Simpson's and composite Simpson's rules

0.1 Question 1:

Consider the Taylor expansion about the mid point $c = (a+b)/2$. Let $h = b-a$, then $a = c-h/2$ and $b = c + h/2$.

$$\begin{aligned} f(a) &= f(c - \frac{h}{2}) \\ &= f(c) - \frac{h}{2}f'(c) + \frac{h^2}{8}f''(c) - \frac{h^3}{48}f'''(c) + \frac{h^4}{384}f^{(iv)}(c) + O(h^5) \end{aligned} \tag{1}$$

$$\begin{aligned} f(b) &= f(c + \frac{h}{2}) \\ &= f(c) + \frac{h}{2}f'(c) + \frac{h^2}{8}f''(c) + \frac{h^3}{48}f'''(c) + \frac{h^4}{384}f^{(iv)}(c) + O(h^5) \end{aligned} \tag{2}$$

$$\begin{aligned} f(x) &= f(c + (x-c)) \\ &= f(c) + (x-c)f'(c) + \frac{(x-c)^2}{2!}f''(c) + \frac{(x-c)^4}{4!}f^{(iv)}(c) + O(h^5) \end{aligned} \tag{3}$$

Substitute $x = c + uh/2$ into (3) and integrate (3) term by term, then

$$M(f) = \int_a^b f(x)\,\mathrm{d}x. = hf(c) + \frac{h^3}{24}f''(c) + \frac{h^5}{1920}f^{(iv)}(c) + O(h^7) \tag{4}$$

Let N(f,h) be the discretization method that approximates M(f). In this case it will be Simpson's rule:

$$N(f,h) = \frac{h}{6}(f(a) + 4f(c) + f(b)) \tag{5}$$

Substitute (1) and (2) into N(f,h):

$$\begin{aligned} N(f,h) &= \frac{h}{6}[6f(c) + \frac{h^2}{4}f''(c) + \frac{h^4}{192}f^{(iv)}(c) + O(h^6)] \\ &= hf(c) + \frac{h^3}{24}f''(c) + \frac{h^5}{1152}f^{(iv)}(c) + O(h^7) \end{aligned} \tag{6}$$

We can now calculate the Error E(h):

$$\begin{aligned}
E(h) &= M(f) - N(f,h) \\
&= \frac{h^5}{1920} f^{(iv)}(c) - \frac{h^5}{1152} f^{(iv)}(c) + O(h^7) \\
&= h^5 f^{(iv)}(c)(\frac{1}{1920} - \frac{1}{1152}) \\
&= \frac{-h^5 f^{(iv)}(c)}{2880} \\
&= \frac{-(b-a)^5 f^{(iv)}(c)}{2880}
\end{aligned} \tag{7}$$

as required.

Question 2:

Question 3: