Monopoly and Cournot Oligopoly under Exponential Demand

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May 2024

Abstract

We consider the case of monopoly and Cournot competition under inverse demand derived from exponential quasilinear preferences. In the case of full information, we find analytic expressions for the equilibrium price and quantity in the case of constant marginal costs, both symmetric and asymmetric, and prove some basic comparative static results. We also consider the case when the firm(s) have uncertainty about the b parameter in the derived inverse demand function $p(x) = ae^{-bx}$. Under some conditions on the moment generating function of b, we can prove the existence and uniqueness of a symmetric cournot equilibrium where firms have identical cost functions and beliefs about the distribution of b. We also use the moment generating function of b to find the monopolists optimal quantity under the continuous uniform, exponential, and Gamma distributions for b. We also find the closed form expression for the symmetric Cournot equilibrium quantity when the firms all have the same constant marginal cost and take b to be exponentially distributed with the same parameter λ.

Introduction and Related Literature

There are a few papers in the literature on Cournot Oligopoly that have results relevant to this paper.

Anderson and Engers investigate Cournot and Stackelberg oligopoly under no costs and the inverse demand structure $p(x) = (1 - \alpha bx)^{\frac{1}{\alpha}}$ for $\alpha > 0$ and $b > 0$, finding closed form expressions for the equilibrium price and quantity and proving some comparative static results. (Anderson and Engers 1991) Importantly, their class of demand functions converges the exponential inverse demand function $p(x) = e^{-bx}$ as $\alpha \to 0$. (Anderson and Engers 1991) While they correctly find that in the case of no costs, each firm's equilibrium output is $\frac{1}{b}$ (which doesn't depend on the number of firms), this does not hold for the case of constant marginal costs, symmetric or asymmetric.

Indeed later in this paper we find that in a Cournot game with n firms, constant marginal costs, and exponential inverse demand, that firm i's output is decreasing in n. However, a drawback of incorporating costs is that finding an expression for the Stackelberg equilibrium quantity is quite a bit more difficult than for Cournot, even with 2 firms and symmetric constant marginal costs. Furthermore, while we find analytic expressions for the Cournot quantities and prices under symmetric and asymmetric constant marginal costs, doing so requires use of the Lambert W function, the inverse of $f(x) = xe^x$ on $[\frac{-1}{e}, \infty)$ (so $W(xe^x) = x$ for $x \in \mathbb{R}_+$). (Dence 2013) This function has some nice properties in addition to its definition that we will make use of in this paper, such as being monotonically increasing for $x \in [0, \infty)$, and that $W(0) = 0$, $W(e) = W(1e^1) = 1$.

Interestingly, while it is the author's belief that this is the first paper to use it in the static Cournot case, the Lambert W shows up in some papers in the Dynamic Cournot literature in continuous time with linear inverse demand. In these papers, it appears as a solution to the reduced set of Hamilton-Jacobi-Bellman equations describing each firm's Value function under the best response path. (Levidna, Fabian and Sircar 2011, Ludkovski and Sircar 2012)

The rest of the paper is laid out as follows: In the first section we derive exponential inverse demand from a consumer with exponential quasilinear preferences, then we consider the case of Cournot competition under certainty, first with symmetric then asymmetric costs. Finally, we consider the case when there is uncertainty in the b parameter, proving existence and uniqueness of a symmetric cournot equilibrium, and deriving an exact expression for the monopoly quantity and symmetric cournot equilibrium under various well known distributions.

Preferences and Demand

Suppose we have an economy with a single price-taking consumer with exponential quasilinear preferences over good x with price p and a numeraire y with price normalized to 1, so:

$$U(x, y) = \frac{-a}{b} e^{-bx} + y$$

Where $a > 0$ and $b > 0$. The consumer also has wealth endowment $w > 0$ and is thus subject to the following budget constraint, which by the concavity of the utility in x and y will hold with equality under the solution to the consumer's problem:[1]

$$px + y \leq w$$

Substituting $y = w - px$ back into the utility function and taking the first order conditions we obtain the inverse demand for good x

$$p = ae^{-bx}$$

Some important things to note about this inverse demand function (and the corresponding demand function $x(p) = ln((\frac{a}{p})^{\frac{1}{b}})$ are that for prices greater than a, demand is negative(or 0, depending on how we choose to define it). Furthermore, as the price goes to 0, quantity demanded goes to ∞:

Proof: Let $M > 0$ be arbitrary, then $\exists \delta > 0$ such that if $0 < p - 0 < \delta$, we have $ln((\frac{a}{p})^{\frac{1}{b}}) > M$

Choose $\delta \leq ae^{-bM}$, then for $0 < p < \delta$ we have

$$p < ae^{-bM}$$

$$pe^{bM} < a$$

and since $p > 0$ we have

$$e^{bM} < \frac{a}{p}$$

since $e^{bM} > 0$, and the logarithm is a monotone increasing function on $(0, \infty)$, taking the log on both sides preserves the inequality, so

$$bM < ln(\frac{a}{p})$$

Since $b > 0$ we have

$$M < ln((\frac{a}{p})^{\frac{1}{b}})$$

It also suffices to consider a single consumer, as these preferences satisfy a particular form of Gorman separability that we denote "closure under aggregation". Closure under aggregation means that if we have an economy with N price-taking consumers where consumer i has exponential quasilinear utility function $U_i(x_i, y_i) = \frac{-a_i}{b_i}e^{-b_i x_i} + y_i$ and budget constraint $px_i + y_i \leq w_i$ (where $a_i > 0$, $b_i > 0$ and $w_i > 0$ for $i = 1, ..., N$), then the market inverse demand curve $p(\tilde{x})$ is exponential,[2] so the market inverse demand faced by firms is $p(\tilde{x}) = \tilde{a}e^{-\tilde{b}\tilde{x}}$,

[1] We can confirm that utility is concave by noting that the Hessian of U is $\mathbf{H} = \begin{pmatrix} -abe^{-bx} & 0 \\ 0 & 0 \end{pmatrix}$ and so for any $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ we have $\mathbf{y}^T\mathbf{H}\mathbf{y} = -y_1^2 abe^{-bx} \leq 0$ (since $a > 0$, $b > 0$, $y_1^2 \geq 0$ and $e^x > 0$ $\forall x \in \mathbb{R}$) and so $\mathbf{H}$ is negative semidefinite for all x, implying U is globally concave.

[2] More specifically, the market inverse demand is $p(x_1 + ...x_N) = p(\tilde{x})$ where $x_i = x_i^*(p, w_i) = \frac{ln(a_i) - ln(p)}{b_i}$ for $i = 1, ..., N$ are the individual demand functions, found by

where $\tilde{a} = e^{\frac{1}{\sum_{i=1}^N \frac{1}{b_i}} \sum_{i=1}^N \frac{ln(a_i)}{b_i}} = (\prod_{i=1}^N a_i^{\frac{1}{b_i}})^{\frac{1}{\sum_{i=1}^N \frac{1}{b_i}}} > 0$ and $\tilde{b} = \frac{1}{\sum_{i=1}^N \frac{1}{b_i}} > 0$ (where we know the inequalities hold since $a_i > 0$ and $b_i > 0$ for $i = 1, ..., N$). This means we can consider competitive equilibrium with a single price taking consumer without loss of generality.

Symmetric Cournot

We will first consider the case where good x is supplied by n firms with the same constant marginal cost $c > 0$ engaged in Cournot competition. Firm i produces quantity x_i of good x. As is standard in the literature, we denote $x_{-i} = \sum_{j \neq i} x_j$

The profit function for firm i is thus :

$$\Pi_i = p(x_1 + ... + x_n)x_i - C_i(x_i) = (ae^{-b(x_1+...+x_n)})x_i - cx_i$$

Taking the first order condition we have

$$\frac{\partial \Pi_i}{\partial x_i} = ae^{-bx_{-i}}(e^{-bx_i} - bx_i e^{-bx_i}) - c = 0$$

By symmetry we know that at the intersection of the best responses we have $x_{-i} = \sum_{j \neq i} x_j = (n-1)x_i$ so

$$\begin{aligned} e^{-b(n-1)x_i}(e^{-bx_i} - bx_i e^{-bx_i}) &= \frac{c}{a} \\ e^{bx_i - bnx_i}(1 - bx_i)e^{-bx_i} &= \frac{c}{a} \\ (1 - bx_i)e^{-bnx_i} &= \frac{c}{a} \\ (n - bnx_i)e^{-bnx_i} &= \frac{nc}{a} \\ (n - bnx_i)e^{n-bnx_i} &= \frac{ne^n c}{a} \end{aligned}$$

Taking the Lambert W function on both sides we have

$$n - bnx_i = W(\frac{ne^n c}{a})$$

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solving consumer i's utility maximization problem: From the FOC for consumer i we get $a_i e^{-b_i x_i} = p \implies x_i^*(p, w_i) = \frac{ln(a_i) - ln(p)}{b_i}$. Summing across the N consumers, we get that market demand is $\tilde{x} = \sum_{i=1}^N \frac{ln(a_i) - ln(p)}{b_i} = \sum_{i=1}^N \frac{ln(a_i)}{b_i} - ln(p) \sum_{i=1}^N \frac{1}{b_i} \implies ln(p) = \frac{1}{\sum_{i=1}^N \frac{1}{b_i}} (\sum_{i=1}^N \frac{ln(a_i)}{b_i} - \tilde{x}) \implies p = e^{\frac{\sum_{i=1}^N \frac{ln(a_i)}{b_i}}{\sum_{i=1}^N \frac{1}{b_i}}} e^{-\frac{1}{\sum_{i=1}^N \frac{1}{b_i}} \tilde{x}}$ and so we have an exponential market inverse demand curve.

Thus we have the Cournot-Nash Equilibrium quantity produced by firm i:

$$x_i^* = \frac{1}{bn}(n - W(\frac{ne^n c}{a}))$$

For $c = 0$ we know $W(0) = 0$ so this simplifies to $\frac{1}{b}$, which is exactly the equilibrium found by Anderson and Engers. (Anderson and Engers 1991)

Note also that for the equilibrium quantity to be positive we require that $c < a$, since $W(.)$ is monotonically increasing on $[\frac{-1}{e}, \infty)$.[3]

The total quantity in equilibrium is thus:

$$X^* = \frac{1}{b}(n - W(\frac{ne^n c}{a}))$$

It's rather simple to show it is increasing in a:

$$\text{suppose } 0 < a < a'$$

then for $n > 0$, $a > c > 0$

$$\frac{ne^n c}{a} > \frac{ne^n c}{a'}$$

Since $W(x)$ is strictly increasing for $x > 0$ we have

$$W(\frac{ne^n c}{a}) > W(\frac{ne^n c}{a'})$$

$$n - W(\frac{ne^n c}{a}) < n - W(\frac{ne^n c}{a'})$$

since $b > 0$ we have

$$\frac{1}{b}(n - W(\frac{ne^n c}{a})) < \frac{1}{b}(n - W(\frac{ne^n c}{a'}))$$

Thus

$$X^*(a) < X^*(a')$$

We can also easily show that it is decreasing in b:

Suppose $0 < b < b'$

Then $0 < \frac{1}{b'} < \frac{1}{b}$, so when $a > c > 0$ we know $n - W(\frac{c}{a}ne^n) > 0$ so

$$\frac{1}{b'}(n - W(\frac{c}{a}ne^n)) < \frac{1}{b}(n - W(\frac{c}{a}ne^n))$$

$$X^*(b') < X^*(b)$$

[3]Indeed if we have $c \geq a$ then $\frac{c}{a} \geq 1$ and since $W(.)$ is increasing we have $W(\frac{ne^n c}{a}) \geq W(ne^n) = n$ implying $0 \geq n - W(\frac{ne^n c}{a})$

To find how the total quantity changes with the number of firms, we will have to do more work:

For simplicity, let $\beta = \frac{c}{a}$. By the above sections, we know $\beta \in (0, 1)$.

Since we know $\frac{dW(x)}{dx} = \frac{W(x)}{x(W(x)+1)}$, taking the derivative of the total quantity with respect to n we have:

$$\frac{\partial}{\partial n}[\frac{1}{b}(n - W(\beta n e^n))] = \frac{1}{b}(1 - \frac{\partial}{\partial n} W(\beta n e^n))$$

$$= \frac{1}{b}(1 - (\beta(n+1)e^n W'(\beta n e^n)))$$

Since we know $W'(\beta n e^n) = \frac{W(\beta n e^n)}{\beta n e^n (W(\beta n e^n)+1)}$ we have

$$= \frac{1}{b}(1 - \frac{n+1}{n} \frac{W(\beta n e^n)}{W(\beta n e^n) + 1})$$

$$= \frac{1}{b} \frac{W(\beta n e^n) + 1 - \frac{n+1}{n} W(\beta n e^n)}{W(\beta n e^n) + 1}$$

$$= \frac{1}{b} \frac{1 + (1 - \frac{n+1}{n}) W(\beta n e^n)}{W(\beta n e^n) + 1}$$

$$= \frac{1}{b} \frac{1 - \frac{1}{n} W(\beta n e^n)}{W(\beta n e^n) + 1}$$

$$= \frac{1}{b[W(\beta n e^n) + 1]}(1 - \frac{1}{n} W(\beta n e^n))$$

But notice that since $W(\beta n e^n) > 0$ for $n > 0$ and $b > 0$ that $\frac{1}{b[W(\beta n e^n)+1]} > 0$. Furthermore, since $0 < \beta < 1$ and $W(.)$ is increasing we know $0 < W(\beta n e^n) < W(n e^n) = n$ and thus $-W(\beta n e^n) > -n$, so $-\frac{W(\beta n e^n)}{n} > -1$, implying

$$\frac{1}{b[W(\beta n e^n) + 1]}(1 - \frac{1}{n} W(\beta n e^n)) > \frac{1}{b[W(\beta n e^n) + 1]}(1 - 1) = 0$$

And thus we know the total quantity is increasing in the number of firms.

Furthermore, for any number of firms the total quantity is always bounded above by the competitive quantity. $X_c = \frac{1}{b} ln(\frac{a}{c}) = \frac{1}{b} ln(\frac{1}{\beta})$, derived in appendix B.

To prove this, we will have to consider two cases of parameter configurations, when $\beta = \frac{c}{a} \in (0, e^{-(n+1)})$ and when $\beta \in [e^{-(n+1)}, 1)$. In the first case, we have that $0 < \beta < e^{-(n+1)}$ and so

$$ln(\beta) < -(n+1)$$

$$ln(\frac{1}{\beta}) > n+1$$

and since $W(\beta n e^n) > 0$ we have

$$\frac{1}{b}(n - W(\beta n e^n)) < \frac{n}{b} < \frac{n+1}{b} < \frac{ln(\frac{1}{\beta})}{b} = X_c$$

Otherwise we have the case when $e^{-(n+1)} \leq \beta < 1$

This implies $-(n+1) < ln(\beta) < 0$, so $n+1 > -ln(\beta)) > 0$
Thus

$$n + 1 + ln(\beta) > 0$$

$$n + ln(\beta) > -1$$

Since $f(x) = xe^x$ is monotone increasing on $[-1, \infty)$ we know the inequality is preserved when applying it, so

$$(n + ln(\beta))e^{(n+ln(\beta))} > -1e^{-1} = \frac{-1}{e}$$

but notice that since $ln(\beta) < 0$ and $W(.)$ is increasing we have

$$W(\beta n e^n) = W(ne^{n+ln(\beta)}) \geq W((n + ln(\beta))e^{n+ln(\beta)}) = n + ln(\beta)$$

By the imposed parameter conditions and since $W(.)$ is well defined and increasing on $[\frac{-1}{e}, \infty)$, we know the lambert W function is well defined for the given input so we have

$$W(\beta n e^n) \geq n + ln(\beta)$$

$$-W(\beta n e^n) \leq -n - ln(\beta) = ln(\frac{1}{\beta}) - n$$

and thus

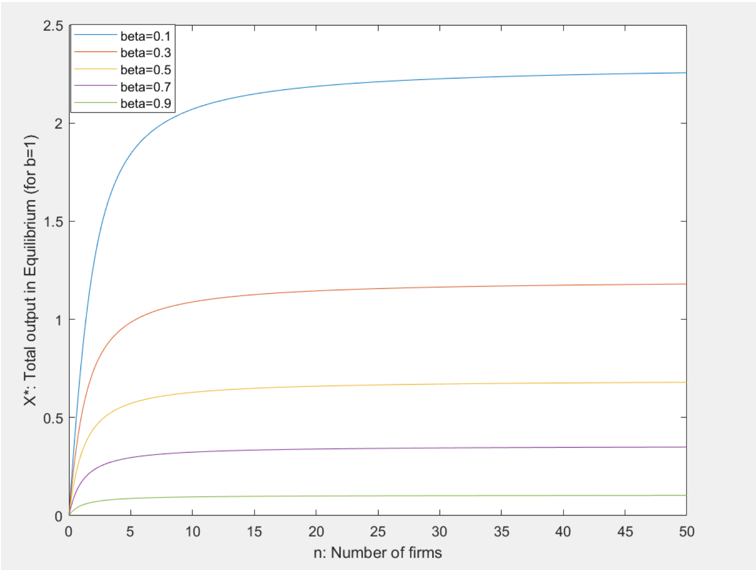
$$n - W(\beta n e^n) \leq n + ln(\frac{1}{\beta}) - n = ln(\frac{1}{\beta})$$

so for positive b we have

$$\frac{1}{b}(n - W(\beta n e^n)) \leq \frac{ln(\frac{1}{\beta})}{b} = X_c$$

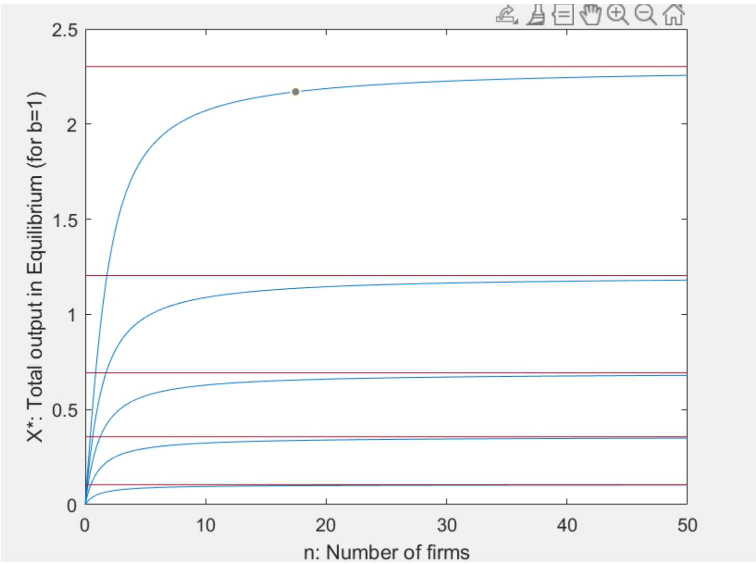
Thus since it is increasing and bounded above, we know that the total quantity as a function of the number of firms must converge to some limit as $n \to \infty$. We conjecture that this is exactly the competitive quantity.

Taking $b = 1$ and using MATLAB's built in Lambert W approximation to plot $X^*(n) = n - W(\beta n e^n)$ for different values of β we get:



So indeed the total quantity seems to be an increasing concave function of the number of firms.

Overlaying the competitive quantity for each β, we get



And so it seems that the total output does converge to the competitive quantity.

Comparing it to the monopoly output, $X_{mon} = \frac{1}{b}(1 - W(\frac{ce}{a}))$ (Derived in Appendix B) , we see that the Monopoly output for a monopolist with marginal cost

$c > 0$ is exactly the total symmetric Cournot output (with common marginal cost $c > 0$) as a function of the number of firms, $X^*(n)$, evaluated at $n = 1$. Since we proved this function is strictly increasing in the number of firms, we know that the symmetric Cournot quantity is always greater than the associated monopoly quantity (where the monopolist faces the same inverse demand $p(x) = ae^{-bx}$ and has the same marginal cost as the common cost among the firms $c(x) = cx$).

Next we turn our attention to each firm's equilibrium output, $x_i^* = \frac{1}{b}(1 - \frac{1}{n}W(\frac{c}{a}ne^n))$. Clearly by similar arguments to above, each firm's output is increasing in a and decreasing in b and c.

To see how firm level output changes with the number of firms, we take the derivative of firm level output with respect to n:

$$\frac{\partial}{\partial n}[\frac{1}{b}(1 - \frac{1}{n}W(\frac{c}{a}ne^n))] = \frac{-1}{b}\frac{\partial}{\partial n}[\frac{1}{n}W(\beta ne^n)]$$

$$= \frac{-1}{b}[\frac{\beta(n+1)e^n W'(\beta ne^n)}{n} - \frac{W(\beta ne^n)}{n^2}]$$

By the above sections we know this simplifies to

$$= \frac{-1}{b}[\frac{\beta(n+1)e^n \frac{W\beta ne^n}{\beta ne^n(W(\beta ne^n)+1)}}{n} - \frac{W(\beta ne^n)}{n^2}]$$

$$= \frac{-1}{b}[\frac{(n+1)\frac{W\beta ne^n}{(W(\beta ne^n)+1)} - W(\beta ne^n)}{n^2}]$$

$$= \frac{1}{b}[\frac{W(\beta ne^n) - (n+1)\frac{W\beta ne^n}{(W(\beta ne^n)+1)}}{n^2}]$$

$$= \frac{W(\beta ne^n)}{bn^2}[1 - (n+1)\frac{1}{(W(\beta ne^n)+1)}]$$

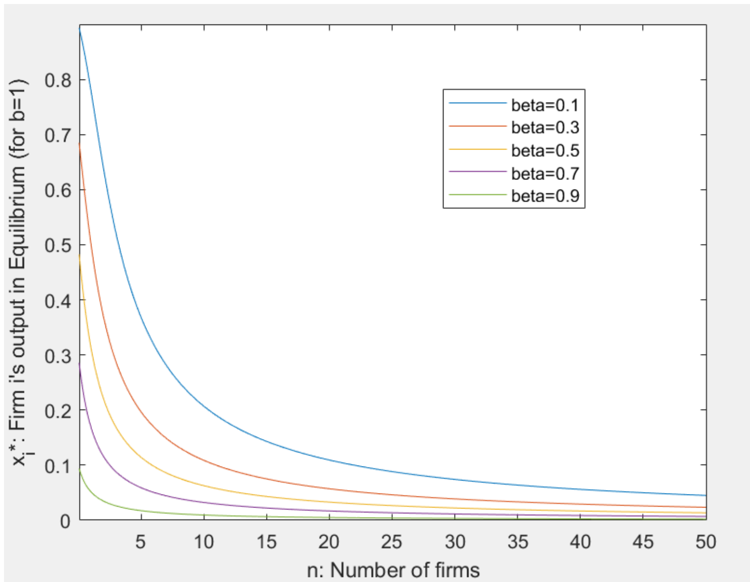
$$= \frac{W(\beta ne^n)}{bn^2}[\frac{W(\beta ne^n) + 1 - (n+1)}{(W(\beta ne^n)+1)}]$$

$$= \frac{W(\beta ne^n)}{bn^2(W(\beta ne^n)+1)}[W(\beta ne^n) - n]$$

But notice that since $b > 0$ and for $n > 0$ we have $W(\beta ne^n) > 0$ we know $\frac{W(\beta ne^n)}{bn^2(W(\beta ne^n)+1)} > 0$. Furthermore, since $\beta \in (0, 1)$ and $W(.)$ increasing we have $W(\beta ne^n) < W(ne^n) = n$ implying $W(\beta ne^n) - n < 0$, so

$$\frac{W(\beta ne^n)}{bn^2(W(\beta ne^n)+1)}[W(\beta ne^n) - n] < \frac{W(\beta ne^n)}{bn^2(W(\beta ne^n)+1)}[0] = 0$$

And so firm level output is decreasing in the number of firms, as alluded to in the introduction. Furthermore, since we know total output converges to some constant in $(X_{mon}, \frac{ln(\frac{1}{\beta})}{b})$ (and intuitively we know this is the competitive) as $n \to \infty$, we know that firm level output must converge to zero.



And so it seems that firm-level output is a decreasing convex function of the number of firms. Code is in Appendix A

Asymmetric costs

Suppose instead that the firms have differing marginal costs, so $C_i(x_i) = c_i x_i$ where $c_i > 0$ for $i = 1, ..., N$. It will be helpful to denote the average cost as $\bar{c} = \frac{1}{n}\sum_{i=1}^{n} c_i$.

The first order condition for firm i is very similar to the symmetric case:

$$\frac{\partial \Pi_i}{\partial x_i} = 0 \implies ae^{-bx_{-i}}(1 - bx_i)e^{-bx_i} = c_i$$

But since the costs are not symmetric, we cannot use the trick used in the above section. We instead have to make use of the fact that at the intersection of the best responses the following must hold:

$$e^{bx_i + bx_{-i}} = e^{b\sum_{i=1}^{n} x_i} = \frac{a(1 - bx_i)}{c_i} = \frac{a(1 - bx_1)}{c_1} = ... = \frac{a(1 - bx_n)}{c_n}$$

Thus, obtaining the best response quantity of firm i in terms of the quantity of firm 1 (x_1) we have

$$(1 - bx_i) = \frac{c_i}{c_1}(1 - bx_1) \implies x_i(x_1) = \frac{1}{b}(1 - \frac{c_i}{c_1} + \frac{bc_i}{c_1}x_1)$$

Then substituting into firm 1's FOC we have

$$e^{bx_1 + (1 - \frac{c_2}{c_1} + \frac{bc_2}{c_1}x_1) + ... + (1 - \frac{c_n}{c_1} + \frac{bc_n}{c_1}x_1)} = \frac{a(1 - bx_1)}{c_1}$$

$$e^{bx_1+(n-1)-\frac{1}{c_1}\sum_{i=2}^{n} c_i+\frac{bx_1}{c_1}\sum_{i=2}^{n} c_i} = \frac{a(1-bx_1)}{c_1}$$

Recalling our definition of $\bar{c}$, we have

$$e^{bx_1+(n-1)-\frac{n\bar{c}-c_1}{c_1}+\frac{bx_1(n\bar{c}-c_1)}{c_1}} = \frac{a(1-bx_1)}{c_1}$$

$$\frac{a}{c_1}(1-bx_1)e^{-bx_1-\frac{b(n\bar{c}-c_1)}{c_1}x_1} = e^{n-1-\frac{nc\bar{-}c_1}{c_1}}$$

$$\frac{a}{c_1}(1-bx_1)e^{-b(1+\frac{(n\bar{c}-c_1)}{c_1})x_1} = e^{n-1-\frac{nc\bar{-}c_1}{c_1}}$$

$$(1+\frac{(n\bar{c}-c_1)}{c_1})\frac{a}{c_1}(1-bx_1)e^{-b(1+\frac{(n\bar{c}-c_1)}{c_1})x_1} = (1+\frac{(n\bar{c}-c_1)}{c_1})e^{n-1-\frac{nc\bar{-}c_1}{c_1}}$$

$$a(1+\frac{(n\bar{c}-c_1)}{c_1})(1-bx_1)e^{-b(1+\frac{(n\bar{c}-c_1)}{c_1})x_1} = (c_1+n\bar{c}-c_1)e^{n-1-\frac{nc\bar{-}c_1}{c_1}} = n\bar{c}e^{n-1-\frac{nc\bar{-}c_1}{c_1}}$$

$$a(1+\frac{(n\bar{c}-c_1)}{c_1})(1-bx_1)e^{(1+\frac{(n\bar{c}-c_1)}{c_1})-b(1+\frac{(n\bar{c}-c_1)}{c_1})x_1} = n\bar{c}e^{(1+\frac{(n\bar{c}-c_1)}{c_1})+n-1-\frac{nc\bar{-}c_1}{c_1}} = n\bar{c}e^{n}$$

$$(1+\frac{(n\bar{c}-c_1)}{c_1})(1-bx_1)e^{(1+\frac{(n\bar{c}-c_1)}{c_1})(1-bx_1)} = \frac{\bar{c}}{a}ne^{n}$$

Applying Lambert W we obtain

$$(1+\frac{(n\bar{c}-c_1)}{c_1})(1-bx_1) = W(\frac{\bar{c}}{a}ne^{n})$$

$$n\bar{c}(1-bx_1) = c_1W(\frac{\bar{c}}{a}ne^{n})$$

$$(1-bx_1) = \frac{c_1}{n\bar{c}}W(\frac{\bar{c}}{a}ne^{n})$$

And thus we have the Cournot-Nash equilibrium quantity of firm 1:

$$x_1^* = \frac{1}{bn}(n-\frac{c_1}{\bar{c}}W(\frac{\bar{c}}{a}ne^{n}))$$

Which is similar to the case of symmetric costs, except for the $\frac{c_1}{\bar{c}}$ term, . Indeed at an interior equilibrium (when every firm is producing a positive quantity) if we sum across all firms to get the total equilibrium quantity we get

$$X^* = \sum_{i=1}^{n} x_i^* = \sum_{i=1}^{n} \frac{1}{bn}(n-\frac{c_i}{\bar{c}}W(\frac{\bar{c}}{a}ne^{n}))$$

$$= \frac{n}{b} - \frac{W(\frac{\bar{c}}{a}ne^{n})}{bn\bar{c}}\sum_{i=1}^{n} c_i = \frac{n}{b} - \frac{W(\frac{\bar{c}}{a}ne^{n})}{bn\bar{c}}n\bar{c}$$

$$= \frac{1}{b}(n-W(\frac{\bar{c}}{a}ne^{n}))$$

Which is exactly the total quantity produced in symmetric Cournot competition with cost parameter $\bar{c}$.

Uncertainty in the b parameter

Another benefit of the exponential demand function is the way it accomodates parameter uncertainty. Consider the case of a monopolist with constant marginal cost facing exponential inverse demand $p = ae^{-bx}$, but with uncertainty about the b parameter. The monopolist takes b to be distributed as a random variable with cumulative distribution $F(x)$ and moment generating function $M_b(t)$. The monopolist's problem is then to choose a quantity to maximize expected profit:

$$\mathbb{E}\pi(x) = \mathbb{E}[axe^{-bx} - cx]$$

By the linearity of expectation, and since the monopolist knows the values of a and c with certainty, we have

$$\mathbb{E}\pi(x) = ax\mathbb{E}[e^{(-x)b}] - cx$$

But notice that this expectation is exactly the moment generating function of b evaluated at $t = -x$, and so the monopolists expected profit is:

$$\mathbb{E}\pi(x) = axM_b(-x) - cx$$

This allows us to find a closed form for the equilibrium quantity when the distribution of b has a "nice" moment generating function. This is the case for many commonly used distributions, such as the bernoulli with probability p, continuous uniform over $[0, b]$ and $[b, 2b]$, the exponential distribution with parameter $\lambda > 0$, and the Gamma distribution with parameters $k = 2$, $\theta > 0$ (for $k = 1$ this is exactly the exponential distribution with parameter $\frac{1}{\theta}$). In the case when b is normally distributed the expected profit function is unbounded above and so has no finite maximum, but if the firm is instead maximizing the p'th quantile of the profit distribution (this would be equivalent to maximizing median profit for $p = 0.5$) then, given some conditions on μ and σ^2, the monopolist will have a positive quantile profit maximizer.

Furthermore, for $a = 1$, if the Moment generating function of b evaluated at $t = -x$ is continuous and twice differentiable, strictly positive for $x = 0$ (which will always be the case, as it is the expectation of an exponential, which maps any realization of $(-x)b$ to a strictly positive real number), and $M_b'(-x) < 0$, $M_b''(-x) \geq 0$ for $x \leq 0$ (and so $-x \geq 0$), and $\lim_{x\to\infty} M_b(x) = 0$, then we can prove the existence of a symmetric Cournot-Nash equilibrium when firms have a common belief about the distribution of b and the same continuous and twice differentiable cost function $c(x)$, such that $0 \leq c'(0) < M_b(0)$, and $c'(x) \geq c > 0$, $c''(x) \geq 0$ for some $c \in \mathbb{R}_+$ and for all $x \geq 0$. Furthermore, if we assume $M_b''(-x) < \frac{c''(x)-(n+1)M_b'(-x)}{nx}$ for all $x \geq 0$ then the equilibrium is unique.

Proof:

For simplicity we will denote $M_b(-x)$ as $P_E(x)$, or the expected inverse demand faced by each firm.

Setting up each firm's expected profit function, we have

$$\Pi_i(x_i) = x_i P_E(x_i + x_{-i}) - c(x_i)$$

We wish to show that there exists a symmetric interior maximizer for each of the firms, so some $x_1^* = x_2^* = ... = x_n^* = x^* > 0$ such that by playing x^* each firm is playing a best response (and so maximizing expected profit, given the actions of the other firms). For $i = 1, ..., n$ we have that the firm's best responses $(x_i(x_{-i})$ are defined implicitly by

$$\frac{d\Pi_i(x_i)}{dx_i} = P_E(x_i + x_{-i}) + x_i P_E'(x_i + x_{-i}) - c'(x_i) = 0$$

At a symmetric equilibrium, $x_{-i} = (n-1)x_i$ for $i = 1, ..., n$, so a symmetric equilibrium exists if there is some $x^* > 0$ such that

$$P_E(nx^*) + x^* P_E'(nx^*) - c'(x^*) = 0$$

Thus satisfying the best response correspondence described by the n first order conditions.

To show such an x^* exists, we will define a function $g(x) = P_E(nx) + xP_E'(nx) - c'(x)$.

Since $P_E(.), P_E'(.)$ and $c'(.)$ are continuous, $g(.)$ is continuous.

Furthermore, notice that by assumption, we have

$$g(0) = P_E(0) + 0 - c'(0) = P_E(0) - c'(0) > 0$$

Since $P_E(.)$ is strictly decreasing, $P_E(0) > 0$, $P_E''(x) \geq 0$ for $x \geq 0$, and $\lim_{x\to\infty} P_E(x) = 0$ we know that $P_E(x) > 0$ for $x \geq 0$. Furthermore, since $\lim_{x\to\infty} P_E(x) = 0$ we know that $\forall \varepsilon > 0 \ \exists N > 0$ such that if $x \geq N$ we have $0 < |P_E(x) - 0| = P_E(x) < \varepsilon$.

Take $\varepsilon = \frac{c}{2}$, so we know $\exists N > 0$ such that for $x \geq N$, $P_E(x) < \frac{c}{2}$.

Then take some $\hat{x} \geq N$ (perhaps $N + 1$), and since $n \geq 2$ we know $n\hat{x} \geq N$ so $P_E(n\hat{x}) < \frac{c}{2}$.

Now consider $g(\hat{x})$, we have

$$g(\hat{x}) = P_E(n\hat{x}) + \hat{x}P_E'(n\hat{x}) - c'(x)$$

Since $P_E'(x) < 0$ for $x \geq 0$ and $P_E(n\hat{x}) < \frac{c}{2}$ we have

$$P_E(n\hat{x}) + \hat{x}P_E'(n\hat{x}) - c'(\hat{x}) < \frac{c}{2} + 0 - c'(\hat{x})$$

$$= \frac{c}{2} - c'(\hat{x})$$

But notice that since $c'(x) \geq c > 0$ for all $x \geq 0$ and $\hat{x} \geq N > 0$, we know $-c'(\hat{x}) < -c$ and thus

$$\frac{c}{2} - c'(\hat{x}) < \frac{c}{2} - c = \frac{-c}{2} < 0$$

Thus since $g(x)$ is continuous, $g(0) > 0$, and $g(\hat{x}) < 0$, by the intermediate value theorem we know there exists some $x^* \in (0, \hat{x})$ such that $g(x^*) = 0$, which means

$$P_E(nx^*) + x^* P_E'(nx^*) - c'(x*) = 0$$

and so a Cournot Nash equilibrium exists.

Furthermore, if as we noted we have $P_E''(x) < \frac{c''(x)-(n+1)P_E'(x)}{nx}$ for all $x \geq 0$ then the equilibrium is unique, as we get that for $x \geq 0$ we have

$$g'(x) = (n+1)P_E'(x) + nxP_E''(x) - c''(x) < (n+1)P_E'(x) + nx\frac{c''(x) - (n+1)P_E'(x)}{nx} - c''(x) = 0$$

so x^* is indeed a maximum of each of the firm's expected profit function, and since g' is decreasing for all x we know x^* is the unique root of $g(.)$, and so the equilibrium is unique. $\square$

We will now look at particular distributions:

Continuous Uniform

There are various surprising results this uncovers. In the case of the monopolist facing $b \sim Unif(0, \hat{b})$ for $\hat{b} > 0$, the monopolist produces exactly the competitive quantity under full information with $b = \hat{b}$. Recall that for $t \neq 0$ the moment generating function of a continuous uniform random variable on $[\alpha, \beta]$ is given by $M_b(t) = \frac{e^{\beta t} - e^{\alpha t}}{t(\beta - \alpha)}$. Thus at $\alpha = 0$, $\beta = \hat{b}$, and $t = -x$, we have that the expected profits of the monopolist are

$$\mathbb{E}\pi(x) = ax[\frac{e^{-\hat{b}x} - 1}{-x\hat{b}}] - cx$$

$$= a[\frac{1 - e^{-\hat{b}x}}{\hat{b}}] - cx$$

Taking the first order condition we get that

$$\frac{d\mathbb{E}\pi}{dx} = ae^{-\hat{b}x} - c = 0$$

$$x^* = \frac{1}{\hat{b}} ln(\frac{a}{c})$$

Which is exactly the competitive quantity under certainty.

We can also find the equilibrium quantity when b is uniform on $[\hat{b}, 2\hat{b}]$ for $\hat{b} > 0$

In this case we have that expected profits are

$$\mathbb{E}\pi(x) = ax[\frac{e^{-2\hat{b}x} - e^{-\hat{b}x}}{-x(2\hat{b} - \hat{b})}] - cx$$

$$= a[\frac{e^{-\hat{b}x} - e^{-2\hat{b}x}}{\hat{b}}] - cx$$

Taking the FOC we have

$$\frac{d\mathbb{E}\pi}{dx} = a[2e^{-2\hat{b}x} - e^{-\hat{b}x}] - c = 0$$

$$2 - e^{\hat{b}x} = \frac{c}{a}e^{2\hat{b}x}$$

Let $y = e^{bx}$, then we have

$$\frac{c}{a}y^2 + y - 2 = 0$$

Since $y = e^{bx} > 0$ for all $x \in \mathbb{R}$, we know the solution is the positive root, so we have

$$y = \frac{-1 + \sqrt{1 - 4\frac{c}{a}(-2)}}{2\frac{c}{a}}$$

$$= \frac{\sqrt{a^2 + 8ac} - a}{2c}$$

And thus the monopolist's expected profit maximizing quantity is

$$x^* = \frac{ln(\frac{\sqrt{a^2+8ac}-a}{2c})}{\hat{b}}$$

In order for this to be positive we require that the quantity inside the logarithm is greater than 1, so

$$\sqrt{a^2 + 8ac} - a \geq 2c$$

$$a^2 + 8ac \geq a^2 + 4ac + 4c^2$$

$$4ac \geq 4c^2$$

$$a \geq c$$

This substitution method can also be used to find a closed form expression for x^* for uniform b on $[\hat{b}, 3\hat{b}]$ and $[\hat{b}, 4\hat{b}]$ using the cubic and quartic formulas, but for $k \geq 5$ and non-integer k the method will not work.

Exponential

In the case when $b \sim exp(\lambda)$, then the moment generating function evaluated at $t = -x$ is $M_b(-x) = \frac{\lambda}{\lambda-(-x)} = \frac{\lambda}{\lambda+x}$. Thus the expected profit of the monopolist is

$$\mathbb{E}\pi = ax[\frac{\lambda}{\lambda + x}] - cx$$

Taking the FOC we have

$$\frac{d\mathbb{E}\pi}{dx} = a\lambda[\frac{-x}{(\lambda + x)^2} + \frac{1}{\lambda + x}] - c = 0$$

$$\frac{c}{a\lambda} = \frac{\lambda + x - x}{(\lambda + x)^2}$$

$$\frac{c}{a\lambda^2} = \frac{1}{(\lambda + x)^2}$$

Taking the positive root we have

$$\lambda + x = \lambda\sqrt{\frac{a}{c}}$$

$$x^* = \lambda(\sqrt{\frac{a}{c}} - 1)$$

Which we know is positive when $c < a$.

In the case of symmetric Cournot Competition with n firms where the firms have constant marginal cost and hold a common belief over the distribution of b (exponential with rate $\lambda > 0$), we have that firm i's expected profit is

$$\mathbb{E}\pi_i(x_i) = ax_i[\frac{\lambda}{\lambda + x_1 + ... + x_n}] - cx_i$$

Taking the first order condition we get

$$\frac{d\mathbb{E}\pi_i}{dx_i} = a\lambda[\frac{-x_i}{(\lambda + x_1 + ... + x_n)^2} + \frac{1}{\lambda + x_1 + ... + x_n}] - c = 0$$

$$\frac{\lambda + x_1 + ... + x_{i-1} + x_{i+1} + ... + x_n}{(\lambda + x_1 + ... + x_n)^2} = \frac{c}{a\lambda}$$

By symmetry of costs and beliefs we know the firms will produce the same quantity in equilibrium, so at the intersection of the best responses we have $x_{-i} = (n-1)x_i$ and thus

$$\frac{\lambda + (n-1)x_i}{(\lambda + nx_i)^2} = \frac{c}{a\lambda}$$

$$\lambda + (n-1)x_i = \frac{c}{a\lambda}(n^2x_i^2 + \lambda^2 + 2n\lambda x) = \frac{n^2c}{a\lambda}x_i^2 + \frac{2nc}{a}x + \frac{c\lambda}{a}$$

$$\frac{n^2c}{\lambda}x_i^2 + (2nc - a(n-1))x + (c-a)\lambda = 0$$
$$n^2cx_i^2 + \lambda(n(2c-a)+a)x_i + (c-a)\lambda^2$$

Thus we have two candidates for our equilibrium

$$x_i = \frac{\lambda(n(a-2c)-a) \pm \sqrt{\lambda^2(n(2c-a)+a)^2 - 4n^2\lambda^2(c^2-ac)}}{2n^2c}$$

We will then have to evaluate the second derivative of the profit function at each to see if it is negative, so we have a maximum.

Gamma

When $b \sim \Gamma(k, \theta)$ for $k > 0$, $\theta > 0$, we know the moment generating function at $t = -x$ is then $M_b(-x) = (1+\theta x)^{-k}$, and so the expected profit function for a monopolist is

$$\mathbb{E}\pi = \frac{ax}{(1+\theta x)^k} - cx$$
$$\frac{d\mathbb{E}\pi}{dx} = a[\frac{-kx}{(1+\theta x)^{k+1}} + \frac{1}{(1+\theta x)^k}] - c = 0$$
$$\frac{c}{a} = \frac{1+(\theta-k)x}{(1+\theta x)^{k+1}}$$

Normal

Interestingly, when b has a normal distribution the expected profit function under constant marginal costs has no finite maximum, as the moment generating function of a normal random variable $b \sim \mathcal{N}(\mu, \sigma^2)$ is $M_b(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ and so $M_b(-x) = e^{-\mu x + \frac{1}{2}\sigma^2 x^2}$, and so $\lim_{x\to\infty} \mathbb{E}\pi(x) = \lim_{x\to\infty} axe^{-\mu x + \frac{1}{2}\sigma^2 x^2} - cx = \infty$

Proof:

Let $M > 0$ be arbitrary, then $\exists N \in \mathbb{N}$ such that if $x \geq N$ we have

$$axe^{-\mu x + \frac{1}{2}\sigma^2 x^2} - cx \geq M$$

Choose $N \geq \max\{1, \frac{2}{\sigma^2} ln(\frac{M+c}{a}e^{\mu})\}$, which exists by the archimedian property. Then since $x \geq N \geq 1$ we have

$$axe^{-\mu x + \frac{1}{2}\sigma^2 x^2} - cx = (ae^{-\mu x + \frac{1}{2}\sigma^2 x^2} - c)x \geq ae^{-\mu x + \frac{1}{2}\sigma^2 x^2} - c$$
$$= a(e^{-\mu + \frac{1}{2}\sigma^2 x})^x - c \geq ae^{-\mu + \frac{1}{2}\sigma^2 x} - c$$

Since $x \geq \frac{2}{\sigma^2} ln(\frac{M+c}{a}e^{\mu})$ we know $\frac{1}{2}\sigma^2 x - \mu \geq ln(\frac{M+c}{a})$, which implies $e^{\frac{1}{2}\sigma^2 x - \mu} \geq \frac{M+c}{a}$ and thus

$$ae^{-\mu + \frac{1}{2}\sigma^2 x} - c \geq a\frac{M+c}{a} - c = M$$

Appendix A

MATLAB Code:

```
X=linspace(0,50,1000)

beta1=0.1
beta2=0.3
beta3=0.5
beta4=0.7
beta5=0.9
f1=@(x) x - lambertw(beta1.*x.*exp(x))
f2=@(x) x - lambertw(beta2.*x.*exp(x))
f3=@(x) x - lambertw(beta3.*x.*exp(x))
f4=@(x) x - lambertw(beta4.*x.*exp(x))
f5=@(x) x - lambertw(beta5.*x.*exp(x))

%
% g1=@(x) 1 - lambertw(beta1.*x.*exp(x))./x
% g2=@(x) 1 - lambertw(beta2.*x.*exp(x))./x
% g3=@(x) 1 - lambertw(beta3.*x.*exp(x))./x
% g4=@(x) 1 - lambertw(beta4.*x.*exp(x))./x
% g5=@(x) 1 - lambertw(beta5.*x.*exp(x))./x

h1=log(1/(beta1)).*ones(1000);
h2= log(1/(beta2)).*ones(1000);
h3=log(1/(beta3)).*ones(1000);
h4=log(1/(beta4)).*ones(1000);
h5= log(1/(beta5)).*ones(1000);

plot(X,f1(X),X,h1,X,f2(X),X,h2,X,f3(X),X, h3,X,f4(X),X,h4,X,f5(X),X, h5)

% plot(X,g1(X),X,g2(X),X,g3(X),X,g4(X),X,g5(X))

xlabel('n: Number of firms')
ylabel('X*: Total output in Equilibrium (for b=1)')

% xlabel('n: Number of firms')
% ylabel('x_i*: Firm i''s output in Equilibrium (for b=1)')
```

Appendix B

Perfect Competition

In the case of perfect competition, firms price at marginal cost (in our case a constant $c > 0$), so in equilibrium we have

$$p(x) = ae^{-bx} = c$$

$$x^* = \frac{ln(\frac{a}{c})}{b}$$

Note that using the Lambert W function, we can obtain the equilibrium quantity when the firms have quadratic cost functions $C(x) = c_1x^2 + c_2x$, since in equilibrium we have

$$ae^{-bx} = 2c_1x + c_2$$

$$a = (2c_1x + c_2)e^{bx}$$
$$\frac{ab}{2c_1} = (bx + \frac{bc_2}{2c_1})e^{bx}$$
$$\frac{ab}{2c_1}e^{\frac{bc_2}{2c_1}} = (bx + \frac{bc_2}{2c_1})e^{bx+\frac{bc_2}{2c_1}}$$
$$bx + \frac{bc_2}{2c_1} = W(\frac{ab}{2c_1}e^{\frac{bc_2}{2c_1}})$$
$$x^* = \frac{1}{b}(W(\frac{ab}{2c_1}e^{\frac{bc_2}{2c_1}}) - \frac{bc_2}{2c_1})$$

Monopoly

For a monopoly with constant marginal costs $c > 0$ facing exponential inverse demand $p(x) = ae^{-bx}$ we know its profit function is given by

$$\Pi = p(x)x - c(x) = axe^{-bx} - cx$$

At an interior maximum we have

$$\frac{d\Pi}{dx} = a(e^{-bx} - bxe^{-bx}) - c = 0$$
$$(1 - bx)e^{-bx} = \frac{c}{a}$$
$$(1 - bx)e^{1-bx} = \frac{ce}{a}$$
$$1 - bx = W(\frac{ce}{a})$$
$$x^* = \frac{1}{b}(1 - W(\frac{ce}{a}))$$

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