Simpson's and composite Simpson's rules

0.1 Question 1:

Consider the Taylor expansion about the mid point $c = (a+b)/2$. Let $h = b-a$, then $a = c-h/2$ and $b = c + h/2$.

$$\begin{aligned} f(a) &= f(c - \frac{h}{2}) \\ &= f(c) - \frac{h}{2}f'(c) + \frac{h^2}{8}f''(c) - \frac{h^3}{48}f'''(c) + \frac{h^4}{384}f^{(iv)}(c) + O(h^5) \end{aligned} \tag{1}$$

$$\begin{aligned} f(b) &= f(c + \frac{h}{2}) \\ &= f(c) + \frac{h}{2}f'(c) + \frac{h^2}{8}f''(c) + \frac{h^3}{48}f'''(c) + \frac{h^4}{384}f^{(iv)}(c) + O(h^5) \end{aligned} \tag{2}$$

$$\begin{aligned} f(x) &= f(c + (x-c)) \\ &= f(c) + (x-c)f'(c) + \frac{(x-c)^2}{2!}f''(c) + \frac{(x-c)^4}{4!}f^{(iv)}(c) + O(h^5) \end{aligned} \tag{3}$$

Substitute $x = c + uh/2$ into (3) and integrate (3) term by term, then

$$M(f) = \int_a^b f(x)\,\mathrm{d}x. = hf(c) + \frac{h^3}{24}f''(c) + \frac{h^5}{1920}f^{(iv)}(c) + O(h^7) \tag{4}$$

Let N(f,h) be the discretization method that approximates M(f). In this case it will be Simpson's rule:

$$N(f,h) = \frac{h}{6}(f(a) + 4f(c) + f(b)) \tag{5}$$

Substitute (1) and (2) into N(f,h):

$$\begin{aligned} N(f,h) &= \frac{h}{6}[6f(c) + \frac{h^2}{4}f''(c) + \frac{h^4}{192}f^{(iv)}(c) + O(h^6)] \\ &= hf(c) + \frac{h^3}{24}f''(c) + \frac{h^5}{1152}f^{(iv)}(c) + O(h^7) \end{aligned} \tag{6}$$

We can now calculate the Error E(h):

$$\begin{aligned}
E(h) &= M(f) - N(f,h) \\
&= \frac{h^5}{1920} f^{(iv)}(c) - \frac{h^5}{1152} f^{(iv)}(c) + O(h^7) \\
&= h^5 f^{(iv)}(c)\left(\frac{1}{1920} - \frac{1}{1152}\right) \\
&= \frac{-h^5 f^{(iv)}(c)}{2880} \\
&= \frac{-(b-a)^5 f^{(iv)}(c)}{2880}
\end{aligned} \tag{7}$$

as required.

0.2 Question 2:

We'll generate N/2 equal sub-intervals of [a,b], for some even N $\in \mathbb{N}$
Let M = N/2
Label the sub-intervals as I_i with end-points $[x_{2i-2}, x_{2i}]$, for i = 1,...,M
The intervals are of equal length $x_{2i} - x_{2i-2} = 2h = \frac{(b-a)}{M}$
Where there are N + 1 quadrature points $x_i = a + ih$, i = 0,...,N
Then the simpson rule on I_i is:

$$\begin{aligned}
\int_{I_i} f(x)\,\mathrm{d}x. &= \frac{2h}{6}\left(f(x_{2i-2}) + 4f\left(\frac{x_{2i-2}+x_{2i}}{2}\right) + f(x_{2i})\right) \\
&= \frac{h}{3}\left(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})\right)
\end{aligned} \tag{8}$$

and the sum of integrals over the sub-intervals is:

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x. &\approx \sum_{i=1}^{M} \int_{I_i} f(x) \\
&= \sum_{i=1}^{M} \frac{h}{3}\left(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})\right) \\
&= \frac{h}{3} \sum_{i=2}^{M}\left(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})\right) \\
&= \frac{h}{3}\left(f(x_0) + f(x_N) + 2\sum_{i=1}^{M} f(x_{2i}) + 4\sum_{i=2}^{M} f(x_{2i})\right)
\end{aligned} \tag{9}$$

Question 3: