Analytical Solutions for the Mutual Inductance of Circular Rings

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Abstract
An analytical solution for mutual inductance of two circular rings in the same plane with identical dimensions offset by a distance beyond the radius, no overlap, is shown. The Neumann formula is used to achieve the analytical approach using discretization along the two loops. The method is confirmed with a comparison between two square loops numerical solution and analytical solution.

***Index Terms*—Inductance, Neumann formula**

I. Introduction and Overview

THE mutual inductance is a result of system geometry. Whether the inductance is being calculated from one loop to the other is irrelevant and the sign is determined by the currents. This results in a positive scalar that can be applied to a current in one of the loops. As one can see knowing the mutual inductance can be a very powerful tool for a rigid geometry.

When designing and building circuit boards being able to model simple geometries that can be found throughout a design, such as lines that form squares and circles, will be useful. The parasitic that arise from these unwanted inductors will alter the utility of the device and possible cause unwanted properties.

II. Methods

A. Exact formula

The exact formula for mutual inductance is achieved with the formula [1]

$$M = \frac{\mu}{4\pi}\left\{c\ ln\left[\frac{a+A}{c+C}\right] + d\ ln\left[\frac{a+A}{d+D}\right] + b\ ln\left[\frac{b+B}{a+A}\right] + (C+D)-(A+B)\right\}$$

Where μ is the vacuum permeability, a is the total length of the two wires, A is the separation vector of the two wires total length, b is the displacement of the two wires horizontally, B is the displacement of the two wires vertically, c is the length of one wire and the x displacement, C is the separation vector of the length of one wire and x displacement, d is the length of the other wire and the x displacement, and D is the separation vector of the length of one wire and x displacement. This formula is able to easily calculate the exact solution for the two square loops.

B. Neumann formula

The analytical solution will be derived using Neumann's formula [1]. This is derived by Maxwell's equations and the electrodynamics of the magnetic field potential

$$M = \frac{1}{I_2}\oint \boldsymbol{A_2}\cdot \boldsymbol{dl_1}$$

Using the concept that geometry is what determines mutual inductance and not current we can rewrite the equation as

$$M = \frac{\mu}{4\pi}\oint\oint \frac{\boldsymbol{dl_1}\cdot\boldsymbol{dl_2}}{\boldsymbol{R}}$$

Where $\boldsymbol{R}$ is the separation vector between the two line segments. With the solution is an all geometric form the order of the loops is irrelevant. We can take the Neumann formula one step further to include the angel of the two-line segments

$$M = \frac{\mu}{4\pi}\oint\oint \frac{dl^2 * cos\theta}{\boldsymbol{R}}$$

This form of the mutual inductance will be used to solve for the analytical value of both the two rectangles and the two circular loops.

III. Results and Discussion

The exact solution was calculated first. Using MATLAB to perform the calculation the value of $1.9937nH$ was found on the two rectangular loops shown below in Figure 1.

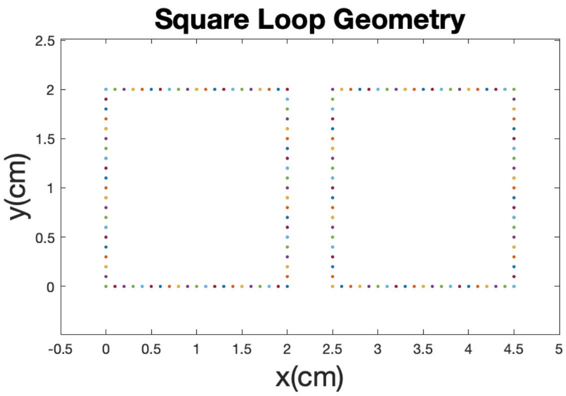

Figure 1-The two squares have the same side lengths of 2cm. The x separation is 0.5cm. There is no y axis separation.

The same geometry was then used with the Neumann formula to find an analytical result with increasing resolution until proper convergence is achieved. The results of this study are shown below in Figure 2.

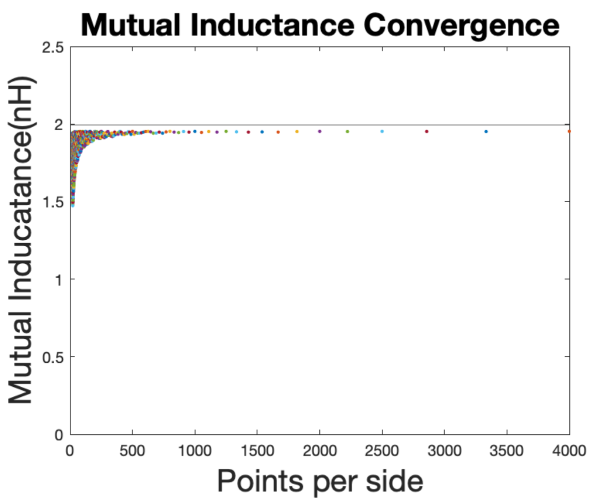

Figure 2- The convergence shows that beyond 2000 points per side the value does not change significantly. This will be utilized to be sure that high enough resolution is used in the circular rings. The points show high contrast based on value early on due to the method used to integrate.

The two values do not reach each other but the analytical approach does approximate the exact solution well. The cause of the difference is the result of needing to approximate the lines. In order to achieve better results a much higher resolution would be needed. This was not performed since computational time did not allow for more than 2 hours which was reached in the final run. The final value for the analytical approach for the mutual inductance is $1.9523nH$.

Now that we have shown what the Neumann formula can do and determined the resolution needed for convergence, we can apply the geometry of two circular loops to the program. When we do so we only need to change the way in which the discrete points change with each step to form the new geometry shown below in Figure 3

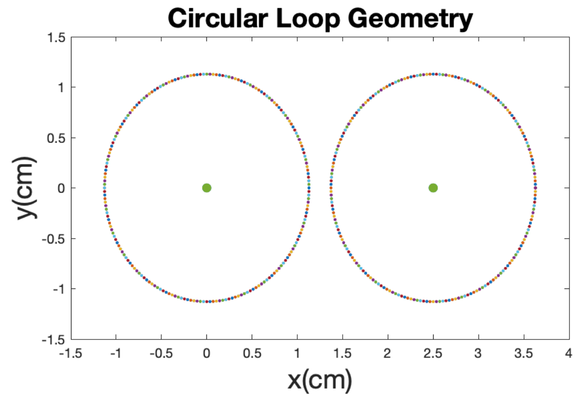

Figure 3-The origin of the primary loop is the x-y plane origin as well. This allows for the use of the separation distance of 2.5cm along with the radius of $\frac{2}{\sqrt{\pi}}$cm to be used in calculating the x y points. The two green dots are to illustrate where the origin is for each circle.

This new circular geometry has a mutual inductance of $1.7020nH$ using the resolution of 1/20000.

IV. Conclusion

The results are within an order of magnitude which shows that the analytical approach is reliable to solve for mutual inductance in same plane simple geometries with no overlap.

This is to say the Neumann formula analytical approach is not viable for self-inductance. The reason for this inconsistency is the method in which the values are computed. As the point of observation and the point of the source near each other the separation vector approaches zero. Once the two discrete points overlap the integral blows up to infinity causing numeric instabilities with every iteration of the reference point.

The error in the analytical approach of the mutual inductance found in the two square loops causes the value of the mutual inductance of the two circular loops to vary based on error of $\pm 2\%$. The reported value of $1.7020nH$ ranges from $1.66796nH - 1.73604nH$. The systematic error that is causing this deviation will also need to be addressed to solve another underlying concern. The two rectangles and the two circles have the same overall area of $\approx 4cm$. This would mean with similar separations and the projection of the angles on the circles should equate to the same mutual inductance for both systems of geometries. Further work needs to be done in the application of the projections on the circular system along with accounting for when contributions are positive or negative to the refrence of total mutual inductance. The lack of precision is not acceptable for crucial components, but this method still makes for an adequate approximation for simple geometries in low frequency systems where the parasitic inductance will not be detrimental for fundamental operations.

Reference

[1] S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics, 3rd ed. New York: Wiley, 1994.*