

# Numerical Methods 4H Assignment 2 - A two-point boundary value problem

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December 2014

## 0.1 Question 1

We use a uniform partition of  $[0,1]$  by  $(N + 1)$  evenly spaced nodes  
 $0 = x_0 < x_1 < \dots < x_n = 1$  where  $x_i = i/N$  then  $h = 1/N$  and  $x_i = hi$ .  
Let  $u_i$  denote  $u(x_i)$

## 0.2 Question 2

At each interior point of  $[0,1]$  we have the following formulas:

$$u''(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2) \quad (1)$$

$$u'(x_i) = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2) \quad (2)$$

The finite difference approximation for  $u(x)$  is now obtained by enforcing the original equation at each interior node of  $[0,1]$  that is:

$$\begin{aligned} \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \exp(u_i) \frac{u_{i+1} - u_{i-1}}{2h} &= \mu \sin(2\pi x_i) \\ \iff 2u_{i+1} - 4u_i + 2u_{i-1} + h \exp(u_i)(u_{i+1} - u_{i-1}) &= 2h^2 \mu \sin(2\pi x_i) \end{aligned} \quad (3)$$

$$\iff u_{i+1}(2 + h \exp(u_i)) - 4u_i + u_{i-1}(2 - h \exp(u_i)) = 2h^2 \mu \sin(2\pi x_i)$$

for  $i \in [1, N - 1]$

## 0.3 Question 3

The  $O(h^2)$  accurate backwards finite difference formula for  $u'$  is:

$$\frac{u_{i-2} - 4u_{i-1} + 3u_i}{2h} = u'(x_i) + O(h^2) \quad (4)$$

Substitute this into the boundary equation  $u'(1) + u^3(1) = 0$ :

$$\begin{aligned} u'(1) + u^3(1) &= 0 \\ \iff u'(x_N) + u_N^3 &= 0 \\ \iff \frac{u_{N-2} - 4u_{N-1} + 3u_N}{2h} + u_N^3 &= 0 \\ \implies u_N(2hu_N^2 + 3) - 4u_{N-1} + u_{N-2} &= 0 \end{aligned} \quad (5)$$

which is the difference equation for node  $x_N$

The difference equation for node  $x_0$  is simply  $u_0 = 1$

## 0.4 Question 4

Let  $F(u) = 0$  be a system of  $N$  equations where

$$\begin{aligned}
 F_1 &= u_2(2 + \exp(u_1)) - 4u_1 + (2 - \exp(u_1)) - 2h^2\mu \sin(2\pi x_1) = 0 \\
 \vdots &= \vdots \\
 F_i &= u_{i+1}(2 + \exp(u_i)) - 4u_i + u_{i-1}(2 - \exp(u_i)) - 2h^2\mu \sin(2\pi x_i) = 0 \\
 \vdots &= \vdots \\
 F_{N-1} &= u_N(2 + \exp(u_{N-1})) - 4u_{N-1} + u_{N-2}(2 - \exp(u_{N-1})) - 2h^2\mu \sin(2\pi x_{N-1}) = 0 \\
 F_N &= u_N(2hu_N^2 + 3) - 4u_{N-1} + U_{N-2} = 0
 \end{aligned} \tag{6}$$

Then calculating the Jacobian matrix  $J$  where entry  $J_{ij} = \frac{\partial F_i}{\partial u_j}$ :

$$J = \begin{bmatrix}
 -4 & 2 + \exp(u_1) & 0 & 0 & \dots & \dots & \dots & 0 \\
 2 - \exp(u_2) & -4 & 2 + \exp(u_2) & 0 & \dots & \dots & \dots & 0 \\
 0 & 2 - \exp(u_3) & -4 & 2 + \exp(u_3) & 0 & \dots & \dots & \dots \\
 \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots & \dots \\
 \vdots & \vdots & \ddots & 2 - \exp(u_i) & -4 & 2 + \exp(u_i) & \dots & \dots \\
 \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \dots & \dots \\
 0 & \dots & \dots & \dots & 0 & 2 - \exp(u_{n-1}) & -4 & 0 \\
 0 & \dots & \dots & \dots & \dots & 1 & -4 & 6hu_n^2 + 3
 \end{bmatrix}$$