Family 48 Problem Set 1

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1 Introduction

So as I explained to all of you before, I'll be throwing some extra problems your way. The entire sequence of these Family 48 sets will be made with a particular specific direction in mind as you'll soon discover for yourself. I will assign these problems maybe once or twice a week.

Contact me if you find errors, don't understand the problem statement or face difficulty with the problems

For this problem set, the first few problems will be trivial or easy, followed by some problems which are a little more difficult.The problems may seem repetitive, don't feel the need to do problems which look nearly the same to you. I just put it there so you can get more comfortable with the style of problems.

Skip a problem if you feel it is too trivial and do anything that interests you.

2 Algebraic numbers

A number $x \in \mathbb{C}$ is said to be algebraic if it is the root of a polynomial with integer coefficients. A number $x \in \mathbb{C}$ is transcendental if it is not algebraic.

All rationals are algebraic.

Now, establishing transcendentality is a little more difficult. A classic proof of Iven Niven shows that e is transcendental.

Example:$\sqrt{3}$, $\sqrt[3]{2}$ are both algebraic numbers which are roots of $x^2 - 3, x^3 - 2$ respectively. Obviously, one can find polynomials of larger degree for which both these numbers are roots. This leads to an obvious definition.

A $\mathbb{Z}$-minimal polynomial(which I'll often refer to as the minimal polynomial) of an algebraic number x is the polynomial of least degree for which x is a root.

A set S of numbers in $\mathbb{C}$ is said to be algebraically dependent if there exists some non-trivial polynomial $f \in \mathbb{Q}[x_1, \cdots, x_n]$ where n is the size of S such that the numbers in S are a solution to f.

Problem 1. *Show that $1, \sqrt{2}, \sqrt{3}$ are linearly independent over $\mathbb{Q}$, i.e there exists no $a, b, c \in \mathbb{Q}$ except $a = b = c = 0$ such that $a + b\sqrt{2} + c\sqrt{3} = 0$.*

Show the same for $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$.

Problem 2. *Show that $\sqrt{5} + \sqrt{7} + \sqrt{11}$ is irrational.*

Problem 3. *Consider $\alpha = \sqrt{3} + \sqrt{5}$(this is algebraic), make an educated guess on what you think the degree of the minimal polynomial of α is. Now, attempt to find the minimal polynomial and prove that it is one.Try to do the same for $\alpha = \sqrt{3} + \sqrt[3]{2}$*

Problem 4. *The set of all algebraic numbers is countable. Try using the fact that the a finite product of countable sets is countable.*

Problem 5. ** Ok, this problem is quite interesting and is useful in thinking about transcendental numbers.*

Let x be a real algebraic numbers whose minimal polynomial has degree n, then prove that there exists some k such that for all rationals $\frac{p}{q}$ with $(p, q) = 1, q > 0$, the following inequality holds true:

$$|x - \frac{p}{q}| > \frac{k}{q^n} \tag{1}$$

Try to play around with the general form of the minimal polynomial to get this one.

Problem 6. ** This problem requires some ring theory, linear algebra.*

It is trivial to show that the sum of two algebraic numbers is algebraic. To show that the product of two algebraic numbers is algebraic is a little more tricky. Here is a quick of guide to do that. If α, β are algebraic with minimal polynomials $f(x), g(y)$ of degree d_1, d_2 respectively, then one could take the polynomial ring in two variables $\mathbb{Q}[x, y]$ and kill relations given by f, g to get the finite-dimensional vector space $V = \mathbb{Q}[x, y]/(f(x), g(y))$. You could show that a basis $x^i y^j; 0 \le i \le d_1, 0 \le j \le d_2$ is obvious. Now, multiplication by x, y is a linear operator on V. Try and finish the proof with Cayley-Hamilton.

There is another way to prove this which doesn't use any ring theory.