Numerical Methods 4H Assignment 2 - A two-point boundary value problem

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0.1 Question 1

We use a uniform partition of [0,1] by (N + 1) evenly spaced nodes
$0 = x_0 < x_1 < \cdots < x_n = 1$ where $x_i = i/N$ then $h = 1/N$ and $x_i = hi$.
Let u_i denote $u(x_i)$

0.2 Question 2

At each interior point of [0,1] we have the following formulas:

$$u''(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2) \tag{1}$$

$$u'(x_i) = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2) \tag{2}$$

The finite difference approximation for $u(x)$ is now obtained by enforcing the original equation at each interior node of [0,1] that is:

$$\begin{aligned} &\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + exp(u_i)\frac{u_{i+1} - u_{i-1}}{2h} = \mu sin(2\pi x_i) \\ \iff\ & 2u_{i+1} - 4u_i + 2u_{i-1} + hexp(u_i)(u_{i+1} - u_{i-1}) = 2h^2\mu sin(2\pi x_i)0 \\ \iff\ & u_{i+1}(2 + hexp(u_i)) - 4u_i + u_{i-1}(2 - hexp(u_i)) = 2h^2\mu sin(2\pi x_i) \end{aligned} \tag{3}$$

for $i \in [1, N-1]$

0.3 Question 3

The $O(h^2)$ accurate backwards finite difference formula for u' is:

$$\frac{u_{i-2} - 4u_{i-1} + 3u_i}{2h} = u'(x_i) + O(h^2) \tag{4}$$

Substitute this into the boundary equation $u'(1) + u^3(1) = 0$:

$$\begin{aligned} &u'(1) + u^3(1) = 0 \\ \iff\ & u'(x_N) + u_N^3 = 0 \\ \iff\ & \frac{u_{N-2} - 4u_{N-1} + 3u_N}{2h} + u_N^3 = 0 \\ \implies & u_N(2hu_N^2 + 3) - 4u_{N-1} + u_{N-2} = 0 \end{aligned} \tag{5}$$

which is the difference equation for node x_N

The difference equation for node x_0 is simply $u_0 = 1$

0.4 Question 4

Let F(u) = 0 be a system of N - 1 equations in N - 1 unknowns where

$$\begin{aligned} F_1 &= u_2(2 + hexp(u_1)) - 4u_1 + (2 - hexp(u_1)) - 2h^2\mu sin(2\pi x_1) = 0 \\ \vdots &= \quad \vdots \\ F_i &= u_{i+1}(2 + hexp(u_i)) - 4u_i + u_{i-1}(2 - hexp(u_i)) - 2h^2\mu sin(2\pi x_i) = 0 \\ \vdots &= \quad \vdots \\ F_{n-1} &= u_n(2 + hexp(u_{n-1})) - 4u_{n-1} + u_{n-2}(2 - hexp(u_{n-1})) - 2h^2\mu sin(2\pi x_{n-1}) = 0 \end{aligned} \tag{6}$$

Then calculating the Jacobian matrix J where entry $J_{ij} = \frac{\partial F_i}{\partial u_j}$:

$$J = \begin{bmatrix} -4 & 2 + hexp(u_1) & 0 & 0 & \dots & \dots & 0 \\ 2 - hexp(u_2) & -4 & 2 + hexp(u_2) & 0 & \dots & \dots & 0 \\ 0 & 2 - hexp(u_3) & -4 & 2 + hexp(u_3) & 0 & \dots & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots \\ \vdots & \vdots & \ddots & 2 - hexp(u_i) & -4 & 2 + hexp(u_i) & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots \\ 0 & \dots & \dots & \dots & 0 & 2 - hexp(u_{n-1}) & -4 \end{bmatrix}$$