Lie Group Reading Group 2023/24

Nicolas Hasse and Ross Ogilvie

June 26, 2024

1 Introduction

In HWS2023 and FSS2024, we held a weekly reading group to work through the proof of the classification of simple Lie groups. Famously, there is a complete classification: every simple Lie group belongs to one of four infinite families or is one of five exceptions. What surprised us is that there was no single book (that we found) who set itself the task of proving this result from the beginning in full. The purpose of this seminar report is to consolidate our understanding by putting together our sources into a single proof. We found that quite often the theorems were written with an eye towards further developments. This was especially true for parts of the representation theory of Lie algebras. We felt certain techniques were avoided or others emphasised because of the role that they play e.g. in linear algebraic groups over finite fields. So a secondary aim is to reduce the ideas to the simplest form necessary to prove the classification. We do not intend to be fanatical; if an idea is clearer in its general form, so be it. Finally, we are geometers and we do not try to hide our biases about what we find interesting or worth exploring.

We imagined audience of this report is ourselves one year ago. If we could send this through time a year into the past, then it would have served as the main source for our reading group. We assume therefore that a reader is a graduate student familiar with manifold theory but has never formally studied Lie groups. Basic linear algebra is also a given.

TODO: Lit review [**?**] We used this for Lie group theory and the bridge to Lie algebras. [**?**] We used this for Lie algebra theory. [**?**] We used this as a supplement for Lie algebra theory, including for the classification of Dynkin diagrams.

[**?**, p. 349] has a nice quote

> The virtue of classification is that it provides a clear indication of the scope of examples in the subject. It is rarely a sound idea to prove a theorem by proving it case-by-case for all simple real Lie algebras. Instead the important thing about classification is the techniques that are involved. Techniques that are subtle enough to identify all the examples are probably subtle enough to help in investigating all semisimple Lie algebras simultaneously.

Citation style: Sharpe [**?**] numbers within chapter and section, but leaves off the chapter. So our Definition 1.1.36 means Chapter 1, Definition 1.36.

2 Preliminaries

2.1 Submanifolds

Often in mathematics there is an obvious notion of a "subobject": given a structure on a set there is a simple way to restrict it to a subset, such that the subset can be said to have the same structure. For example, the structure on a group is the identity, inversion, and multiplication; if there is a subset containing the identity and which is preserved under inversion and multiplication, then we have a subgroup. Or for a topological space X, any subset A has a topology given by intersection of open sets of X with A. The structures on the subset can often be characterised by the fact that they make the inclusion map $A \hookrightarrow X$ a structure preserving map (group homomorphism and continuous map respectively for the two examples).

In the case of manifolds however the definition of a submanifold is not as trivial and there are several notions that have strong claims for the title. To complicate matters, the notion that is the most widely taught and used (embedded submanifold) is not the one that is most appropriate for Lie group theory (immersed submanifold). Further, the two perspectives of the first paragraph, restriction and inclusion, each have their advantages. It is perhaps more intuitive to work directly with a subset, but a manifold structure is an atlas, and it unpleasant to consider different atlases on the same subset. The alternative is to work with inclusion maps. Then different manifold structures of the subset are realised as inclusion maps from different manifolds with the same image.

Definition 2.1. [**?**, Def 1.27, Rem 1.33],[**?**, Defs 1.1.36, 1.1.40, 1.2.10, 1.2.21]
Let $\phi : N \to M$ be a smooth map of manifolds.

- *a. ϕ is called an* immersion *if $d\phi_p : T_pN \to T_{\phi(p)}M$ is injective at every point $p \in N$. The pair (N, ϕ) is called an* immersed manifold *in M.*
- *b. If ϕ is an injective immersion then the pair is called an* immersed submanifold*.*
- *c. Two immersed submanifolds (N_1, ϕ_1) and (N_2, ϕ_2) are called* equivalent *if there is a diffeomorphism $\varphi : N_1 \to N_2$ such that $\phi_1 = \phi_2 \circ \varphi$.*
- *d. If an injective immersion ϕ has the property that for every smooth map $f : S \to M$ with $f[S] \subset \phi[N]$ the map $\phi^{-1} \circ f : S \to N$ is smooth, then we call ϕ a* weak embedding *and the pair a* weakly embedded submanifold*.*
- *e. If an immersion ϕ is a homeomorphism from N to $\phi(N)$, the latter with the subspace topology of M, then we call it an* embedding *and the pair is called an* embedded submanifold*.*
- *f. A continuous function between Hausdorff spaces is called* proper *if the preimage of a compact set is always a compact set. A proper immersion submanifold is called a* proper submanifold*.*

This is a lot of terminology, but it harmonises the definitions in Sharpe and Warner, see below table. It is in fact a strict hierarchy: each type of submanifold is a subtype of the previous.

These definitions are given in terms of a smooth map into M. So the question from the other perspective is whether the manifold structure of N is determined or can be recovered solely

	Sharpe	Warner
immersed manifold	immersed manifold	
immersed submanifold		submanifold
weakly embedded submanifold		
plaque submanifold	submanifold	
embedded submanifold	regular submanifold	imbedding
proper submanifold	proper submanifold	proper submanifold

from the image $\phi[N]$. There are simple examples that show that not even the topology of N is determined for an immersed submanifold: consider the subset $\{x^2 = y^2\} \subset \mathbb{R}^2$. We can split this into a line and two rays in two ways. Therefore immersed submanifolds must always be given with the immersion ϕ.

Let us give a construction that can construct an atlas on a subset $N' \subset M$.

Definition 2.2. [**?**, Def 1.2.1,1.2.2,Thm 1.2.7] *Given a chart $\varphi : U \subset M \to \mathbb{R}^m$, the connected components of $N' \cap U$ are called plaques. The intersection of open sets of M with plaques gives N' the* submanifold topology*. In general the submanifold topology is finer (has more open sets) than the subspace topology [**?**, Def 1.2.4]. For any plaque W, if $\varphi(W)$ lies in an n-dimensional affine subspace of $\mathbb{R}^m$ then we call this plaque flat and $\varphi|_W$ a plaque chart of W. If there is a collection of charts U_α of M that cover $\bar{N}'$ such that all components $U_\alpha \cap N'$ are flat plaques we call N' a plaque submanifold of M; the plaque charts constitute a smooth atlas.*

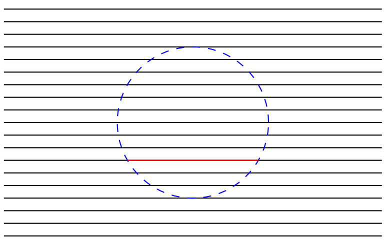


Figure 1: N' is the collection of black lines. The blue circle is an open subset of the plane, and the red segment is a plaque.

The plaque manifold structure on N' makes the inclusion map into a weak embedding [**?**, Thm 1.2.7]. Unfortunately, the reverse is not true. Consider N as a countable collection of spheres and take $\phi : N \to \mathbb{R}^3$ to be the map that embeds the kth sphere as $\partial B(k^{-1}, 0.5(k+1)^{-2})$. Overall one has a sequence of non-overlapping spheres decreasing in size with the origin as a limit point. In particular any neighbourhood U of the origin must contain a sphere as a connected component of $U \cap N'$. But a sphere cannot be a flat plaque and therefore N' is not a plaque submanifold. It would be interesting to consider relaxations of the conditions to be a plaque submanifold.

The difference between plaque submanifold and embedded submanifold is that there is a covering of N' by open sets of M such that each chart has a single flat plaque. This means that there can be no 'strange' limits arising from the immersion. For embedded submanifolds the subspace and submanifold topologies coincide [**?**, Prop 1.2.9]. The example of the dense wrapping of the line around a torus is the classic example of a plaque submanifold that is not embedded.

Finally, by [**?**, Thm 1.2.11] proper submanifolds are automatically embedded, so we have a strict hierarchy of conditions. The standard example of an embedded submanifold that is not proper is $(0,1) \subset \mathbb{R}$. We see that a sequence in N may converge to a point of $M \setminus N$.

These constructions answer the question of how to endow a subset with a manifold structure. There is the possibility however that there are different constructions. Warner addresses these concerns with the following theorem:

Theorem 2.3. [**?**, Remark 1.33]

a. *Let M be a manifold and A a subset of M. Fix a topology on A. Then there is at most one manifold structure on A such that (A, ι) is an immersed submanifold of M, where ι is the inclusion map.*

b. *Again let A be a subset of M. If A with the subspace topology has a manifold structure such that (A, ι) is an immersed submanifold of M, then it is the unique topology on A such that there exits a manifold structure for which (A, ι) is an immersed submanifold of M.*

2.2 Distributions

TODO: Revise this section, include proofs as necessary.

It is common in a course on manifolds to study vector fields and their integral curves. The key local result is

Theorem 2.4 (Picard-Lindelöff)**.** *Let $F : J \times U \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a smooth time-dependent vector field. Assume $0 \in J$. Consider $u : \mathbb{R} \to U$ the system of ODEs $u'(t) = F(t, u(t))$. For any $c \in U$ there exists a $\varepsilon > 0$ such that there is a unique solution smooth solution on $(-\varepsilon, \varepsilon)$ with $u(0) = c$. Moreover, for any $p \in U$ there is an open neighbourhood $p \in V \subset U$, $\varepsilon > 0$ and smooth map $u : (-\varepsilon, \varepsilon) \times V \to U$ such that $u(;c)$ is the unique solution with initial condition c.* [**?**, Theorem 2.1.1]*[**?**, Theorem 1.2.1]*

If one has a smooth vector field on a manifold, then this theorem provides for the existence of integral curves of the vector field in every coordinate chart, and uniqueness means that they can be patched together to give unique maximal integral curves through every point.

We will need a generalisation of this result that deals with multiple vector fields. To motivate why this is geometrically interesting and not just generalisation for its own sake, consider a submanifold N inside M. At each point $p \in N$ we can consider the vector subspace $T_pN \subset T_pM$. At least locally, we can describe these subspaces as the span of independent vector fields. The natural question is the converse: given a set of independent vector fields on M, does there exist a submanifold N whose tangent space is their span? For a single vector field, the answer is affirmative, namely the integral curve.

Definition 2.5. *An r-dimensional distribution $\mathcal{D}$ on M is a choice of an r-dimensional subspace $\mathcal{D}_p$ of T_pM at every point $p \in M$. It is called smooth if every point has a neighbourhood and smooth vector fields $\{X_1, \ldots, X_r\}$ that span the subspaces. It is called integrable if every point has a coordinate neighbourhood in which the distribution is spanned by coordinate vector fields. A set of vector fields is called algebraically involutive if their Lie brackets are contained in their*

span. Given two vector fields with the same span, either one is algebraically involutive if and only if the other is. Therefore algebraically involutive is a property that is defined for distributions. A connected r-dimensional immersed submanifold N is called an integral manifold of $\mathcal{D}$ if at every point $T_pN = \mathcal{D}_p$.
[**?**, 2.2.1,.2.2.2,2.3.2]

Consider the example of nested spheres centered at the origin in $\mathbb{R}^3 \setminus \{0\}$. Their tangent planes define a 2-distribution. In a neighbourhood of any point, suitable polar coordinates show that it is smooth and integrable. However there are not global vector fields spanning this distribution, due to the hairy ball theorem (every vector field on a sphere vanishes at least once). By definition, any sphere is an integral manifold of this distribution.

The important example of a non-integrable distribution is given in $\mathbb{R}^3$ by the planes normal to the vector field $n(x, y, z) = (y, -x, 1)$ [**?**, Ex 3.1]. There can be no integral manifold containing the origin. Suppose there were. Because the tangent plane at the origin is the xy-plane, we know that there is a neighbourhood where is a graph over x and y. Now take a small loop in the integral manifold around the origin lying in this neighbourhood. It's height is monotonic, it doesn't close up. This is a contradiction.

If a distribution is integrable, then every point has an integral submanifold through it, just by taking a coordinate plane in an appropriate chart. The important theorem is Frobenius' theorem [**?**, 2.4.1], which states a distribution is integrable if and only if it is algebraically involutive. The proof inductively applies the Picard-Lindelöff theorem.

There is also a formulation of Frobenius' theorem in terms of differential forms. We can define the annihilator of a distribution as the algebraic ideal generated by the one-forms such that $\omega(v) = 0$ for all $v \in \mathcal{D}_p$. By this we mean all C^∞-linear combinations and wedge products. Conversely, the kernel of an algebraic ideal of differential forms generated locally by $m - r$ independent one-forms is a distribution. The theorem then says that $\mathcal{D}$ is integrable if and only if the ideal contains all its exterior derivatives (it is a differential ideal). The proof comes down to the simple formula

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

TODO: Sharpe does foliations in general, and the argument look a lot like the arguments Warner uses for Lie subgroups. I think it might be advantageous to follow Sharpe here and separate what is truly manifold theory from Lie theory.

TODO: I don't know where else to put this theorem. Ah, Sharpe's submanifolds implies weakly embedded, so have this property already. There is a concern that if we have a smooth map $\psi : N \to M$ that happens to lie in a submanifold P, then is the induced map $\psi : N \to P$ given by restriction on the codomain a smooth map with respect to the manifold structure ono the submanifold. The following theorem of Warner answers this affirmatively:

Theorem 2.6. [**?**, 1.62]
Suppose that $\psi\colon N \to M^d$ is C^∞, that (P^c, φ) is an integral manifold of an involutive distribution $\mathcal{D}$ on M, and that ψ factors through (P, φ), that is, $\psi[N] \subset \varphi[P]$. Let $\psi_0 : N \to P$ be the (unique) mapping such that $\varphi \circ \psi_0 = \psi$. Then ψ_0 is continuous and therefore C^∞.

2.3 Eigenvalues and Weights

Eigenvalues and eigenvectors are ubiquitous in linear algebra. If we are working over $\mathbb{C}$, then every linear endomorphism (linear map from a vector space V to itself) has an eigenvalue (root of the characteristic polynomial) and every eigenvalue has an eigenvector $Av_\lambda = \lambda v_\lambda$. If we can find a basis of eigenvectors, then with respect to this basis the linear operator is a diagonal matrix. In general however, there may be fewer eigenvectors than the dimension of the vector space. A standard result in linear algebra says that every matrix is conjugate to a matrix in Jordan normal form, unique up to reordering of the blocks.

Going further, we may ask what can be said of two linear endomorphisms A, B. The key observation is to consider commuting operators. If A and B commute then B preserves the eigenspaces eigenspaces of A:

$$(A - \lambda I)(Bv) = B(A - \lambda I)v = 0.$$

Therefore B restricts to an endomorphism on each of the eigenspaces of A. Imposing different conditions on A restricts the possible decompositions of B. For example, if A is diagonalizable (so V decomposes as the direct sum of the eigenspaces of A) then on each eigenspace of A we can choose a basis that puts B into Jordan normal form. Hence A and B can simultaneously conjugated to normal form. Or if A is diagonalizable and has no repeated eigenvalues, ie its eigenspaces are one dimensional, then these must also be eigenspaces of B. Hence A and B are simultaneously diagonalizable.

This argument can be applied inductively to a finite set $\{A_1, \ldots, A_k\}$ of pairwise commuting endomorphisms. It also extends to a commuting family of operators $\mathcal{A} = \mathrm{span}\{A_1, \ldots, A_k\}$, a linear subspace of $\mathrm{End}(V)$ such that all operators are pairwise commuting. These are effectively equivalent, since a family is pairwise commuting if and only if a basis is pairwise commuting. Likewise, v is a simultaneous eigenvector for $\{A_1, \ldots, A_k\}$ if and only if it is a simultaneous eigenvector for every operator of $\mathcal{A}$. The eigenvalues are not completely independent:

$$\lambda v = Av = (a_1 A_1 + \cdots + a_k A_k)v = a_1 \lambda_1 v + \cdots + a_k \lambda_k v = (a_1 \lambda_1 + \cdots + a_k \lambda_k)v.$$

We understand the eigenvalue of v to be a linear function

$$\lambda : \mathcal{A} \to \mathbb{K}, \quad \lambda(A) = a_1 \lambda_1 + \cdots + a_k \lambda_k.$$

Understood in this way, it is more common to call λ a *weight* of the commuting family $\mathcal{A}$, v a *weight vector*, and the set of vectors v with $Av = \lambda(A)v$ the *weight* space [**?**, Definition A.14].

As an aside, the descriptor "weight" should probably replace "eigen-" even in the single operator case. Consider an operator A with a 2-weight vector v and a 3-weight vector w. Then of course $A(av + bw) = 2v + 3w$, which is a weighted sum.

We finish with an example. Let $\mathcal{A}$ be the set of diagonal $n \times n$ matrices. Then a basis for this family is $A_i = e_{ii}$, with 1 at the ith position on the diagonal. e_i is a 1-weight vector of A_i while all vectors of $\mathrm{span}\{e_1, \ldots, \hat{e}_i, \ldots, e_n\}$ are 0-weight (in the kernel). These are all diagonal (in particular simultaneously diagonal) so there should be a basis of weight vectors. Indeed, this is just the standard basis $\{e_1, \ldots, e_n\}$. The weight of e_i is the linear form

$$A = \mathrm{diag}(a_1, \ldots, a_n) \mapsto a_i$$

because $Ae_i = a_i$.

3 Lie Groups

Historically and in practice, Lie groups arise first as the study of the transformation of geometric objects. Let us consider the example of a sphere in euclidean space. There are in a sense three ways to rotate a sphere. Choose a point on the equator of a sphere. You can rotate the sphere so that this point moves towards a pole (y-axis rotation), so that this point moves along the equator (z-axis rotation), or so that the point is stationary (x-axis rotation). Moreover these rotations are continuous in a way that rotating an equilateral triangle is not, because at each stage of rotation the sphere as a whole occupies the same space.

How should we describe the rotations of a sphere? First observe that antipodal points remain antipodal, so the rotation of a sphere extends to a linear transformation of $\mathbb{R}^3$. Hence any rotation R can be described by a real 3×3-matrix. Moreover rotation is length and angle preserving, so $\langle Rx, Ry \rangle = \langle x, y \rangle$. This holds exactly if $R^T R = I$, which gives us the orthogonal group

$$\mathrm{O}(3) = \{R \in \mathrm{Mat}(3, \mathbb{R}) \mid R^T R = I\}.$$

We can understand the defining equation as saying that the columns of R are an orthonormal basis of $\mathbb{R}^3$. Indeed they are the images of the standard basis of $\mathbb{R}^3$ under R. We were discussing proper rotations, which are by definition orientation preserving, so we want that the columns of R are a right handed basis. That leads to the special orthogonal group

$$\mathrm{SO}(3) = \{R \in \mathrm{O}(3) \mid \det R = 1\}.$$

The group operation is composition of operators, ie matrix multiplication. Both groups clearly contain the identity I. The property $R^T R = I$ implies $(\det R)^2 = 1$, so these are invertible matrices. Therefore they really are groups. Moreover the sign of the determinant can be used to distinguish a proper rotation from an improper one.

Heuristically, we have nine choices for the matrix of R subject to the restriction each of the three columns must be unit length and the restriction that pairs of columns must be orthogonal (six restrictions total). This agrees with the three degrees of freedom we argued for above. To see this is a manifold however we consider a function f from $\mathrm{Mat}(3, \mathbb{R})$ to the symmetric matrices given by $f(R) = R^T R$. The orthogonal matrices are exactly $\mathrm{O}(3) = f^{-1}(I)$. The derivative at R in the direction S is $f'(R)(S) = R^T S + S^T R = R^T S + (R^T S)^T$. At a point $R \in \mathrm{O}(3)$ this is surjective in S: for any symmetric matrix Y let $S = \frac{1}{2} RY$. Hence f is full rank at every point $R \in \mathrm{O}(3)$ and by the implicit function theorem $\mathrm{O}(3)$ is an embedded submanifold of $\mathrm{Mat}(3, \mathbb{R}) \cong \mathbb{R}^9$. The symmetric matrices are dimension 6, so in fact our heuristic has been formalised to a rigorous argument.

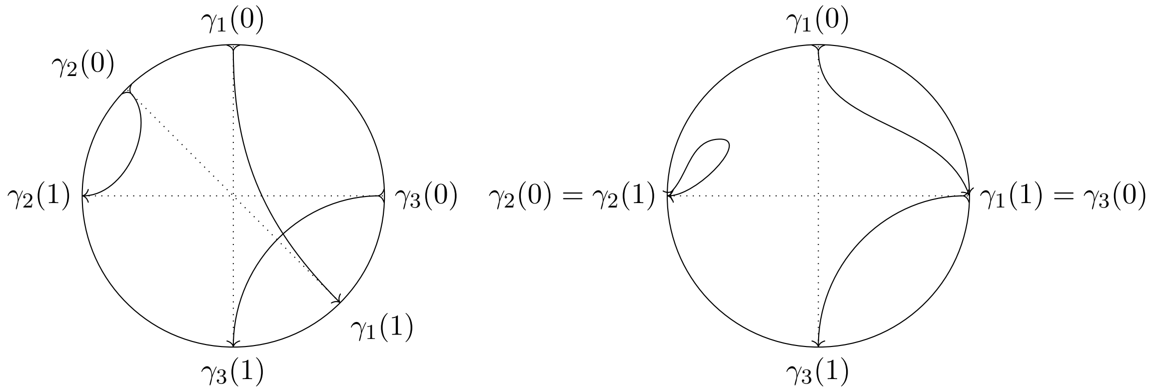
The group operation, matrix multiplication, is a smooth operation on the set of matrices because it is polynomial. Therefore it is also smooth when restricted to an embedded submanifold. Similarly inversion is an everywhere defined rational function on the open set of invertible matrices, so also smooth on $\mathrm{O}(3)$. This makes $\mathrm{O}(3)$ and $\mathrm{SO}(3)$ Lie groups.

Now that we know that $\mathrm{O}(3)$ is a manifold, we can ask about its connected components. Intuitively we can rotate any right handed frame $R \in \mathrm{SO}(3)$ to the standard basis I, so $\mathrm{SO}(3)$ is connected. Because it contains the identity, it is called the identity component. On the other hand, the reflection in the plane $x_1 = 0$ has determinant -1 whereas $\det I = 1$. Determinant is continuous (polynomial) function on matrices and as already noted $R \in \mathrm{O}(3)$ implies

$\det R = \pm 1$, so this reflection is not in the identity component. Composing an improper rotation with this reflection gives a proper rotation and vice-versa. Therefore the subset of O(3) with $\det R = -1$ is also a connected component of O(3). In conclusion, SO(3) is connected and O(3) has two diffeomorphic components.

To understand the topology of SO(3) an alternate description is useful. Every rotation of $\mathbb{R}^3$ is rotation around an axis. More precisely, we can describe the rotation axis by a unit vector u such that the rotation is right handed by an angle θ in the range $[0, \pi]$. Thus the rotations can be described by the closed unit ball $\theta u \in \overline{B(0,1)}$. The origin is rotation by angle 0 with the axis of rotation irrelevant. But similarly, rotation by π around u and $-u$ are the same rotation. Thus SO(3) can be modelled as the closed unit ball with antipodal points on the boundary identified, the real projective space $\mathbb{RP}^3$.

In this model it is easy to understand the fundamental group. Take any closed loop in SO(3). If it lies entirely in $B(0,1)$ then it can be contracted to a point. Otherwise it can be divided into a collection of segments $\gamma_i : (0,1) \to B(0,1)$ with $\gamma_{i+1}(0) = \pm\gamma_i(1) \in \partial B(0,1)$ and $\gamma_1(0) = \pm\gamma_n(1) \in \partial B(0,1)$. These conditions ensure that the segments connect up to a loop in $\mathbb{RP}^3$. We call the number of negative signs the index of the loop. If $\gamma_{i+1}(0) = \gamma_i(1)$ then it is possible to move this point into the interior of $B(0,1)$ and fuse these two segments together into a single segment. This doesn't change the index of the loop. As an extreme case, if the index is zero, then we can move all the endpoints of the segments off $\partial B(0,1)$ and contract the loop to a point. So without loss of generality, assume that all the signs are negatives.



If there is more than one segment, we can move γ_1 and γ_2 such that $\gamma_1(0)$ and $\gamma_2(1)$ remain fixed (so no other segments are affected) but $\gamma_2(0)$ is moved to $\gamma_2(1)$ (so necessarily $\gamma_1(1) \to -\gamma_2(1)$). Then γ_2 can be contracted to the constant map $\gamma_1(1) = -\gamma_2(0) = -\gamma_2(1) = \gamma_3(0)$. This means that we can eliminate γ_2 and fuse γ_1 and γ_3 into a single segment. In particular, the index has decreased by two. This argument has not quite proved that the parity of the index (even or odd) is a homotopy invariant, since there is still the possibility that some other tricky operation can decrease the index by an odd amount. But it should convince you none-the-less that the fundamental group of $\mathbb{RP}^3$, and hence SO(3), is $\mathbb{Z}_2$: any loop with an even number of segments can be contracted to a point, whereas any loop with an odd number of segments can be reduced to a diameter.

3.1 Definition

Definition 3.1. [**?**, 3.1][**?**, Definition 1.20]
A Lie Group *is a G manifold with a group structure such that multiplication $\mu : (g, h) \mapsto gh$ and inversion $\iota : g \mapsto g^{-1}$ are smooth.*

Many familiar manifolds are also Lie groups in natural ways. For example, the reals $\mathbb{R}$ under addition, the multiplicative group of the complex numbers $\mathbb{C}^\times$, and the circle $\mathbb{S}^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$. The product of two Lie groups is a Lie group, using the product manifold and product group structures. This gives us euclidean space $\mathbb{R}^n$ with vector addition and the torus $\mathbb{T}^n = (\mathbb{S}^1)^n$ as further examples.

Left and right actions. We can interchange left and right actions using inversion to make the opposite group. Convention is to work primarily with left actions. Hold off talking about general actions until quotients.

Many definition carry over naturally by requiring both a manifold-theory and a group-theory property. For example

Definition 3.2. [**?**, 3.13]
A homomorphism of Lie groups *is a smooth map $\phi : G \to H$ that is also group homomorphism. If it is also a diffeomorphism, then we say ϕ is an* isomorphism of Lie groups.

On the other hand, sometimes a different concept is more appropriate for Lie theory. In manifold theory one is mostly concerned with embedded submanifolds, while in Lie theory immersed submanifolds are more useful:

Definition 3.3. [**?**, 3.17], contrast [**?**, §7.1]
A Lie subgroup *(H, φ) of a Lie group G is a Lie group H and an injective immersion $\varphi : H \to G$ that is also a homomorphism. It is called a* closed Lie subgroup *if $\varphi(H)$ is further closed.*

An immersed manifold that is closed is an embedded submanifold.

TODO: What is the proper order of material? In particular, the order of introducing Lie algebras and subgroups. Presumably you want to introduce examples early, which are mostly subgroups. Do you introduce subgroups before the examples? Is it possible to introduce subgroups, quotients and covers without talking about the Lie algebra? Then introduce the Lie algebra and show how many of these it can grok. That is certainly a differential geometry supremacist approach.

3.2 Examples

There are many interesting properties that Lie groups can possess, and we give a quick tour of them with examples.

All finite groups are also Lie groups using the discrete topology to make them 0-dimensional manifolds. These are not central examples of Lie groups, whose essential character is their 'continuity', and they could reasonably be excluded by definition. However we do not do so because they arise naturally. For example, we have seen that the SO(3) is the component of

O(3) that contains the identity. In fact O(3) is the product of SO(3) and C_2 the group with two elements. Generalising, the identity component G_0 of a Lie group G is a Lie group. To prove this, note that $II = I$ and $I^{-1} = I$, so the images of $G_0 \times G_0$ under multiplication and G_0 under inversion are both contained in G_0. If g belongs to another connected component G_1 then multiplication with g is a diffeomorphism between G_0 and G_1. In this way, every Lie group with finitely many connected components is the product of its identity component and a finite group. For this reason we usually consider connected Lie groups of positive dimension.

Perhaps the most important category of Lie group are the matrix Lie groups [**?**, Definition 1.4]. First we have the general linear group $\mathrm{GL}(n, \mathbb{C})$, the set of $n \times n$ invertible matrices with complex entries. This can be considered as an open subset of $\mathbb{C}^{n^2}$, so it is a manifold. And just as for O(3) the group operation is polynomial and group inversion is rational without zeroes of the denominator, hence both are smooth. A matrix group is any closed Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$. As a special case we have the real matrix groups, which are subsets of the (real) matrix group $\mathrm{GL}(n, \mathbb{R})$.

We have already seen the real matrix groups O(3) and SO(3). As the notation suggests, these belong to families indexed by the size of the matrices. We have the following families of matrix groups

$$\begin{aligned}
\mathrm{SL}(n, \mathbb{C}) &= \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \det A = 1\} \\
\mathrm{SL}(n, \mathbb{R}) &= \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det A = 1\} \\
\mathrm{U}(n) &= \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \bar{A}^T A = I\} \\
\mathrm{SU}(n) &= \{A \in \mathrm{U}(n) \mid \det A = 1\} \\
\mathrm{O}(n) &= \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\} \\
\mathrm{SO}(n) &= \{A \in \mathrm{O}(n) \mid \det A = 1\}.
\end{aligned}$$

If we give $\mathbb{C}^n$ the standard inner product $\langle v, w \rangle = \bar{v}^T w$ then unitary matrices are exactly the linear transformations that preserve it. In this way the orthogonal groups are the real counterparts to the unitary groups. The following trick shows that $\mathrm{U}(n)$ is compact: As a vector in $\mathbb{C}^{n^2}$ the square of the norm of $A \in \mathrm{U}(n)$ is $\mathrm{tr}(\bar{A}^T A) = \mathrm{tr}\, I = n$, thus $\mathrm{U}(n)$ is bounded. Thus all closed subsets, such as $\mathrm{SU}(n), \mathrm{O}(n), \mathrm{SO}(n)$, are also compact.

There are also the symplectic groups. Like U and O they preserve a bilinear form. Let

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

be a $2n \times 2n$ matrix in block form and define

$$\begin{aligned}
\mathrm{Sp}(2n, \mathbb{C}) &= \{A \in \mathrm{GL}(2n, \mathbb{C}) \mid A^T \Omega A = \Omega\} \\
\mathrm{Sp}(2n, \mathbb{R}) &= \{A \in \mathrm{GL}(2n, \mathbb{R}) \mid A^T \Omega A = \Omega\} \\
\mathrm{Sp}(n) &= \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n).
\end{aligned}$$

The notation around $\mathrm{Sp}(n)$ is a bit confusing, but the point is to make a compact group. Indeed $\mathrm{Sp}(n)$ is called the compact symplectic group.

Together, these examples are called the classical groups and they will figure prominently in the classification of Lie groups. There are of course many other matrix Lie groups. One could consider groups of matrices preserving other bilinear forms. For a concrete example, the subset of diagonal matrices of any of the classical groups is again a matrix group. In $\mathrm{U}(n)$ the diagonal subgroup is $\{\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\}$ with $|\lambda_i| = 1$. We see that this is isomorphic to $\mathbb{T}^n$. The standard terminology is that a Lie group that is isomorphic to a matrix group is called a linear group. In other words, $\mathbb{T}^n$ which is defined as the product of circles, is a linear group but not a matrix group. Similarly $\mathbb{R}$ is a linear group because we can consider real matrices of the form

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

The result of multiplying two such matrices is the add the off-diagonal term.

TODO: Complex Lie groups. $\mathrm{SL}(n, \mathbb{C})$ is a complex Lie group but $\mathrm{U}(n)$ is not. A similar direction that we will not explore is linear algebraic groups. These are matrix groups whose defining equations are polynomial. This means that they can be defined over any field, not just $\mathbb{C}$ and $\mathbb{R}$.

As in group theory, we have abelian and non-abelian groups. Abelian Lie groups include $\mathbb{R}^n$ and $\mathbb{T}^n$ and $\mathrm{O}(3)$ and $\mathrm{SO}(3)$ are examples of non-abelian groups.

3.3 Subgroups

First, we remind ourselves of definition 3.3 to see what a Lie subgroup is. First we start with a Proposition that is used quite often:

Proposition. 3.4. [**?**, 3.18]
Let G be a connected Lie group, and let U be a neighborhood of the identity e. Then

$$G = \bigcup_{n=1}^{\infty} U^n$$

where U^n consists of all n-fold products of elements of U.

Proof. We will outline the idea of the proof. We consider an open subset $V \subset U$ s.t. $V = V^{-1}$, for example choosing $V = U \cap U^{-1}$. Then we define

$$H = \bigcup_{n=1}^{\infty} V^n \subset \bigcup_{n=1}^{\infty} U^n.$$

By choice of V, H satisfies the subgroup condition. Further H is an open subset of G as a union of open sets. In fact for any $g \in G$ the coset gH is open in G, since it preimage of H under $(g^{-1}\cdot) : G \to G$. Now we want to prove that H is also closed. But the complement of H can be written

$$G \setminus H = \bigcup_{g \in G \setminus H} gH,$$

a union of open sets. □

Next we have a theorem considering when an abstract subgroup of a Lie group is also a Lie subgroup. This is an analogy of 2.3

Theorem 3.5. [?, 3.20]
If an abstract subgroup A of a Lie group G has a manifold structure which makes (A, ι) into an immersed (sub-)manifold of G, where ι is the inclusion map, then it has a unique manifold structure, and in this manifold structure, A is a Lie group and hence, (A, ι), is a Lie subgroup of G.

Proof. Step 1: We define $\mathcal{D}$ to be the distribution on G determined by left translations of the tangent space to A at the identity e. Then prove that (A, ι) is an integral manifold of $\mathcal{D}$ at e. In particular, ι is a smooth immersion.

Step 2: Likewise we know that for any $\sigma \in G$, (A, l_σ) is an integral manifold of $\mathcal{D}$ through σ. Now the map $\beta\colon A \ \times A \to G$ sending $(\sigma, \tau) \to \iota(\sigma)\iota(\tau)^{-1} = \sigma\tau^{-1}$ is C^∞ due to Theorem 2.6. Thus we have a manifold structure such that A is a Lie group.

Step 3: We show uniqueness. If there are two structures (A_1, ι) and (A_2, ι) then we can lift ι to a map $A_1 \to A_2$ and vice versa, giving a diffeomorphism. □

From this, we get the following result

Theorem 3.6. [?, 3.21]
Let (H^d, φ) be a Lie subgroup of G^c. Then φ is an embedding (a homeomorphism of H with $\varphi[H]$ in the relative topology) if and only if (H, φ) is a closed subgroup of G (meaning $\varphi[H]$ is closed in G).

Proof. Step 1: We want to construct an embedding φ from H onto $\varphi[H]$.
By Frobenius theorem we can choose a cubic-centered coordinate system (U, τ) for $e \in G$ s.t. $\varphi[H] \cap U$ is an at most countable union of slices

$$\tau_i \equiv \mathit{const} \text{ for all } i \in \{d+1, \dots, c\},$$

which includes the slice through e. So we define $C \subset U$ tp be the subset containing e and whose image under τ is a cube. Then $S \subset C$ is the slice $\tau_1 = \cdots = \tau_d = 0$. So we have constructed $\tau[\varphi[H] \cap S]$ to be a non-empty, closed and countable subset of $\mathbb{R}^{c-d}$. Since the set is countable, it has at least one isolated point. So there is an isolated slice S_0 included in $\varphi[H] \cap U$. But then the pre-image $\varphi^{-1}[S_0]$ needs to be open in H and hence φ gives an embedding into $\varphi[H]$.

Step 2: Now we assume that φ is an embedding and prove that $\varphi[H]$ is closed.
So let $\{x_n\}_{n\in\mathbb{N}} \subset \varphi[H]$ be a sequence converging to $x \in G$. Since φ now is an embedding we choose a cubic coordinate system (U, τ) including $e \in G$ s.t. $\varphi[H] \cap U$ is a single slice S. We again choose cubic neighborhoods $V \subset W \subset U$ s.t. $V^{-1}V \subset \bar{W} \subset U$. By convergence of x_n to x we choose $N \in \mathbb{N}$ s.t. $x_n \in xV$ for all $n \geq N$. That implies $x_N^{-1}x_n \in \bar{W}$ for $n \geq N$. Since $x_N^{-1}x_n \in \varphi[H]$ holds as well, $x_N^{-1}x_n \in S \cap \bar{W}$ too. By construction, this converges to $x_N^{-1}x$ which therefore again lies in $S \cap \bar{W}$. Therefore $x_N^{-1}x \in \varphi[H]$ as well. But that implies $x \in \varphi[H]$, proving closedness.

□

3.4 Quotients

We start with a defining theorem:

Theorem 3.7. *Let H be a closed subgroup of a Lie Group G and let G/H be defined to be the set $\{\sigma H \colon \sigma \in G\}$ of left cosets modulo H. Let $\pi \colon G \to G/H$ denote the natural projection $pi(\sigma) = \sigma H$. Then G/H has a unique manifold structure s.t.*

- *π is C^∞*
- *There exist local smooth sections of G/H in G, meaning that if $\sigma H \in G/H$, there is a neighborhood W of σH and a C^∞ map $\tau \colon W \to G$ s.t. $\pi \circ \tau = id|_W$.*

This gives us the following definition:

Definition 3.8. *Manifolds of the form G/H, where G is a closed Lie group, H is a closed subgroup of H and the manifold structure of G/H is as in theorem 3.7 are called* ***homogeneous manifolds****.*

Definition 3.9. *Let $\eta \colon G \times M \to M$ be an action of G on M on the left and let $\eta_\sigma(m) := \eta(\sigma, m)$. The action map is called* ***effective*** *if e is the only element of G for which η_e is the identity map on M. The action is called* ***transitive*** *if whenever m and n belong to M there exists a $\sigma \in G$ s.t. $\eta_\sigma(m) = n$. Let $m_0 \in M$ and let*

$$H = \{\sigma \in G \,|\, \eta_\sigma(m_0) = m_0\}.$$

H is a closed subgroup of G called ***isotropy group at*** *m_0.*

Theorem 3.10. *Let $\eta \colon G \times M \to M$ be a transitive action of the Lie group G on the manifold M on the left. Let $m_0 \in M$, and let H be the isotropy group at m_0. Define a mapping*

$$\tilde{\beta} \colon G/H \to M \text{ by } \tilde{\beta}(\sigma H) = \eta_\sigma(m_0).$$

Then $\tilde{\beta}$ is a diffeomorphism.

We finally arrive at a statement that tells us when quotients of Lie groups are again Lie groups.

Theorem 3.11. *Let G be a Lie group and H a closed normal subgroup of G. Then the homogeneous manifold G/H with its natural group structure is a Lie group.*

3.5 Covers

3.6 Mauer-Cartan forms

Should we have such a section? Sharpe and Ivey has it, but I don't think Warner puts as much emphasis on it.

3.7 Simple Lie groups

We should give a refined version of the classification problem: to find simply-connected simple Lie groups.

4 Lie algebra

TODO: maybe the angle here should be that Lie groups are homogeneous manifolds, so it makes sense to focus on neighbourhoods of the identity. Maybe the result that a neighbourhood of the identity generates a connected Lie group [**?**, 3.18]. Then talk about the exponential map which is a local diffeomorphism from $\mathfrak{g}$ to G at e. This frames the section: we have the internal generation of a Lie group, how do we externally generate it from the tangent space?

4.1 Lie Bracket

Lie bracket coming from the adjoint action. There are lots of names for conjugation, eg the power notation, Sharpe uses **Ad**, Warner a. I think a two letter operator, eg Cn(g) would be best. Wikipedia uses Ψ. There's also the question of $\mathrm{ad}(x)$ or ad_x. I think this one shows most clearly how the bracket is encodes some infinitesimal information of the group operation.

This is Lie bracket of left-invariant vector fields.

4.2 Examples

Matrix vs abstract Lie algebras Bracket of matrix Lie algebras is commutator

Lie algebras of all the classical groups.

4.3 Correspondences

Stuff like Lie group homomorphisms inducing Lie algebra homomorphisms. ideals and subalgebras

I found this pdf https://www.cis.upenn.edu/~cis6100/cis610-15-sl17.pdf that connects metrics on the Lie group with inner products on the Lie algebra.

4.4 Ado's theorem

https://terrytao.wordpress.com/2011/05/10/ados-theorem/ This seems elementary, and introduces all the important classes of Lie algebras: sovlable, nilpotent, etc. But it defines complex Lie algebras at the outset? Maybe we already need to consider real vs complex Lie algebras, complexifications, etc.

Maybe we only need a weaker version, that all simple Lie algebras are matrix? Indeed, the adjoint representation works for centerless Lie algebras.

5 Classification of Lie algebras

TODO: make sections.

I would like to gather up at the start all the structure that we need. I am thinking here in particular of inner products and Weyl reflections.

The introduction of https://terrytao.wordpress.com/2013/04/27/notes-on-the-classification-of-c is nice.

From https://en.wikipedia.org/wiki/Killing_form there seems to be a close connection between the Killing form and decompositions of Lie algebras:

- On a simple Lie algebra any invariant symmetric bilinear form is a scalar multiple of the Killing form.
- The Killing form is also invariant under automorphisms of the Lie algebra
- The Cartan criterion states that a Lie algebra is semisimple if and only if the Killing form is non-degenerate.
- The Killing form of a nilpotent Lie algebra is identically zero.
- Ideals that intersect only trivially are orthogonal.

This paper https://math.uchicago.edu/~may/REU2012/REUPapers/Bosshardt.pdf from Uchicago undergrad summer projects 2012 seems to go hard on the Killing form.

If we are focused on simple Lie algebras, then can we avoid the other types?

I don't want to build representation theory. I think we only really need the adjoint representation and $\mathfrak{sl}(2, \mathbb{C})$-representations. Maybe we can understand them as a special type of subspace.

5.1 Cartan subalgebra

A point of difficulty seems to be Cartan subalgebras. In fact there are several, somewhat incompatible definitions in the literature (in different cases) to suit different circumstances. Hall takes the simplest definition: a maximal commuting subalgebra of diagonalizable elements. This immediately gives you a root space decomposition, commuting implies simultaneously diagonalizable. The existence is rather easy too, but only because he begins with a real compact form for the complex semisimple algebra. FH takes the same definition, but in Appendix D proves existence through regular elements.

Knapp takes a more principled approach, asking what we should require of a subalgebra to get a decomposition like we see in the standard examples, p83. It takes the more general starting point of a nilpotent subalgebra h. Not all of them lead to a really nice decomposition, but somehow maximal ones do, and these we call Cartan subalgebras. The drawback is that we don't have diagonalizability at the start, so we have to work with generalised eigenspaces, see Prop 2.5 p88. Here too the existence relies on regular elements Theorem 2.9.

Knapp Chapter 4 comes back to Hall's point: Cor 4.26 a compact Lie group has negative definite Killing form. Prop 4.27, gives an argument for the converse!

What drew me to Knapp though was the discussion of real simple Lie algebras. In particular, how complex Lie algebras can arise from real ones. The two extremes are the split real form and the compact real form. These are constructed explicitly Cor 6.10 and Thm 6.11, but they require the root decomposition. Other real forms (and these ones too) are described by a Cartan involution. It is a vector space sum on which the Killing form is resp pos and neg definite (6.26) and the brackets are behaved (6.24). Perhaps this could be the basis of a direct proof of a compact real form. After some googling, this was suggested by Cartan 1929 to simplify the proof, and there's a paper by Richardson (?) that carries it out. But it's not as simple as it sounds.

Another thing to consider would be to only classify compact lie groups. This has the advantage that then the compact real form is a natural part of the set-up. You get basically the same list at the end. It has most of the features you want to consider. And then maybe you can carry the classification all the way back to the level of real Lie groups, without so much of the messy involution stuff at the end. Go deeper on compact Lie groups; apparently their universal covers are compact, you can understand the quotient theory well. It does feel like a limitation, when we are so close to getting all complex semisimple Lie algebras.