University of Glasgow
School of Mathematics and Statistics
Mathematics 5M Advanced Numerical Methods

Exercise Sheet 2

Computing questions 2

C1 Write a MATLAB function which implements Gaussian Elimination without pivoting for computing the LU decomposition of a matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$

$$
\begin{aligned}
&U = A, L = I\\
&\textbf{for}\, k = 1\,\textbf{to}\, m-1\\
&\qquad \textbf{for}\, j = k+1\,\textbf{to}\, m\\
&\qquad\qquad l_{jk} = u_{jk}/u_{kk}\\
&\qquad\qquad u_{j,k:m} = u_{j,k:m} - l_{jk}u_{k,k:m}
\end{aligned}
$$

C2 Write a MATLAB function which solves the matrix problem $\mathbf{Ax} = \mathbf{b}$, for $\mathbf{x} \in \mathbb{C}^m$ where $\mathbf{A} \in \mathbb{C}^{m\times m}$ and $\mathbf{b} \in \mathbb{C}^m$ are given.

C3 Use your algorithm to solve the four examples in T3 below. What do you notice?

C4 Write a MATLAB function which implements Gaussian Elimination with partial pivoting for computing the LU decomposition of a matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$.

$$
\begin{aligned}
&U = A, L = I, P = I\\
&\textbf{for}\, k = 1\,\textbf{to}\, m-1\\
&\qquad \textbf{Select}\, i \geq k\, \textbf{to maximise}\, |u_{ik}|\\
&\qquad u_{k,k:m} \leftrightarrow u_{i,k:m}\ (\textbf{interchange two rows})\\
&\qquad l_{k,1:k-1} \leftrightarrow l_{i,1:k-1}\\
&\qquad p_{k,:} \leftrightarrow p_{i,:}\\
&\qquad \textbf{for}\, j = k+1\,\textbf{to}\, m\\
&\qquad\qquad l_{jk} = u_{jk}/u_{kk}\\
&\qquad\qquad u_{j,k:m} = u_{j,k:m} - l_{jk}u_{k,k:m}
\end{aligned}
$$

C5 Use this modified algorithm to solve two examples in T3 below. Note how the problem identified in C3(iv) has now been resolved.

C6 Write a MATLAB function which implements Cholesky Factorisation of a square matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$

$$
\begin{aligned}
&R = A\\
&\textbf{for } k = 1 \textbf{ to } m\\
&\qquad \textbf{for } j = k+1 \textbf{ to } m\\
&\qquad\qquad R_{j,j:m} = R_{j,j:m} - R_{k,j:m}\overline{R}_{kj}/R_{kk}\\
&\qquad R_{k,k:m} = R_{k,k:m}/\sqrt{R_{kk}}
\end{aligned}
$$

C7 Compute the Cholesky Factorisation of the hermitian positive definite matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Tutorial questions 1

T1 Use LU decomposition to solve the following systems $\mathbf{Ax} = \mathbf{b}$,

$$\text{(i) } \mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \qquad \text{(ii) } \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \\ -1 & 2 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\text{(iii) } \mathbf{A} = \begin{bmatrix} 2 & 1 & 5 \\ 5 & 1 & -5 \\ 1 & 2 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 9 \\ 4 \end{bmatrix}, \qquad \text{(iv) } \begin{bmatrix} 0 & 6 & 4 \\ -2 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

Consider the general matrix system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{A} \in \mathbb{C}^{m\times m}$, $\mathbf{b} \in \mathbb{C}^m$.

T2 Show that the number of flops required for forward substitution and backward substitution is $O(m^2)$ respectively.

T3 How many flops would it require to compute the pivot at each step of Gaussian elimination if every available element needed to be tested against some criterion?

T4 Show that the number of flops required for Gaussian elimination with partial pivoting (algorithm above) is $O(\frac{2}{3}m^3)$

T5 Show that the number of flops required for Cholesky factorisation of a hermitian positive definite matrix is half as many as Gaussian elimination.