$\mathbb{Z}[\sqrt{-5}]$ is not a UFD

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Ok, so I forgot to mention this fact during today's meeting. But now that we've seen that $\mathbb{Z}, F[x]$ are unique factorization domains, it would be good to observe an example of a ring that isn't.

Consider the ring $\mathbb{Z}[\sqrt{-5}]$.I claim this is not a UFD. To see this, we need two different decompositions into irreducibles/primes. Remember that we have proven before that if $N(\alpha)$ is prime then α is prime/irreducible. We'll use this fact.

Consider $9 \in \mathbb{Z}[\sqrt{-5}]$, we get the following two decompositions of 9

$9 = 3 \times 3$ and

$9 = (2+\sqrt{-5})(2-\sqrt{-5})$. You can verify that $3, 2 \pm \sqrt{-5}$ is irreducible using the above fact.

To finish this proof, we finally see that a quick calculation yields that the only units in $\mathbb{Z}[\sqrt{-5}]$ are ± 1 and obviously $3, q + \sqrt{-5}$ are not equal upto(i.e multiplication) either of these units.

Try and think of more examples of rings that are not UFDs. What about the Gaussian integers $\mathbb{Z}[i]$.