rightarrow |x_n - x_m| < \varepsilon$

Point of Clarification. The Wednesday class before the assigment is due We will cover all the material needed for the assignment. Also we can use calculus 2. Alot of this course is in the language of Calculus and Real Analysis

Prop Convergent $\Rightarrow$ Cauchy

Proof: Let $\varepsilon > 0$ be given and Suppose (X_n) is convergent. Say $x_n \to x \in \mathbb{R}$. There exist $N \in \mathbb{N}$ $n \geq N$

$\Rightarrow |x_n - x| < \boxed{\frac{\varepsilon}{2}}$ Then for $n, m \geq N$

both are $< \varepsilon$ which gives information on what should be in the box

$$|x_n - x_m| = |X_n - X + X - X_m| \leq |X_n - X| + |X - X_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon \quad \square$$

take ε divided by # of parts

Fact: $(X_n) \subseteq \mathbb{R}$, (X_n) Cauchy $\Leftrightarrow$ (X_n) Convergent This is also apart of the Completeness of $\mathbb{R}$

Big Idea: To prove (X_n) is Cauchy you do not have to guess the limit! $\rightarrow$ This will be useful for fixed points

Def n $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, $a \in A$. we say f is **continuous** at a iff $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|F(x) - F(a)| < \varepsilon$ whenever $x \in A$ and $|x - a| < \delta$.

$\rightarrow$ close values in the domain gives close values in function

Fact: f is continuous at a iff $\forall (x_n) \subseteq A$ (sequence) with $x_n \to a$, we have $f(x_n) \to f(a)$ (function at the terms in the sequence) Provides link between convergence and continuity.

$\hookrightarrow$ can prove as an excersize

highly dependent on that f goes from $[a,b] \to [a,b]$

Prop: If $f: [a,b] \to [a,b]$ is Continuous then $f(x)$ has a fixed points

Proof: we know $f(a) \geq a$ and $f(b) \leq b$. $\Leftrightarrow$ $f(a) - a \geq 0$ and $f(b) - b \leq 0$

By appling the IVT to the continuous function $g(x) = f(x) - x$ $\exists x \in [a,b]$ such that $g(x) = 0 \Leftrightarrow f(x) = x$ $\square$

$\hookrightarrow$ a preview of why Calculus is applicable to this course. Epsilons will come back.

is a type of Lipshitz fcn $\rightarrow$ would that mean fcn. is below line $x = y$ graphically?

Def'n $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$. We say that f is a contraction iff $\exists C \in [0,1)$ s.t. $\forall x, y \in A$, $|f(x) - f(y)| \leq C|x - y|$

Prop Contractions are Continuous at every point

Proof Let $\varepsilon > 0$ be given Suppose $|f(x) - f(y)| \leq C|x - y|$ as before. Fix $y \in A$, Consider $\delta = \boxed{\frac{\varepsilon}{C+1}}$ $\rightarrow$ to make sure we arent dividing by 0 and Assume $x \in A$ s.t. $|x-y| < \delta$. Then

$$|f(x) - f(x)| \leq C|x-y| < C\delta < \varepsilon \quad \square$$

Sqrt function a problematic, not a contract b/c it drops off a to 0

Defn We say $A \subseteq \mathbb{R}$ is close iff whenever $(x_n) \subseteq A$ with $x_n \to x \in \mathbb{R}$ then $x \in A$.

Ex. $[a,b]$ is closed

Ex $(0,1]$ not closed $\frac{1}{n} \to 0 \notin (0,1]$

Thereom [Banach Contraction Mapping theorem] Suppose $A \subseteq \mathbb{R}$ is closed and $f: A \to A$ is a contraction. Then there exist a unique fixed point $a \in A$ for f. Moreover $\forall x \in A$ $f^n(x) \longrightarrow a$

has a cute proof. allegedy.
will prove on monday

Ex $f: [0,1] \to [0,1]$ $f(x) = \frac{1}{3-x}$

Note: $\frac{1}{3} \leq f(x) \leq \frac{1}{2}$ shows it maps back into $[0,1]$
also not an onto function

$$f'(x) = \frac{1}{(3-x)^2}, \qquad \frac{1}{9} \leq |f'(x)| \leq \frac{1}{4}$$

By the MVT, $\forall x,y \in [0,1]$ $\exists c \in (0,1)$ s.t. $f(x) - f(y) = f'(c)(x-y)$

$$\Rightarrow |f(x) - f(y)| = |f'(c)|\,|x-y|$$
$$\leq \underbrace{\tfrac{1}{4}}_{C}|x-y|$$

$\therefore$ $f(x)$ is a contraction.

$$\frac{1}{3-x} = x$$

$$\Leftrightarrow 1 = 3x - x^2$$

$$\Leftrightarrow x^2 - 3x + 1 = 0$$

$$\Leftrightarrow x = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\Leftrightarrow x = \frac{3-\sqrt{5}}{2}$$

$\therefore$ $\forall x \in [0,1]$ $f^n(x) \to \frac{3-\sqrt{5}}{2}$

Lec 4 - Jan 15th 2023

Note: Graphical Analysis tool posted on piazza. Assignment 1 due next week.

Remark: The Banach Contraction Mapping is almost like a black hole

Recall: $(a_n) \subseteq \mathbb{R}$ we say (a_n) is strongly Cauchy if $\exists\ (\varepsilon_n) \subseteq [0,\infty)$ s.t.

① $\sum_{n=1}^{\infty} \varepsilon_n < \infty$

② $\forall n,\ |a_n - a_{n+1}| < \varepsilon_n$

This is on assigment.

Hint: $\sum_{n=1}^{\infty} a_n = L$, $\underbrace{\sum_{k=1}^{n} a_k}_{S_n} \xrightarrow{n\to\infty} L$

$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ s.t $n \geq N \Rightarrow \underbrace{|S_n - L|}_{\left|\sum_{k=n+1}^{\infty} a_k\right|} < \varepsilon$ → the fact you can make up a tail end, a series of them, and adding up is arbitrally small

Proof of Banach Contraction Mapping

Let $A \subseteq \mathbb{R}$ be closed and suppose $\exists C \in [0,1)$ s.t. $|f(x) - f(y)| \leq c|x-y|$ for all $x, y \in A$. Take a point $x_0 \in A$

and construct $\{x_1 = f(x_0),\ x_2 = f(x_1), \dots, x_n = f(x_{n-1}) = f^n(x_0)\}$ this is the orbit. → we arent including x_0 in the orbit but it doesnt matter

For $n \in \mathbb{N}$ $|x_{n+1} - x_n| \overset{\text{defn of orbit}}{=} |f(x_n) - f(x_{n-1})| \overset{\text{contraction assumption}}{\leq} c|x_n - x_{n-1}| = c|f(x_{n-1}) - f(x_{n-2})| \leq c^2|x_{n-1} - x_{n-2}|$

$\vdots$

$\leq c^n|x_1 - x_0|$ → $c \in [0,1)$ → this is geometric

Since $\sum_{n=1}^{\infty} c^n|x_1 - x_0|$ is a convergent geometric series, we have that (x_n) is strongly Cauchy. Hence $x_n \to a$ for some $a \in A$

↳ This is because domain is closed

Since f is continuous, $\underbrace{f(x_n)}_{x_{n+1}} \to f(a)$ $\therefore f(a) = a$ → B/c $x_n \to a$, $f(x_n) \to f(a) \Rightarrow f(a) = a$

→ whats significant about this?

Suppose $a, b \in A$, s.t $f(a) = a$ and $f(b) = b$. Then $|f(a) - f(b)| \leq c|a-b| \Rightarrow |a-b| \leq c|a-b|$. Since $c<1$,

$|a-b| = 0$ and so $a = b$. This is our strongest fixed point theorem. This is the big analytic results for existence and uniqueness of ODEs

Chapter 2 Graphical Analysis

to Visualize the orbit of a under f:

① Superimpose $y = f(x)$, $y = x$

② Use a vertical line: (a, a) — $(a, f(a))$

③ Use a horizontal line: $(a, f(a))$ — $(f(a), f(a))$

④ Vertical line: $(f(a), f(a))$ — $(f(a), f^2(a))$

⑤ Use a horizontal line: $(f(a), f^2(a))$ — $(f^2(a), f^2(a))$

Etc...

Ex) Using online tool and $f(x) = x^2 - x + 1$, fixed points $x = 1$

Orbit Analysis:

① $x \in [0, 1]$ we have, $f^n(x) \to 1$

② otherwise, $x \notin [0, 1]$ $f^n(x) \to \infty$ In assignment, only expecting this sort of an answer.