Deformations of Harmonic Maps

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The spaces

- A torus is a doughnut.

- A 3-sphere $\mathbb{S}^4$ is unit sphere in $\mathbb{R}^4$ or $\mathbb{C}^2$. For us, we will think of it as $SU(2)$, which looks like

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \text{ with } \alpha\bar{\alpha} + \beta\bar{\beta} = 1$$

Harmonic Map

- Let $f : T^2 \to SU(2)$. Use f to pull pack the connection on $SU(2)$ to get a connection A. Automatically we have $d_A(df) = 0$.
- A harmonic map is one that minimises the energy. It must satisfy $d_A^*(df) = 0$.
- Translate the values of df so it is a map to the Lie algebra, and split it up as $\frac{1}{2} f^{-1} df = \Phi - \Phi^*$.
- We can then rearrange our two equations to be about A and Φ.

$$d_A^{''} \Phi = 0, \qquad\qquad F_A = [\Phi, \Phi^*]$$

Φ is called the Higgs field.

Family of Flat Connections

- Given a pair (Φ, A), we can make a family of connections in the following way. Let $\zeta \in \mathbb{C}^\times$ and define

$$\nabla_\zeta := \nabla_A + \zeta^{-1}\Phi - \zeta\Phi^*$$

- Every connection in this family is flat (zero curvature) since

$$F = (d_A + \zeta^{-1}\Phi - \zeta\Phi^*)^2 = d_A^2 - [\Phi, \Phi^*] = 0$$

- The connections are $SL(2, \mathbb{C})$ valued. When $\zeta = 1, -1$ the result is the left and right connections on $SU(2)$

Holonomy

- Because the connections are flat, we can define holonomy for them. Pick a base point, and take two loops around the torus.
- Parallel translating a vector with ∇_ζ around one loop gives a linear map the vector space at the base point. Call this $H(\zeta)$. Around the other loop call the transformation $\tilde{H}(\zeta)$.
- With some analysis and the fact that these two matrices commute (because the fundamental group of the torus is abelian), one can show that $(\operatorname{tr} H)^2 - 4$ (a function in ζ) vanishes to odd order only finitely many times.

Spectral Curve

- The eigenvalues of $H(\zeta)$ are $\mu(\zeta), \mu(\zeta)^{-1}$. The characteristic polynomial is
$$\mu^2 - (\operatorname{tr} H)\mu + 1 = 0$$
- What we want to do is to build a curve over $\zeta \in \mathbb{C}^\times$ where μ is single valued.
- We can use the characteristic equation to define a two sheeted cover of $\mathbb{C}^\times$ with finite branching.
- Looking at the behaviour of μ as $\zeta \to 0$ (ditto $\zeta \to \infty$), we can fill in the missing points to make a two sheeted cover Σ of $\mathbb{C}\mathrm{P}^1$, called the spectral curve.
- It is an algebraic curve, with several nice additional structures.

The Differentials

- The spectral curve provides a natural setting to talk about the eigenvalues, but they are not so nice themselves.
- However, $\log \mu, \log \tilde{\mu}$ are holomorphic on $\mathbb{C}^{\times}$, have simple poles above $\zeta = 0, \infty$ and nowhere else, but are only locally defined.
- $d \log \mu$ removes the additive ambiguity of log. Thus we consider a pair of meromorphic differentials $\Theta = d \log \mu, \tilde{\Theta} = \log \tilde{\mu}$, which have residue free double poles over $\zeta = 0, \infty$.
- In order for the eigenvalues to be well defined, one requires that the periods of the differentials lie in $2\pi i \mathbb{Z}$.

Spectral Data

- We now have a curve Σ and a pair of differentials $\Theta, \tilde{\Theta}$. There is one more piece of information needed: a line bundle.
- Then a theorem of Hitchin gives conditions for such data to correspond to a harmonic map.
- An important part of this result is that for fixed $(\Sigma, \Theta, \tilde{\Theta})$ we can choose from many line bundles. These are called isospectral deformations.

Genus Zero example

- Take the spectral curve to be genus zero. By the various symmetric constraints, its two branch points must be a pair $\alpha, \overline{\alpha}^{-1}$, with $\alpha \in D$

$$\eta^2 = P(\zeta) = (\zeta - \alpha)(1 - \bar{\alpha}\zeta)$$

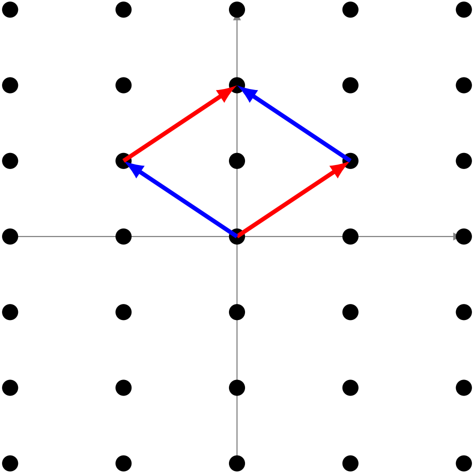
- All differentials on it are exact, and for the Θ's to have the right poles they must be of the form

$$\log \mu = (b\zeta^{-1} - \bar{b})\eta, \qquad b \in \mathbb{C}$$

- The constraints to be a harmonic map further imply that

$$b = \frac{\pi}{2}\left(\frac{n}{|1+\alpha|} + i\frac{m}{|1-\alpha|}\right), \qquad n, m \in \mathbb{Z}$$

Genus Zero Example

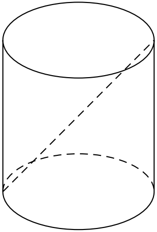


Moduli Space

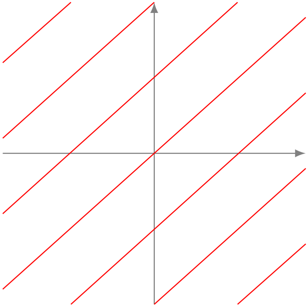
- In general, Σ of genus g curve can be described by $g+1$ pair of conjugate inverse points.
- The differentials become much harder to pin down. However they must be of the form
$$\Theta = \frac{d\zeta}{\zeta^2\eta} b(\zeta)$$
for some polynomials of degree $g+3$. If you just require symmetries and imaginary periods, there is a plane of differentials.
- Whitham deformations tell us that the space of $(\Sigma, \Theta, \tilde{\Theta})$ is 2-dimensional.

Genus One

- Spectral curves have two pairs of branch points $\alpha, \beta, \overline{\alpha}^{-1}, \overline{\beta}^{-1}$, $(\alpha, \beta) \in D \times D \setminus \Delta$.



- Period conditions mean that there are a collection of lines of differentials, one for each $2\pi i\mathbb{Z}$.



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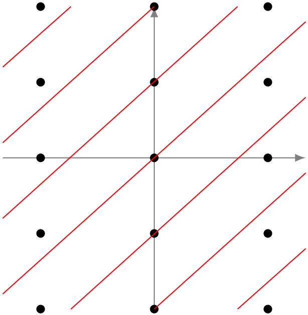

- Not every spectral curve has differentials that meet all the conditions.
- As one traverses around the diagonal, the lattice shifts.

- On the exterior boundary, where one branch point lies in $\mathbb{S}^1$, normalisation of the spectral curve correspond to genus zero spectral data.