Moduli Space of Harmonic Tori in S^3

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Harmonic Map

- ▶ Let $f : T^2 \to S^3 = SU(2)$ be a harmonic map.
- ▶ A harmonic map is a critical point of the energy functional.
- ▶ Long historical interest in minimal and constant curvature surfaces. A surface is CMC iff its Gauss map is harmonic.
- ▶ Minimal surfaces = conformal harmonic = CMC with zero mean curvature.
- ▶ Thought to be quite rare; Hopf Conjecture. Wente (1984) constructed immersed CMC tori.
- ▶ A classification of such maps is given by spectral data $(\Sigma, \Theta, \tilde{\Theta}, E)$ (Hitchin, Pinkall-Sterling, Bobenko).

Spectral Data $(\Sigma, \Theta, \tilde{\Theta}, E)$

▶ Spectral curve Σ is a real (possibly singular) hyperelliptic curve,

$$\eta^2 = \prod(\zeta - \alpha_i)(1 - \bar{\alpha}_i \zeta)$$

▶ $\Theta, \tilde{\Theta}$ are differentials with double poles and no residues over $\zeta = 0, \infty$.

▶ Period conditions: The periods of $\Theta, \tilde{\Theta}$ must lie in $2\pi i \mathbb{Z}$.

▶ Closing conditions: for γ_+ a path in Σ between the two points over $\zeta = 1$, and γ_- between the points over $\zeta = -1$ then

$$\int_{\gamma_+} \Theta, \int_{\gamma_-} \Theta, \int_{\gamma_+} \tilde{\Theta}, \int_{\gamma_-} \tilde{\Theta} \in 2\pi i \mathbb{Z}.$$

▶ E is a quaternionic line bundle of a certain degree.

CMC Moduli Space (Kilian-Schmidt-Schmitt)

- One can vary the line bundle E, so called isospectral deformations.
- CMC non-isospectral deformations. Maps come in one dimensional families.
- $\mathcal{M}_0^{CMC}$ is disjoint lines parametrised by $H \in \mathbb{R}$
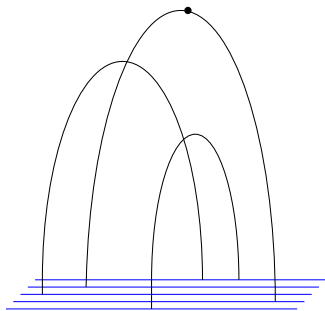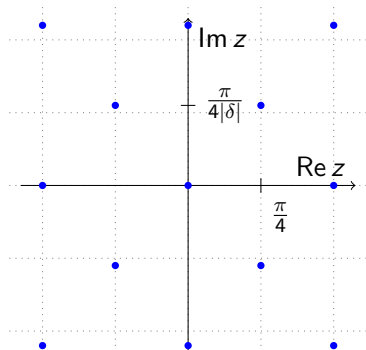

- Components $\mathcal{M}_1^{CMC}$ end in either $\mathcal{M}_0^{CMC}$ or bouquet of spheres.

Harmonic Map Example

- $f(x + iy) = \exp(-4x\mathbf{X})\exp(4y\mathbf{Y})$, for

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}, \qquad \mathfrak{Im}\,\delta > 0$$

- This map is periodic. Formula well-defined on any torus $\mathbb{C}/\Gamma$, where Γ is a sublattice of this periodicity lattice.

Constructing Spectral Data

- ▶ Up to translations, f is determined by the Lie algebra valued map $f^{-1}df$, the pullback of the Mauer-Cartan form.
- ▶ Decompose into its dz and $d\bar{z}$ parts $f^{-1}df = 2(\Phi - \Phi^*)$.
- ▶ Use f to pull pack the Levi-Civita connection on $SU(2)$ to get a connection A.
- ▶ Given a pair (Φ, A), we can make a family of flat $SL(2, \mathbb{C})$ connections. Let $\zeta \in \mathbb{C}^\times$ be the *spectral parameter* and define

$$d_\zeta := d_A + \zeta^{-1}\Phi - \zeta\Phi^*$$

Family of connections is

$$
\begin{aligned}
d_\zeta = d &- \left[(\mathbf{X} - i\mathbf{Y}) + \zeta^{-1}(\mathbf{X} + i\mathbf{Y}) \right] dz \\
&- \left[(\mathbf{X} + i\mathbf{Y}) + \zeta(\mathbf{X} - i\mathbf{Y}) \right] d\bar{z} \\
= d &- \zeta^{-1} \left[(\mathbf{X} + i\mathbf{Y}) + \zeta(\mathbf{X} - i\mathbf{Y}) \right] \left[dz + \zeta d\bar{z} \right]
\end{aligned}
$$

Holonomy

- Because the connections are flat, we can define holonomy for them.
- Pick a base point and generators for the fundamental group, ie take two loops around the torus.
- Parallel translating vectors with d_ζ around one loop gives a linear map on the tangent space at the base point. Call this $H(\zeta)$. Around the other loop call the transformation $\tilde{H}(\zeta)$.

$$H_\tau(\zeta) = \exp\left\{\zeta^{-1}\left[(\mathbf{X} + i\mathbf{Y}) + \zeta(\mathbf{X} - i\mathbf{Y})\right]\left[\tau + \zeta\bar{\tau}\right]\right\}$$

Spectral curve

- The fundamental group of T^2 is abelian, so H and $\tilde{H}$ commute. Therefore they have common eigenspaces.

- Define

  $$\Sigma = \text{closure } \left\{ (\zeta, L) \in \mathbb{C}^\times \times \mathbb{C}P^1 \mid L \text{ is an eigenline for } H(\zeta) \right\}$$

- The eigenvalues of $H(\zeta)$ are $\mu(\zeta), \mu(\zeta)^{-1}$. The characteristic polynomial is

  $$\mu^2 - (\text{tr } H)\mu + 1 = 0$$

- Using the compactness of the torus, one can show that $(\text{tr } H)^2 - 4$ vanishes to odd order only finitely many times. The spectral curve is always finite genus for harmonic maps $T^2 \to \mathbb{S}^3$.

- From example

$$\Sigma = \left\{ \left(\zeta, \left[\pm\sqrt{(1 - i\delta)(\zeta - \alpha)} : \sqrt{-(1 + i\bar{\delta})(1 - \bar{\alpha}\zeta)} \right] \right) \right\}$$

for

$$\alpha = \frac{1 + i\delta}{-1 + i\delta}$$

- Can write equation for Σ as

$$\eta^2 = (\zeta - \alpha)(1 - \bar{\alpha}\zeta)$$

The Differentials

- The differentials come from the eigenvalues $\mu(\zeta), \tilde{\mu}(\zeta)$ of $H(\zeta), \tilde{H}(\zeta)$. These functions have essential singularities.

- However $\log \mu, \log \tilde{\mu}$ are holomorphic on $\mathbb{C}^{\times}$ and have simple poles above $\zeta = 0, \infty$.

- $d \log \mu$ removes the additive ambiguity of log. Thus we set $\Theta = d \log \mu$ and $\tilde{\Theta} = d \log \tilde{\mu}$

- In order to recover the eigenvalues, one requires residue free double poles over $\zeta = 0, \infty$ and that the periods of the differentials lie in $2\pi i \mathbb{Z}$.

- The eigenvalues of $H_\tau(\zeta)$ are

$$\mu_\tau(\zeta, \eta) = \exp\left[i\left|1 - i\delta\right|(\tau + \bar\tau\zeta)\eta\zeta^{-1}\right].$$

- The corresponding differential is therefore

$$\Theta_\tau = i\left|1 - i\delta\right| d\left[(\tau + \bar\tau\zeta)\eta\zeta^{-1}\right].$$

- On any given spectral curve, there is a lattice of differentials that may be used in spectral data. Different choices corresponds to coverings of the same image.

Moduli Space $\mathcal{M}_0$

- Every spectral curve in genus zero arises from this class of examples.
- Choice amounts to branch point $\alpha \in D^2$ and choice of pair of differentials from a lattice

$$\mathcal{M}_0 = \coprod D^2$$

- Two dimensional (in contrast to CMC case).

Moduli Space $\mathcal{M}_g$

Theorem
At a point $(\Sigma, \Theta^1, \Theta^2) \in \mathcal{M}_g$ corresponding to a nonconformal harmonic map, if Σ is nonsingular, and Θ^1 and Θ^2 vanish simultaneously at most four times on Σ and never at a ramification point of Σ, then $\mathcal{M}_g$ is a two-dimensional manifold in a neighbourhood of this point.

Theorem
At a point $(\Sigma, \Theta^1, \Theta^2) \in \mathcal{M}_g$ corresponding to a conformal harmonic map, if Σ is nonsingular, and Θ^1 and Θ^2 never vanish simultaneously on Σ then $\mathcal{M}_g$ is a two-dimensional manifold in a neighbourhood of this point.
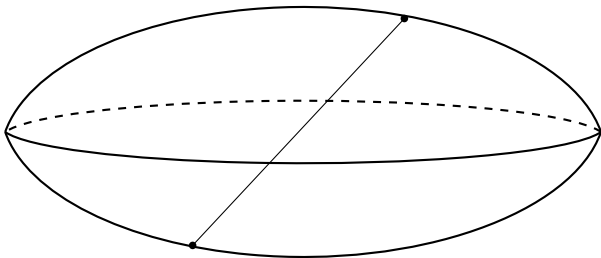
▶ Proof uses Whitham deformations.

Genus One

- ▶ Spectral curves have two pairs of branch points $\alpha, \beta, \overline{\alpha}^{-1}, \overline{\beta}^{-1}$. Let $\mathcal{A}_1 = \left\{ (\alpha, \beta) \in D^2 \times D^2 \mid \alpha \neq \beta \right\}$.
- ▶ Not every genus 1 curve has differentials that meet all the conditions.
- ▶ There is an exact differential Θ^E meeting the closing condition if and only if

$$S(\alpha, \beta) := \frac{|1 - \alpha| \, |1 - \beta|}{|1 + \alpha| \, |1 + \beta|} \in \mathbb{Q}^+$$

▶ Fix a value of $p \in \mathbb{Q}^+$. Let $\mathcal{A}_1(p) = S^{-1}(p)$. It is an open three-ball with a line removed.


▶ Rugby football shaped. Seams are points with both α, β in $\mathbb{S}^1$.

- There is a second differential Θ^P with periods 0 and $2\pi i$. Every differential that meets period conditions is a combination $\mathbb{R}\Theta^E + \mathbb{Z}\Theta^P$.

- Define T, up to periods of Θ^P, by

$$2\pi i T := p \int_{\gamma_-} \Theta^P - \int_{\gamma_+} \Theta^P$$

- A curve admits spectral data if and only if both $S \in \mathbb{Q}^+$ and $T \in \mathbb{Q}$ (and the latter is well-defined).

- The connected components of the space of spectral curves are annuli if $S = 1$ and strips $(0, 1) \times \mathbb{R}$ if $S \neq 1$.

- The connected components of the space of spectral data $\mathcal{M}_1$ are all strips $(0, 1) \times \mathbb{R}$.
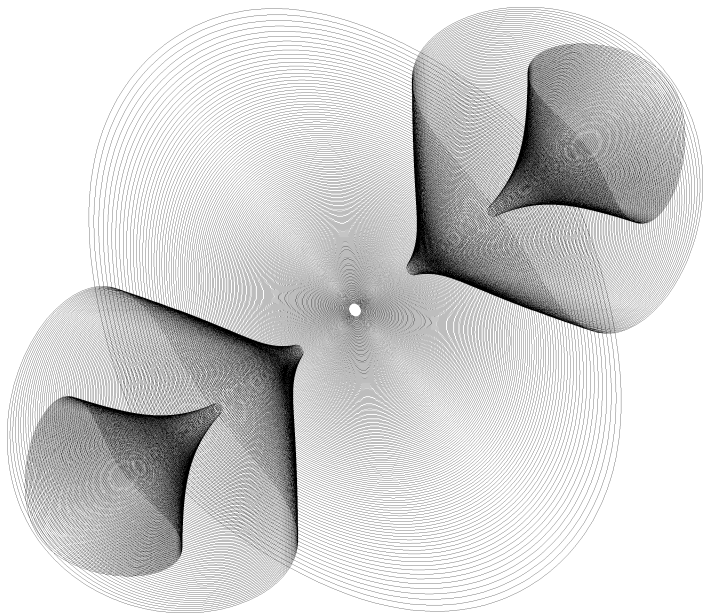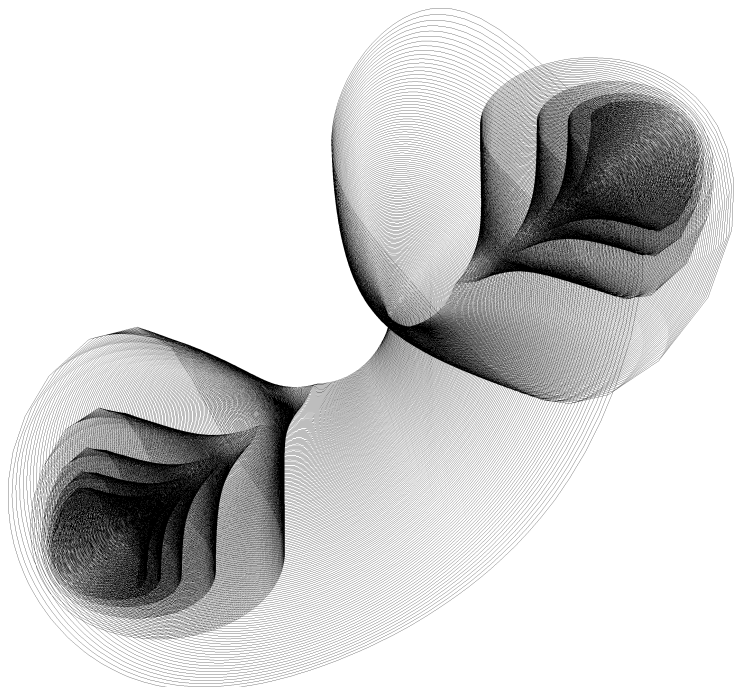
Boundary of $\mathcal{M}_1$

- ▶ Consider boundary where α or β tends to $\mathbb{S}^1$.
- ▶ A singular curve with a double point over the unit circle corresponds to genus zero spectral data via normalisation (blow-up).
- ▶ We can consider $\mathcal{M}_0 \subset \partial\mathcal{M}_1$.
- ▶ Each face of the football $\overline{\mathcal{A}_1(p)}$ is a disc, identified with the space of genus zero spectral curves.
- ▶ Edges of $\overline{\mathcal{A}_1(p)}$ correspond to all branch points on unit circle, ie a map to a circle.

Further questions

- Can we identify geometric properties that parameterise $\mathcal{M}$?
- Is $\mathcal{M}_0 \cup \mathcal{M}_1$ connected? No. What other maps need to be included to make it connected?
- Can one deform a harmonic tori of any spectral degree to one of low genus?
- How does $\mathcal{M}_g$ sit inside the moduli space of harmonic cylinders? Harmonic planes?