Deformations of Harmonic Maps

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Harmonic Map

- Let $f : T^2 \to S^3 = SU(2)$ be a harmonic map.
- A harmonic map is a critical point of the energy functional.
- Long historical interest in minimal and constant curvature surfaces. A surface is CMC iff its Gauss map is harmonic.
- Thought to be quite rare; Hopf Conjecture. Wente (1984) constructed immersed CMC tori.
- A classification of such maps is given by spectral data $(\Sigma, \Theta, \tilde{\Theta}, E)$ (Hitchin, Pinkall-Sterling, Bobenko).

Spectral Data $(\Sigma, \Theta, \tilde{\Theta}, E)$

- Spectral curve Σ is a real (possibly singular) hyperelliptic curve,

$$\eta^2 = \prod (\zeta - \alpha_i)(1 - \bar{\alpha}_i \zeta)$$

- $\Theta, \tilde{\Theta}$ are differentials with double poles and no residues over $\zeta = 0, \infty$.
- Period conditions: The periods of $\Theta, \tilde{\Theta}$ must lie in $2\pi i \mathbb{Z}$.
- Closing conditions: for γ_+ a path in Σ between the two points over $\zeta = 1$, and γ_- between the points over $\zeta = -1$ then

$$\int_{\gamma_+} \Theta, \int_{\gamma_-} \Theta, \int_{\gamma_+} \tilde{\Theta}, \int_{\gamma_-} \tilde{\Theta} \in 2\pi i \mathbb{Z}.$$

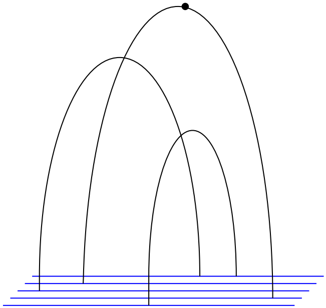
- E is a quaternionic line bundle of a certain degree.

CMC Spectral Data

- Compatible spectral curve construction. Very similar conditions.
- Spectral curve is branched over $\zeta = 0, \infty$.
- Same symmetry and periodicity conditions on differentials Θ and $\tilde{\Theta}$.
- Closing conditions apply over points $\zeta = 1, \lambda$ for some $\lambda \in \mathbb{S}^1$.
- Minimal surfaces = conformal harmonic = CMC with zero mean curvature.

CMC Moduli Space (Kilian-Schmidt-Schmitt)

- One can vary the line bundle E, so called isospectral deformations.
- CMC non-isospectral deformations. Maps come in one dimensional families.
- $\mathcal{M}_0^{CMC}$ is disjoint lines parametrised by $H \in \mathbb{R}$



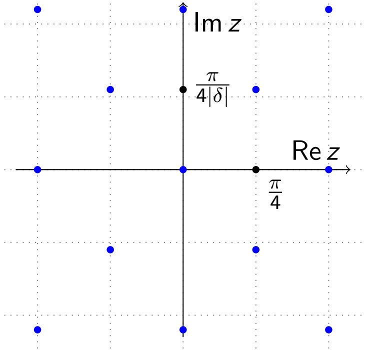
- Components $\mathcal{M}_1^{CMC}$ end in either $\mathcal{M}_0^{CMC}$ or bouquet of spheres.

Harmonic Map Example

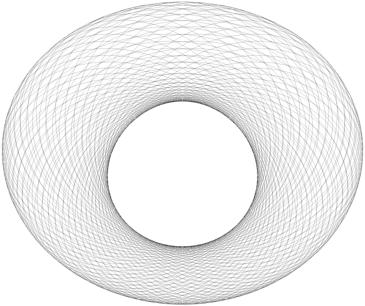
- $f(x+iy) = \exp(-4x\mathbf{X})\exp(4y\mathbf{Y})$, for

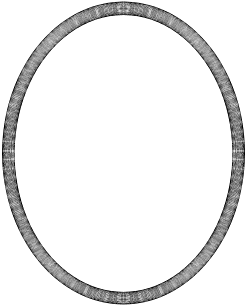
$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix}, \qquad \Im\mathrm{m}\,\delta > 0$$

- This map is periodic



- Formula well-defined on any torus $\mathbb{C}/\Gamma$, where Γ is a sublattice of this periodicity lattice.
- Holding either x or y constant gives circles.
- As $\delta \to \mathbb{R}^\times$, image collapses to a circle.
- As $\delta \to 0, \infty$, the periodicity lattice degenerates.





Constructing Spectral Data

- Up to translations, f is determined by the Lie algebra valued map $f^{-1}df$, the pullback of the Mauer-Cartan form.
- Decompose into its dz and $d\bar{z}$ parts $f^{-1}df = 2(\Phi - \Phi^*)$.
- Use f to pull pack the Levi-Civita connection on $SU(2)$ to get a connection A.
- Given a pair (Φ, A), we can make a family of flat $SL(2,\mathbb{C})$ connections. Let $\zeta \in \mathbb{C}^\times$ be the *spectral parameter* and define

$$d_\zeta := d_A + \zeta^{-1}\Phi - \zeta\Phi^*$$

Family of connections is

$$\begin{aligned} d_\zeta &= d - \left[(\mathbf{X} - i\mathbf{Y}) + \zeta^{-1}(\mathbf{X} + i\mathbf{Y})\right] dz \\ &\quad - \left[(\mathbf{X} + i\mathbf{Y}) + \zeta(\mathbf{X} - i\mathbf{Y})\right] d\bar{z} \\ &= d - \zeta^{-1}\left[(\mathbf{X} + i\mathbf{Y}) + \zeta(\mathbf{X} - i\mathbf{Y})\right]\left[dz + \zeta d\bar{z}\right] \end{aligned}$$

Holonomy

- Because the connections are flat, we can define holonomy for them.
- Pick a base point and generators for the fundamental group, ie take two loops around the torus.
- Parallel translating vectors with d_ζ around one loop gives a linear map on the tangent space at the base point. Call this $H(\zeta)$. Around the other loop call the transformation $\tilde{H}(\zeta)$.

$$H_\tau(\zeta) = \exp\left\{\zeta^{-1}\left[(\mathbf{X} + i\mathbf{Y}) + \zeta(\mathbf{X} - i\mathbf{Y})\right]\left[\tau + \zeta\bar{\tau}\right]\right\}$$

Spectral curve

- The fundamental group of T^2 is abelian, so H and $\tilde{H}$ commute. Therefore they have common eigenspaces.
- Define

$$\Sigma = \text{closure } \left\{ (\zeta, L) \in \mathbb{C}^\times \times \mathbb{C}\mathrm{P}^1 \mid L \text{ is an eigenline for } H(\zeta) \right\}$$

- The eigenvalues of $H(\zeta)$ are $\mu(\zeta), \mu(\zeta)^{-1}$. The characteristic polynomial is

$$\mu^2 - (\operatorname{tr} H)\mu + 1 = 0$$

- Using the compactness of the torus, one can show that $(\operatorname{tr} H)^2 - 4$ vanishes to odd order only finitely many times. The spectral curve is always finite genus for harmonic maps $T^2 \to \mathbb{S}^3$.

- From example

$$\Sigma = \left\{ \left(\zeta, \left[\pm\sqrt{(1-i\delta)(\zeta-\alpha)} : \sqrt{-(1+i\bar{\delta})(1-\bar{\alpha}\zeta)} \right] \right) \right\}$$

for

$$\alpha = \frac{1+i\delta}{-1+i\delta} \qquad \Leftrightarrow \qquad \delta = i\frac{1+\alpha}{1-\alpha}$$

- Can write equation for Σ as

$$\eta^2 = (\zeta-\alpha)(1-\bar{\alpha}\zeta)$$

The Differentials

- The differentials come from the eigenvalues $\mu(\zeta), \tilde{\mu}(\zeta)$ of $H(\zeta), \tilde{H}(\zeta)$. These functions have essential singularities.
- However $\log \mu, \log \tilde{\mu}$ are holomorphic on $\mathbb{C}^\times$ and have simple poles above $\zeta = 0, \infty$.
- $d \log \mu$ removes the additive ambiguity of log. Thus we set $\Theta = d \log \mu$ and $\tilde{\Theta} = d \log \tilde{\mu}$
- In order to recover the eigenvalues, one requires residue free double poles over $\zeta = 0, \infty$ and that the periods of the differentials lie in $2\pi i \mathbb{Z}$.

- The eigenvalues of $H_\tau(\zeta)$ are
$$\mu_\tau(\zeta,\eta) = \exp\left[i\,|1-i\delta|\,(\tau+\bar{\tau}\zeta)\eta\zeta^{-1}\right].$$
- The corresponding differential is therefore
$$\Theta_\tau = i\,|1-i\delta|\; d\left[(\tau+\bar{\tau}\zeta)\eta\zeta^{-1}\right].$$

- On any given spectral curve, there is a lattice of differentials that may be used in spectral data. Different choices corresponds to coverings of the same image.

Moduli Space $\mathcal{M}_0$

- Every spectral curve in genus zero arises from this class of examples.
- Choice amounts to branch point $\alpha \in D^2$ and choice of pair of differentials from a lattice

$$\mathcal{M}_0 = \coprod D^2$$

- Image degenerates: $\delta \to \mathbb{R}^\times \quad \Leftrightarrow \quad \alpha \to \mathbb{S}^1 \setminus \{\pm 1\}$.
- Lattice degenerates: $\delta \to 0, \infty \quad \Leftrightarrow \quad \alpha \to \pm 1$.
- Two dimensional (in contrast to CMC case).

Moduli Space $\mathcal{M}_g$

Theorem

At a point $(\Sigma, \Theta^1, \Theta^2) \in \mathcal{M}_g$ corresponding to a nonconformal harmonic map, if Σ is nonsingular, and Θ^1 and Θ^2 vanish simultaneously at most four times on Σ and never at a ramification point of Σ, then $\mathcal{M}_g$ is a two-dimensional manifold in a neighbourhood of this point.

Theorem

At a point $(\Sigma, \Theta^1, \Theta^2) \in \mathcal{M}_g$ corresponding to a conformal harmonic map, if Σ is nonsingular, and Θ^1 and Θ^2 never vanish simultaneously on Σ then $\mathcal{M}_g$ is a two-dimensional manifold in a neighbourhood of this point.

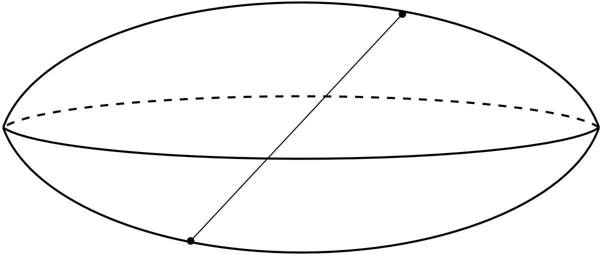
- Proof uses Whitham deformations.

Genus One

- Spectral curves have two pairs of branch points $\alpha, \beta, \overline{\alpha}^{-1}, \overline{\beta}^{-1}$. Let $\mathcal{A}_1 = \{(\alpha, \beta) \in D^2 \times D^2 \mid \alpha \neq \beta\}$.
- Not every spectral curve has differentials that meet all the conditions.
- There is always an exact differential Θ^E that meets all conditions except closing condition.
- A multiple of Θ^E meets the closing condition if and only if

$$S(\alpha, \beta) := \frac{|1-\alpha|\,|1-\beta|}{|1+\alpha|\,|1+\beta|} \in \mathbb{Q}^+$$

- Fix a value of $p \in \mathbb{Q}^+$. Let $\mathcal{A}_1(p) = S^{-1}(p)$. It is an open three-ball with a line removed.

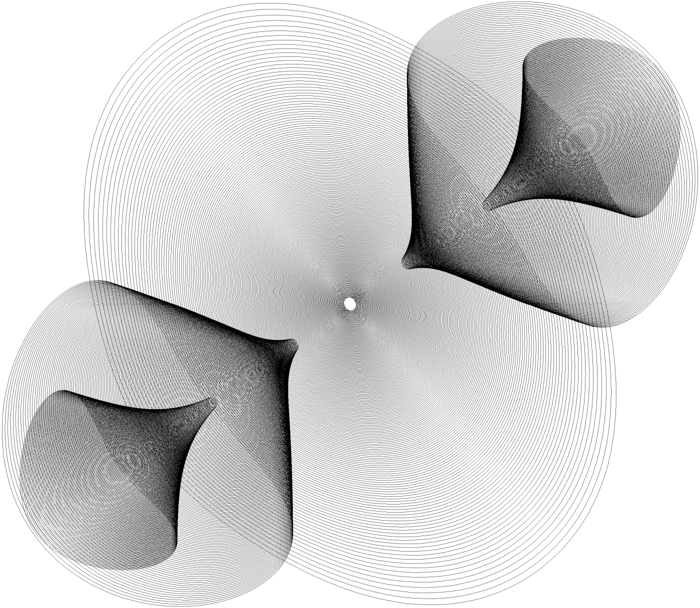


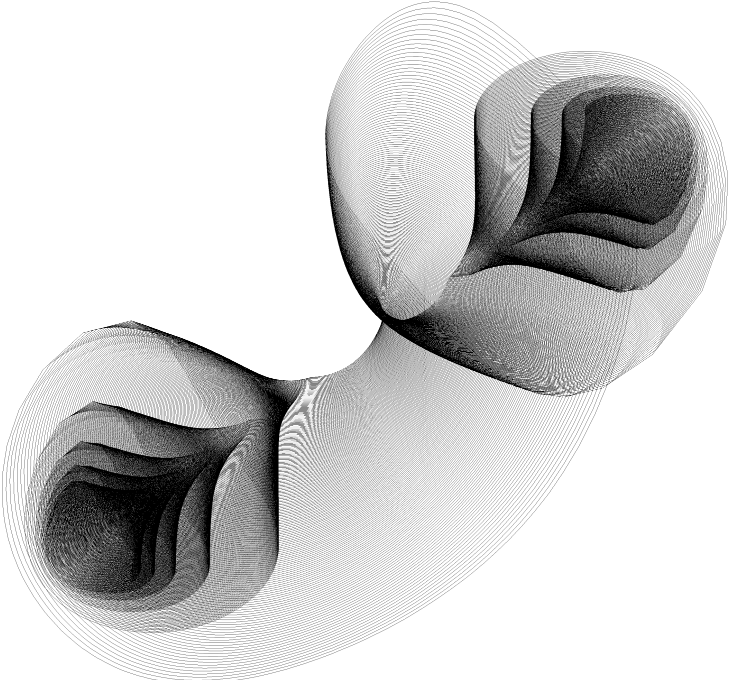
- Rugby football shaped. Ends are $(\alpha, \beta) = (1, -1), (-1, 1)$. Seams are points with both α, β in $\mathbb{S}^1$.

- There is a second differential Θ^P with periods 0 and $2\pi i$. Every differential that meets period conditions is a combination $\mathbb{R}\Theta^E + \mathbb{Z}\Theta^P$.
- Define T, up to periods of Θ^P, by

$$2\pi i T := p \int_{\gamma_-} \Theta^P - \int_{\gamma_+} \Theta^P$$

- A curve admits spectral data if and only if both $S \in \mathbb{Q}^+$ and $T \in \mathbb{Q}$ (and the latter is well-defined).
- The connected components of the space of spectral curves are annuli if $S = 1$ and strips $(0,1) \times \mathbb{R}$ if $S \neq 1$.
- The connected components of the space of spectral data $\mathcal{M}_1$ are all strips $(0,1) \times \mathbb{R}$.





Method of Proof

- Move to the universal cover of the parameter space

$$\pi_p : \tilde{\mathcal{A}}_1(p) \to \mathcal{A}_1(p).$$

- Define a function $\tilde{T}$ on $\tilde{\mathcal{A}}_1(p)$ such that $\tilde{T} = T \circ \pi_p$.
- In the right coordinates, the level sets of $\tilde{T}$ are graphs over $(0,1) \times \mathbb{R}$.
- Quotient by deck transformations to recover space of spectral curves.
- Consider how the lattice of differentials change as you change the spectral curve.

Interior Boundary $\mathcal{M}_1$

- $\mathcal{M}_1 \cap \mathcal{A}_1(p)$ spirals around the diagonal line $\{\alpha = \beta\} \cap \mathcal{A}_1(p)$.
- Just a single point on this diagonal line is reachable along a finite path.
- This limit seems not to be well-defined.

Exterior Boundary $\mathcal{M}_1$

- This boundary is where α or β tends to $\mathbb{S}^1$.
- A singular curve with a double point over the unit circle corresponds to genus zero spectral data via normalisation (blow-up).
- We can consider $\mathcal{M}_0 \subset \partial\mathcal{M}_1$.
- Each face of the football $\overline{\mathcal{A}_1(p)}$ is a disc, identified with the space of genus zero spectral curves.
- Edges of $\overline{\mathcal{A}_1(p)}$ correspond to all branch points on unit circle, ie a map to a circle.

Further questions

- Can we identify geometric properties that parameterise $\mathcal{M}$?
- Is $\mathcal{M}_0 \cup \mathcal{M}_1$ connected? No. What other maps need to be included to make it connected?
- Can one deform from a harmonic map to a circle to any harmonic map of a torus? ie what about $\mathcal{M} = \bigcup \mathcal{M}_g$?
- How does $\mathcal{M}$ sit inside the moduli space of harmonic cylinders? Harmonic planes?
- What deformations lead to topological changes of the image of the harmonic map?