

Simpson's and composite Simpson's rules

0.1 Question 1:

Consider the Taylor expansion about the mid point $c = (a+b)/2$. Let $h = b-a$, then $a = c-h/2$ and $b = c+h/2$.

$$\begin{aligned} f(a) &= f\left(c - \frac{h}{2}\right) \\ &= f(c) - \frac{h}{2}f'(c) + \frac{h^2}{8}f''(c) - \frac{h^3}{48}f'''(c) + \frac{h^4}{384}f^{(iv)}(c) + O(h^5) \end{aligned} \quad (1)$$

$$\begin{aligned} f(b) &= f\left(c + \frac{h}{2}\right) \\ &= f(c) + \frac{h}{2}f'(c) + \frac{h^2}{8}f''(c) + \frac{h^3}{48}f'''(c) + \frac{h^4}{384}f^{(iv)}(c) + O(h^5) \end{aligned} \quad (2)$$

$$\begin{aligned} f(x) &= f(c + (x - c)) \\ &= f(c) + (x - c)f'(c) + \frac{(x - c)^2}{2!}f''(c) + \frac{(x - c)^4}{4!}f^{(iv)}(c) + O(h^5) \end{aligned} \quad (3)$$

Substitute $x = c + uh/2$ into (3) and integrate (3) term by term, then

$$M(f) = \int_a^b f(x) dx = hf(c) + \frac{h^3}{24}f''(c) + \frac{h^5}{1920}f^{(iv)}(c) + O(h^7) \quad (4)$$

Let $N(f,h)$ be the discretization method that approximates $M(f)$. In this case it will be Simpson's rule:

$$N(f, h) = \frac{h}{6}(f(a) + 4f(c) + f(b)) \quad (5)$$

Substitute (1) and (2) into $N(f,h)$:

$$\begin{aligned} N(f, h) &= \frac{h}{6}\left[6f(c) + \frac{h^2}{4}f''(c) + \frac{h^4}{192}f^{(iv)}(c) + O(h^6)\right] \\ &= hf(c) + \frac{h^3}{24}f''(c) + \frac{h^5}{1152}f^{(iv)}(c) + O(h^7) \end{aligned} \quad (6)$$

We can now calculate the Error $E(h)$:

$$\begin{aligned}
E(h) &= M(f) - N(f, h) \\
&= \frac{h^5}{1920} f^{(iv)}(c) - \frac{h^5}{1152} f^{(iv)}(c) + O(h^7) \\
&= h^5 f^{(iv)}(c) \left(\frac{1}{1920} - \frac{1}{1152} \right) \\
&= \frac{-h^5 f^{(iv)}(c)}{2880} \\
&= \frac{-(b-a)^5 f^{(iv)}(c)}{2880}
\end{aligned} \tag{7}$$

as required.

0.2 Question 2:

We'll generate N equal sub-intervals of $[a, b]$, for some $N \in \mathbb{N}$
Label the sub-intervals as I_i with end-points $[x_{2i-2}, x_{2i}]$, for $i = 1, \dots, N$
The intervals are of equal length $x_{2i} - x_{2i-2} = 2h = \frac{(b-a)}{N}$
Where there are $2N + 1$ quadrature points $x_i = a + ih$, $i = 0, \dots, 2N$
Then the Simpson rule on I_i is:

$$\begin{aligned}
\int_{I_i} f(x) \, dx &= \frac{2h}{6} (f(x_{2i-2}) + 4f(\frac{x_{2i-2} + x_{2i}}{2}) + f(x_{2i})) \\
&= \frac{h}{3} (f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i}))
\end{aligned} \tag{8}$$

and the sum of integrals over the sub-intervals is:

$$\begin{aligned}
\int_a^b f(x) \, dx &\approx \sum_{i=1}^N \int_{I_i} f(x) \\
&= \sum_{i=1}^N \frac{h}{3} (f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})) \\
&= \frac{h}{3} \sum_{i=1}^N (f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})) \\
&= \frac{h}{3} (f(x_0) + f(x_{2N}) + 2 \sum_{i=1}^{N-1} f(x_{2i}) + 4 \sum_{i=1}^N f(x_{2i-1}))
\end{aligned} \tag{9}$$

Question 3:

Note that:

$$\begin{aligned}
Sc(f; a, b, 1) &= \frac{h}{3}(f(x_0) + f(x_2) + 2 \sum_{i=1}^0 f(x_{2i}) + 4 \sum_{i=1}^1 f(x_{2i-1})) \\
&= \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)) \\
&= \frac{h}{3}(f(a) + 4f(\frac{a+b}{2}) + f(b)) \\
&= S(f; a, b)
\end{aligned} \tag{10}$$

$$\implies Ec(f; a, b, 1) = E(f; a, b) \tag{11}$$

As f^{iv} is constant: $f^{(iv)}(c) = \alpha$ for $\forall c \in [a, b]$ where α is a constant
Then:

$$E(f; a, b) = -\frac{(b-a)^5}{2880}\alpha \tag{12}$$

$$Ec(f; a, b, 2) = -\frac{(b-a)^5}{2880 * 2^4}\alpha \tag{13}$$

$$\implies E(f; a, b) = 16Ec(f; a, b, 2) \tag{14}$$

Let $M(f) = \int_a^b f(x) dx$

From definition of Numerical Schemes

$$\begin{aligned}
M(f) &= Sc(f; a, b, 2) + Ec(f; a, b, 2) \\
M(f) &= Sc(f; a, b, 1) + Ec(f; a, b, 1)
\end{aligned} \tag{15}$$

Then:

$$\begin{aligned}
Sc(f; a, b, 2) - Sc(f; a, b, 1) &= M(f) - Ec(f; a, b, 2) - M(f) + Ec(f; a, b, 1) \\
&= Ec(f; a, b, 1) - Ec(f; a, b, 2) \\
&= E(f; a, b) - Ec(f; a, b, 2) \\
&= 16Ec(f; a, b, 2) - Ec(f; a, b, 2) \\
&= 15Ec(f; a, b, 2)
\end{aligned} \tag{16}$$

$$\iff \frac{1}{15}(Sc(f; a, b, 2) - Sc(f; a, b, 1)) = Ec(f; a, b, 2) \tag{17}$$

as required.

Let $E(f; a, b, 2) = E_c(f; a, b, 2)$ and by $M(f) = S_c(f; a, b, 2) + E_c(f; a, b, 2)$ it follows that:

$$\int_a^b f(x) \, dx = S_c(f; a, b, 2) + E(f; a, b, 2) \tag{18}$$