HARMONIC MAPS FROM T^2 TO S^3

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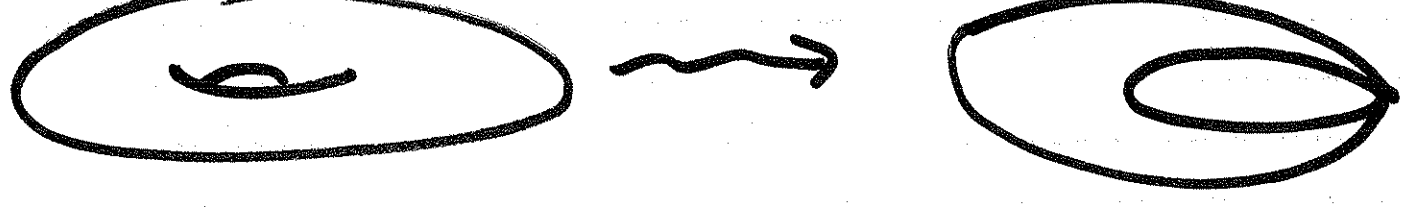
- Hitchen classified harmonic maps from $T^2 \to S^3$.
- Constructed a spectral curve
- Shows how harmonic maps can be deformed
- A pinched torus (aka nodal curve) is a degeneration of the torus



$$f : T^2 \longrightarrow S^3 = SU(2)$$

$$d_A \, df = 0$$

$$d_A^* \, df = 0$$

write $\frac{1}{2} f^{-1} df = \Phi - \Phi^*$

$$\left.\begin{aligned} d_A'' \Phi &= 0 \\ F_A &= [\Phi, \Phi^*] \end{aligned}\right\} \text{Harmonic Map Equations}$$

Sol^n is a pair (A, Φ)

Family of Flat Connections

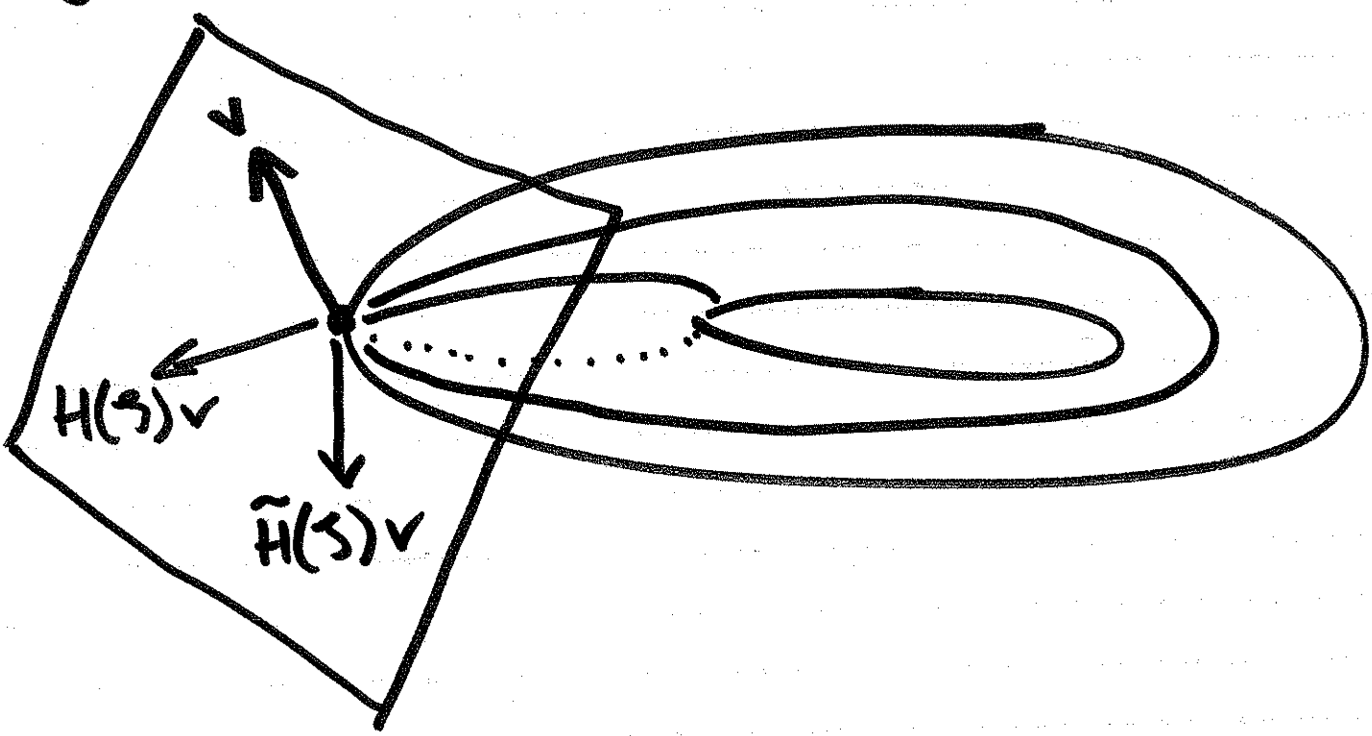
$$\nabla_\zeta = \nabla_A + \zeta^{-1}\Phi - \zeta\Phi^*$$

for $\zeta \in \mathbb{C}^\times$

Are $SL(2,\mathbb{C})$-connections, so

$$\pi_1(T^2) \longrightarrow SL(2,\mathbb{C})$$

given by the holonomy representation



Look at eigenvalues

- $H, \tilde{H}$ commute

$$\Rightarrow$$

common $\mu, \tilde{\mu}$

- PDE theory

$$\Rightarrow$$

$(\mathrm{tr}\, H)^2 - 4$ has finitely many odd order zeroes

$$\mu^2 - \mathrm{tr} H\, \mu + 1 = 0$$

- $\Theta = \dfrac{d\mu}{\mu}$ has double poles w/ no residues @ $\zeta = 0, \infty$

Make double cover $\hat{\Sigma}$ for μ to live on.

But extra info in the eigenspace bundle. Add in extra zeroes to get

$$\Sigma = \{ \eta^2 = P(\zeta) \}$$

Spectral curve.

Thm (Hitchin '90)
From $(\Sigma, \Theta, \tilde{\Theta})$ there is a sol$^{\underline{n}}$ to the harmonic map eq^n for every quaternionic line bundle

Moreover,

i) get harmonic map

$\Longleftrightarrow$

$\forall \xi \in \pi^{-1}(1,-1)$

$$\mu(\xi) = \tilde{\mu}(\xi) = 1$$

ii) conformal $\Longleftrightarrow$ $P(\theta) = 0$

iii) Maps to 2-sphere $\subset S^3$

$\Longleftrightarrow$

bunch of stuff on $(\Sigma, \theta, \bar{\theta})$

Simplest case : genus 0.

$$\Sigma = \{ \eta^2 = -\bar{\alpha}\zeta^2 + (|\alpha|^2+1)\zeta - \alpha \}$$

for pair of branch pts $\alpha, \bar{\alpha}^{-1}$

$$\Rightarrow \log \mu = (b - \bar{b}\zeta^{-1})\eta + \pi i k$$

$b \in \mathbb{C}$, $k \in \mathbb{Z}$

To get actual map, need

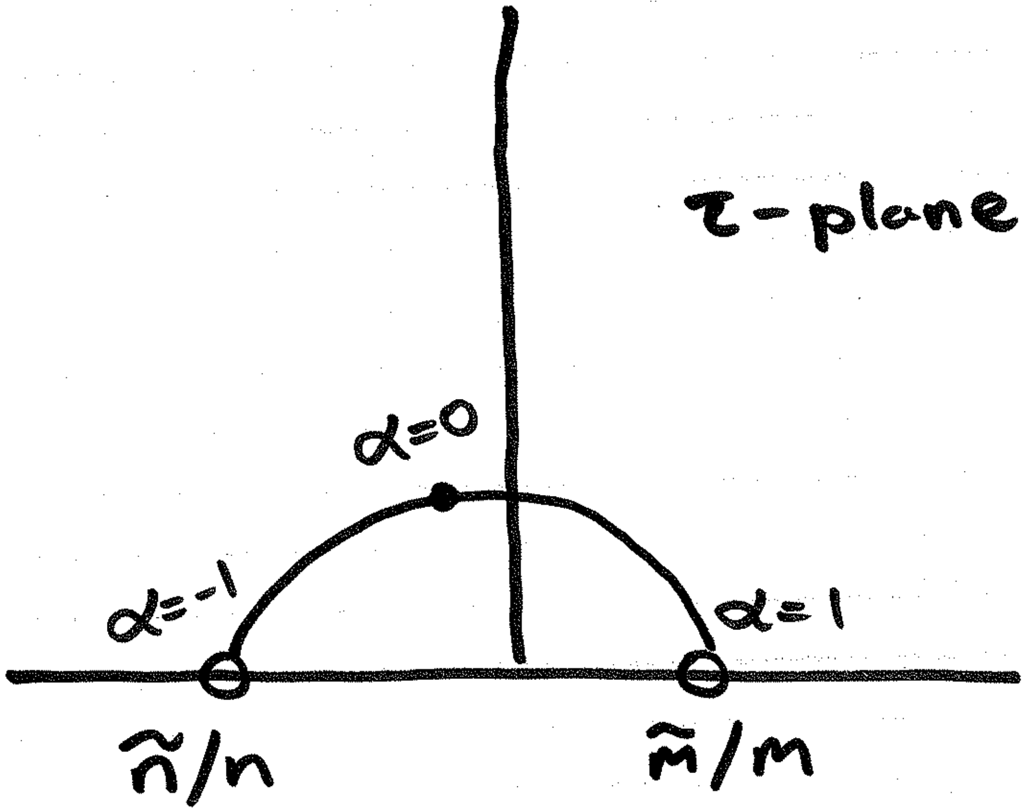
$$b = \frac{\pi}{2}\left(\frac{n}{r} + i\frac{m}{s}\right)$$

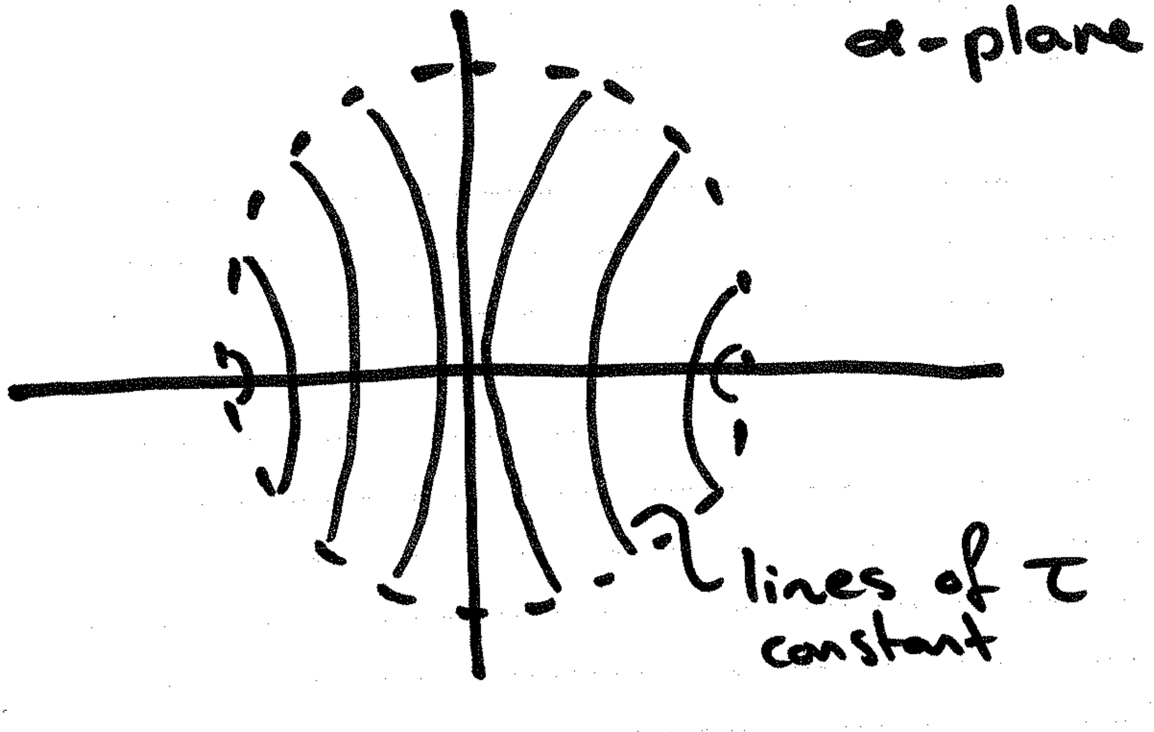
r, s functions of α

n, m same parity as k.

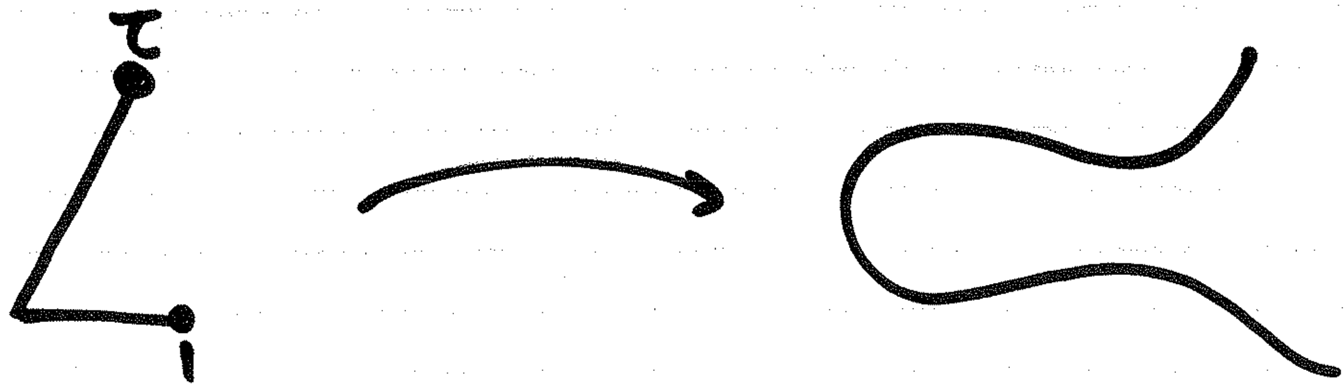
What conformal structure?

$$\tau = \overline{\tilde{b}/b}$$

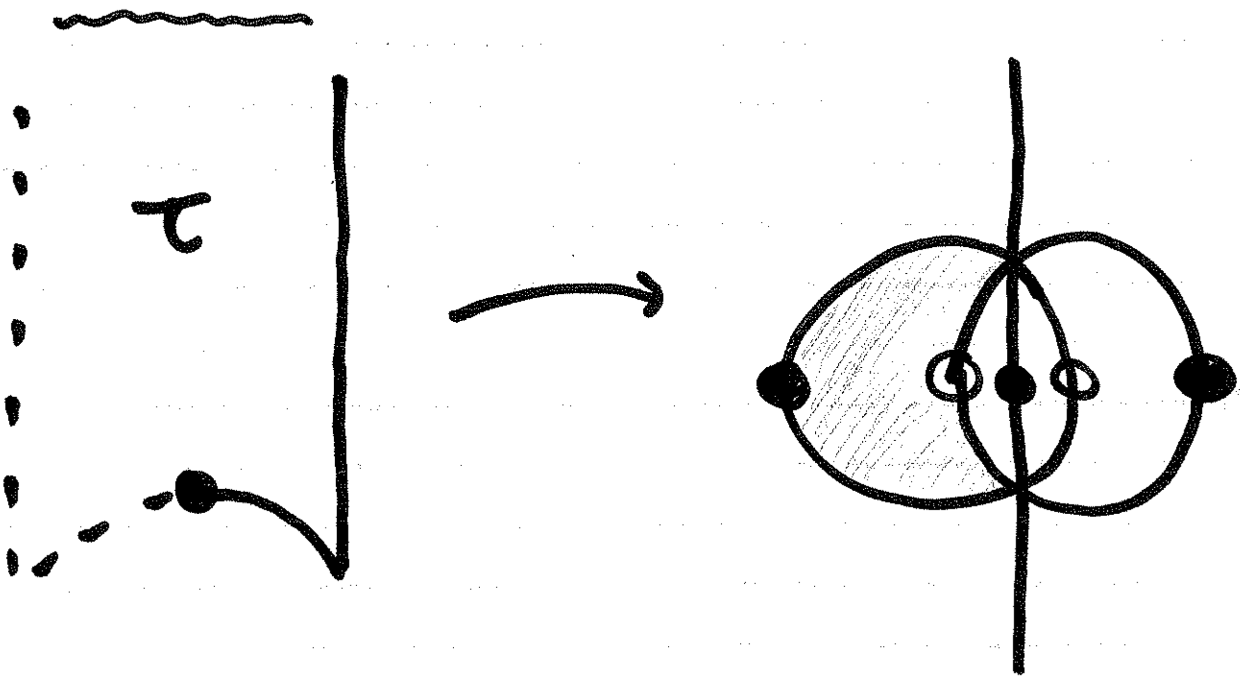




What conformal structure does a pinched torus have?



$\mathbb{C}/\mathbb{Z}\{1,\tau\}$ $\qquad$ $y^2 = x(x-1)(x-\lambda)$



$$\lambda \to 0 \iff \tau \to \infty$$

Take $m = \tilde{n} = 0$

$\alpha = 1 - \varepsilon$ $\quad$ ε real, small

Write $\Phi = \begin{pmatrix} c & p \\ p & -c \end{pmatrix} dz = \phi \, dz$

$p \in \mathbb{R}$, $c \in \mathbb{C}$

$$d_S'' = d_A'' + \psi \, d\bar{z}$$

$[\phi, \psi] = 0 \quad \Rightarrow \quad \psi = E \phi$

$[\phi, \phi^*] = [\psi, \psi^*]$

$= |E|^2 [\phi, \phi^*]$

$\Rightarrow |E^2| = 1$

Need to solve c, p, E in terms of $\alpha, n, \tilde{m}, k$

Trace through holonomy, spectral curve construction

$$\log \mu = \pi i k' \pm \sqrt{-\pi^2 k'^2 - \det B}$$

$$B = (E + J)\phi + (\bar{E} + J^{-1})\phi^*$$

$\ulcorner$ Side note: $H = e^{-B}$ $\lrcorner$

Get $k = k'$ and

$$-\bar{b}^2 \alpha = c^2 + p^2$$

$$\bar{b}^2(|\alpha|^2+1) + 2|b|^2\alpha = 2E(c^2+p^2) - 2\bar{E}(|c|^2+p^2)$$

$$-b^2\alpha - 2|b|^2(|\alpha|^2+1) - \bar{b}^2\bar{\alpha}$$

$$= -\pi^2 k + E^2 c^2 - 4|c|^2 + \bar{E}^2\bar{c}^2$$

$$+ 2p^2 - (E - \bar{E})^2 p^2$$

Has sol$^{\underline{n}}$

$$E = -1$$

$$p = \frac{n\pi}{4} \frac{\varepsilon}{2-\varepsilon}$$

$$c = i\frac{n\pi}{4}$$

$$k = 0.$$

Leading to a harmonic map f.

$f(y_1, y_2) =$

$$\begin{pmatrix} \exp n\pi i y_1 & -\exp -n\pi i y_2 \, \varepsilon/2-\varepsilon \\ \exp n\pi i y_2 \frac{\varepsilon}{2-\varepsilon} & \exp -n\pi i y_1 \end{pmatrix}$$

$$0 \leq y_1 < 1$$
$$0 \leq y_2 < \operatorname{Im} \tau$$

$$0 \leq n\pi y_1 < n\pi$$

$$0 \leq n\pi y_2 \,{}^{\varepsilon}/_{2-\varepsilon} < \tilde{m}\pi$$

Covers a square torus in S^3 $\frac{n}{2}$ times in one direction and $\tilde{m}/2$ times in the other.

Where to from here?

- Get full solⁿ, look at path dependence as $a \to 1$

- Look at path dependence of $\tau \to \infty$

- Higher genus solⁿs have deformations of the image

Check out:

Hitchen, N.J. Harmonic maps from a 2-torus to the 3-sphere
J. Differential Geom. 31 (1990)