University of Glasgow
School of Mathematics and Statistics
Mathematics 5M Advanced Numerical Methods

Exercise Sheet 1

Computing questions 1

C1 Write a MATLAB function which implements the Classical Gram–Schmidt algorithm for computing QR decomposition of a square matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$. *ie* your function should accept $\mathbf{A}$ as an input and matrices $\mathbf{Q}$ and $\mathbf{R}$ [This algorithm is numerically unstable].

$$
\begin{aligned}
&\textbf{for}\; j = 1\;\textbf{to}\; m\\
&\quad v_j = a_j\\
&\quad \textbf{for}\; i = 1\;\textbf{to}\; j-1\\
&\quad\quad r_{ij} = q_i^* a_j\\
&\quad\quad v_j = v_j - r_{ij} q_i\\
&\quad r_{jj} = ||v_j||_2\\
&\quad q_j = v_j / r_{jj}
\end{aligned}
$$

C2 Apply this algorithm to the matrix in T5 and compare your answer.

C3 Write a MATLAB function which implements the modified Gram–Schmidt algorithm for computing QR decomposition of a matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$ [This algorithm is numerically stable].

$$
\begin{aligned}
&\textbf{for}\; i = 1\;\textbf{to}\; m\\
&\quad v_i = a_i\\
&\textbf{for}\; i = 1\;\textbf{to}\; m\\
&\quad r_{ii} = ||v_i||_2\\
&\quad q_i = v_i / r_{ii}\\
&\quad \textbf{for}\; j = i+1\;\textbf{to}\; m\\
&\quad\quad r_{ij} = q_i^* v_j\\
&\quad\quad v_j = v_j - r_{ij} q_i
\end{aligned}
$$

C4 Apply this algorithm to the matrix in T5 and compare your answer.

C5 Use both methods to compute the QR factorisation of a random matrix of size 80×80 with exponentially graded singular values *ie:* $\sigma_j = 2^{-j}$, $j = 1, \cdots 80$. *[HINT: Think carefully how you can use the SVD to construct such a matrix].*

What happens to the elements on the diagonal of $\mathbf{R}$, r_{jj}, as j becomes large?

Tutorial questions 1

T1 Compute the SVD of the following matrices

$$\text{(i)} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

T2 For $\mathbf{A} \in \mathbb{C}^{m \times m}$, prove that the non-zero singular values of $\mathbf{A}$ are the square roots of the eigenvalues of $\mathbf{A}^*\mathbf{A}$ or $\mathbf{A}\mathbf{A}^*$. *[Hint: Similar matrices]*

T3 For $\mathbf{A} \in \mathbb{C}^{m \times m}$, prove that if $\mathbf{A} = \mathbf{A}^*$ then the singular values of $\mathbf{A}$ are the absolute values of the eigenvalues of $\mathbf{A}$.

T4 For $\mathbf{A} \in \mathbb{C}^{m \times m}$, prove that $|\det(\mathbf{A})| = \prod_{i=1}^{m} \sigma_i$ where σ_i are the singular values of $\mathbf{A}$.

T5 Compute the QR decompostion of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

using the Gram–Schmidt process.