Numerical Methods 4H Assignment 2 - A two-point boundary value problem

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0.1 Question 1

We use a uniform partition of [0,1] by (N + 1) evenly spaced nodes
$0 = x_0 < x_1 < \cdots < x_n = 1$ where $x_i = i/N$ then $h = 1/N$ and $x_i = hi$.
Let u_i denote $u(x_i)$

0.2 Question 2

At each interior point of [0,1] we have the following formulas:

$$u''(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2) \tag{1}$$

$$u'(x_i) = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2) \tag{2}$$

The finite difference approximation for $u(x)$ is now obtained by enforcing the original equation at each interior node of [0,1]
that is:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + exp(u_i)\frac{u_{i+1} - u_{i-1}}{2h} = \mu sin(2\pi x_i)$$

$$\iff 2u_{i+1} - 4u_i + 2u_{i-1} + hexp(u_i)(u_{i+1} - u_{i-1}) = 2h^2\mu sin(2\pi x_i)0 \tag{3}$$

$$\iff u_{i+1}(2 + hexp(u_i)) - 4u_i + u_{i-1}(2 - hexp(u_i)) = 2h^2\mu sin(2\pi x_i)$$

for $i \in [1, N-1]$

0.3 Question 3

The $O(h^2)$ accurate backwards finite difference formula for u' is:

$$\frac{u_{i-2} - 4u_{i-1} + 3u_i}{2h} = u'(x_i) + O(h^2) \tag{4}$$

Substitute this into the boundary equation $u'(1) + u^3(1) = 0$:

$$u'(1) + u^3(1) = 0$$

$$\iff u'(x_N) + u_N^3 = 0 \tag{5}$$

$$\iff \frac{u_{N-2} - 4u_{N-1} + 3u_N}{2h} + u_N^3 = 0$$
$$\implies u_N(2hu_N^2 + 3) - 4u_{N-1} + u_{N-2} = 0$$

which is the difference equation for node x_N

The difference equation for node x_0 is simply $u_0 = 1$

0.4 Question 4

Let F(u) = 0 be a system of N equations where

$$F_1 = u_2(2 + hexp(u_1)) - 4u_1 + (2 - hexp(u_1)) - 2h^2\mu sin(2\pi x_1) = 0$$

$$\vdots = \vdots$$

$$F_i = u_{i+1}(2 + hexp(u_i)) - 4u_i + u_{i-1}(2 - hexp(u_i)) - 2h^2\mu sin(2\pi x_i) = 0$$

$$\vdots = \vdots$$

$$F_{N-1} = u_N(2 + hexp(u_{N-1})) - 4u_{N-1} + u_{N-2}(2 - hexp(u_{N-1})) - 2h^2\mu sin(2\pi x_{N-1}) = 0$$

$$F_N = u_N(2hu_N^2 + 3) - 4u_{N-1} + U_{N-2} = 0$$

(6)

Then calculating the Jacobian matrix J where entry $J_{ij} = \frac{\partial F_i}{\partial u_j}$:

Define the following equations for i $\in [2, n-1]$:

$$L_i = \frac{\partial F_i}{\partial u_{i-1}} = 2 - hexp(u_i)$$
$$U_i = \frac{\partial F_i}{\partial u_{i+1}} = 2 + hexp(u_i)$$
$$D_i = \frac{\partial F_i}{\partial u_i} = (u_{i+1} - u_{i-1})hexp(u_i) - 4$$

$$J = \begin{bmatrix} (u_2 - 1)hexp(u_1) - 4 & 2 + hexp(u_1) & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ L_2 & D_2 & U_2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & L_3 & D_3 & U_3 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \vdots & \vdots & \ddots & L_i & D_i & U_i & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & L_{n-1} & D_{n-1} & U_{n-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 & -4 & 6hu_n^2 + 3 \end{bmatrix}$$