

Deformations of Harmonic Maps

Ross Ogilvie

School of Mathematics and Statistics
University of Sydney

June 2017

Harmonic Map

- ▶ Let $f : T^2 \rightarrow S^3 = SU(2)$ be a harmonic map.
- ▶ A harmonic map is a critical point of the energy functional.
- ▶ Long historical interest in minimal and constant curvature surfaces. A surface is CMC iff its Gauss map is harmonic.
- ▶ Thought to be quite rare; Wente (1984) constructed immersed CMC tori.
- ▶ A classification of such maps is given by spectral data $(\Sigma, \Theta, \tilde{\Theta}, E)$ (Hitchin, Pinkall-Sterling, Bobenko).

Spectral Data $(\Sigma, \Theta, \tilde{\Theta}, E)$

- ▶ Spectral curve Σ is a real (possibly singular) hyperelliptic curve,

$$\eta^2 = \prod (\zeta - \alpha_i)(1 - \bar{\alpha}_i \zeta)$$

- ▶ $\Theta, \tilde{\Theta}$ are differentials with double poles and no residues over $\zeta = 0, \infty$.
- ▶ Period conditions: The periods of $\Theta, \tilde{\Theta}$ must lie in $2\pi i\mathbb{Z}$.
- ▶ Closing conditions: for γ_+ a path in Σ between the two points over $\zeta = 1$, and γ_- between the points over $\zeta = -1$ then

$$\int_{\gamma_+} \Theta, \int_{\gamma_-} \Theta, \int_{\gamma_+} \tilde{\Theta}, \int_{\gamma_-} \tilde{\Theta} \in 2\pi i\mathbb{Z}.$$

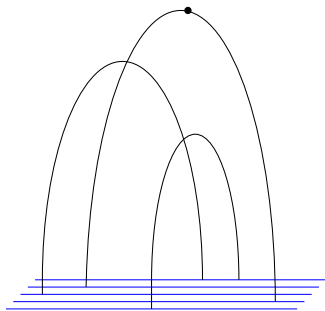
- ▶ E is a quaternionic line bundle of a certain degree.

CMC Spectral Data

- ▶ Compatible spectral curve construction. Very similar conditions.
- ▶ Spectral curve is branched over $\zeta = 0, \infty$.
- ▶ Same symmetry and periodicity conditions on differentials Θ and $\tilde{\Theta}$.
- ▶ Closing conditions apply over points $\zeta = 1, \lambda$ for some $\lambda \in \mathbb{S}^1$.
- ▶ Minimal surfaces = conformal harmonic = CMC with zero mean curvature.

CMC Moduli Space (Kilian-Schmidt-Schmitt)

- ▶ One can vary the line bundle E , so called isospectral deformations.
- ▶ CMC non-isospectral deformations. Maps come in one dimensional families.
- ▶ \mathcal{M}_0^{CMC} is disjoint lines parametrised by $H \in \mathbb{R}$



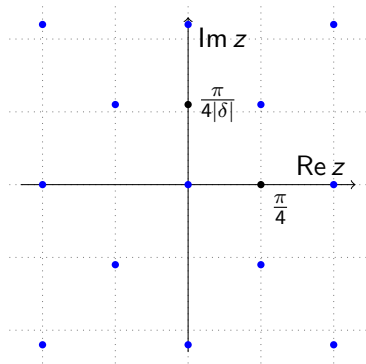
- ▶ Components \mathcal{M}_1^{CMC} end in either \mathcal{M}_0^{CMC} or bouquet of spheres.

Harmonic Map Example

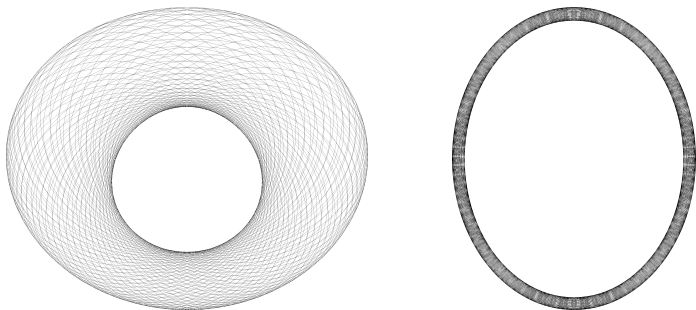
- $f(x + iy) = \exp(-4x\mathbf{X}) \exp(4y\mathbf{Y})$, for

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}, \quad \Im \delta > 0$$

- This map is periodic



- ▶ Formula well-defined on any torus \mathbb{C}/Γ , where Γ is a sublattice of this periodicity lattice.
- ▶ Holding either x or y constant gives circles.
- ▶ As $\delta \rightarrow \mathbb{R}^\times$, image collapses to a circle.
- ▶ As $\delta \rightarrow 0, \infty$, the periodicity lattice degenerates.



Constructing Spectral Data

- ▶ Up to translations, f is determined by the Lie algebra valued map $f^{-1}df$, the pullback of the Maurer-Cartan form.
- ▶ Decompose into its dz and $d\bar{z}$ parts $f^{-1}df = 2(\Phi - \Phi^*)$.
- ▶ Use f to pull back the Levi-Civita connection on $SU(2)$ to get a connection A .
- ▶ Given a pair (Φ, A) , we can make a family of flat $SL(2, \mathbb{C})$ connections. Let $\zeta \in \mathbb{C}^\times$ be the *spectral parameter* and define

$$d_\zeta := d_A + \zeta^{-1}\Phi - \zeta\Phi^*$$

Family of connections is

$$\begin{aligned} d_\zeta &= d - [(\mathbf{X} - i\mathbf{Y}) + \zeta^{-1}(\mathbf{X} + i\mathbf{Y})] dz \\ &\quad - [(\mathbf{X} + i\mathbf{Y}) + \zeta(\mathbf{X} - i\mathbf{Y})] d\bar{z} \\ &= d - \zeta^{-1} [(\mathbf{X} + i\mathbf{Y}) + \zeta(\mathbf{X} - i\mathbf{Y})] [dz + \zeta d\bar{z}] \end{aligned}$$

Holonomy

- ▶ Because the connections are flat, we can define holonomy for them.
- ▶ Pick a base point and generators for the fundamental group, ie take two loops around the torus.
- ▶ Parallel translating vectors with d_ζ around one loop gives a linear map on the tangent space at the base point. Call this $H(\zeta)$. Around the other loop call the transformation $\tilde{H}(\zeta)$.

$$H_\tau(\zeta) = \exp \left\{ \zeta^{-1} [(\mathbf{X} + i\mathbf{Y}) + \zeta(\mathbf{X} - i\mathbf{Y})] [\tau + \zeta\bar{\tau}] \right\}$$

Spectral curve

- ▶ The fundamental group of T^2 is abelian, so H and \tilde{H} commute. Therefore they have common eigenspaces.
- ▶ Define

$$\Sigma = \text{closure } \{(\zeta, L) \in \mathbb{C}^\times \times \mathbb{CP}^1 \mid L \text{ is an eigenline for } H(\zeta)\}$$

- ▶ The eigenvalues of $H(\zeta)$ are $\mu(\zeta), \mu(\zeta)^{-1}$. The characteristic polynomial is

$$\mu^2 - (\text{tr } H)\mu + 1 = 0$$

- ▶ Using the compactness of the torus, one can show that $(\text{tr } H)^2 - 4$ vanishes to odd order only finitely many times. The spectral curve is always finite genus for harmonic maps $T^2 \rightarrow \mathbb{S}^3$.

- From example

$$\Sigma = \left\{ \left(\zeta, \left[\pm \sqrt{(1 - i\delta)(\zeta - \alpha)} : \sqrt{-(1 + i\bar{\delta})(1 - \bar{\alpha}\zeta)} \right] \right) \right\}$$

for

$$\alpha = \frac{1 + i\delta}{-1 + i\delta} \quad \Leftrightarrow \quad \delta = i \frac{1 + \alpha}{1 - \alpha}$$

- Can write equation for Σ as

$$\eta^2 = (\zeta - \alpha)(1 - \bar{\alpha}\zeta)$$

The Differentials

- ▶ The differentials come from the eigenvalues $\mu(\zeta), \tilde{\mu}(\zeta)$ of $H(\zeta), \tilde{H}(\zeta)$. These functions have essential singularities.
- ▶ However $\log \mu, \log \tilde{\mu}$ are holomorphic on \mathbb{C}^\times and have simple poles above $\zeta = 0, \infty$.
- ▶ $d \log \mu$ removes the additive ambiguity of \log . Thus we set $\Theta = d \log \mu$ and $\tilde{\Theta} = d \log \tilde{\mu}$
- ▶ In order to recover the eigenvalues, one requires residue free double poles over $\zeta = 0, \infty$ and that the periods of the differentials lie in $2\pi i\mathbb{Z}$.

- ▶ The eigenvalues of $H_\tau(\zeta)$ are

$$\mu_\tau(\zeta, \eta) = \exp \left[i |1 - i\delta| (\tau + \bar{\tau}\zeta) \eta \zeta^{-1} \right] .$$

- ▶ The corresponding differential is therefore

$$\Theta_\tau = i |1 - i\delta| d \left[(\tau + \bar{\tau}\zeta) \eta \zeta^{-1} \right] .$$

- ▶ On any given spectral curve, there is a lattice of differentials that may be used in spectral data. Different choices corresponds to coverings of the same image.

Moduli Space \mathcal{M}_0

- ▶ Every spectral curve in genus zero arises from this class of examples.
- ▶ Choice amounts to branch point $\alpha \in D^2$ and choice of pair of differentials from a lattice

$$\mathcal{M}_0 = \coprod_{\text{Mat}_2^* \mathbb{Z}} D^2$$

- ▶ Image degenerates: $\delta \rightarrow \mathbb{R}^\times \Leftrightarrow \alpha \rightarrow \mathbb{S}^1 \setminus \{\pm 1\}$.
- ▶ Lattice degenerates: $\delta \rightarrow 0, \infty \Leftrightarrow \alpha \rightarrow \pm 1$.
- ▶ Two dimensional (in contrast to CMC case).

Moduli Space \mathcal{M}_g

Theorem

At a point $(\Sigma, \Theta^1, \Theta^2) \in \mathcal{M}_g$ corresponding to a nonconformal harmonic map, if Σ is nonsingular, and Θ^1 and Θ^2 vanish simultaneously at most four times on Σ and never at a ramification point of Σ , then \mathcal{M}_g is a two-dimensional manifold in a neighbourhood of this point.

Theorem

At a point $(\Sigma, \Theta^1, \Theta^2) \in \mathcal{M}_g$ corresponding to a conformal harmonic map, if Σ is nonsingular, and Θ^1 and Θ^2 never vanish simultaneously on Σ then \mathcal{M}_g is a two-dimensional manifold in a neighbourhood of this point.

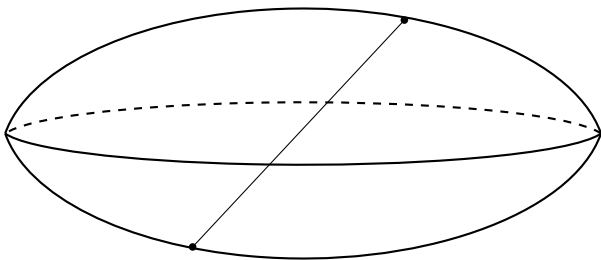
- Proof uses Whitham deformations.

Genus One

- ▶ Spectral curves have two pairs of branch points $\alpha, \beta, \bar{\alpha}^{-1}, \bar{\beta}^{-1}$. Let $\mathcal{A}_1 = \{(\alpha, \beta) \in D^2 \times D^2 \mid \alpha \neq \beta\}$.
- ▶ Not every spectral curve has differentials that meet all the conditions.
- ▶ There is always an exact differential Θ^E that meets all conditions except closing condition.
- ▶ A multiple of Θ^E meets the closing condition if and only if

$$S(\alpha, \beta) := \frac{|1 - \alpha| |1 - \beta|}{|1 + \alpha| |1 + \beta|} \in \mathbb{Q}^+$$

- Fix a value of $p \in \mathbb{Q}^+$. Let $\mathcal{A}_1(p) = S^{-1}(p)$. It is an open three-ball with a line removed.

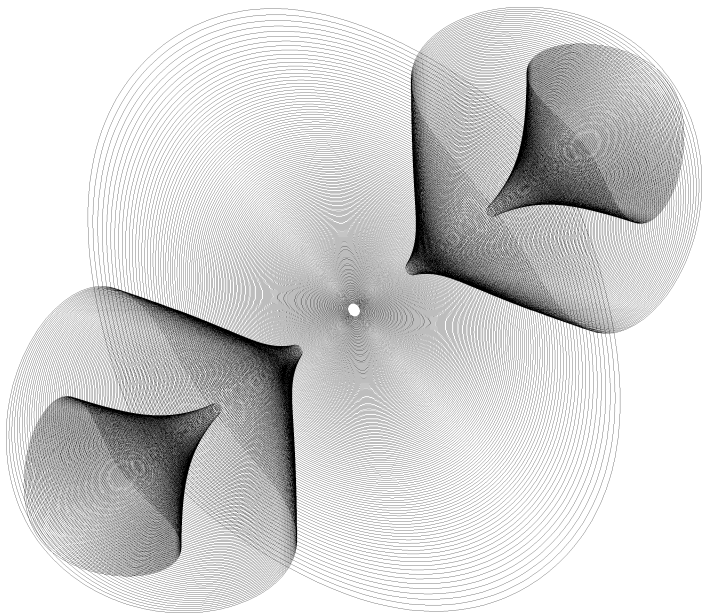


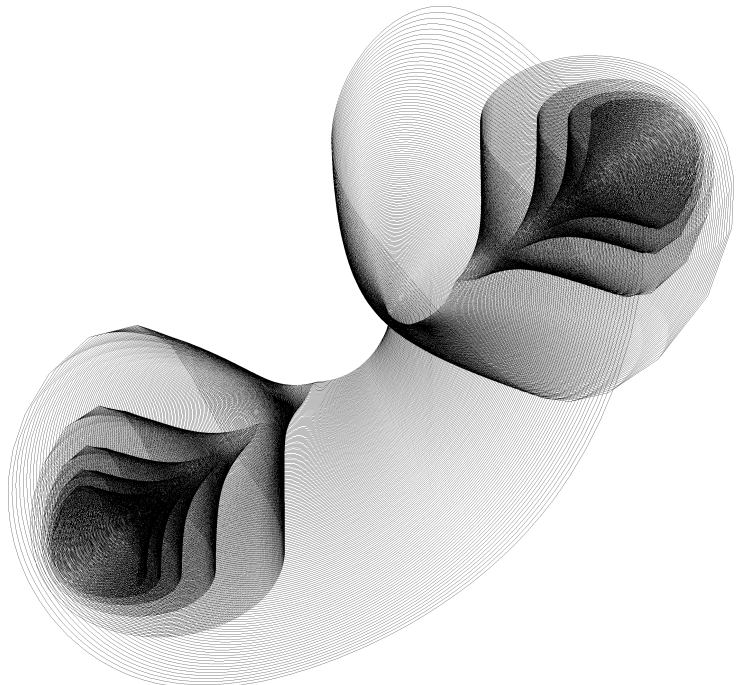
- Australian football shaped. Ends are $(\alpha, \beta) = (1, -1), (-1, 1)$. Seams are points with both α, β in \mathbb{S}^1 .

- ▶ There is a second differential Θ^P with periods 0 and $2\pi i$. Every differential that meets period conditions is a combination $\mathbb{R}\Theta^E + \mathbb{Z}\Theta^P$.
- ▶ Define T , up to periods of Θ^P , by

$$2\pi iT := p \int_{\gamma_-} \Theta^P - \int_{\gamma_+} \Theta^P$$

- ▶ A curve admits spectral data if and only if both $S \in \mathbb{Q}^+$ and $T \in \mathbb{Q}$ (and the latter is well-defined).
- ▶ The connected components of the space of spectral curves are annuli if $S = 1$ and strips $(0, 1) \times \mathbb{R}$ if $S \neq 1$.
- ▶ The connected components of the space of spectral data \mathcal{M}_1 are all strips $(0, 1) \times \mathbb{R}$.





Method of Proof

- ▶ Move to the universal cover of the parameter space

$$\pi_p : \tilde{\mathcal{A}}_1(p) \rightarrow \mathcal{A}_1(p).$$

- ▶ Define a function \tilde{T} on $\tilde{\mathcal{A}}_1(p)$ such that $\tilde{T} = T \circ \pi_p$.
- ▶ In the right coordinates, the level sets of \tilde{T} are graphs over $(0, 1) \times \mathbb{R}$.
- ▶ Quotient by deck transformations to recover space of spectral curves.

Interior Boundary \mathcal{M}_1

- ▶ $\mathcal{M}_1 \cap \mathcal{A}_1(p)$ spirals around the diagonal line $\{\alpha = \beta\} \cap \mathcal{A}_1(p)$.
- ▶ Just a single point on this diagonal line is reachable along a finite path.
- ▶ This limit seems not to be well-defined.

Exterior Boundary \mathcal{M}_1

- ▶ This boundary is where α or β tends to \mathbb{S}^1 .
- ▶ A singular curve with a double point over the unit circle corresponds to genus zero spectral data via normalisation (blow-up).
- ▶ We can consider $\mathcal{M}_0 \subset \partial\mathcal{M}_1$.
- ▶ Each face of the football $\overline{\mathcal{A}_1(p)}$ is a disc, identified with the space of genus zero spectral curves.
- ▶ Edges of $\overline{\mathcal{A}_1(p)}$ correspond to all branch points on unit circle, ie a map to a circle.

Further questions

- ▶ Can we identify geometric properties that parameterise \mathcal{M} ?
- ▶ Is $\mathcal{M}_0 \cup \mathcal{M}_1$ connected? No. What other maps need to be included to make it connected?
- ▶ Can one deform from a harmonic map to a circle to any harmonic map of a torus? ie what about $\mathcal{M} = \bigcup \mathcal{M}_g$?
- ▶ How does \mathcal{M} sit inside the moduli space of harmonic cylinders? Harmonic planes?
- ▶ What deformations lead to topological changes of the image of the harmonic map?