Numerical Methods 4H Assignment 2 - A two-point boundary value problem

1107023m

December 2014

0.1 Question 1

We use a uniform partition of [0,1] by (N + 1) evenly spaced nodes
$0 = x_0 < x_1 < \cdots < x_n = 1$ where $x_i = i/N$ then $h = 1/N$ and $x_i = hi$.
Let u_i denote $u(x_i)$

0.2 Question 2

At each interior point of [0,1] we have the following formulas:

$$u''(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2) \tag{1}$$

$$u'(x_i) = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2) \tag{2}$$

The finite difference approximation for $u(x)$ is now obtained by enforcing the original equation at each interior node of [0,1] that is:

$$\begin{aligned} &\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + exp(u_i)\frac{u_{i+1} - u_{i-1}}{2h} = \mu sin(2\pi x_i) \\ &\iff 2u_{i+1} - 4u_i + 2u_{i-1} + hexp(u_i)(u_{i+1} - u_{i-1}) = 2h^2 \mu sin(2\pi x_i)0 \\ &\iff u_{i+1}(2 + hexp(u_i)) - 4u_i + u_{i-1}(2 - hexp(u_i)) = 2h^2 \mu sin(2\pi x_i) \end{aligned} \tag{3}$$

for $i \in [1, N-1]$

0.3 Question 3

The $O(h^2)$ accurate backwards finite difference formula for u' is:

$$\frac{u_{i-2} - 4u_{i-1} + 3u_i}{2h} = u'(x_i) + O(h^2) \tag{4}$$

Substitute this into the boundary equation $u'(1) + u^3(1) = 0$:

$$\begin{aligned} &u'(1) + u^3(1) = 0 \\ &\iff u'(x_N) + u_N^3 = 0 \\ &\iff \frac{u_{N-2} - 4u_{N-1} + 3u_N}{2h} + u_N^3 = 0 \\ &\implies u_N(2hu_N^2 + 3) - 4u_{N-1} + u_{N-2} = 0 \end{aligned} \tag{5}$$

which is the difference equation for node x_N

The difference equation for node x_0 is simply $u_0 = 1$

0.4 Question 4

Let F(u) = 0 be a system of N equations where

$$
\begin{aligned}
F_1 &= u_2(2+hexp(u_1)) - 4u_1 + (2-hexp(u_1)) - 2h^2\mu sin(2\pi x_1) = 0\\
\vdots &= \quad \vdots\\
F_i &= u_{i+1}(2+hexp(u_i)) - 4u_i + u_{i-1}(2-hexp(u_i)) - 2h^2\mu sin(2\pi x_i) = 0\\
\vdots &= \quad \vdots\\
F_{N-1} &= u_N(2+hexp(u_{N-1})) - 4u_{N-1} + u_{N-2}(2-hexp(u_{N-1})) - 2h^2\mu sin(2\pi x_{N-1}) = 0\\
F_N &= u_N(2hu_N^2+3) - 4u_{N-1} + U_{N-2} = 0
\end{aligned} \tag{6}
$$

Then applying the ficticious node procedure at node i = N, this gives:

$$\frac{u_{N+1} - 2u_N + u_{N-1}}{h^2} + exp(u_i)\frac{u_{N+1} - u_{N-1}}{2h} = \mu sin(2\pi x_N) \tag{7}$$

Centred difference approximation of the derivative boundary condition gives:

$$\frac{u_{N+1} - u_{N-1}}{2h} + u_N^3 = 0 \tag{8}$$

Elimination of U_{N+1} gives :

$$
\begin{aligned}
&\frac{u_{N+1} - 2u_N + u_{N-1}}{h^2} - exp(u_i)u_N^3 = \mu sin(2\pi x_N)\\
\iff &\frac{2u_{N+1} - 4u_N + 2u_{N-1}}{2h^2} - exp(u_N)u_N^3 = \mu sin(2\pi x_N)\\
\iff &\frac{2u_{N-1}}{2h^2} - \frac{2u_N^3}{h} + \frac{-4u_N + 2u_{N-1}}{2h^2} - exp(u_N)u_N^3 = \mu sin(2\pi x_N)\\
\iff &\frac{2u_{N-1}}{h^2} - u_N(exp(u_N)u_N^2 + \frac{2}{h^2}) - \mu sin(2\pi x_N) = 0
\end{aligned} \tag{9}
$$

which gives us our new equation for F_N
Then calculating the Jacobian matrix J where entry $J_{ij} = \frac{\partial F_i}{\partial u_j}$:

$$
J = \begin{bmatrix}
-4 & 2+hexp(u_1) & 0 & 0 & \dots & \dots & \dots & 0\\
2-hexp(u_2) & -4 & 2+hexp(u_2) & 0 & \dots & \dots & \dots & 0\\
0 & 2-hexp(u_3) & -4 & 2+hexp(u_3) & 0 & \dots & \dots & \dots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots & \dots\\
\vdots & \vdots & \ddots & 2-hexp(u_i) & -4 & 2+hexp(u_i) & \dots & \dots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots & \dots\\
0 & \dots & \dots & \dots & 0 & 2-hexp(u_{n-1}) & -4 & 0\\
0 & \dots & \dots & \dots & \dots & 0 & \frac{2}{h^2} & -\frac{2}{h^2} - u_N^2 exp(u_N)(u_N+3)
\end{bmatrix}
$$