Simpson's and composite Simpson's rules

0.1 Question 1:

Consider the Taylor expansion about the mid point $c = (a+b)/2$. Let $h = b-a$, then $a = c-h/2$ and $b = c + h/2$.

$$\begin{aligned} f(a) &= f(c-\frac{h}{2}) \\ &= f(c) - \frac{h}{2}f'(c) + \frac{h^2}{8}f''(c) - \frac{h^3}{48}f'''(c) + \frac{h^4}{384}f^{(iv)}(c) + O(h^5) \end{aligned} \tag{1}$$

$$\begin{aligned} f(b) &= f(c+\frac{h}{2}) \\ &= f(c) + \frac{h}{2}f'(c) + \frac{h^2}{8}f''(c) + \frac{h^3}{48}f'''(c) + \frac{h^4}{384}f^{(iv)}(c) + O(h^5) \end{aligned} \tag{2}$$

$$\begin{aligned} f(x) &= f(c + (x-c)) \\ &= f(c) + (x-c)f'(c) + \frac{(x-c)^2}{2!}f''(c) + \frac{(x-c)^4}{4!}f^{(iv)}(c) + O(h^5) \end{aligned} \tag{3}$$

Substitute $x = c + uh/2$ into (3) and integrate (3) term by term, then

$$M(f) = \int_a^b f(x)\,\mathrm{d}x. = hf(c) + \frac{h^3}{24}f''(c) + \frac{h^5}{1920}f^{(iv)}(c) + O(h^7) \tag{4}$$

Let N(f,h) be the discretization method that approximates M(f). In this case it will be Simpson's rule:

$$N(f,h) = \frac{h}{6}(f(a) + 4f(c) + f(b)) \tag{5}$$

Substitute (1) and (2) into N(f,h):

$$\begin{aligned} N(f,h) &= \frac{h}{6}[6f(c) + \frac{h^2}{4}f''(c) + \frac{h^4}{192}f^{(iv)}(c) + O(h^6)] \\ &= hf(c) + \frac{h^3}{24}f''(c) + \frac{h^5}{1152}f^{(iv)}(c) + O(h^7) \end{aligned} \tag{6}$$

We can now calculate the Error E(h):

$$\begin{aligned}
E(h) &= M(f) - N(f,h) \\
&= \frac{h^5}{1920} f^{(iv)}(c) - \frac{h^5}{1152} f^{(iv)}(c) + O(h^7) \\
&= h^5 f^{(iv)}(c)\left(\frac{1}{1920} - \frac{1}{1152}\right) \\
&= \frac{-h^5 f^{(iv)}(c)}{2880} \\
&= \frac{-(b-a)^5 f^{(iv)}(c)}{2880}
\end{aligned} \tag{7}$$

as required.

0.2 Question 2:

We'll generate N equal sub-intervals of [a,b], for some N $\in \mathbb{N}$
Label the sub-intervals as I_i with end-points $[x_{2i-2}, x_{2i}]$, for i = 1,...,N
The intervals are of equal length $x_{2i} - x_{2i-2} = 2h = \frac{(b-a)}{N}$
Where there are 2N + 1 quadrature points $x_i = a + ih$, i = 0,...,2N
Then the simpson rule on I_i is:

$$\begin{aligned}
\int_{I_i} f(x)\,\mathrm{d}x. &= \frac{2h}{6}\left(f(x_{2i-2}) + 4f\left(\frac{x_{2i-2} + x_{2i}}{2}\right) + f(x_{2i})\right) \\
&= \frac{h}{3}(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i}))
\end{aligned} \tag{8}$$

and the sum of integrals over the sub-intervals is:

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x. &\approx \sum_{i=1}^{N} \int_{I_i} f(x) \\
&= \sum_{i=1}^{N} \frac{h}{3}(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})) \\
&= \frac{h}{3} \sum_{i=1}^{N}(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})) \\
&= \frac{h}{3}\left(f(x_0) + f(x_{2N}) + 2\sum_{i=1}^{N} f(x_{2i}) + 4\sum_{i=2}^{N} f(x_{2i})\right)
\end{aligned} \tag{9}$$

Question 3:
As f^{iv} is constant: $f^{(iv)}(c) = \alpha$ for $\forall c \in [a,b]$ where α is a constant

Then:

$$E(f;a,b) = -\frac{(b-a)^5}{2880}\alpha \tag{10}$$

$$E_c(f;a,b,2) = -\frac{(b-a)^5}{2880 * 2^4}\alpha \tag{11}$$

$$\implies E(f;a,b) = 16E_c(f;a,b,2) \tag{12}$$

Let $M(f) = \int_a^b f(x)\,\mathrm{d}x$
From definition of errors

$$\begin{aligned} M(f) &= S_c(f;a,b,2) + E_c(f;a,b,2) \\ M(f) &= S(f;a,b) + E(f;a,b) \end{aligned} \tag{13}$$

Then:

$$\begin{aligned} S_c(f;a,b,2) - S(f;a,b) &= M(f) - E_c(f;a,b,2) - M(f) + E(f;a,b) \\ &= E(f;a,b) - E_c(f;a,b,2) \\ &= 16E_c(f;a,b,2) - E_c(f;a,b,2) \\ &= 15E_c(f;a,b,2) \end{aligned} \tag{14}$$

$$\iff \frac{1}{15}(S_c(f;a,b,2) - S(f;a,b)) = E_c(f;a,b,2) \tag{15}$$

as required.

Let $E(f;a,b,2) = E_c(f;a,b,2)$ and by $M(f) = S_c(f;a,b,2) + E_c(f;a,b,2)$ it follows that:

$$\int_a^b f(x)\,\mathrm{d}x = S_c(f;a,b,2) + E(f;a,b,2) \tag{16}$$