

DEFORMATIONS OF HARMONIC TORI IN \mathbb{S}^3

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Statement of Originality

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

A handwritten signature in blue ink, reading "R Ogilvie". The signature is written in a cursive style with a large, stylized 'R' and 'O'.

Ross Ogilvie

Abstract

In this thesis we investigate the topology of the moduli space of spectral data of harmonic maps from the torus into the 3-sphere. Harmonic tori in the 3-sphere are in bijective correspondence with their spectral data, which consists of an algebraic curve (called a spectral curve), a pair of differentials, and a line bundle. Deformations of the spectral data correspond to deformations of the tori themselves. There are two classes of deformations; isospectral deformations vary only the line bundle, whereas non-isospectral deformations change the spectral curve itself. This thesis explores the latter. We use the theory of Whitham deformations to show that the moduli space of spectral data is a surface. For spectral curves of genus zero and one, the global topology of the moduli space is treated through explicit parametrisation. We enumerate the path connected components and show them to be simply connected, and prove that the moduli space of these adjacent spectral genera connect to one another in an appropriate limit.

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Preliminaries

The study of harmonic maps is a old and broad topic within differential geometry. Many familiar and foundational concepts are types of harmonic maps, such as closed geodesics and holomorphic maps. A harmonic map $f : M \rightarrow N$ is an extrema for the energy functional

$$E(f) = \frac{1}{2} \int_M \|df\|^2 \omega_M.$$

A type of harmonic map that is of particular interest to us are minimal surfaces. A minimal surface may be characterised as a conformal harmonic map. These surfaces are common in everyday life. A loop of wire dipped in soapy water will produce a film that naturally adopts the least area given that fixed boundary, and this is a minimal surface. This configuration can be understood physically through the surface tension, which causes the film to stretch or contract until the forces on each small piece of it are balanced, minimising the elastic potential energy. The mathematical analogy to surface tension is the Euler-Lagrange equation for the energy functional. This translates the integral formulation above into a second-order semi-linear elliptic system of partial differential equations.

Though this analytic framing has been the subject of much research [EL78, MP11], which has been successful in proving the existence of harmonic maps between various classes of Riemannian manifolds [ES64, EW83, CM08], we here shall focus instead on a more geometric approach. This approach originates from the study of integrable systems. The core idea is that to each harmonic map it is possible to assign an algebraic curve, called a spectral curve, a pair of meromorphic differentials of the second kind, and a line bundle of a particular degree. From this collection of spectral data it is possible to reconstruct the harmonic map, which thereby provides a classification.

There have been several methods developed to make this assignment, most of which begin by associating the harmonic map to a family of flat connections.

This family of connections may be thought of as a generalisation of associated S^1 -family of harmonic maps. For harmonic maps into Lie groups, one can identify this family of connections with a loop in the group and use a decomposition of the group to examine them, [Uhl89, DPW98]. Alternatively, by reformulating the family of connections as a Lax pair one can seek out a set of polynomial Killing fields (a set of parallel sections of the family). These act as an intermediary allowing one to go back and forth between the spectral curve and the harmonic map, [BFPP93, KS10]. A novel approach for building spectral curves for the Lawson surface of genus two proceeds by first constructing a square torus that covers the moduli of complex structures and then showing that the moduli of families of flat connections is a branched cover of this square torus, [Hel14]. The inverse problem is generally substantively different, but a variety of methods exist [McI01].

We however will follow primarily the methodology of [Hit90]. The approach there is specific to harmonic tori in $S^3 = SU_2$. For each member of the family of flat connections, one considers the corresponding holonomy representation of the fundamental group. The fundamental group of the torus is abelian, so the two generators of the representation must commute and therefore share eigenspaces. Generically the eigenspaces split into two eigenlines. The family of flat connections is parametrised by $\zeta \in \mathbb{C}^\times$, so by varying this parameter the eigenlines form a line bundle on a Riemann surface which double covers \mathbb{C}^\times . By the compactness of the torus, this surface is only ramified at a finite number of points and may be completed to a hyperelliptic curve called the spectral curve.

Taking the domain of the harmonic map to be a torus and the codomain to be S^3 yields several interesting properties. Of the three common three-dimensional spaces forms \mathbb{R}^3 , \mathbb{H}^3 and S^3 , by the maximum principle only the last can admit compactly embedded minimal surfaces. The study of embedded minimal surfaces in S^3 has a long history. All immersed minimal spheres are congruent to the equator [Alm66], and more recently Brendle [Bre13] has shown the Clifford torus to be the only embedded minimal surface of genus one. This result has also been proved via integrable systems methods in [HKS15, HKS16a]. On the other hand, Lawson [Law70] proved that every compact surface (except the projective plane) may be minimally immersed in S^3 and that such immersions are not unique if the genus of the surface is not prime. The spectral curve theory is also rich, in that for every genus there exists a harmonic tori with a spectral curve of that genus [Car07]. Therefore S^3 is fertile ground to study the moduli space of harmonic tori. We will consider the moduli space of harmonic tori via their spectral data classification; it is entirely natural to consider families of algebraic curves, differentials and line bundles.

A close cousin of harmonic maps are surfaces of constant mean curvature (CMC). Indeed, minimal surfaces may be characterised equivalently as a CMC surface with mean curvature zero. To give a physical interpretation, CMC surfaces describe soap bubbles and the pressure difference between the inside and outside

determines the mean curvature. The Gauss map of a CMC surface is harmonic [RV70], so they are a source of many examples of harmonic maps. Important examples are provided by the Gauss maps of the Delaunay surface [Del41] and Wente torus [Wen86]. The classification of CMC surfaces [PS89, Bob91] strongly resembles that of harmonic maps. The work that has been done on understanding the moduli of CMC surfaces [HKS16a, KSS15, CS16] is an inspiration to this thesis.

In this thesis we demonstrate two main results. We shall prove a general result for the moduli space of harmonic tori in \mathbb{S}^3 , independent of the genus of the spectral curve, namely that it is (generically) two-dimensional. This is shown in Chapter 1 through application of Whitham deformations, deformations which preserve the periods of differentials. Also, we will investigate the structure of the moduli space in detail for spectral curves of genus one (Chapter 3). This inquiry will lead us to an enumeration of the path connected components of this space and will show that every component is naturally a ribbon $(0, 1) \times \mathbb{R}$. The chapter on spectral curves of genus zero, Chapter 2, serves a three-fold purpose: it provides a long worked example of the construction of the spectral data, it gives a flavour of the result we are aiming to achieve for genus one spectral curves, and it develops some formulae that are necessary in the final chapter. To conclude, in Chapter 4 we examine two boundaries of the moduli space of spectral data. On one boundary we show that there is accumulation at a point. The other boundary we identify with points of the moduli space of genus zero spectral curves.

0.1 Harmonic Maps to Lie Groups

Suppose that M is a compact Riemann surface, and G is a Lie group with a bi-invariant metric. Given any smooth map $f : M \rightarrow G$, we can pull back the bi-invariant metric connection on TG to a connection we shall denote A on $f^*(TG)$, with associated covariant exterior differential d_A . Using the complex structure of M we have the Hodge star operator $*$: $\Omega^1(M) \rightarrow \Omega^1(M)$. The map f is harmonic if and only if $d_A^* df$ is zero. Using the Hodge star to express the adjoint, this is equivalent to

$$d_A(*df) = 0.$$

The Maurer-Cartan form ω of a Lie group is the unique \mathfrak{g} -valued one-form that is invariant under the left group action and acts trivially on the tangent space at the identity. This forces, for any $X \in T_g G$, $\omega(X) = (L_{g^{-1}})^* X$, where L is left multiplication. For linear groups, such as we will be considering, if use the identification map $g : G \rightarrow \text{Mat}_{n \times n}(\mathbb{R})$, at $a \in G$

$$\omega_a = g(a)^{-1} dg_a.$$

The importance of the Maurer-Cartan form is its property of characterising maps to the Lie group. If φ is a \mathfrak{g} -value one-form on a simply connected manifold

U satisfying the Maurer-Cartan equation

$$d\varphi + \frac{1}{2}[\varphi \wedge \varphi] = 0,$$

where $[(X \otimes \alpha) \wedge (Y \otimes \beta)] = [X, Y] \otimes (\alpha \wedge \beta)$ for elements $X, Y \in \mathfrak{g}$ and differential forms α, β , then there is a map $f : U \rightarrow G$ such that $f^*\omega = \varphi$, unique up to left translation.

To apply this to a harmonic map $f : M \rightarrow G$, pull back the Maurer-Cartan form of G to $\varphi = 2(\Phi - \Phi^*)$, where 2Φ is the $(1, 0)$ part of the form φ . The differential Φ is known as a Higgs field. Harmonicity of f implies that

$$d_A'' \Phi = 0.$$

Suppose that d_L is the trivial connection on $f^*(TG)$ arising from the left trivialisation. The Levi-Civita connection d_A is then $d_L + \frac{1}{2}\varphi$, because the Maurer-Cartan form is the difference between the left and right connections and the Levi-Civita connection is their average. Computing the curvature

$$F_A = d_A^2 = (d_L + \frac{1}{2}\varphi)^2 = d_L^2 + \frac{1}{2}[(\Phi - \Phi^*) \wedge (\Phi - \Phi^*)] = [\Phi \wedge \Phi^*].$$

Together, these two equations

$$d_A'' \Phi = 0, \quad F_A = [\Phi \wedge \Phi^*] \tag{0.1}$$

are [Hit90, (1.7)]. One may consider these equations directly, without the context of a map f and its pullback of the tangent bundle $f^*(TG)$. For a matrix Lie group, consider instead a trivial complex vector bundle V over M . Let Φ be a $(1, 0)$ section of $\mathfrak{g}(V) \subset \text{End}(V)$ and d_A a connection on V . To the same effect, one may also use a gauge theoretical setup, taking a principal G -bundle P . Then we may take d_A to be a connection on P , and Φ to be a $(1, 0)$ section of the vector bundle $\text{ad } P$ associated to P via the adjoint representation. The two approaches are reconciled by taking P to be the G -frame bundle of V . If a pair (A, Φ) satisfies (0.1) then the connections

$$d_{-1} := d_A - \Phi + \Phi^* \text{ and } d_1 := d_A + \Phi - \Phi^*$$

are flat. If further they are trivial then it is possible to recover the harmonic map from M to G . This is certainly a necessary condition, as if such a pair arises from a harmonic map then the above two connections are the left and right connections respectively. Conversely, given such a pair, if the two connections are trivial then there exists sections X and Y of P that are parallel with respect to d_1 and d_{-1} respectively. Let $f : M \rightarrow G$ be the map such that $X = Y \cdot f$. As the difference $d_1 - d_{-1}$ is $\varphi = 2(\Phi - \Phi^*)$ and

$$(d_1 - d_{-1})(X) = d_1(Yf) - d_{-1}(X) = X(f^{-1}d_{-1}f), \tag{0.2}$$

it follows that φ is the pull back of the Maurer-Cartan form by f . The first of (0.1) then gives that f is harmonic. We note that even if the two connections

d_1 and d_{-1} are not trivial, we may take the universal cover $\tilde{M} \rightarrow M$ and pull back both Φ and d_A . In the pullback those connections will be trivial, and so we will have a harmonic map from \tilde{M} into G . In this vein, maps of the torus can be seen to lie among maps of the plane.

0.2 Spectral Curves

Consider now the case where M a torus and $G = \mathrm{SU}(2)$. Hitchin [Hit90] investigated solutions of (0.1) and characterised them in terms of a spectral curve construction. We summarise that construction now. From a pair (A, Φ) we construct a \mathbb{C}^\times family of flat $\mathrm{SL}_2\mathbb{C}$ connections

$$d_\zeta := d_A + \zeta^{-1}\Phi - \zeta\Phi^*, \quad (0.3)$$

where flatness follows from the two equations (0.1). Fix a base point in the torus and take a pair of loops that are a basis for its fundamental group. The holonomy representation is generated by two matrices $H(\zeta)$ and $\tilde{H}(\zeta)$ in $\mathrm{SL}_2\mathbb{C}$ corresponding to transporting vectors parallel to d_ζ along those loops. The eigenvalues μ and μ^{-1} of H satisfy the quadratic equation

$$\mu^2 - (\mathrm{tr} H)\mu + 1 = 0.$$

As the matrices H and \tilde{H} commute, they share eigenspaces. We therefore define the spectral curve Σ to be the closure of

$$\left\{ (\zeta, L) \in \mathbb{C}^\times \times \mathbb{CP}^1 \mid L \text{ is an eigenline of } H(\zeta) \right\}, \quad (0.4)$$

in $\mathbb{CP}^1 \times \mathbb{CP}^1$. This curve Σ double covers \mathbb{C}^\times via projection onto the first factor. Hitchin establishes (Prop 2.3) that the eigenvalues μ and $\tilde{\mu}$ of H and \tilde{H} considered as holomorphic functions over \mathbb{C}^\times are branched at only finitely many points, and further that (except in the trivial case of a conformal map to a 2-sphere) these functions have the same branch points (Prop 2.10). Thus the spectral curve is of finite genus.

Having constructed a curve, one may furnish it with additional objects. By analysis of the singularities of $\mathrm{tr} H$ and $\mathrm{tr} \tilde{H}$ as $\zeta \rightarrow 0$, it can be shown that $\log \mu$ and $\log \tilde{\mu}$ are meromorphic functions in a neighbourhood of 0. Specifically, they have a simple pole and the eigenlines may coincide to at most first order. Using the real structure, one may transport this information to $\zeta = \infty$ also. We can differentiate $\log \mu$ and $\log \tilde{\mu}$ to get differentials of the second kind $\Theta, \tilde{\Theta}$ respectively. One can also consider the trivial \mathbb{C}^2 bundle over Σ and construct the eigenline bundle E over Σ as the pullback of the tautological bundle on \mathbb{CP}^1 . By the construction of Σ , its arithmetic genus obeys $g = \deg E^* - 1$.

Having assembled this data $(\Sigma, \Theta, \tilde{\Theta}, E)$, [Hit90, Theorem 8.1] characterises spectral data that arises from pairs (A, Φ) and shows that the correspondence

unique up to tensoring with a flat Z_2 bundle. The proof proceeds by direct reconstruction. [Hit90, Theorem 8.20] firstly reformulates the requirement that d_1 and d_{-1} be flat connections into a requirement about the values of μ , and identifies certain geometric features of a harmonic map with properties that spectral data may possess. For example, a harmonic map is conformal (and therefore a minimal surface) exactly when the spectral curve is branched over 0. The following sections will introduce the various conditions that characterise the spectral data.

Throughout this thesis, we will consider only nonsingular spectral curves. To a large extent this is without loss of generality because when the genus of the spectral curve is low they are all nonsingular, as we demonstrate in Lemma 0.14. We make this assumption without further comment.

Definition 0.5. *Consider a (nonsingular) hyperelliptic curve Σ over \mathbb{CP}^1 with hyperelliptic involution σ and projection $\pi : \Sigma \rightarrow \mathbb{CP}^1$, such that $\pi \circ \sigma = \pi$. We call a tuple $(\Sigma, \{\xi_0, \sigma(\xi_0)\}, \{\xi_1, \sigma(\xi_1)\})$ a marked curve if*

$$(P.1') \quad \pi(\xi_0) = 0 \text{ and } \pi(\xi_1) = 1.$$

$$(P.2') \quad \text{there is a real involution } \rho : \Sigma \rightarrow \Sigma \text{ such that } \rho \circ \sigma \text{ is without fixed points} \\ \text{and } \pi(\rho(\xi)) = \overline{\pi(\xi)}^{-1}.$$

As is common, we will metonymically refer Σ as a marked curve. Points where $\xi = \sigma(\xi)$ are the ramification points of the curve. Note that ξ_0 and $\sigma(\xi_0)$ may or may not be distinct points. The condition that $\rho \circ \sigma$ is without fixed points ensures that the curve is not ramified at any point over the unit circle. Thus there are always two points of both $\pi^{-1}(1) = \{\xi_1, \sigma(\xi_1)\}$ and $\pi^{-1}(-1)$. We shall use ρ to also denote the real involution $\pi(\zeta) = \bar{\zeta}^{-1}$ on \mathbb{CP}^1 .

We may build a model for marked curves, unique up to biholomorphism. We will construct Σ as an algebraic curve in the total space of a certain line bundle over \mathbb{CP}^1 . To that end, let us now describe the line bundles over \mathbb{CP}^1 . Let $[z_0 : z_1]$ be homogeneous coordinates on \mathbb{CP}^1 . We define $\mathcal{O}(k)$ to be the line bundle over \mathbb{CP}^1 whose sections are given by homogeneous polynomials of degree k in two variables.

In homogeneous coordinates the reality structure ρ is $\rho([z_0 : z_1]) = [\bar{z}_1 : \bar{z}_0]$. This reality structure ρ on \mathbb{CP}^1 pulls back to give a reality structure on each line bundle $\mathcal{O}(k)$, namely

$$\rho^*q(z_0, z_1) = q(\bar{z}_1, \bar{z}_0).$$

A real section of this bundle is one such that $\rho^*q = \bar{q}$. Writing $q = q_0(z_0)^k + q_1(z_0)^{k-1}z_1 + \dots + q_k(z_1)^k$ and expanding the reality condition out gives

$$q_0(\bar{z}_1)^k + q_1(\bar{z}_1)^{k-1}\bar{z}_0 + \dots + q_k(\bar{z}_0)^k = \overline{q_0(z_0)^k + q_1(z_0)^{k-1}z_1 + \dots + q_k(z_1)^k},$$

so a section is real its coefficients obey $q_i = \overline{q_{k-i}}$, for $0 \leq i \leq k$. Often will work in an affine coordinate $\zeta = z_1/z_0$. We may identify a section of $\mathcal{O}(k)$ with a polynomial via

$$q_0(z_0)^k + q_1(z_0)^{k-1}z_1 + \cdots + q_k(z_1)^k = (z_0)^k (q_0 + q_1\zeta + \cdots + q_k\zeta^k).$$

Under this identification, the reality structure acts as

$$\rho^*q = (z_0)^k (\overline{q_0}\zeta^k + \overline{q_1}\zeta^{k-1} + \cdots + \overline{q_k}) = \overline{\zeta}^k q(\overline{\zeta}^{-1}).$$

Definition 0.6. Let \mathcal{P}^k be the space of polynomials of degree at most k . We define the real polynomials $\mathcal{P}_{\mathbb{R}}^k$ to be

$$\mathcal{P}_{\mathbb{R}}^k = \{ q = q_0 + \cdots + q_k\zeta^k \in \mathcal{P}^k \mid q_i = \overline{q_{k-i}} \text{ for all } 0 \leq i \leq k \}.$$

This is a real vector space of dimension $k+1$.

This reality relationship between the coefficients of a polynomial of $\mathcal{P}_{\mathbb{R}}^k$ implies a relationship between the roots of the polynomial. Let q be a real polynomial of $\mathcal{P}_{\mathbb{R}}^k$. Let the k roots of q , counted with multiplicity, be $\{\alpha_i\}$. Then for some scalar $a \in \mathbb{C}$, q factors as $q(\zeta) = a(\zeta - \alpha_1) \cdots (\zeta - \alpha_k)$. Applying the real involution,

$$\begin{aligned} q = \rho^*q &= \overline{\zeta}^{-k} \overline{a}(\overline{\zeta} - \overline{\alpha_1}) \cdots (\overline{\zeta} - \overline{\alpha_k}) \\ &= (-1)^k \overline{a} \left(\prod_{i=1}^k \overline{\alpha_i} \right) (\overline{\zeta}^{-1} - \overline{\alpha_1}^{-1}) \cdots (\overline{\zeta}^{-1} - \overline{\alpha_k}^{-1}). \end{aligned}$$

Therefore the set of roots must be invariant under $\zeta \mapsto \overline{\zeta}^{-1}$. Every root must either lie on the unit circle or else come in conjugate-inverse pairs. As seen above, writing the factors as $\zeta - \alpha$ does not lead to a nice expression for scaling factor a . However, we can choose a normalisation of the factors that does better. If β is a point on the unit circle, then

$$\beta + \overline{\beta}\zeta$$

is a real polynomial of $\mathcal{P}_{\mathbb{R}}^1$, which has a root on the unit circle at $-\beta^2$. If α is a point inside the unit disk then

$$(\zeta - \alpha)(1 - \overline{\alpha}\zeta)$$

is a real polynomial of $\mathcal{P}_{\mathbb{R}}^2$. Given any real polynomial, we may construct another with the same roots where each factor is of the form above. The quotient of these two polynomials is a real polynomial of $\mathcal{P}_{\mathbb{R}}^0$, a real number. Hence every real polynomial may be written as a product of factors of these forms, multiplied by a real scalar.

Armed with these line bundles over \mathbb{CP}^1 , we may now provide a description of a marked curve Σ . Suppose that Σ has genus g . Take a section $P(\zeta)$ of $\mathcal{O}(2g+2)$ and consider the curve Σ' in the total space of $\mathcal{O}(g+1)$ defined by $\eta^2 = P(\zeta)$, where η is the coordinate on the fibre. It is hyperelliptic, with involution $\sigma(\zeta, \eta) = (\zeta, -\eta)$ and projection $\pi(\zeta, \eta) = \zeta$. Suppose that $P(\zeta)$ has the following properties

- (P.1) Real spectral curve: $P(\zeta)$ is a real section of $\mathcal{O}(2g+2)$ with respect to the real structure induced by ρ . That is, it is an element of $\mathcal{P}_{\mathbb{R}}^{2g+2}$.
- (P.2) No real zeroes: $P(\zeta)$ has no zeroes on the unit circle $\mathbb{S}^1 \subset \mathbb{CP}^1$.
- (P.3) Simple zeroes: $P(\zeta)$ has only simple zeroes.

Then Σ' is a marked curve. The holomorphic involution ρ is the restriction of σ on $\mathcal{O}(g+1)$. The fixed points of σ are the roots of P , so (P.2) ensures that $\rho \circ \sigma$ is fixed point free and (P.3) provides that the curve is not singular.

Hyperelliptic curves are determined by their branch points in \mathbb{CP}^1 , up to automorphism of \mathbb{CP}^1 . To see this, first choose a non-branch point of \mathbb{CP}^1 and take the complement. Over this affine space, the function field of Σ is a quadratic extension and so must be Galois. Take the nontrivial automorphism σ of the extension and find an element η such that $\sigma(\eta) = -\eta$ (this may be done by taking any non-fixed element $\tilde{\eta}$ and choosing $\eta = \tilde{\eta} - \sigma(\tilde{\eta})$). If we consider its square, $\sigma(\eta^2) = \eta^2$, so we have that $\eta^2 = P(\zeta)$ for some polynomial P . This determines Σ up to scaling of y and automorphism of \mathbb{CP}^1 . The choice of automorphism of \mathbb{CP}^1 has however been removed by the imposition of the marked points, which fix $0, 1$ and ∞ . Every marked curve therefore corresponds to some section $P(\zeta)$, determined uniquely up to scaling.

Suppose we have a marked curve of genus g . Denote its branch points inside the unit circle as $\alpha_0, \dots, \alpha_g$. We fix the following scaling of P ,

$$P(\zeta) = \prod_{i=0}^g (\zeta - \alpha_i)(1 - \bar{\alpha}_i \zeta). \quad (0.7)$$

Notice the absence of any factors of the form $\beta + \bar{\beta}\zeta$ because P has no roots on the unit circle. The nice feature of this scaling is that it is well-behaved if one branch point is zero; the corresponding factor becomes simply ζ .

We already mentioned that the differentials in the spectral data must be of the second kind. This means that they have at most double poles, but are residue free. Such differential are also required to possess certain symmetries arising from their origin as eigenvalues of connections. For instance, at a point $\zeta \in \mathbb{C}^\times$ that is not a branch point of the spectral curve, the two eigenvalues of $H(\zeta)$ are $\mu(\zeta)$ and $\mu(\zeta)^{-1}$. Hence $\sigma^* \mu = \mu^{-1}$, and so for $\Theta = d \log \mu$

$$\sigma^* \Theta = d \log (\mu^{-1}) = -\Theta.$$

On a marked curve Σ , a differential Θ must satisfy the following conditions.

- (P.4) Poles: The differential has double poles with no residues at $\pi^{-1}\{0, \infty\}$, where π is the projection $\pi : \Sigma \rightarrow \mathbb{CP}^1$, but are otherwise holomorphic.
- (P.5) Symmetry: The differential satisfies $\sigma^*\Theta = -\Theta$.
- (P.6) Reality: The differential satisfies $\rho^*\Theta = -\bar{\Theta}$.
- (P.7) Imaginary Periods: The differentials have purely imaginary periods.

These conditions restrict the form that a differential Θ may take. On any hyperelliptic curve $d\zeta/\eta$ is a holomorphic differential, so meromorphic differentials on Σ are of the form $(f(\zeta) + \eta g(\zeta))d\zeta/\eta$ for rational functions f and g on \mathbb{CP}^1 [Mir95, Prop III.1.10]. Applying the hyperelliptic involution gives

$$\sigma^* \left((f(\zeta) + \eta g(\zeta)) \frac{d\zeta}{\eta} \right) = (-f(\zeta) + \eta g(\zeta)) \frac{d\zeta}{\eta},$$

so the symmetry condition forces $g \equiv 0$. Suppose first that Σ does not have a branch point at 0. Then ζ is a local coordinate at both points of $\pi^{-1}(0)$, and for the differential to have double poles above $\zeta = 0$ and ∞ , the function f must be of the form $\zeta^{-2}b(\zeta)$ for a polynomial b of degree $g+3$. To handle the residue condition, expand the differential as a series at zero to see that the residue is

$$b_1 - \frac{1}{2} \frac{P_1}{P_0} b_0, \quad (0.8)$$

where subscripts denote coefficients of the polynomials. This quantity must therefore vanish.

Reality of the differential means that this polynomial is real, that is $\overline{b(\zeta)} = \bar{\zeta}^{g+3} b(\bar{\zeta}^{-1})$.

Now suppose that P has a root at $\zeta = 0$. Because it must be a simple root, $P_1 \neq 0$. Let ξ be a local coordinate of the point $\pi^{-1}(0)$ with $\xi^2 = \zeta$ and expand

$$\Theta = \frac{d\zeta}{\eta} f(\zeta) \sim \frac{2\xi d\xi}{\xi} \frac{1}{\sqrt{P_1}} \left(1 - \frac{1}{2} \frac{P_2}{P_1} \xi^2 + O(\xi^4) \right) f(\xi^2)$$

So to have double poles requires that $f(\zeta) = \zeta^{-1}a(\zeta)$ for some polynomial $a(\zeta)$ of degree $g+1$ and note that this is automatically residue free. Reality of the differential means that this polynomial a is also real, that is $\overline{a(\zeta)} = \bar{\zeta}^{g+1} a(\bar{\zeta}^{-1})$. If we write $b(\zeta) = \zeta a(\zeta)$, we see that this is a special case of the form above, where if P_0 vanishes so too must b_0 . In light of this, we may rephrase equation 0.8 among the coefficients to be

$$P_1 b_0 - 2P_0 b_1 = 0. \quad (0.9)$$

In summary, every differential meeting the three conditions (P.4), (P.5) and (P.6) can be written as

$$\Theta = b(\zeta) \frac{d\zeta}{\zeta^2 \eta} = (b_0 + b_1 \zeta + \dots + b_{g+3} \zeta^{g+3}) \frac{d\zeta}{\zeta^2 \eta}, \quad (0.10)$$

for some real degree $g + 3$ polynomial $b(\zeta) \in \mathcal{P}_{\mathbb{R}}^{g+3}$, meeting the condition $P_1 b_0 - 2P_0 b_1 = 0$ on its coefficients.

Finally, having purely imaginary periods imposes further linear relations on the coefficients of b . Take a basis of the homology of Σ denoted $A_1, \dots, A_g, B_1, \dots, B_g$. Let A_i be the difference of the two lifts to Σ of the arc connecting α_i and $\bar{\alpha}_i^{-1}$. Such cycles have the property that $\rho_*(A_i) = -A_i$, and therefore

$$\int_{A_i} \Theta = - \int_{\rho_*(A_i)} \Theta = - \int_{A_i} \rho^* \Theta = \overline{\int_{A_i} \Theta}.$$

Thus the A_i -period of a real differential is real. If the periods are to be imaginary, these periods must all vanish, which imposes g real constraints. In the choice of b we have initially $2(g + 4)$ real degrees of freedom, but they are reduced by half due to the reality condition, and by a further two because of the relationship (0.9) between b_0 and b_1 . The g constraints from the imaginary periods leave just 2 real degrees of freedom in the choice of differential satisfying (P.4)–(P.7). Thus there is a real 2-plane of differentials with purely imaginary periods, which we shall call \mathcal{B}_{Σ} .

We are now in a position to state fully the conditions a tuple $(\Sigma, \Theta, \tilde{\Theta}, E)$ must meet in order to correspond to a pair (A, Φ) solving the equations (0.1). In addition to a marked curve Σ with properties (P.1)–(P.3) and a pair of differentials Θ and $\tilde{\Theta}$ satisfying (P.4)–(P.7) we require the following.

(P.8) Periods: The periods of the differentials Θ and $\tilde{\Theta}$ lie in $2\pi i\mathbb{Z}$.

(P.9) Linear independence: The principal parts of the differentials Θ and $\tilde{\Theta}$ are real linearly independent.

(P.10) Quaternionic: E^* is a line bundle of degree $g + 1$ that is quaternionic with respect to the involution $\rho \circ \sigma$.

A theorem of Hitchin [Hit90, Theorem 8.1] provides a correspondence between solutions of (0.1) and tuples $(\Sigma, \Theta, \tilde{\Theta}, E)$ (though such a correspondence necessarily includes singular spectral curves, which we are not considering). If the choice of curve and differentials is fixed, one is free to choose E subject to only to condition (P.10). There are many such choices; they form a real g dimensional torus in the Jacobian of Σ . If one varies E this varies the harmonic map, and such deformations are called isospectral deformations. Conversely, if we have a

triple $(\Sigma, \Theta, \tilde{\Theta})$ satisfying the above conditions, then there always exists such a line bundle E completing the tuple. Thus we focus our attention on the problem of deforming the triple $(\Sigma, \Theta, \tilde{\Theta})$, so called non-isospectral deformations, and in particular to the structure of the space of such triples. Though non-standard, we will refer to these triples as spectral data.

Moreover, [Hit90, Theorem 8.20] gives an additional condition on a triple $(\Sigma, \Theta, \tilde{\Theta})$ to ensure that the two connections are trivial, and so that the pair (A, Φ) correspond to a harmonic map from the torus into \mathbb{S}^3 . The Theorem also states that the harmonic map is uniquely determined by its spectral data, up to the action of $\mathrm{SO}(4)$ on $\mathrm{SU}(2)$.

(P.11) Closing conditions: μ has value 1 at $\pi^{-1}\{\pm 1\}$, where μ is a meromorphic function on $\Sigma \setminus \pi^{-1}\{0, \infty\}$, $\Theta = d \log \mu$ and $\mu \sigma^* \mu = 1$. Likewise for $\tilde{\mu}$ and $\tilde{\Theta} = d \log \tilde{\mu}$.

As our interest is foremost in harmonic maps, not solutions of (0.1), we consider this condition as on par with the ones above. The closing condition can be reformulated as a period type constraint also. Consider for a given Θ if we had a function μ such that $\Theta = d \log \mu$. Then suppose that γ_+ was a path in Σ connecting the two points of $\pi^{-1}(1) = \{\xi_1, \sigma(\xi_1)\}$. Integrating,

$$\int_{\gamma_+} \Theta = \log \mu(\xi_1) - \log \mu(\sigma(\xi_1)) \in 2\pi i\mathbb{Z},$$

and likewise for γ_- connecting the two points $\{\xi_{-1}, \sigma(\xi_{-1})\}$ over -1 ,

$$\int_{\gamma_-} \Theta = \log \mu(\xi_{-1}) - \log \mu(\sigma(\xi_{-1})) \in 2\pi i\mathbb{Z}.$$

Conversely, suppose that

$$\int_{\gamma_+} \Theta \quad \text{and} \quad \int_{\gamma_-} \Theta \in 2\pi i\mathbb{Z} \tag{0.11}$$

Then we could define μ to be $\exp(\int \Theta)$. The integral of a differential is defined up to a constant, periods and residues. In this case, the latter are zero, the constant of integration is fixed by the condition $\mu \sigma^* \mu = 1$, and the periods are all in $2\pi i\mathbb{Z}$ by (P.8) so do not change μ . These integrals then imply that

$$\mu(\xi_1)/\mu(\sigma(\xi_1)) = \exp\left(\int_{\gamma_+} \Theta\right) = 1.$$

Together with $\mu(\xi_1)\mu(\sigma(\xi_1)) = \mu(\xi_1)\sigma^*\mu(\xi_1) = 1$, this implies that $\mu(\xi_1)$ and $\mu(\sigma(\xi_1))$ are equal to ± 1 . Looking at the construction of solutions of (0.1) in [Hit90, Section 8], we are free to tensor with an element of $H^1(M, \mathbb{Z}_2)$, which

we can use to fix the signs to be $+1$ in a unique way. Thus for any Θ that satisfy (0.11), there is a unique corresponding μ that satisfies the closing conditions. We therefore use the term closing conditions to describe either formulation.

Given a tuple of spectral data, we can detect certain features of the corresponding map. The most important example of this has occurred several times already. The harmonic map f is a conformal map if and only if $P(0) = 0$. Another example that we shall come across in Section 3.5 is the case where f takes its image in a totally geodesic 2-sphere. Such a case occurs exactly when P is an even polynomial, and Θ , $\tilde{\Theta}$, and E are invariant under the extra involution $(\zeta, \eta) \mapsto (-\zeta, -\eta)$.

In the study of CMC surfaces similar spectral data is obtained. Again one has a hyperelliptic curve, a pair of meromorphic differentials, and a line bundle; but the conditions that the data must satisfy are subtly different. In particular the spectral curve of a CMC torus is always branched at zero and infinity. However the closing conditions for the spectral data of a CMC torus are less restrictive than for its harmonic tori brethren. The two points at which μ is required to be ± 1 must lie over the unit circle, not specifically over $\zeta = 1$ and $\zeta = -1$. In [KSS15] these points λ_1, λ_2 are called *sym points*.

We have already noted that minimal tori may be characterised as both conformal harmonic tori and CMC tori with vanishing mean curvature H . The spectral data of minimal tori are exactly the spectral data that satisfies the conditions required of harmonic maps and of CMC maps simultaneously. This can be seen in the following way. The mean curvature of a CMC torus may be deduced from its spectral data using [KSS15, (2.3)]:

$$H = \left| \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \right|.$$

By rotation we may fix $\lambda_1 = 1$. Then $H = 0$ exactly when $\lambda_2 = -1$. In the other direction, we have already stated that a harmonic map is conformal if and only if its spectral curve is branched over zero and infinity.

0.3 Moduli Space

In order to speak of the moduli of spectral data $(\Sigma, \Theta, \tilde{\Theta})$, we shall parametrise the spaces of marked curves and differentials satisfying (P.4)–(P.7) and then identify the moduli space as a subset. Consider (nonsingular) marked curves and their genus. Recall that by the choice of scaling of P , equation (0.7), the marked curve Σ is determined uniquely by the roots of P . Let D be the open unit disc and define

$$\mathcal{A}_g = \{(\alpha_0, \alpha_1, \dots, \alpha_g) \in D^{g+1} \mid \alpha_i \neq \alpha_j\}. \quad (0.12)$$

Every point of \mathcal{A}_g determines a marked curve $\Sigma(\alpha_0, \dots, \alpha_g)$ via a polynomial P with roots $\{\alpha_0, \bar{\alpha}_0^{-1}, \dots, \alpha_g, \bar{\alpha}_g^{-1}\}$. However, different permutations of components of a point of \mathcal{A}_g yield the same marked curve. Let $\text{Sym}(g+1)$ be the symmetric group on $g+1$ elements, and have it act on \mathcal{A}_g by permutation. By excluding singular curves, that is curves with multiple roots, the action of the symmetric group has no fixed points and so the quotient is also a smooth manifold.

Definition 0.13. We define \mathcal{C}_g to be $\mathcal{A}_g/\text{Sym}(g+1)$ and we call this the space of marked curves.

Before we proceed to differentials, let us give the promised motivation for the exclusion of singular curves from the definition of a marked curve. We shall prove that in the case of singular curves of arithmetic genus two or lower, there are no differentials with real linearly independent principal parts. The proof requires the following lemma about marked curves of genus zero.

Lemma 0.14. On a marked curve of genus zero, differentials satisfying conditions (P.4)–(P.6) with linearly independent principal parts do not have common roots.

Proof. We distinguish between two cases: whether or not the marked curve is branched over $\zeta = 0$. If $\zeta = 0$ is a branch point, then we note the following more general proof. Suppose the marked curve has genus g . Then from (0.10) we have that any differential may be written as

$$a(\zeta) \frac{d\zeta}{\zeta \eta},$$

for some real polynomial a of degree $g+1$. Any real polynomial is determined up real scaling to its $g+1$ roots, so if two such differentials have $g+1$ roots in common, then they are real linearly dependent. Letting $g=0$ shows that two differentials may not share any roots.

In the nonconformal case, there is no similar elegant generalisation. Instead we consider the specific form of the differentials. Let the spectral curve be given by

$$\eta^2 = -\alpha + (1 + \alpha \bar{\alpha})\zeta - \bar{\alpha}\zeta^2,$$

for $\alpha \neq 0$ in the unit disc, and let $x = -\frac{1}{2}\alpha^{-1}(1 + \alpha \bar{\alpha})$. Then the differential is determined by a nonzero constant y ,

$$(y + xy\zeta + \bar{x}\bar{y}\zeta^2 + \bar{y}\zeta^3) \frac{d\zeta}{\zeta^2 \eta}.$$

Take two differentials Θ and $\tilde{\Theta}$ with real linearly independent principal parts. We shall compute their greatest common divisor. Observe

$$\bar{y}\tilde{\Theta} - \bar{x}\Theta = (\bar{y}x - \bar{x}y)(x\zeta + 1) \frac{d\zeta}{\zeta^2 \eta}.$$

By assumption, x/y is not real, so this is nonzero. Its only root is $-x^{-1}$, but at $\zeta = -x^{-1}$

$$y + xy(-x^{-1}) + \overline{xy}(-x^{-1})^2 + \overline{y}(-x^{-1})^3 = \overline{y}x^{-3}(x\overline{x} - 1),$$

which is only zero if $|x| = 1$, which itself only occurs when $\alpha = 0$. But we are considering the nonconformal case, $\alpha \neq 0$, so Θ and $x\zeta + 1$ have no common factors. Hence

$$\gcd(\Theta, \tilde{\Theta}) = \gcd(\Theta, \overline{y}\tilde{\Theta} - \overline{x}\Theta) = \gcd(\Theta, x\zeta + 1) = 1$$

□

We can now give the argument that there are no singular spectral curves with genus two or less. Suppose we have a singular spectral curve Σ with normalisation $\tilde{\Sigma}$. Because there can be no singular points on the unit circle, all singular points come in pairs, so the (arithmetic) genus of Σ and $\tilde{\Sigma}$ differ by at least two. This excludes the possibility of singular spectral curves of genus zero or one. If we add in the fact that at a point of $\tilde{\Sigma}$ that maps to Σ with multiplicity m , the differentials Θ and $\tilde{\Theta}$ both have a common zero of order $m - 1$, the the above lemma shows that no singular spectral curve has a genus zero normalisation. This rules out singular genus two spectral curves also.

Over \mathcal{C}_g , there is a bundle \mathcal{B}_g of differentials with imaginary periods. As argued above it is a rank two vector bundle; the fibre over Σ is \mathcal{B}_Σ . We define \mathcal{M}_g to be the space of spectral data $(\Sigma, \Theta, \tilde{\Theta})$ with integral periods and satisfying the closing conditions, a subspace of the total space of the fibrewise product $\mathcal{B}_g \times \mathcal{B}_g$. Importantly, it is not a bundle over \mathcal{C}_g , because not every marked curve admits spectral data. Instead, we use \mathcal{S}_g to denote the space of spectral curves, marked curves that do admit spectral data. It is a subspace of \mathcal{C}_g , namely the projection of \mathcal{M}_g .

Deformations of Harmonic Maps

In this chapter we investigate the tangent space to the moduli space \mathcal{M}_g of spectral data with a spectral curve of genus g . We will prove there is an open set \mathcal{U}' such that $\mathcal{M}_g \cap \mathcal{U}'$ is a two dimensional manifold. To show this, we use (0.7) and (0.10) to consider \mathcal{M}_g as a subset of \mathcal{U} , where \mathcal{U} is itself an open subset of an affine space. Then by considering paths in \mathcal{M}_g we develop equations (1.4) and (1.9) that characterise the tangent vectors to these paths. On an open subset $\mathcal{U}' \subset \mathcal{U}$, we find all solutions to these equations to demonstrate that $T_p \mathcal{M}_g$ is two-dimensional at these points (Lemmata 1.23, 1.27 and 1.29). Having established that the dimension is constant, it naturally follows that $\mathcal{M}_g \cap \mathcal{U}'$ is a manifold (Theorem 1.32).

A comment about notation. Throughout we will use diacritical marks to indicate the factors that a polynomial does or does not have. A polynomial with a circumflex (hat) will be shown to have a factor of $\zeta^2 - 1$, and a tilde will indicate that any common factors have been removed, cf. (1.11). We shall use a dash to denote differentiation with respect to ζ and a dot for differentiation with respect to t evaluated at $t = 0$.

When giving the solutions to equations, we will use bold to signify a particular solution, which may or may not be unique, whereas a solution without bold signifies any solution from the set of potential solutions. Given a tuple of polynomials, such as (X, Y) , we also give their degrees as a tuple, eg (x, y) for $x = \deg X$ and $y = \deg Y$. Finally, we shall use i and j for indices ranging over $\{1, 2\}$, with the understanding that they are not equal. For example, if $i = 1$, then we take $j = 2$ and vice versa.

1.1 Whitham Deformations

Let us consider infinitesimal deformations of the spectral data $(\Sigma, \Theta^1, \Theta^2)$ within the space \mathcal{M}_g of spectral data where the spectral curves have a fixed genus g .

Recall that this triple consists of a marked curve Σ and differentials Θ^1 and Θ^2 satisfying the conditions (P.1)–(P.11). A deformation of spectral data is a path $\ell : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}$, parametrised by t . An infinitesimal deformation is the tangent vector of such a curve at $t = 0$. There is an established methodology for finding the deformations that preserve the periods of differentials, the so called Whitham deformations. This method was first discovered for the Korteweg-de Vries equation [FFM80, LL83], before being developed generally for other integrable systems [Kri95]. The form of Whitham deformations we use here resemble their application in the theory of constant mean curvature surfaces [KSS15, CS16].

Suppose that we are at a point of \mathcal{M}_g that admits a deformation ℓ . If we write $\ell(t) = (\Sigma(t), \Theta^1(t), \Theta^2(t))$, then we know from (0.7) that every spectral curve $\Sigma(t)$ along this deformation may be written as $\eta^2 = P(t, \zeta)$, for $P(t, \zeta)$ a polynomial in ζ belonging to $\mathcal{P}_{\mathbb{R}}^{2g+2}$, and from (0.10) that each differential can be identified with $b^i(t, \zeta)$ such that

$$\Theta^i(t) = \frac{1}{\zeta^2 \eta} b^i(t, \zeta) d\zeta,$$

for $i = 1, 2$, where for each t we have that $b^i(t, \zeta)$ a member of $\mathcal{P}_{\mathbb{R}}^{g+3}$.

Equivalently then, we may describe ℓ in terms of these polynomials, $\ell(t) = (P(t, \zeta), b^1(t, \zeta), b^2(t, \zeta))$. We let $\Sigma = \Sigma(0)$ and likewise $b^i(\zeta) = b^i(0, \zeta)$. More generally, omission of the parameter t will correspond to evaluation at the point $t = 0$. We use this representation of the spectral data as polynomials to consider the moduli space of spectral data \mathcal{M}_g as a subspace of the affine space of real polynomials $\mathbb{R}^{4g+11} = \mathbb{R}^{2g+3} \times \mathbb{R}^{g+4} \times \mathbb{R}^{g+4}$. Because of the conditions spectral data must satisfy, we can in fact be more precise. The moduli space \mathcal{M}_g is a subset of the following open set \mathcal{U} .

Definition 1.1. *Let \mathcal{U} be the open subset of $\mathcal{P}_{\mathbb{R}}^{2g+2} \times \mathcal{P}_{\mathbb{R}}^{g+3} \times \mathcal{P}_{\mathbb{R}}^{g+3}$ of triples of real polynomials (P, b^1, b^2) where P has only simple zeroes (cf. (P.3)) and no zeroes on the unit circle (cf. (P.2)), and the polynomials b^i have at most a simple root at $\zeta = 0$ (cf. (P.4)).*

Recall from Section 0.2 that the differentials of the spectral data are the derivatives of the logarithms of the eigenvalues μ and $\tilde{\mu}$ of the holonomy matrices H and \tilde{H} . With this in mind, we introduce the notation q^i for $q^1 = \log \mu$ and $q^2 = \log \tilde{\mu}$. Because the differential $\Theta^i = dq^i$ has nonzero periods, it is only possible to define q^i locally, and even then only up a constant. However, along ℓ a triple of spectral data satisfies (P.8), the period integrality conditions. This forces the periods to take fixed values. In particular then, the derivative of dq^i with respect to t is an exact differential, and thus \dot{q}^i is a well defined meromorphic function on Σ .

The functions \dot{q}^i are of interest because they encode infinitesimal deformations of the spectral data that preserve the integrality of the periods. If we could

determine the location and order of the poles of \dot{q}^i , we could characterise it as the quotient of a polynomial and a fixed holomorphic function, analogously to the characterisation of Θ^i by polynomials b^i . To this end, we derive now in generality how a function may acquire additional poles when it is differentiated with respect to t .

Consider a function that varies smoothly with t and is meromorphic on each curve $\Sigma(t)$. In a neighbourhood of a point of $\Sigma(0)$ that is not a ramification point, for small t , $(t, \zeta - \zeta_0)$ are local coordinates, for a fixed point $\zeta_0 \in \mathbb{CP}^1$. Thus we may expand the function as $(\zeta - \zeta_0)^k f(t, \zeta - \zeta_0)$, for a function f holomorphic in its second parameter and non-vanishing at $f(0, 0)$. As $\zeta - \zeta_0$ is independent of t , the order k of this function cannot decrease under differentiation by t . However, at a ramification point lying over a branch point α we must instead take (t, ξ) as local coordinates, for $\xi(t)^2 = \zeta - \alpha(t)$ where $\alpha(t)$ a root of $P(t)$ and $\alpha(0) = \alpha$. Any meromorphic function may be written locally as $\xi(t)^k f(t, \xi(t))$. Differentiating with respect to t yields

$$\left. \frac{d}{dt} \xi^k f(t, \xi) \right|_{t=0} = -\frac{k}{2} \xi^{k-2} \dot{\alpha} f - \frac{1}{2} \xi^{k-1} \dot{\alpha} f' + \xi^k \dot{f}. \quad (1.2)$$

If the function had a pole at a ramification point, then its derivative with respect to t may have a pole up to two orders worse. If the function was holomorphic, then the derivative has at worst a simple pole.

As dq^i has double poles without residues over $\zeta = 0$ and ∞ , it follows that \dot{q}^i has simple poles at those same points and is holomorphic at all other points. Applying (1.2) to q^i , we see that \dot{q}^i may have simple poles at the nonzero roots of P . If the curve Σ is branched over $\zeta = 0$ and ∞ , then \dot{q}^i may have a triple pole there. Otherwise \dot{q}^i has at worst simple poles over $\zeta = 0$ and ∞ .

The consequence of this is that $\zeta \eta \dot{q}^i$ is holomorphic. This expression is invariant under the hyperelliptic involution σ , so from [Mir95, Prop III.1.10] we deduce that

$$\dot{q}^i = \frac{1}{\zeta \eta} \tilde{c}^i(\zeta) \quad (1.3)$$

for some degree $g+3$ polynomial \tilde{c}^i . This gives a parametrisation of the functions \dot{q}^i by a vector space, the space of polynomials \tilde{c}^i .

We shall now demonstrate how the functions \dot{q}^i relate to an infinitesimal deformation $(\dot{P}, \dot{b}^1, \dot{b}^2)$, by using the equality of mixed partial derivatives. We compute the derivatives of dq and \dot{q} with respect to the variables t and ζ respectively.

$$\begin{aligned} \left. \frac{d}{dt} dq^i \right|_{t=0} &= \frac{d\zeta}{\zeta^2 \eta} \left(-\frac{1}{2} \frac{\dot{P}}{P} b^i + \dot{b}^i \right) \\ d(\dot{q}^i) &= \frac{d\zeta}{\zeta^2 \eta} \left(-\tilde{c}^i - \frac{1}{2} \frac{P'}{P} \zeta \tilde{c}^i + \zeta \tilde{c}^{i'} \right). \end{aligned}$$

After equating and simplifying, one produces an equation that ensures compatibility between these two derivatives,

$$\dot{P}b^i - 2P\dot{b}^i = 2P(\hat{c}^i - \zeta\hat{c}^{i'}) + P'\zeta\hat{c}^i. \quad (1.4)$$

These two equations provide a link between \dot{b}^i and \hat{c}^i for each $i = 1, 2$. Indeed,

Lemma 1.5. *Given a point (P, b^1, b^2) of \mathcal{M}_g for which there are deformations, the polynomials \hat{c}^i are determined uniquely by a tangent vector $(\dot{P}, \dot{b}^1, \dot{b}^2)$ to \mathcal{M}_g .*

Proof. Since the equations (1.4) are linear in the components of the tangent vector, we need only demonstrate that the zero tangent vector uniquely corresponds to $\hat{c}^i = 0$. For the zero tangent vector,

$$0 = 2P(\hat{c}^i - \zeta\hat{c}^{i'}) + P'\zeta\hat{c}^i. \quad (1.6)$$

The polynomial P is either of the form L or ζL , where L has only nonzero roots. L has degree either $2g + 2$ or $2g$ respectively. The assumption of a nonsingular spectral curve requires that P and P' have no common factors, hence evaluation of (1.6) at any root α of L shows that $P'(\alpha)\alpha\hat{c}^i(\alpha) = 0$, and hence α is a root of \hat{c}^i . This shows that L divides \hat{c}^i , and for $g \geq 4$ the inequality $\deg L \geq 2g \geq g + 4$ is sufficient to show that \hat{c}^i is the zero polynomial, as it is a degree $g + 3$ polynomial that is divisible by a polynomial of greater degree. To handle the remaining cases, $g < 4$, we substitute in this factorisation of \hat{c}^i and then remove the factor of L ,

$$\begin{aligned} 0 &= 2L \frac{P}{L} \left(L \frac{\hat{c}^i}{L} - \zeta \left[L' \frac{\hat{c}^i}{L} + L \left(\frac{\hat{c}^i}{L} \right)' \right] \right) + \left[L' \frac{P}{L} + L \left(\frac{P}{L} \right)' \right] \zeta L \frac{\hat{c}^i}{L} \\ 0 &= L \left[2 \frac{P}{L} \frac{\hat{c}^i}{L} - 2\zeta \frac{P}{L} \left(\frac{\hat{c}^i}{L} \right)' + \zeta \left(\frac{P}{L} \right)' \frac{\hat{c}^i}{L} \right] - \zeta L' \frac{P}{L} \frac{\hat{c}^i}{L}. \end{aligned}$$

Again, this shows that L divides \hat{c}^i/L . If $\deg L = 2g + 2$, this shows that \hat{c}^i is divisible by a polynomial of degree $4g + 4$, and so must be zero for any g . If $\deg L = 2g$, then we have only shown that \hat{c}^i vanishes for $g \geq 2$. We treat the two remaining cases, $\deg L = 2g$ and $g = 0$ or 1 , individually.

If $\deg L = 2g$ and $g = 1$, then \hat{c}^i is a scalar multiple of L^2 . Let $\hat{c}^i = aL^2$ and equation (1.6) simplifies to

$$0 = 2aL^3(3L - 2\zeta L'),$$

which forces $a = 0$. If $g = 0$, then $P = \zeta$ and \hat{c}^i is a cubic polynomial. After removing the factor of ζ ,

$$0 = 2(\hat{c}^i - \zeta\hat{c}^{i'}) + \hat{c}^i = 3\hat{c}_0^i + \hat{c}_1^i\zeta - \hat{c}_2^i\zeta^2 - 3\hat{c}_3^i\zeta^3,$$

which again shows that \hat{c}^i is zero. Hence, the polynomials \hat{c}^i are uniquely determined by a tangent vector $(\dot{P}, \dot{b}^1, \dot{b}^2)$ to \mathcal{M}_g as claimed. \square

The converse result, that a pair (\hat{c}^1, \hat{c}^2) determines an infinitesimal deformation $(\dot{P}, \dot{b}^1, \dot{b}^2)$, does not hold in general. In the next section we shall take up the task of describing this failure and the information, additional to (\hat{c}^1, \hat{c}^2) , that must be supplied in order to determine a unique tangent vector.

First though, let us return to our line of inquiry. Recall that we have supposed that we are at a point (P, b^1, b^2) of \mathcal{M}_g that admits a deformation ℓ , from which we have defined polynomials \hat{c}^1 and \hat{c}^2 and derived the pair of equations (1.4). Note that the two equations (1.4) are not independent of one another, for they both contain P and its derivatives. If we multiply the equations by \hat{c}^2 and \hat{c}^1 respectively and take the difference, we observe

$$\dot{P}(b^1\hat{c}^2 - b^2\hat{c}^1) = 2P(\dot{b}^1\hat{c}^2 - \dot{b}^2\hat{c}^1 - \zeta\hat{c}^{1'}\hat{c}^2 + \zeta\hat{c}^{2'}\hat{c}^1). \quad (1.7)$$

We will prove that $b^1\hat{c}^2 - b^2\hat{c}^1$ is divisible by P by showing that it vanishes at every root of P . If α is a root of P and not a root of \dot{P} , we see it is a root of $b^1\hat{c}^2 - b^2\hat{c}^1$ immediately from (1.7). Suppose then that P and \dot{P} have a common root α . If $\alpha = 0$, then we know from (0.9) that $b_0^i = 0$ and so ζ divides b^i . If $\alpha \neq 0$, from (1.4) we have that

$$\begin{aligned} \dot{P}(\alpha)b^i(\alpha) &= 2P(\alpha)(\dot{b}^i(\alpha) + \hat{c}^i(\alpha) - \alpha\hat{c}^{i'}(\alpha)) + P'(\alpha)\alpha\hat{c}^i(\alpha) \\ 0 &= 0 + P'(\alpha)\alpha\hat{c}^i(\alpha) \end{aligned}$$

But the assumption that the spectral curve is nonsingular forces $P'(\alpha) \neq 0$. Thus we may conclude that $\hat{c}^i(\alpha) = 0$. Hence P divides $b^1\hat{c}^2 - b^2\hat{c}^1$ and there is some polynomial \hat{Q} , of degree at most four, such that

$$b^1\hat{c}^2 - b^2\hat{c}^1 = \hat{Q}P. \quad (1.8)$$

Thus far, we have only placed two conditions on the points along the deformation ℓ . First, that it must preserve the integral periods of Θ^1 and Θ^2 , which allowed us to produce well-defined meromorphic functions q^i . And second, that the differentials must have double poles over $\zeta = 0$ and ∞ with no residues, which allowed us to write q^i as the quotient of a polynomial \hat{c}^i by $\zeta\eta$. There two additional properties that the polynomials \hat{c}^i must satisfy arising from (P.6) and (P.11). From the former, the eigenvalues satisfy $\rho^*\mu = \bar{\mu}^{-1}$. Applying log and differentiating with respect to t shows that the polynomials \hat{c}^i are imaginary (that is, $i\hat{c}^i$ is a real polynomial with respect to the involution ρ).

Next, consider the closing condition (P.11) in its integral form (0.11). For some consistent choice of q^i along γ_+

$$\int_{\gamma_+} dq^i = q^i(\sigma(\xi_1)) - q^i(\xi_1) \in 2\pi i\mathbb{Z},$$

where ξ_1 is one of the points in Σ over $\zeta = 1$. Hence, as for the periods, the derivative with respect to t of this integral is zero. If we differentiate the above,

we find that $\dot{q}^i(\sigma(\xi_1)) = \dot{q}^i(\xi_1)$. But

$$\dot{q}^i(\sigma(\xi_1)) = \sigma^* \dot{q}^i(\xi_1) = -\dot{q}^i(\xi_1).$$

Thus \tilde{c}^i has a factor of $\zeta - 1$. The same reasoning applied to γ_- leads to a factor of $\zeta + 1$. Therefore let $\hat{c}^i(\zeta) = (\zeta^2 - 1)c^i(\zeta)$, for c^i a real polynomial of degree $g + 1$.

As $\zeta^2 - 1$ is a factor of both polynomials \tilde{c}^i , and P has no zeroes on the unit circle, it follows from (1.8) that $\zeta^2 - 1$ must be a factor of \hat{Q} . Define $\hat{Q} = (\zeta^2 - 1)Q$ to give

$$b^1 c^2 - b^2 c^1 = QP \quad (1.9)$$

for some real quadratic polynomial Q . This equation is of central importance; we shall use it as our starting point for finding infinitesimal deformations, and we shall see that it ensures that the solutions to the two equations (1.4) are consistent with one another.

The final condition on the spectral data that we are yet to satisfy is condition (P.4): that the differentials Θ^1 and Θ^2 are residue free. We shall require $P_1(t)b_0^i(t) - 2P_0(t)b_1^i(t) = 0$ to hold at every point of the deformation ℓ (from (0.9)). Taking derivatives, we see that

$$\dot{P}_1 b_0^i + P_1 \dot{b}_0^i - 2\dot{P}_0 b_1^i - 2P_0 \dot{b}_1^i = 0 \quad (1.10)$$

holds for any for tangent vector $(\dot{P}, \dot{b}^1, \dot{b}^2)$ to \mathcal{M}_g .

In summary, any infinitesimal deformation of the spectral data, in other words a tangent vector $(\dot{P}, \dot{b}^1, \dot{b}^2)$ to \mathcal{M}_g , must give rise to a pair of degree $g + 1$ real polynomials (c^1, c^2) , such that $\hat{c}^i = (\zeta^2 - 1)c^i$ satisfy (1.4), and also satisfy (1.10). The polynomials (c^1, c^2) must themselves satisfy (1.9) for some real quadratic polynomial Q .

1.2 The Tangent Space to \mathcal{M}_g

In the preceding section, we elucidated several properties that a deformation of spectral data necessarily possesses. Now we turn our attention to the converse; under what conditions is it possible to solve (1.4) and (1.9) to find an infinitesimal deformation? Firstly, we examine whether it is possible construct polynomials c^i solving (1.9) for a given Q , and whether this construction is unique. Then secondly, we shall insert the polynomials c^i into the right hand side of (1.4) and solve it to recover \dot{b}^i and \dot{P} .

For each equation, the main obstacle to the existence of a solution is common factors among the polynomials (P, b^1, b^2) . If there are too many common factors, then it will not be possible to deform the spectral data. Even when it is possible to deform, the form of the solution of the equations (1.4) and (1.9) is dependent

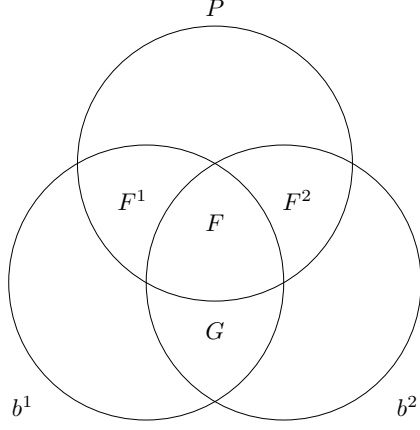


Figure 1.1
The common factors of P , b^1 and b^2 are represented as overlaps between the three circles.

on those common factors. Thus we will need to divide our approach into several cases.

Suppose that $\ell(t) = (P(t), b^1(t), b^2(t))$ is a path in the space of spectral data \mathcal{M}_g , and that at $t = 0$ we have the following common factors

$$\begin{aligned} \gcd(P, b^1, b^2) &= F, \\ \gcd(P/F, b^1/F) &= F^1, \quad \gcd(P/F, b^2/F) = F^2, \\ \gcd(b^1/FF^1, b^2/FF^2) &= G, \end{aligned} \tag{1.11}$$

where we first find the common factor of all three polynomials, then remove any further factors that the differentials and P share, and then finally remove any remaining factors common to b^1 and b^2 . An graphic representation of this process is given in Figure 1.1. We write

$$P = FF^1F^2\tilde{P}, \quad b^1 = FF^1G\tilde{b}^1, \quad b^2 = FF^2G\tilde{b}^2.$$

Because the spectral curve is nonsingular, P has no repeated factors, and so the polynomials F , F^1 , F^2 , \tilde{P} , \tilde{b}^1 and \tilde{b}^2 are pairwise coprime. Be aware that the polynomials b^1 and b^2 may have higher order roots, so it is not possible to say if G is coprime to F , F^1 or F^2 . The common factor of b^1 and b^2 , and therefore any differential on Σ satisfying conditions (P.4)–(P.7), is the product FG . We denote the degrees of the polynomials F , F^1 , F^2 and G as d_F , d_1 , d_2 and d_G respectively.

We may ask, given these common factors among P , b^1 and b^2 , what, if any, factors do the polynomials c^i and Q possess? Inserting these factorisations into

(1.4), we observe that

$$\dot{P}FF^iG\tilde{b}^i = 2FF^1F^2\tilde{P}(\dot{b}^i + \tilde{c}^i - \zeta\tilde{c}^{i'}) + \zeta(\zeta^2 - 1)P'c^i.$$

Again, by the assumption of that the spectral curve is nonsingular, P' does not share any common factors with P . Further P has no roots on the unit circle. Hence we see that FF^i divides ζc^i . Conversely, given an arbitrary c^i , it would not be possible to solve this equation for (\tilde{P}, \tilde{b}^i) unless FF^i divides ζc^i , otherwise we would have a contradiction. We would like to say that FF^i divides c^i alone, but because F may have a factor of ζ , we must treat the conformal and nonconformal cases separately.

Assume first that $P(0) \neq 0$, which corresponds to a nonconformal harmonic map. Then ζ is not a factor of P and so cannot be a factor of FF^i . Therefore FF^i divides c^i . Applying this to (1.9),

$$FF^1G\tilde{b}^1c^2 - FF^2G\tilde{b}^2c^1 = QFF^1F^2\tilde{P}.$$

By definition, neither F nor G divide \tilde{P} , demonstrating that FG divides Q . This provides a bound on the number of coincident roots that are allowed if a deformation is to exist; Q is quadratic so $FG = \gcd(b^1, b^2)$ must be degree two or less. Moreover, because all of P , b^1 , b^2 are real, and P has no roots on the unit circle, any common roots of the three polynomials must come in conjugate inverse pairs and so the degree of F will always be even.

In the conformal case, where the spectral curve is branched over $\zeta = 0$, we know that $P_0(t)$, $b_0^1(t)$ and $b_0^2(t)$ all vanish at $t = 0$. Thus F includes a factor of ζ . This time, we conclude that it is $\zeta^{-1}FF^i$ that divides c^i and so $\zeta^{-1}FG$ divides Q , by (1.9). However, the residue condition (1.10) in this case simplifies in a way that forces another constraint on Q . At $t = 0$, (1.10) becomes

$$P_1\dot{b}_0 - 2\dot{P}_0b_1 = 0,$$

and the terms of linear degree in equation (1.4) read

$$\dot{P}_0b_1^i - 2P_1\dot{b}_0^i = 3P_1\tilde{c}_0^i.$$

Combining these two expressions shows that

$$3P_1\tilde{c}_0^i = \dot{P}_0b_1^i - 4\dot{P}_0b_1^i = -3\dot{P}_0b_1^i.$$

Substituting this into the linear degree of (1.9), we finally arrive at

$$\begin{aligned} Q_0P_1 &= b_1^1c_0^2 - b_1^2c_0^1 \\ Q_0(P_1)^2 &= b_1^1(P_1c_0^2) - b_1^2(P_1c_0^1) \\ &= b_1^1(\dot{P}_0b_1^2) - b_1^2(\dot{P}_0b_1^1) = 0. \end{aligned} \tag{1.12}$$

A spectral curve must be nonsingular at $\zeta = 0$, so if $P_0 = 0$ we can be sure that $P_1 \neq 0$. Hence Q_0 must vanish. As Q is a real quadratic polynomial, it must be

of the form $Q = Q_1\zeta$ for some real number Q_1 . Immediately it follows that if a deformation exists at a point corresponding to a conformal map then $F = \zeta$ and $G = 1$, as the polynomials b^i are not permitted to have multiple roots at $\zeta = 0$.

Thus we have divided our analysis into five cases organised according to the common roots of b^1 and b^2 . There are four nonconformal cases and the conformal case, which are summarised and numbered in the following table.

Case	P_0	$\deg F$	$\deg G$
i	$P_0 \neq 0$	0	0
ii		0	1, 2
iii		2	0
iv		$FG \in \mathcal{P}_{\mathbb{R}}^k, k > 2$	
v	$P_0 = 0$		

The four nonconformal cases may also be described purely in terms of the greatest common factor of b^1 and b^2 .

Case	Description
i	$\gcd(b^1, b^2) = 1$
ii	$\gcd(b^1, b^2)$ is linear or quadratic and does not divide P
iii	$\gcd(b^1, b^2)$ is quadratic and divides P
iv	The degree of $\gcd(b^1, b^2)$ is greater than two

The five sets corresponding to these five cases are disjoint and cover the space \mathcal{U} . We have seen that the spectral data in case (iv) do not admit any deformations. With regards to case (v), by equation (0.9), if P_0 vanishes at a point of \mathcal{M}_g then so too must b_0^1 and b_0^2 . Thus the intersection of \mathcal{M}_g and the points of case (v) is contained in the set the points where ζ divides $\gcd(b^1, b^2)$. The discussion following (1.12) shows that deformations are only possible at a point in case (v) if $\gcd(b^1, b^2) = \zeta$ exactly, otherwise no deformations can exist.

In general, points of \mathcal{M}_g where case (iii) holds are singularities of a deformation [HKS16b] and are not considered further in this thesis. We shall note only that there are no such points of \mathcal{M}_g when the genus g of the spectral curve is zero or one. When the genus is zero, we have seen that the differentials cannot have any common roots at all (Lemma 0.14).

We shall now give a similar short proof to show that the differentials of a genus one spectral curve may not have a common root at the branch points. Suppose that Σ is genus one with branch points at $\alpha, \beta, \bar{\alpha}^{-1}, \bar{\beta}^{-1}$, none of which are zero, and that $\gcd(P, b^1, b^2) = F$ is quadratic. Without loss of generality, let $F = (\zeta - \alpha)(1 - \bar{\alpha}\zeta)$ and $b^1 = (c + d\zeta + \bar{c}\zeta^2)F$ for some complex number c and

real number d . Expanding this and applying (0.9) shows that

$$\begin{aligned} 0 &= (-\alpha(1 + \beta\bar{\beta}) - \beta(1 + \alpha\bar{\alpha}))(-\alpha c) - 2(\alpha\beta)(-\alpha d + (1 + \alpha\bar{\alpha})c) \\ &= (\alpha^2(1 + \beta\bar{\beta}) - \alpha\beta(1 + \alpha\bar{\alpha}))c + 2\alpha^2\beta d. \end{aligned}$$

The coefficient of c above, the bracketed expression, is never zero and so c is determined by d . Hence b^1 is determined up to a real scalar. This demonstrates any two differentials with the same factor F are real linearly dependent, which contradicts (P.9).

Definition 1.13. Let \mathcal{U}' be the subset of points of the two cases (i) and (ii). We denote the points of case (i) as $\mathcal{U}^{(i)}$ and the points of case (v) where $\gcd(b^1, b^2) = \zeta$ as \mathcal{U}'' .

The remainder of the chapter seeks to prove that $\mathcal{M}_g \cap \mathcal{U}'$ is a manifold. We also show that the points of $\mathcal{M}_g \cap \mathcal{U}''$ are smooth points of \mathcal{M}_g . We may characterise \mathcal{U}' as the set

$$\mathcal{U}' = \{ (P, b^1, b^2) \in \mathcal{U} \mid \gcd(P, b^1, b^2) = 1 \text{ and } \gcd(b^1, b^2) \in \mathcal{P}_{\mathbb{R}}^l \text{ for } l \leq 2 \}. \quad (1.14)$$

The set \mathcal{U}' is open because together the points of cases (iii), (iv) and (v) form a closed subset of the space \mathcal{U} , the space of triples of real polynomials satisfying the conditions of Definition 1.1. To see that this complement $\mathcal{U} \setminus \mathcal{U}'$ is closed, we write it as

$$\begin{aligned} &\{ (P, b^1, b^2) \in \mathcal{U} \mid \gcd(b^1, b^2) \in \mathcal{P}_{\mathbb{R}}^k, k > 2 \} \\ &\cup \{ (P, b^1, b^2) \in \mathcal{U} \mid \gcd(P, b^1, b^2) \in \mathcal{P}_{\mathbb{R}}^l, l \geq 2 \} \\ &\cup \{ (P, b^1, b^2) \in \mathcal{U} \mid P_0 = 0 \}, \end{aligned}$$

which is the union of three closed sets.

At each point of \mathcal{U}' and \mathcal{U}'' that lies in \mathcal{M}_g , F^i must divide c^i and G must divide Q . Let us therefore define

$$c^i = F^i \tilde{c}^i, \text{ and } Q = G \tilde{Q}. \quad (1.15)$$

We can then remove the common factor F^i from (1.4) to arrive at the reduced equations

$$\dot{P}G\tilde{b}^i - 2F^j\tilde{P}\dot{b}^i = 2F^j\tilde{P}(\tilde{c}^i - \zeta\tilde{c}^j) + \frac{\zeta}{F}(\zeta^2 - 1)P'\tilde{c}^i, \quad (1.16)$$

for $i = 1, 2$ and $j \neq i$. In the same manner, the Q equation (1.9) reduces to

$$\begin{aligned} FF^1G\tilde{b}^1F^2\tilde{c}^2 - FF^2G\tilde{b}^2F^1\tilde{c}^1 &= G\tilde{Q}FF^1F^2\tilde{P} \\ \tilde{b}^1\tilde{c}^2 - \tilde{b}^2\tilde{c}^1 &= \tilde{Q}\tilde{P}. \end{aligned} \quad (1.17)$$

As commented on previously, these equations (1.16) and (1.17) are necessary conditions to be able to solve (1.4) and (1.9) respectively. If the polynomials c^i or Q did not have the factors indicated by (1.15), then there would be factors on the left hand sides of (1.4) and (1.9) that did not appear on the right hand sides, and this contradiction would preclude the possibility of a solution.

Both of these equations are in the form $AX - BY = C$, to which Bézout's identity for polynomials applies. Bézout's identity asserts that if $\gcd(A, B) = 1$, then there is a unique solution (\mathbf{X}, \mathbf{Y}) of minimal degree, for $\deg \mathbf{X} < \deg B$ and $\deg \mathbf{Y} < \deg A$. Recall the notation introduced at the start of the chapter that given a pair of polynomials (X, Y) , we also give their degrees as a pair (x, y) , for $x = \deg X$ and $y = \deg Y$.

One may prove the existence of these minimal solutions using the Euclidean algorithm [Mor03]. Because we will have need of the specific formula for the leading coefficients of \mathbf{X} and \mathbf{Y} , we shall give an effective method to construct this minimal solution. We shall also prove some simple corollaries, about when real solutions exist, and given the unique minimal solution, how to generate all solutions less than a given degree.

We shall construct the minimal solution by finding a linear system of equations that its coefficients must satisfy, and showing that there is a unique solution to this linear system. The solution to the linear system and the solution to Bézout's identity must therefore be the same. Assume that A and B are coprime and let (\mathbf{X}, \mathbf{Y}) be the solution of minimal degree. If β is a root of B of multiplicity r and the degree of B is $n + 1$, then

$$X_0 + X_1\beta + X_2(\beta)^2 + \cdots + X_n\beta^n = \frac{C(\beta)}{A(\beta)},$$

is a linear equation in the coefficients X_i of \mathbf{X} . Note that this is well-defined because $A(\beta) \neq 0$. If β is a higher order root of B , then we may differentiate to obtain another relation,

$$\begin{aligned} \mathbf{X} &= \frac{C}{A} - B \frac{\mathbf{Y}}{A} \\ \mathbf{X}' &= \left(\frac{C}{A}\right)' - B' \frac{\mathbf{Y}}{A} - B \left(\frac{\mathbf{Y}}{A}\right)' \\ \mathbf{X}'(\beta) &= X_1 + 2X_2\beta + \cdots + nX_n\beta^{n-1} = \left(\frac{C}{A}\right)'(\beta), \end{aligned}$$

Differentiating repeatedly, we will obtain r linearly independent equations at the root β . If we label the distinct roots of B as β_i and their multiplicities as

r_i , then the full system of equations is

$$\begin{bmatrix} 1 & \beta_1 & (\beta_1)^2 & \dots & (\beta_1)^{r_1-1} & \dots & (\beta_1)^n \\ 0 & 1 & 2\beta_1 & \dots & (r_1-1)(\beta_1)^{r_1-2} & \dots & n(\beta_1)^{n-1} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & \dots & \frac{n!}{(n+1-r_1)!}(\beta_1)^{n+1-r_1} \\ 1 & \beta_2 & (\beta_2)^2 & \dots & (\beta_2)^{r_2-1} & \dots & (\beta_2)^n \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & \dots & \frac{n!}{(n+1-r_2)!}(\beta_2)^{n+1-r_2} \\ 1 & \beta_3 & (\beta_3)^2 & \dots & (\beta_3)^{r_2-1} & \dots & (\beta_3)^n \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & \dots & \frac{n!}{(n+1-r_k)!}(\beta_k)^{n+1-r_k} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} (C/A)(\beta_1) \\ (C/A)'(\beta_1) \\ \vdots \\ (C/A)^{(r_1-1)}(\beta_1) \\ (C/A)(\beta_2) \\ \vdots \\ (C/A)^{(r_2-1)}(\beta_2) \\ (C/A)(\beta_3) \\ \vdots \\ \vdots \\ (C/A)^{(r_k-1)}(\beta_k) \end{bmatrix}, \quad (1.18)$$

The $(n+1) \times (n+1)$ coefficient matrix on the left is called the confluent Vandermonde matrix at the roots of B , and we shall denote it $V(B)$. We shall denote the vector on the right by $h(B, C/A)$. A confluent Vandermonde matrix is always nonsingular [Kal84], therefore there is a unique solution to this system. One can likewise consider the system of equations at the roots of A to construct the solution \mathbf{Y} .

We have shown how to construct the minimal solution to the equation $AX - BY = C$. But in fact we will be interested in all solutions with degree less than a fixed value. To this end, consider the related equation $AX - BY = 0$. As we have assumed that A and B are coprime, it must be that B divides X . Likewise A divides Y . Return now to the equation $AX - BY = C$ and suppose then that (\mathbf{X}, \mathbf{Y}) the minimal solution. If (X, Y) is any other solution, then $A(\mathbf{X} - X) - B(\mathbf{Y} - Y) = 0$. Therefore $\mathbf{X} - X = rB$ and $\mathbf{Y} - Y = rA$ for some polynomial r . Conversely, given any solution (X, Y) , it is clear that $(X + rB, Y + rA)$ is again a solution for every polynomial r . Thus polynomials of this form are exactly the solutions to $AX - BY = C$.

Finally, we show that if the polynomials A, B, C are real, and the solution (X, Y) has the appropriate degree, then there are real solutions of the same degree. Let the polynomials A, B and C lie in $\mathcal{P}_{\mathbb{R}}^a, \mathcal{P}_{\mathbb{R}}^b$ and $\mathcal{P}_{\mathbb{R}}^c$ respectively, for integers a, b and c . Further suppose that there exists a solution (X, Y) exist of degree $(c - a, c - b)$ or less. Then observe

$$\begin{aligned} C(\zeta) &= \overline{\zeta^c C(\bar{\zeta}^{-1})} = \overline{\zeta^c A(\bar{\zeta}^{-1}) X(\bar{\zeta}^{-1})} - \overline{\zeta^c B(\bar{\zeta}^{-1}) Y(\bar{\zeta}^{-1})} \\ &= A(\zeta) \overline{\zeta^{c-a} X(\bar{\zeta}^{-1})} - B(\zeta) \overline{\zeta^{c-b} Y(\bar{\zeta}^{-1})} \\ C(\zeta) &= A(\zeta) \frac{1}{2} \left(X(\zeta) + \overline{\zeta^{c-a} X(\bar{\zeta}^{-1})} \right) - B(\zeta) \frac{1}{2} \left(Y(\zeta) + \overline{\zeta^{c-b} Y(\bar{\zeta}^{-1})} \right), \end{aligned}$$

which demonstrates that X lies in $\mathcal{P}_{\mathbb{R}}^{c-a}$ and $Y \in \mathcal{P}_{\mathbb{R}}^{c-b}$.

To see how this applies to the current situation, as each of b^1, b^2, P and Q are real, (1.9) will have real solutions. It is not obvious that the right hand side of (1.4) is real however. In particular, we must see how to take the real involution of a derivative. We compute the following, supposing f is real of degree k

$$\begin{aligned}\zeta^k \bar{f}(\zeta^{-1}) &= f(\zeta) \\ k\zeta^{k-1} \bar{f}(\zeta^{-1}) - \zeta^{k-2} \bar{f}'(\zeta^{-1}) &= f'(\zeta) \\ \rho^*(\zeta f') &= \zeta^{k-1} \bar{f}'(\zeta^{-1}) = kf(\zeta) - \zeta f'(\zeta).\end{aligned}$$

Thus we can compute the involution of the right hand side of (1.4).

$$\begin{aligned}\rho^*(2P\hat{c}^i - 2P\zeta\hat{c}^{i'} + \zeta P'\hat{c}^i) &= -2P\hat{c}^i - 2P(-(g+3)\hat{c}^i + \zeta\hat{c}^{i'}) - ((2p+2)P - \zeta P')\hat{c}^i \\ &= (-2+2g+6-2g-2)P\hat{c}^i - 2P\zeta\hat{c}^{i'} + \zeta P'\hat{c}^i \\ &= 2P\hat{c}^i - 2P\zeta\hat{c}^{i'} + \zeta P'\hat{c}^i\end{aligned}$$

We have demonstrated that this is a real polynomial whenever P is real and \hat{c}^i is imaginary, not just when these polynomials arise from a deformation.

In case (i), given an arbitrary quadratic polynomial Q , it will not always be possible to solve (1.17) for polynomials $\tilde{c}^i(\zeta)$ corresponding to an infinitesimal deformation $(\dot{P}, \dot{b}^1, \dot{b}^2)$. To see why, consider the linear system of equations in the coefficients of \tilde{c}^2 that arises from evaluating (1.17) at the roots of \tilde{b}^2 ,

$$V(\tilde{b}^2) \begin{bmatrix} \tilde{c}_0^2 \\ \vdots \\ \tilde{c}_n^2 \end{bmatrix} = h \left(\tilde{b}^2, \frac{Q\tilde{P}}{\tilde{b}^1} \right). \quad (1.19)$$

We know that the degree of the unique minimal solution could be as high as

$$n := \deg \tilde{b}^2 - 1 = g + 2 - d_2.$$

But c^2 is degree $g+1$, and $c^2 = F^2 \tilde{c}^2$. Therefore the degree of \tilde{c}^2 is $g+1-d_2 = n-1$. If every solution of (1.17) is degree n or more, then there can be no solutions that correspond to infinitesimal deformations. Thus we must introduce a restriction on our choice of Q , so that a solution to (1.17) of the correct degree exists. We will express this restriction in terms of the vanishing of a function R .

Definition 1.20. Recall the confluent Vandermonde matrix $V(B)$ and vector $h(B, C/A)$ defined in (1.18). We define the function R to be

$$\begin{aligned}R : \mathcal{U}^{(i)} \times \mathbb{R}^3 &\rightarrow \mathbb{C} \\ ((P, b^1, b^2), Q) &\mapsto \text{the last entry of } [V(\tilde{b}^2)]^{-1} h \left(\tilde{b}^2, \frac{Q\tilde{P}}{\tilde{b}^1} \right).\end{aligned}$$

When the point of $\mathcal{U}^{(i)}$ is understood, we shall abbreviate this to $R(Q)$.

This function R is simply the function that gives the value of the degree n coefficient of \tilde{c}^2 ; the condition that $R((P, b^1, b^2), Q) = 0$ is equivalent to the condition that there is a solution \tilde{c}^2 to (1.17) of degree $n - 1$ or less. Likewise, evaluating (1.17) at the roots of \tilde{b}^1 leads to a solution \tilde{c}^1 of degree $g + 2 - d_1$. From the highest order term of (1.17), if \tilde{c}^2 is degree $n - 1$ or lower, then \tilde{c}^1 will be degree $g + 1 - d_1$ or lower without any further restrictions on Q .

It is important to note that R is linear function in the coefficients of Q , so that the solutions of $R(Q) = 0$ form a linear space. Intuitively, because R is a complex valued function, we would expect this linear space of Q to be real-codimension two. However, R satisfies the following reality type condition, and thus $R(Q) = 0$ puts only one real constraint on Q . In other words, at any point of $\mathcal{U}^{(i)}$ there is a two real-dimensional plane of real quadratic polynomials Q that satisfy $R(Q) = 0$.

Lemma 1.21. *At every point (P, b^1, b^2) of $\mathcal{U}^{(i)}$, $R((P, b^1, b^2), Q)$ satisfies the relation*

$$\overline{R} = (-1)^n \left(\prod_{i=1}^{n+1} \beta_i \right) R, \quad (1.22)$$

where β_i are the $n + 1 = g + 3 - d_2$ roots of \tilde{b}^2 counted with multiplicity.

Proof. We shall demonstrate this property first at points where the roots of \tilde{b}^2 are distinct. Let \tilde{b}^2 have $n + 1 = g + 3 - d_2$ distinct roots β_i . In this case, the explicit form of the solution to the linear system of equations (1.19) is elegant. Consider the Lagrange polynomials at the roots of \tilde{b}^2 ,

$$L_i(\zeta) := \prod_{j \neq i} \frac{\zeta - \beta_j}{\beta_i - \beta_j}.$$

Each of these polynomials is degree n and has the property that $L_i(\beta_j) = \delta_{ij}$. The unique polynomial of degree at most n solving the linear system is

$$\tilde{c}^2(\zeta) = \sum_{i=1}^{n+1} \left(\frac{Q\tilde{P}}{\tilde{b}^1} \right) (\beta_i) L_i(\zeta),$$

and in particular the highest coefficient is R ,

$$R = \sum_{i=1}^{n+1} \left(\frac{Q\tilde{P}}{\tilde{b}^1} \right) (\beta_i) \prod_{j \neq i} (\beta_i - \beta_j)^{-1}.$$

Because \tilde{b}^2 is a real polynomial, its set of roots is invariant under $\zeta \mapsto \bar{\zeta}^{-1}$. This creates an involution on the set of roots. Let τ be the involution on the integers $\{1, 2, \dots, n + 1\}$ such that $\beta_{\tau(i)} = \bar{\beta}_i^{-1}$. Note that

$$\overline{\left(\frac{Q\tilde{P}}{\tilde{b}^1} \right) (\beta_i)} = \frac{Q(\beta_i)\tilde{P}(\beta_i)}{\tilde{b}^1(\beta_i)} = \bar{\beta}_i^{g+1-d_2} \frac{Q(\bar{\beta}_i^{-1})\tilde{P}(\bar{\beta}_i^{-1})}{\tilde{b}^1(\bar{\beta}_i^{-1})} = \bar{\beta}_i^{n-1} \left(\frac{Q\tilde{P}}{\tilde{b}^1} \right) (\beta_{\tau(i)}),$$

and

$$\begin{aligned} \prod_{j \neq i} (\bar{\beta}_i - \bar{\beta}_j)^{-1} &= \prod_{j \neq i} \bar{\beta}_i^{-1} \bar{\beta}_j^{-1} (\bar{\beta}_j^{-1} - \bar{\beta}_i^{-1})^{-1} \\ &= (\bar{\beta}_i^{-1})^n \left(\prod_{j \neq i} \beta_{\tau(j)} \right) (-1)^n \prod_{j \neq i} (\beta_{\tau(i)} - \beta_{\tau(j)})^{-1}, \end{aligned}$$

so that the conjugate of R is

$$\begin{aligned} \bar{R} &= \sum_{i=1}^{n+1} \left(\frac{Q\tilde{P}}{\tilde{b}^1} \right) (\beta_i) \prod_{j \neq i} (\bar{\beta}_i - \bar{\beta}_j)^{-1} \\ &= \sum_{i=1}^{n+1} \left(\frac{Q\tilde{P}}{\tilde{b}^1} \right) (\beta_{\tau(i)}) \bar{\beta}_i^{-1} \left(\prod_{j \neq i} \beta_{\tau(j)} \right) (-1)^n \prod_{j \neq i} (\beta_{\tau(i)} - \beta_{\tau(j)})^{-1} \\ &= (-1)^n \left(\prod_{i=1}^{n+1} \beta_i \right) R. \end{aligned}$$

To extend this argument to a point $p = (P, b^1, b^2)$ where the roots of \tilde{b}^2 are not distinct, we may construct a sequence of points $p_k = (P(p_k)(\zeta), b^1(p_k)(\zeta), b^2(p_k)(\zeta))$ of $\mathcal{U}^{(i)}$ that converge to the point p , but such that $b^2(p_k)(\zeta)$ has distinct roots for every k . We will show that on this sequence $\lim_{k \rightarrow \infty} R(p_k, Q) = R(p, Q)$, and therefore

$$\begin{aligned} \overline{R(p, Q)} &= \lim_{k \rightarrow \infty} \overline{R(p_k, Q)} = \lim_{k \rightarrow \infty} (-1)^n \left(\prod_{i=1}^{n+1} \beta_i(p_k) \right) R(p_k, Q) \\ &= (-1)^n \left(\prod_{i=1}^{n+1} \beta_i \right) R(p, Q). \end{aligned}$$

Suppose that we are at a point $p = (P, b^1, b^2)$ of $\mathcal{U}^{(i)}$ where \tilde{b}^2 has a double root β . Considering the subvariety of $\mathcal{U}^{(i)}$ where $F^2 = \gcd(P, b^2)$ is fixed, we may find a sequence of points $p_k = (P(p_k)(\zeta), b^1(p_k)(\zeta), b^2(p_k)(\zeta))$ converging to p with the property that $\gcd(P(p_k), b^2(p_k)) = F^2$ and such that the roots of each polynomial $\tilde{b}^2(p_k)$ are distinct. Let us label the two simple roots of $\tilde{b}^2(p_k)$ that coalesce at p to form the double root β as $\beta_1(k), \beta_2(k)$. In other words, these are the two roots of $\tilde{b}^2(p_k)$ such that $\lim_{k \rightarrow \infty} \beta_1(p_k) = \lim_{k \rightarrow \infty} \beta_2(p_k) = \beta$. The corresponding rows of the Vandermonde matrix $V(\tilde{b}^2(p_k))$ in the system (1.19)

are

$$\begin{bmatrix} 1 & \beta_1 & (\beta_1)^2 & \cdots & (\beta_1)^n \\ 1 & \beta_2 & (\beta_2)^2 & \cdots & (\beta_2)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} \tilde{c}_0^2 \\ \tilde{c}_1^2 \\ \vdots \\ \tilde{c}_n^2 \end{bmatrix} = \begin{bmatrix} \left(\frac{Q\tilde{P}}{b^1}\right)(\beta_1) \\ \left(\frac{Q\tilde{P}}{b^1}\right)(\beta_2) \\ \vdots \end{bmatrix}.$$

Performing elementary row operations does not change the solution to this system, and so we may subtract the first row from the second and scale it by $(\beta_2 - \beta_1)^{-1}$. This gives

$$\begin{bmatrix} 1 & \beta_1 & (\beta_1)^2 & \cdots & (\beta_1)^n \\ 0 & 1 & \beta_2 + \beta_1 & \cdots & \sum_{j=0}^{n-1} (\beta_1)^j (\beta_2)^{n-1-j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} \tilde{c}_0^2 \\ \tilde{c}_1^2 \\ \vdots \\ \tilde{c}_n^2 \end{bmatrix} = \begin{bmatrix} \left(\frac{Q\tilde{P}}{b^1}\right)(\beta_1) \\ (\beta_2 - \beta_1)^{-1} \left[\left(\frac{Q\tilde{P}}{b^1}\right)(\beta_2) - \left(\frac{Q\tilde{P}}{b^1}\right)(\beta_1) \right] \\ \vdots \end{bmatrix}.$$

The limit of the above as $k \rightarrow \infty$ is precisely the confluent Vandermonde matrix at $\tilde{b}^2(p)$. The calculation for higher order roots is similar. If \tilde{b}^2 has more than one higher order root, then we may perform this operation concurrently for each of them.

Combining these row operations with the fact that inversion of a matrix is continuous and that the roots of a polynomial are continuous functions of coefficients [Whi72, Theorem V.4A], this shows that limit of the solutions $\tilde{c}^2(p_k)$ is just the solution $\tilde{c}^2(p)$. In particular, the last component of $\tilde{c}^2(p)$ is $R(p, Q)$, and this is therefore the limit of $R(p_k, Q)$ as $k \rightarrow \infty$.

Hence we have established (1.22) at all points of $\mathcal{U}^{(i)}$. \square

In cases (ii) and (v), the degree of \tilde{b}^2 is

$$g + 3 - (d_2 + d_F + d_G) \leq g + 2 - d_2,$$

as either F or G will be nontrivial. This means that the minimal solution to (1.17), which has degree strictly less than $\deg \tilde{b}^2$, will have the correct degree without needing to impose any extra conditions on Q .

At this point we have shown how to solve equations such as (1.16) and (1.17), when the solutions will be have the correct degree, how to determine all of the solutions when the solution is not unique, and the existence of real solutions. We are therefore ready to solve these equations in each of the three cases (i), (ii), and (v), and to use the solutions to solve (1.4) and (1.9). At the end of that process, we will have constructed a tangent vector $(\dot{P}, \dot{b}^1, \dot{b}^2)$ to the moduli space of spectral data \mathcal{M}_g .

Lemma 1.23 (Case (i)). *Take a triple of spectral data $(P, b^1, b^2) \in \mathcal{M}_g$ associated with a nonconformal harmonic map, with a nonsingular spectral curve given by $\eta^2 = P$ of genus g . Suppose that $\gcd(b^1, b^2) = 1$. Then for every polynomial $Q \in \mathcal{P}_{\mathbb{R}}^2$ with $R(Q) = 0$, there exist unique real polynomials $c^i \in \mathcal{P}_{\mathbb{R}}^{g+1}$ that satisfy (1.9). Further, for each such pair (c^1, c^2) , there is a unique triple $(\dot{P}, \dot{b}^1, \dot{b}^2) \in \mathcal{P}_{\mathbb{R}}^{2g+2} \times \mathcal{P}_{\mathbb{R}}^{g+3} \times \mathcal{P}_{\mathbb{R}}^{g+3}$ that satisfies (1.4) and (1.10). This triple is therefore a tangent vector to the space of spectral data \mathcal{M}_g .*

Proof. In order to solve (1.4) or (1.9), one must first solve their reduced counterparts (1.16) and (1.17).

We consider (1.17) as a linear system in the coefficients of \tilde{c}^i , using a confluent Vandermonde matrix as above. Thus there is a unique solution $(\tilde{c}^1, \tilde{c}^2)$ of degree at most $(g+2-d_1, g+2-d_2)$, where $d_i = \deg F^i$. We note that $\tilde{c}_{g+2-d_2}^2$ is $R(Q)$ by definition, which by assumption is zero. By consideration of the leading order of (1.17), if $\tilde{c}_{g+2-d_2}^2$ vanishes, so too must $\tilde{c}_{g+2-d_1}^1$. Multiplying (1.17) through by $F^1 F^2$, we arrive at unique solutions $c^i = F^i \tilde{c}^i$ for (1.9). Both of these polynomials are degree at most $g+1$. We similarly define $\hat{c}^i = (\zeta^2 - 1)c^i$.

Next we must solve (1.16), for which (1.17) was a necessary condition. It reads

$$\dot{P}\tilde{b}^i - 2F^j \tilde{P}\dot{b}^i = 2F^j \tilde{P}(\hat{c}^i - \zeta \hat{c}^{i'}) + \zeta(\zeta^2 - 1)P'\tilde{c}^i. \quad (1.24)$$

This too can be solved using a (confluent) Vandermonde matrix, if \tilde{b}^i is non-vanishing at the roots of $F^j \tilde{P}$ and vice versa. But by definition they are coprime. This was why it was necessary to force c^i to have F^i as a factor, so that the common factor F^1 of b^i and P could be removed from (1.4).

One should be concerned that the two equations for $i = 1, 2$ may give different solutions for \dot{P} , and indeed in general they do. However, neither of the equations have unique solutions, and we shall use the freedom in the choice of solution to find a common solution to both. Let a solution to each equation (1.24) be (\dot{P}^1, \dot{b}^1) and (\dot{P}^2, \dot{b}^2) . The sets of solutions of degree $(2g+2, g+3)$ are

$$\left\{ \left(\dot{P}^1 + 2rF^2 \tilde{P}, \dot{b}^1 + r\tilde{b}^1 \right) \mid r \text{ a real polynomial of degree } d_1 \right\},$$

and

$$\left\{ \left(\dot{P}^2 + 2sF^1 \tilde{P}, \dot{b}^2 + s\tilde{b}^2 \right) \mid s \text{ a real polynomial of degree } d_2 \right\},$$

respectively. First note that every element of both of these sets take the same value at any root α of \tilde{P} . This follows from (1.24), which reads

$$\dot{P}^i(\alpha)\tilde{b}^i(\alpha) = \alpha(\alpha^2 - 1)P'(\alpha)\tilde{c}^i(\alpha).$$

By definition, roots of \tilde{P} are roots of P that are not common to either \tilde{b}^1 or \tilde{b}^2 . From (1.17) we see that

$$\tilde{b}^1(\alpha)\tilde{c}^2(\alpha) - \tilde{b}^2(\alpha)\tilde{c}^1(\alpha) = Q(\alpha)\tilde{P}(\alpha) = 0,$$

and thus

$$\dot{P}^1(\alpha) = \alpha(\alpha^2 - 1)P'(\alpha)\frac{\tilde{c}^1(\alpha)}{\tilde{b}^1(\alpha)} = \alpha(\alpha^2 - 1)P'(\alpha)\frac{\tilde{c}^2(\alpha)}{\tilde{b}^2(\alpha)} = \dot{P}^2(\alpha).$$

At the d_1 roots of F^1 , we see that every solution \dot{P}^2 takes the same value. Let β be such a root, then

$$\dot{P}^2(\beta) = \dot{P}^2(\beta) + 2s(\beta)F^1(\beta)\tilde{P}(\beta) = \dot{P}^2(\beta) = \beta(\beta^2 - 1)P'(\beta)\frac{\tilde{c}^2(\beta)}{\tilde{b}^2(\beta)},$$

where we can be sure that $\tilde{b}^2(\beta) \neq 0$ because it cannot be a root of b^2 (if it were, $F \neq 1$). However, the other solutions have different values at β , and this provides the following constraint on the choice of r :

$$\dot{P}^1(\beta) + 2r(\beta)F^2(\beta)\tilde{P}(\beta) = \dot{P}^2(\beta) = \beta(\beta^2 - 1)P'(\beta)\frac{\tilde{c}^2(\beta)}{\tilde{b}^2(\beta)}.$$

This constraint is nontrivial because β is not a root of \tilde{P} or F^2 by the assumption of the nonsingularity of the spectral curve. As F^1 has d_1 distinct roots, there are d_1 constraints.

Likewise, at the d_2 roots of F^2 , we acquire constraints on the choice of s . It is always possible to meet these constraints (because, for example, the degree of s is d_2 and there are only d_2 roots of F^2), so we see that there is a common solution $(\dot{P}, \dot{b}^1, \dot{b}^2)$ to (1.16). It is also a solution to (1.4).

This solution is still not unique; there remains one degree of freedom. For any real number s , we have solutions to (1.4) of the form

$$\begin{aligned} \dot{P} &= \dot{P} + 2sP \\ \dot{b}^i &= \dot{b}^i + sb^i \end{aligned} \tag{1.25}$$

However, this freedom is simply the freedom to rescale P . We have chosen a preferred scaling of P , so our choice of s is determined. Explicitly, if we were to allow other scalings, the formula for P would be, (cf. (0.7))

$$P(t) = r(t) \prod_k (\zeta - \alpha_k(t))(1 - \bar{\alpha}_k(t)\zeta),$$

where α_k are the roots inside the unit circle and $r(t)$ is some real function. Then from any solution \dot{P} we can determine the derivatives of the roots α_k at $t = 0$. Simply differentiate P and evaluate at α_k

$$\begin{aligned} \dot{P} &= \dot{r} \prod_k (\zeta - \alpha_k)(1 - \bar{\alpha}_k\zeta) \\ &\quad + r(0) \sum_k (-\dot{\alpha}_k + (\dot{\alpha}_k \bar{\alpha}_k + \alpha_k \dot{\bar{\alpha}}_k)\zeta - \dot{\bar{\alpha}}_k \zeta^2) \prod_{m \neq k} (\zeta - \alpha_m)(1 - \bar{\alpha}_m \zeta) \end{aligned} \tag{1.26}$$

$$\dot{P}(\alpha_k) = -\dot{\alpha}_k(1 + \alpha_k \bar{\alpha}_k) \prod_{m \neq k} (\alpha_k - \alpha_m)(1 - \bar{\alpha}_m \alpha_k).$$

Thus we know the values of $\dot{\alpha}_k$, independent of choice of s in our solution (1.25), because any two solutions \dot{P} differ by a multiple of P , which vanishes at every root α_k . Alternatively, if we take the lowest order of (1.26),

$$\dot{P}_0 = \dot{P}_0 + 2sP_0 = \dot{r}P_0 + \sum_k (-\dot{\alpha}_k) \prod_{m \neq k} (-\alpha_m),$$

so we may ensure that $r \equiv 1$ by choosing s so that $\dot{r} = 0$. In short, if we fix a scaling of the spectral curve, then there is a unique solution to (1.4).

Finally then there is a second necessary condition that must be satisfied by our solution $(\dot{P}, \dot{b}^1, \dot{b}^2)$. We must satisfy (1.10), so that (0.8) holds along the path. But this condition is satisfied automatically. Observe

$$\begin{aligned} \dot{P}_1 \dot{b}_0^i + P_1 \dot{b}_0^i - 2(\dot{P}_0 \dot{b}_1^i + P_0 \dot{b}_1^i) &= P_1 \dot{b}_0^i - 2\dot{P}_0 \dot{b}_1^i + 3P_1 \hat{c}_0 - P_0 \dot{b}_1^i + 2P_1 \dot{b}_0^i \\ &= 3(P_1 \dot{b}_0^i - \dot{P}_0 \dot{b}_1^i + P_1 \hat{c}_0) \\ &= \frac{3}{P_0} \left(P_0 P_1 \dot{b}_0^i - P_0 \dot{P}_0 \dot{b}_1^i + P_1 \left(\frac{1}{2} \dot{P}_0 \dot{b}_0^i - P_0 \dot{b}_0^i \right) \right) \\ &= \frac{3\dot{P}_0}{P_0} \left(-P_0 \dot{b}_1^i + \frac{1}{2} P_1 \dot{b}_0^i \right) \\ &= 0 \end{aligned}$$

The substitution in the first line comes from the ζ^1 terms of (1.9), the third line from the constant terms of (1.9) and the last line comes from the fact that the quantity in the bracket is exactly the residue at $\zeta = 0$, which is zero by the assumption that (P, b^1, b^2) lies in \mathcal{M}_g , the space of spectral data.

Hence $(\dot{P}, \dot{b}^1, \dot{b}^2)$ is a tangent vector to \mathcal{M}_g at (P, b^1, b^2) . \square

Lemma 1.27 (Case (ii)). *Take a triple of spectral data $(P, b^1, b^2) \in \mathcal{M}_g$ associated with a nonconformal harmonic map, with a nonsingular spectral curve given by $\eta^2 = P$ of genus g . Suppose that $G = \gcd(b^1, b^2)$ is a non-constant polynomial, that is real with respect to the involution ρ and that does not divide P . If G lies in $\mathcal{P}_{\mathbb{R}}^1$ then for every polynomial $\tilde{Q} \in \mathcal{P}_{\mathbb{R}}^1$, or if G lies in $\mathcal{P}_{\mathbb{R}}^2$ then for every pair of real numbers (\tilde{Q}, r) , there is a unique triple $(\dot{P}, \dot{b}^1, \dot{b}^2) \in \mathcal{P}_{\mathbb{R}}^{2g+2} \times \mathcal{P}_{\mathbb{R}}^{g+3} \times \mathcal{P}_{\mathbb{R}}^{g+3}$ that satisfies (1.4) and (1.10). This triple is therefore a tangent vector to the space of spectral data \mathcal{M}_g .*

Proof. This lemma, and the lemma to follow, are similar to the first proof. We proceed by first solving the reduced equations (1.16) and (1.17) and using the solutions to those equations to establish solutions to (1.4) and (1.9). Regardless of the degree of G , which recall is denoted d_G , we must set $Q = G\tilde{Q}$. (1.17) reads

$$\tilde{b}^1 \tilde{c}^2 - \tilde{b}^2 \tilde{c}^1 = \tilde{Q} \tilde{P}.$$

There is a unique solution to this equation $(\tilde{c}^1, \tilde{c}^2)$ of degree at most $(g+2-d_1-d_G, g+2-d_2-d_G)$. If we multiply these by F^1 and F^2 respectively, we have solutions to (1.9) of degree at most $g+2-d_G$ each. If G is linear therefore, this is the unique solution of degree at most $(g+1, g+1)$, but if G is quadratic the space of solutions to (1.9) is

$$\left\{ (F^1 \tilde{c}^1 + r F^1 \tilde{b}^1, F^2 \tilde{c}^1 + r F^2 \tilde{b}^2) \mid r \text{ a real scalar} \right\}.$$

Hence for every $r \in \mathbb{R}$ there is a unique solution (c^1, c^2) to equation (1.9). In either case, it was not necessary to have a condition similar to $R(Q) = 0$, but conversely the choice of Q was restricted by $Q = G\tilde{Q}$.

Next we must solve (1.16), but the proof above applies again, essentially without modification. The equation in this case is

$$\dot{P}G\tilde{b}^i - 2F^j \tilde{P}\dot{b}^i = 2F^j \tilde{P}(\tilde{c}^i - \zeta \tilde{c}^{i'}) + \zeta(\zeta^2 - 1)P'\tilde{c}^i. \quad (1.28)$$

This has a solution because $\gcd(F^j \tilde{P}, G\tilde{b}^i) = 1$. Analysis at the roots of $F^1 F^2 \tilde{P}$ shows that there is a common solution $(\tilde{P}, \tilde{b}^1, \tilde{b}^2)$. Again, a choice of scaling of P forces a unique solution. This solution also satisfies (1.10). Hence it is a tangent vector to \mathcal{M}_g at (b^1, b^2, P) . \square

Lemma 1.29 (Case (v)). *Take a triple of spectral data $(P, b^1, b^2) \in \mathcal{M}_g$ associated with a conformal harmonic map, with a nonsingular spectral curve given by $\eta^2 = P$ of genus g . Suppose that $\gcd(b^1, b^2) = \zeta$. Then for every pair of real numbers (Q_1, r) , there is a unique triple $(\tilde{P}, \tilde{b}^1, \tilde{b}^2) \in \mathcal{P}_{\mathbb{R}}^{2g+2} \times \mathcal{P}_{\mathbb{R}}^{g+3} \times \mathcal{P}_{\mathbb{R}}^{g+3}$ that satisfies (1.4) and (1.10). This triple is therefore a tangent vector to the space of spectral data \mathcal{M}_g .*

Proof. Recall that the condition for a triple of spectral data to be associated to a conformal harmonic map is that the spectral curve is branched over $\zeta = 0$ and ∞ . Thus $P(0) = P_0 = 0$. From (0.9), $b_0^i = 0$ also. We may write therefore that $P = \zeta F^1 F^2 \tilde{P}$ and $b^i = \zeta F^i \tilde{b}^i$, where \tilde{P} is a real polynomial of degree $2g-d_1-d_2$ and the polynomials \tilde{b}^i are real, coprime and degree $g+1-d_i$.

We have already demonstrated in (1.12) that ζ necessarily divides Q . Thus (1.17) is simply

$$\tilde{b}^1 \tilde{c}^2 - \tilde{b}^2 \tilde{c}^1 = \zeta Q_1 \tilde{P}. \quad (1.30)$$

This is similar to the above case where G was quadratic. The space of solutions is

$$\left\{ (\tilde{c}^1 + r \tilde{b}^1, \tilde{c}^2 + r \tilde{b}^2) \mid r \in \mathbb{R} \right\},$$

where $(\tilde{c}^1, \tilde{c}^2)$ is the unique solution of degree at most $(g+1-d_1, g+1-d_2)$. For every such solution, let $c^i = F^i \tilde{c}^i$ and consider the corresponding (1.16), namely

$$\dot{P}\tilde{b}^i - 2F^j \tilde{P}\dot{b}^i = 2F^j \tilde{P}(\tilde{c}^i - \zeta \tilde{c}^{i'}) + (\zeta^2 - 1)P'\tilde{c}^i. \quad (1.31)$$

In what is by now a familiar story, for the dotted quantities there is a common solution $(\dot{\mathbf{P}}, \dot{\mathbf{b}}^1, \dot{\mathbf{b}}^2)$. The space of solutions is however

$$\left\{ (\dot{\mathbf{P}} + 2sF^1F^2\tilde{\mathbf{P}}, \dot{\mathbf{b}}^1 + sF^1\tilde{\mathbf{b}}^1, \dot{\mathbf{b}}^2 + sF^2\tilde{\mathbf{b}}^2) \mid s \text{ a real quadratic polynomial} \right\}.$$

There appears not to be a unique tangent vector corresponding to each choice (Q, r) . However, unlike the previous two cases, condition (1.10) is not automatically satisfied. Let $s = s_0 + s_1\zeta + \bar{s}_0\zeta^2$. For $i = 1$, we see that the condition implies that

$$2\dot{P}_0b_1^1 - P_1\dot{b}_0^1 + 3s_0P_1b_1^1 = 0,$$

which fully determines s_0 . Can we therefore simultaneously satisfy the condition for $i = 2$? Note that (1.31) in the lowest degree reads

$$\dot{P}_0b_1^i - 2P_1\dot{b}_0^i = -3P_1c_0^i,$$

and (1.30) in the lowest degree yields

$$\begin{aligned} b_1^1c_0^2 &= b_1^2c_0^1 \\ b_1^1(\dot{P}_0b_1^2 - 2P_1\dot{b}_0^2) &= b_1^2(\dot{P}_0b_1^1 - 2P_1\dot{b}_0^1) \\ 2b_1^1\dot{b}_0^2 &= 2b_1^2\dot{b}_0^1. \end{aligned}$$

Condition (1.10) for $i = 2$ is therefore

$$\begin{aligned} b_1^1(2\dot{P}_0b_1^2 - P_1\dot{b}_0^2 + 3s_0P_1b_1^2) &= 2\dot{P}_0b_1^1b_1^2 - P_1b_1^1\dot{b}_0^2 + 3s_0P_1b_1^1b_1^2 \\ &= 2\dot{P}_0b_1^1b_1^2 + P_1b_1^2\dot{b}_0^1 + 3s_0P_1b_1^1b_1^2 \\ &= b_1^2(2\dot{P}_0b_1^1 - P_1\dot{b}_0^1 + 3s_0P_1b_1^1) \\ &= 0. \end{aligned}$$

Hence we have demonstrated that the condition holds for $i = 2$ also. Having cleared this hurdle, there is still one free parameter. For any Q_1 and r , the corresponding tangent vectors that solve (1.4) are

$$\left\{ \begin{aligned} &(\dot{\mathbf{P}} + 2(s_0 + \bar{s}_0\zeta^2)F^1F^2\tilde{\mathbf{P}} + 2s_1\zeta F^1F^2\tilde{\mathbf{P}}, \\ &\dot{\mathbf{b}}^1 + (s_0 + \bar{s}_0\zeta^2)F^1\tilde{\mathbf{b}}^1 + s_1\zeta F^1\tilde{\mathbf{b}}^1, \\ &\dot{\mathbf{b}}^2 + (s_0 + \bar{s}_0\zeta^2)F^2\tilde{\mathbf{b}}^2 + s_1\zeta F^2\tilde{\mathbf{b}}^2) \end{aligned} \mid s_1 \in \mathbb{R} \right\}.$$

But our free choice of $s_1 \in \mathbb{R}$ is only adding multiples of $(2P, b^1, b^2)$, which as in the nonconformal case is a rescaling of the spectral curve, and so also determined uniquely. \square

Though the details of the above three lemmata vary, we see that in every case they tell essentially the same story: starting with a polynomial Q it is possible to recover a tangent vector to \mathcal{M}_g . It is also interesting to note that in each

case there was a choice of two real parameters, and for each choice of those parameters there was a unique tangent vector. Conversely, given any tangent vector $(\dot{P}, \dot{b}^1, \dot{b}^2)$ to \mathcal{M}_g there is a unique pair of polynomials (\hat{c}^1, \hat{c}^2) , as remarked in the paragraph following (1.4), and thus a unique polynomial Q from (1.9). Hence this pairing between parameters and tangent vectors is bijective, and we may identify the tangent space to \mathcal{M}_g with these two real parameters. This suggests that $\mathcal{M}_g \cap \mathcal{U}'$ itself is a surface.

Theorem 1.32. *The open subset $\mathcal{M}_g \cap \mathcal{U}'$ of the space of spectral data \mathcal{M}_g is a two dimensional manifold. Further, the points of $\mathcal{M}_g \cap \mathcal{U}''$ are also smooth points of \mathcal{M}_g .*

Proof. Recall the definition of \mathcal{U}' , equation (1.14), as the open set whose points correspond to cases (i) or (ii). That is, \mathcal{U}' is the open set of points (P, b^1, b^2) where P has only simple roots, none of which are on the unit circle, the polynomials b^i have at most a simple roots at $\zeta = 0$, $\gcd(P, b^1, b^2)$ is one and $\gcd(b^1, b^2)$ lies in $\mathcal{P}_{\mathbb{R}}^k$ for some $k \leq 2$.

At any point $p \in \mathcal{M}_g \cap \mathcal{U}'$, take a simply connected open neighbourhood $\mathcal{V} \subset \mathcal{U}$. On this neighbourhood, define the map $\Psi : \mathcal{V} \rightarrow \mathbb{R}^{4g+9}$ by

$$\begin{aligned} \Psi(P, b^1, b^2) = & \left(\int_{A_1} \Theta^1, \dots, \int_{A_g} \Theta^1, \int_{B_1} \Theta^1, \dots, \int_{B_g} \Theta^1, \right. \\ & \int_{A_1} \Theta^2, \dots, \int_{A_g} \Theta^2, \int_{B_1} \Theta^2, \dots, \int_{B_g} \Theta^2, \\ & \int_{\gamma_+} \Theta^1, \int_{\gamma_-} \Theta^1, \int_{\gamma_+} \Theta^2, \int_{\gamma_-} \Theta^2, \\ & \left. P_1 b_0^1 - 2P_0 b_1^1, P_1 b_0^2 - 2P_0 b_1^2, (P_0)^{-1} \prod_k (-\alpha_k) \right) \end{aligned}$$

where A_i, B_i are the real and imaginary periods of the curve $\eta^2 = P(\zeta)$, γ_+ and γ_- are the paths in the curve between the points over $\zeta = 1$ and $\zeta = -1$, and α_k are the roots of P inside the unit circle. Because V is simply connected, the choice of paths $\{A_k\}, \{B_k\}, \gamma_+, \gamma_-$ may be made smoothly. The components of Ψ are the conditions that spectral data must satisfy. In particular, the first $4g$ components of Ψ are the periods of the differentials, the next four components are the integrals in the closing conditions (0.11), followed by the conditions to have no residues (0.9) and the last component of Ψ is our preferred scaling of the spectral curve.

Hence each connected components of $\mathcal{M}_g \cap \mathcal{V}$ is contained in a level set of Ψ ,

$$\begin{aligned} \mathcal{M}_g \cap \mathcal{V} \subset \Psi^{-1} & (0, \dots, 0, 2\pi i\mathbb{Z}, \dots, 2\pi i\mathbb{Z}, 0, \dots, 0, 2\pi i\mathbb{Z}, \dots, 2\pi i\mathbb{Z}, \\ & 2\pi i\mathbb{Z}, 2\pi i\mathbb{Z}, 2\pi i\mathbb{Z}, 2\pi i\mathbb{Z}, 0, 0, 1) . \end{aligned}$$

The point p of $\mathcal{M}_g \cap \mathcal{V}$ falls under either Lemma 1.23 or 1.27. In both cases we computed that the kernel of $d\Psi_p$ is two dimensional. The differential of Ψ is a map from \mathbb{R}^{4g+11} to $\mathbb{R}^{4g+9} = \mathbb{R}^{4g+4+2+2+1}$, and so is full rank at every such point p . Therefore, by the implicit function theorem, $\mathcal{M}_g \cap \mathcal{U}'$ is a two dimensional manifold.

The proof of the second part of the theorem is entirely similar. Recall that \mathcal{U}'' is the subset of points of \mathcal{U} where $P(0) = 0$ and $\gcd(b^1, b^2) = \zeta$. Take a point $p \in \mathcal{M}_g \cap \mathcal{U}''$ and a simply connected open neighbourhood $\mathcal{V} \subset \mathcal{U}$ of p as before. We may use the same definition of Ψ , and by the implicit function theorem and Lemma 1.29 it follows that p is also a smooth point of \mathcal{M}_g . \square

By Lemma 0.14, spectral data with a spectral curve of genus zero fall entirely within $\mathcal{U}^{(i)}$ and \mathcal{U}'' , while those with a genus one spectral curve may also be case (ii). Either way, \mathcal{M}_0 and \mathcal{M}_1 are smooth at every point and therefore are surfaces. The subsequent chapters of this thesis focus their investigation on these low genus cases.

To close this chapter, let us give a geometric interpretation to the polynomial Q . Recall that the conformal type of the domain of a harmonic map is given by the ratios of the principal parts of the differentials of its spectral data. Let the conformal type be denoted τ . For a nonconformal harmonic map we have that $b_0^2 = \tau b_0^1$. Consideration of the constant terms of (1.4) reveal that

$$\dot{P}_0 b_0^i - 2P_0 \dot{b}^i = 2P_0 \hat{c}_0^i.$$

Noting that $c_0^i = -\hat{c}_0^i$, we substitute this into (1.9) to arrive at

$$Q_0 P_0 = b_0^1 c_0^2 - b_0^2 c_0^1 = b_0^1 b_0^2 - \dot{b}_0^1 b_0^2.$$

Differentiating the relationship $b_0^2 = \tau b_0^1$ gives $\dot{b}_0^2 = \dot{\tau} b_0^1 + \tau \dot{b}_0^1$. Rearranging yields that

$$Q_0 = \frac{\dot{\tau}}{\tau} \frac{b_0^1 b_0^2}{P_0}. \quad (1.33)$$

We see therefore that Q_0 controls the change in the conformal type of the domain of the harmonic map.

The Genus Zero Moduli Space

The moduli space of spectral data where the spectral curve has genus zero is an instructive case. It is possible to describe this space using only elementary functions, so we will give explicit formulae for all such harmonic maps f , their associated solutions (A, Φ) to (0.1), and spectral data $(\Sigma, \Theta^1, \Theta^2)$. These formulae will allow us to illustrate the correspondences between certain properties of the map and its spectral data. The process of deriving the spectral data from a formula of a harmonic map also serves as a demonstration of the spectral curve construction that was outlined in Section 0.2. Finally, we may describe the moduli space \mathcal{M}_0 as a disjoint union of discs to both give an example of the results of the previous chapter and as a guide for the description of \mathcal{M}_1 in the subsequent chapter.

The derivations within this chapter may be divided into three parts. In the first part, we start with equations (0.1) and find all translation invariant solutions. Among these solutions, we determine which correspond to harmonic maps from a torus by forcing a periodicity constraint. We then write an explicit formula for each harmonic map, and bring it into a standard form by applying rotations. In the second part, we take these harmonic maps and work through the steps of Section 0.2 to produce the associated spectral data. Further calculations give rise to formulae for the infinitesimal deformations in terms of derivative of the branch point, and an expression of the energy of the harmonic map.

Let us start then by finding all translation invariant pairs (A, Φ) solving equations (0.1). By translation invariance we mean that, for $z \in \mathbb{C}$ a uniformising coordinate on the torus $M = \mathbb{C}/\mathbb{Z}\langle 1, \tau \rangle$, we may write the Higgs field as $\Phi = F dz$ and the $(0, 1)$ part of the connection as $d_A'' = d'' + G d\bar{z}$ with respect to some trivialisation d , for constant traceless matrices F and G (cf. [Hit90, (9.11)]). With these definitions, equations (0.1) reduce to

$$[F, G] = 0 \quad \text{and} \quad [G, G^*] = [F, F^*].$$

If F commutes with its conjugate-transpose then so too does G , and they are

simultaneously diagonalisable by an $SU(2)$ matrix. This case corresponds to a conformal map from the torus to a 2-sphere, which does not produce a spectral curve [Hit90, Prop 3.14]. Assume therefore that $[F, F^*] \neq 0$. The first equation implies that F and G commute, so we may write $G = \lambda F$. Then $(|\lambda|^2 - 1)[F, F^*] = 0$ implies that $|\lambda| = 1$. Let $\kappa \in \mathbb{S}^1$ be such that $\bar{\kappa}^2 = \lambda$.

Hence a translation invariant solution to (0.1) is given by a constant traceless matrix F and a complex number $\kappa \in \mathbb{S}^1$, with $G = \bar{\kappa}^2 F$. Recall equation (0.3), that for any solution for (0.1) there is an \mathbb{S}^1 family of flat unitary connections, which are given in this case by

$$d_\zeta := d_A + \zeta^{-1} \Phi - \zeta \Phi^* = d + G d\bar{z} - G^* dz + \zeta^{-1} F dz - \zeta F^* d\bar{z}, \quad (2.1)$$

for $\zeta \in \mathbb{S}^1$. As the matrices F and G are constant, for each ζ one can solve the parallel transport equation $d_\zeta X = 0$ explicitly by exponentiation. One may then recover the associated harmonic map from \mathbb{C} to $SU(2)$ as the change of gauge between two parallel vector fields for the connections d_1 and d_{-1} , as in (0.2). We do not yet know whether this map will descend to a map on M , because we have not required d_1 and d_{-1} to be trivial. After factorising, we write the harmonic map as

$$f(z) = \exp[(\kappa z + \bar{\kappa} \bar{z})(-\bar{\kappa} F + \kappa F^*)] \exp[(\kappa z - \bar{\kappa} \bar{z})(-\bar{\kappa} F - \kappa F^*)]. \quad (2.2)$$

We may simplify this formula by making the following substitutions. On \mathbb{C} , make the change of coordinates $w = \kappa z$ and let $w = w_R + iw_I$ be the sum of its real and imaginary parts. We note that $\mathfrak{sl}_2 \mathbb{C}$ can be decomposed as a direct sum $\mathfrak{su}_2 \oplus i\mathfrak{su}_2$. If we decompose $\bar{\kappa} F = X + iY$ for $X, Y \in \mathfrak{su}_2$ then we may replace the two expressions $-\bar{\kappa} F + \kappa F^*$ and $-\bar{\kappa} F - \kappa F^*$ by the real and imaginary parts of κF . With these substitutions, the above formula becomes

$$f(w) = \exp(-4w_R X) \exp(4w_I Y). \quad (2.3)$$

The matrix exponential of an \mathfrak{su}_2 matrix is of course an $SU(2)$ matrix, and a line through the origin of \mathfrak{su}_2 , such as $\{-4w_R X \mid w_R \in \mathbb{R}\}$, is mapped to a one parameter subgroup of $SU(2)$, a circle. As we vary w_R , the image of $\exp(-4w_R X)$ is a circle and so for a fixed value of w_I , the image of f is the right translation of this circle by $\exp(4w_I Y)$. The same is true if we fix w_R and vary w_I . Immediately therefore we can see (2.3) as the product of two circles, and hence the image is a torus.

Not all solutions to (0.1) correspond to harmonic maps of the torus $M = \mathbb{C}/\mathbb{Z}\langle 1, \tau \rangle$, and thus far this map f is only a map from the plane \mathbb{C} to $SU(2)$. It induces a map on the torus M when it is periodic with respect to the lattice $\mathbb{Z}\langle 1, \tau \rangle$. Thus we must show firstly that the map is periodic and then determine for which matrices X, Y and rotations κ these periods lie in the lattice. To do so we will need to know how to compute explicitly the matrix exponential of an \mathfrak{su}_2 matrix.

Calculations with \mathfrak{su}_2 matrices are far more pleasant when one leverages their underlying geometry. We may identify \mathfrak{su}_2 with $\mathbb{R}^3 = \mathbb{R}\langle i, j, k \rangle$ via the standard basis of \mathfrak{su}_2

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto i, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto j, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mapsto k. \quad (2.4)$$

If $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ and $\times_{\mathbb{R}^3}$ are the usual inner and cross products of \mathbb{R}^3 then the product of two matrices $A, B \in \mathfrak{su}_2$ may be computed as

$$AB = -\langle A, B \rangle_{\mathbb{R}^3} I + (A \times_{\mathbb{R}^3} B). \quad (2.5)$$

Further, this identification puts an inner product on \mathfrak{su}_2 , given by

$$\langle A, B \rangle_{\mathfrak{su}_2} = -\frac{1}{2} \operatorname{tr} AB.$$

This inner product is actually the same as the one that arises by considering $\mathrm{SU}(2)$ as \mathbb{S}^3 with the standard metric. From this formula, the norm $\|\cdot\|$ of an \mathfrak{su}_2 matrix is the square root of its determinant. We define the unit matrix \hat{Z} in direction Z to be Z divided by its norm. With these definitions, note that $Z^2 = -\|Z\|^2 I$ and hence the matrix exponential may be written concisely as

$$\exp Z = I \cos \|Z\| + \hat{Z} \sin \|Z\|.$$

Given this explicit formula for the matrix exponential, we now compute the periods of the map f . We will show that f is always periodic, and that its periods are precisely the points where it takes the value I . By the assumption that the matrix $F = \kappa(X + iY)$ does not commute with its conjugate-transpose, X and Y are linearly independent. If $a + ib$ is a point such that $f(a + ib) = I$ then $\exp(4aX) = \exp(4bY)$, and so by linear independence they must both equal $\pm I$. For any $u \in \mathbb{C}$ it follows that

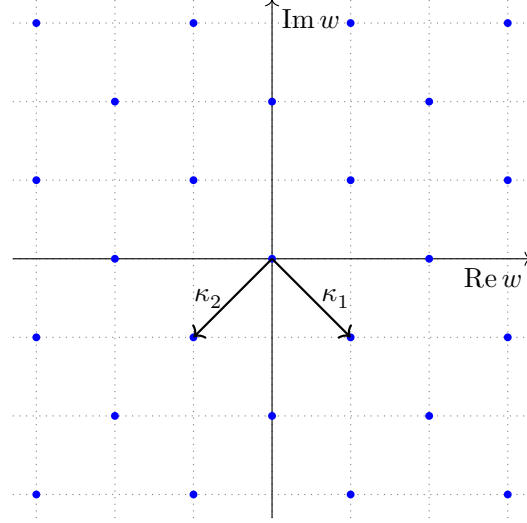
$$\begin{aligned} f(u + a + ib) &= \exp(-4(u_R + a)X) \exp(4(u_I + b)Y) \\ &= \exp(-4aX) \exp(-4u_R X) \exp(4bY) \exp(4u_I Y) \\ &= (\pm I)^2 \exp(-4u_R X) \exp(4u_I Y) \\ &= f(u), \end{aligned}$$

and thus $a + ib$ is a period. The converse, that $f(w) = I$ if w is a period, is trivial.

We have reduced the task of finding the periods of f to that of solving $f(w) = I$. Since X and Y are linearly independent matrices, it follows from (2.5) that the set $\{I, X, Y, XY\}$ is as well. An element $w \in \mathbb{C}$ is a period of f exactly when

$$\begin{aligned} I = f(w) &= I \cos(4w_R \|X\|) \cos(4w_I \|Y\|) - \hat{X} \sin(4w_R \|X\|) \cos(4w_I \|Y\|) \\ &\quad + \hat{Y} \cos(4w_R \|X\|) \sin(4w_I \|Y\|) - \hat{X} \hat{Y} \sin(4w_R \|X\|) \sin(4w_I \|Y\|) \end{aligned}$$

Figure 2.1
The lattice of periods of f .



Squaring the coefficients of \hat{Y} and $\hat{X}\hat{Y}$ and adding them together shows that $\sin^2(4w_I\|Y\|) = 0$. Doing likewise for the coefficients of \hat{X} and $\hat{X}\hat{Y}$ shows that $\sin^2(4w_R\|X\|) = 0$. Let $4w_R\|X\| = \pi k$ and $4w_I\|Y\| = \pi l$. Then the remaining term, the coefficient of I , forces

$$1 = \cos(4w_I\|Y\|) \cos(4w_R\|X\|) = (-1)^k (-1)^l = (-1)^{k+l}.$$

Thus the lattice of the periods of f , expressed in the variable w , is generated by

$$\kappa_1 := \frac{\pi}{4} \left(\frac{1}{\|X\|} - i \frac{1}{\|Y\|} \right), \quad \text{and} \quad \kappa_2 := -\frac{\pi}{4} \left(\frac{1}{\|X\|} + i \frac{1}{\|Y\|} \right). \quad (2.6)$$

The geometric meaning of κ , $\|X\|$, and $\|Y\|$ are now apparent. The parameter κ mediates between the z and w coordinates on \mathbb{C} and so is rotational offset angle of the lattice of periods of f with respect to $\langle 1, \tau \rangle$. More simply, the parameters $\|X\|$ and $\|Y\|$ determine the size of the lattice of periods by (2.6). The three parameters must be chosen so that 1 and τ are points of this lattice of periods $\mathbb{Z}\langle \bar{\kappa}\kappa_1, \bar{\kappa}\kappa_2 \rangle$. That is, there must be integers n^1, m^1, n^2, m^2

$$\kappa = n^1 \kappa_1 + m^1 \kappa_2, \quad \kappa \tau = n^2 \kappa_1 + m^2 \kappa_2. \quad (2.7)$$

Eliminating κ and solving for τ yields

$$\tau = \frac{(n^2 + m^2) + ix(n^2 - m^2)}{(n^1 + m^1) + ix(n^1 - m^1)}, \quad \text{for } x = \frac{\|Y\|}{\|X\|}. \quad (2.8)$$

To turn this around, if one begins with the parameters $\kappa, \|X\|$ and $\|Y\|$ then this shows that the conformal type of domain of the map f depends on x and

four integers. These four integers may be interpreted as winding numbers of the map. The parallelogram spanned by κ_1 and κ_2 covers the image exactly once. Thus in (2.7) the integers n^1 and m^1 may be interpreted as how many times the loop $[0, 1] \subset \mathbb{C}/\langle 1, \tau \rangle$ is wrapped around the image, and likewise for n^2 and m^2 .

Before proceeding to the second part of this example, where we compute the spectral data associated to the map f , we may further simplify (2.3). The correspondence between spectral data and harmonic maps does not distinguish between maps that differ by an $\mathrm{SO}(4)$ rotation of $\mathbb{S}^3 = \mathrm{SU}(2)$. We may use this freedom of rotation to align X in the direction of σ_2 and have Y lie in the plane spanned by σ_2 and σ_3 such that (X, Y) carries the same orientation as (σ_2, σ_3) . After this rotation $X = \|X\|\sigma_2$, and for some $\delta \in (0, \pi)$

$$Y = \|Y\| \begin{pmatrix} 0 & e^{i\delta} \\ -e^{-i\delta} & 0 \end{pmatrix} = \|X\| \begin{pmatrix} 0 & xe^{i\delta} \\ -xe^{-i\delta} & 0 \end{pmatrix}. \quad (2.9)$$

Using the inner product on \mathfrak{su}_2 , we may consider δ as the angle between X and Y . This transformation of f into a standard form shows that the image of f is determined by the angle δ , up to rotation of \mathbb{S}^3 . The image is not determined uniquely by δ though, as can be seen in Figure 2.5: the tori for δ and $\pi - \delta$ are the same. Indeed, if one reverses w_R , then this has the effect of changing X to $-X$, and the angle between $-X$ and Y is the supplement of δ .

We turn now to computing the spectral data associated to one of these harmonic maps. We may unwind the substitutions to express F as

$$F = \kappa \|X\| \begin{pmatrix} 0 & 1 + ixe^{i\delta} \\ -1 - ixe^{-i\delta} & 0 \end{pmatrix},$$

and likewise $G = \bar{\kappa}^2 F$. Recall the family of flat connections (2.1). Again, because we can explicitly solve the parallel transport equations, we can easily compute the holonomy matrices. For a connection d_ζ , the holonomy matrix for the loop from $z = 0$ to 1 is

$$H^1(\zeta) = \exp(-G + G^* - \zeta^{-1}F + \zeta F^*),$$

and for the loop from $z = 0$ to τ it is

$$\begin{aligned} H^2(\zeta) &= \exp(-G\bar{\tau} + G^*\tau - \zeta^{-1}F\tau + \zeta F^*\bar{\tau}) \\ &= \exp \left\{ \zeta^{-1}(\kappa\tau + \bar{\kappa}\bar{\tau}\zeta)\|X\| \right. \\ &\quad \times \begin{pmatrix} 0 & -(1 + ixe^{i\delta}) + (-1 + ixe^{i\delta})\zeta \\ (1 + ixe^{-i\delta}) + (1 - ixe^{-i\delta})\zeta & 0 \end{pmatrix} \left. \right\}. \end{aligned}$$

We note that setting $\tau = 1$ above recovers the formula for $H^1(\zeta)$, so we shall do all our computations with H^2 . Also, by rotating the vectors X and Y into

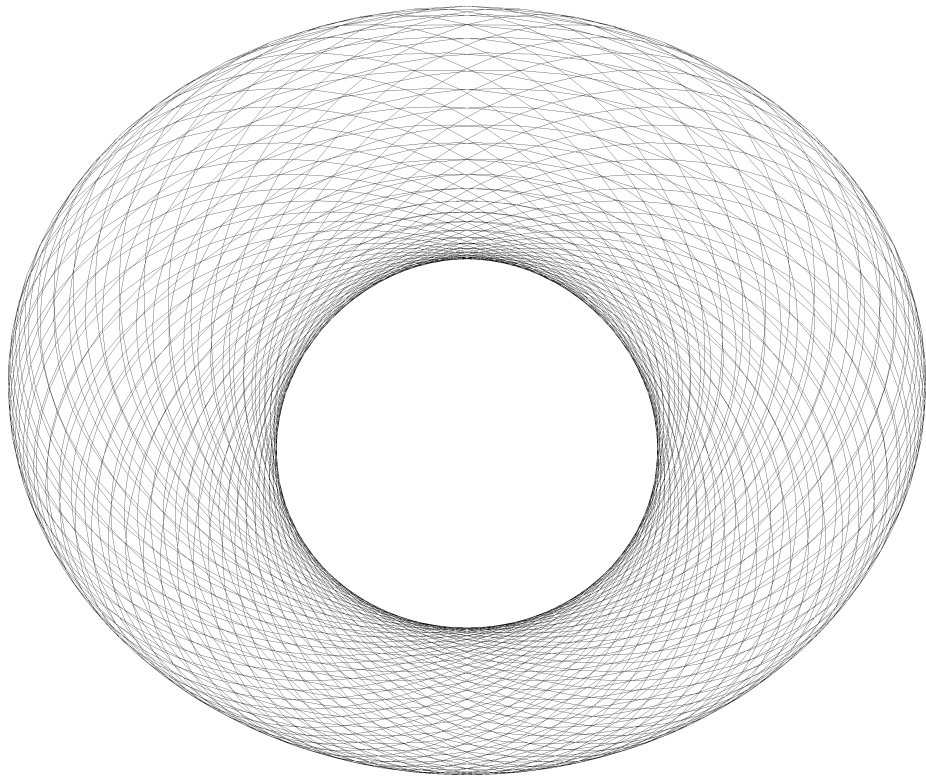


Figure 2.2
The image of a harmonic map of the form (2.3) with $\delta = \pi/4$. It is visualised by stereographic projection of \mathbb{S}^3 .

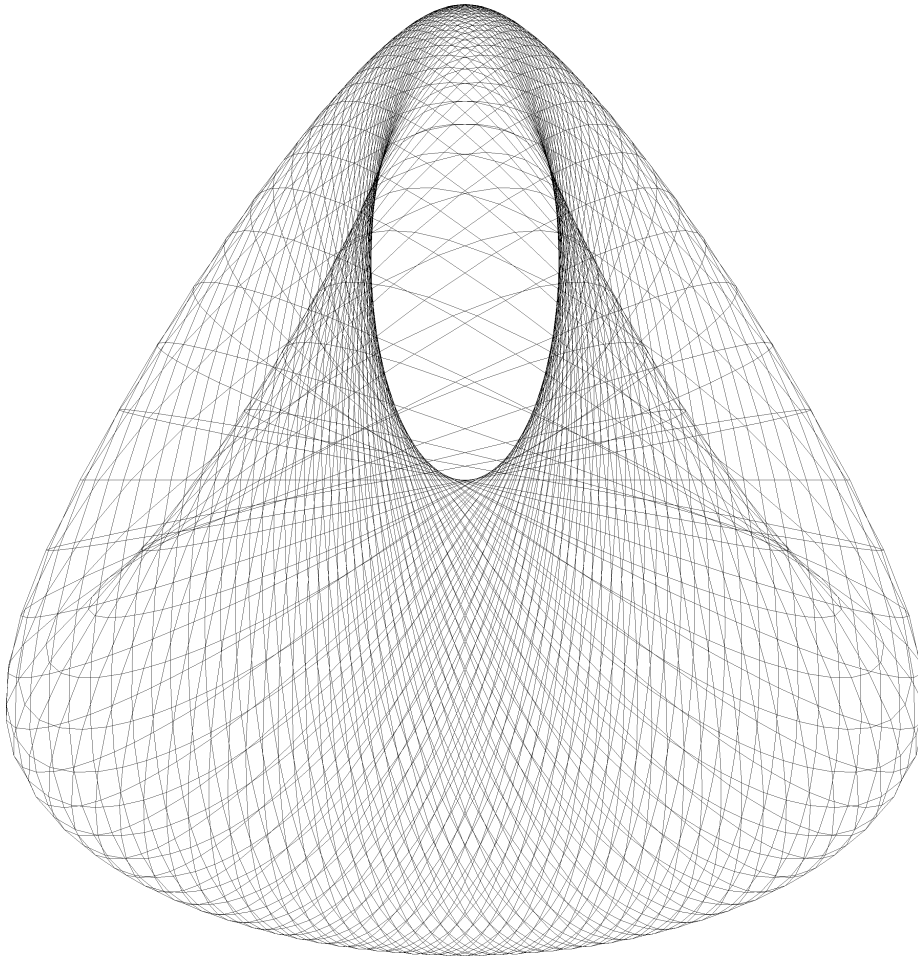


Figure 2.3

The image of a harmonic map of the form (2.3) with $\delta = \pi/2$. The image is congruent to the Clifford torus, and it divides \mathbb{S}^3 into two congruent solid tori.

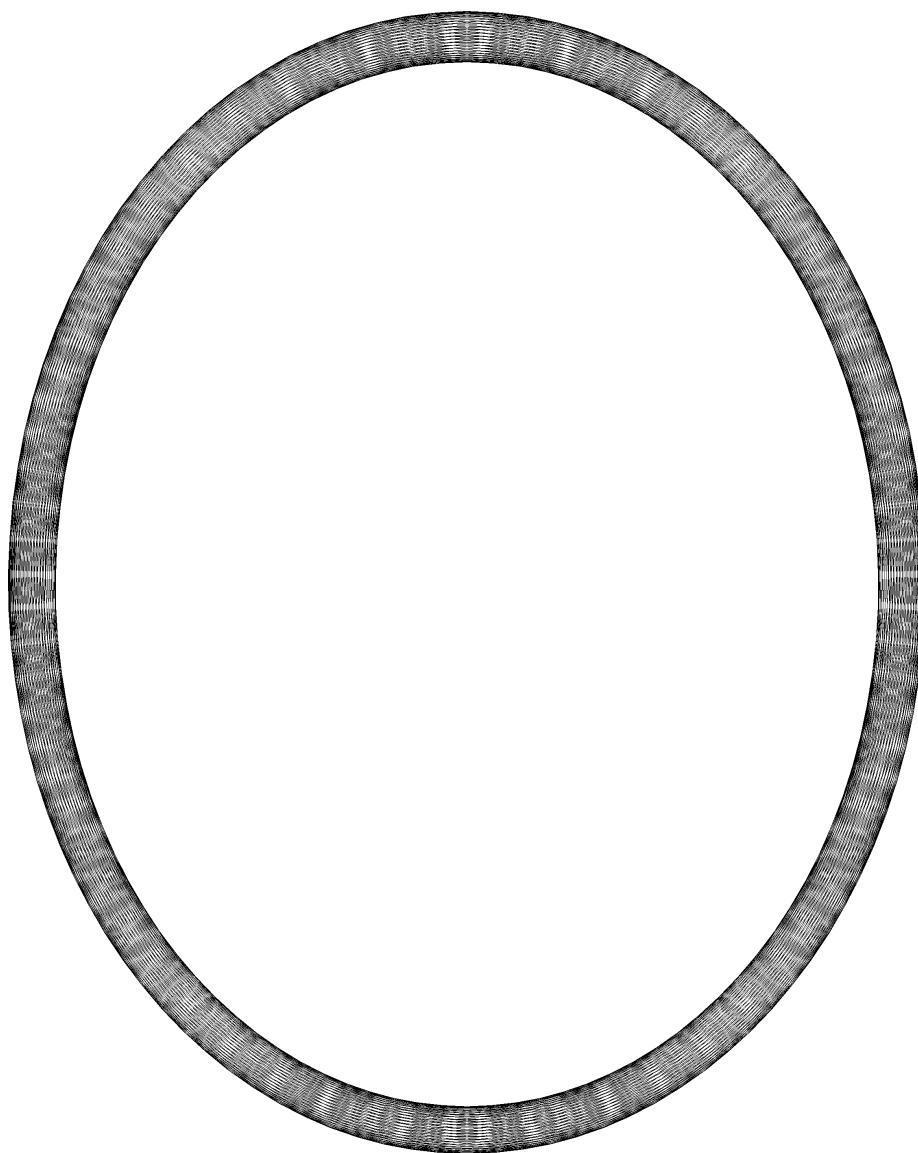


Figure 2.4

The image of a harmonic map of the form (2.3) with $\delta = \pi/32$. Notice how it is very thin. In the limit as $\delta \rightarrow 0$, the image collapses to a circle.

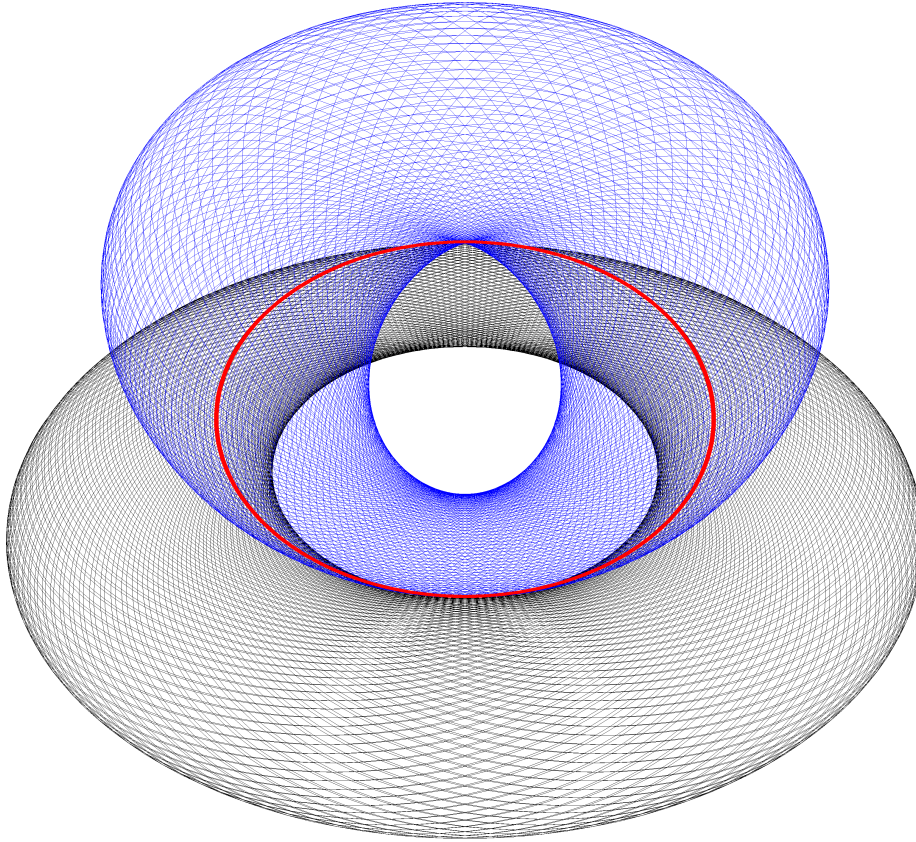


Figure 2.5

The image of two harmonic maps of the form (2.3) with $\delta = \pi/4$ (black) and $3\pi/4$ (blue). The two images are interlinked, congruent and share a common circle $\{ \exp(\omega\sigma_2) \mid \omega \in \mathbb{R} \}$, drawn in red. In fact all tori of the form (2.3) are tangent along this circle. It is the circle towards which the torus in Figure 2.4 is tending. Every point of \mathbb{S}^3 except those on this circle belongs to exactly one image, so varying the parameter δ sweeps out all of \mathbb{S}^3 .

the $\sigma_2\sigma_3$ -plane we have ensured that the matrices are off-diagonal. Thus their eigenvalues and eigenspaces are simple to write down.

To find the spectral curve, we find the values of ζ for which the two eigenlines of $H^2(\zeta)$ coincide. If $B(\zeta)$ is defined by $H^2(\zeta) = \exp B(\zeta)$, then the eigenspaces of $H^2(\zeta)$ and $B(\zeta)$ are the same, so we may do our computation with the latter. The matrix $B(\zeta)$ is off-diagonal, so $u(\zeta) = (u_1(\zeta), u_2(\zeta))^T$ is an eigenvector if and only if

$$\begin{aligned} -\zeta^{-1}(\kappa\tau + \overline{\kappa\tau}\zeta)\|X\|(1 - ixe^{i\delta})(\zeta - \alpha)u_2(\zeta)^2 \\ = \zeta^{-1}(\kappa\tau + \overline{\kappa\tau}\zeta)\|X\|(1 + ixe^{-i\delta})(1 - \overline{\alpha}\zeta)u_1(\zeta)^2. \end{aligned}$$

where α is a point that is always inside the unit circle, given by

$$\alpha = \frac{1 + ixe^{i\delta}}{-1 + ixe^{i\delta}} = \frac{xe^{i\delta} - i}{xe^{i\delta} + i}. \quad (2.10)$$

Following (0.4), points of the eigenline curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ are of the form

$$\begin{aligned} \left(\zeta, \left[\pm \sqrt{-\zeta^{-1}(\kappa\tau + \overline{\kappa\tau}\zeta)\|X\|(1 - ixe^{i\delta})(\zeta - \alpha)} \right. \right. \\ \left. \left. : \sqrt{\zeta^{-1}(\kappa\tau + \overline{\kappa\tau}\zeta)\|X\|(1 + ixe^{-i\delta})(1 - \overline{\alpha}\zeta)} \right] \right) \\ = \left(\zeta, \left[\pm \sqrt{-(1 - ixe^{i\delta})(\zeta - \alpha)} : \sqrt{(1 + ixe^{-i\delta})(1 - \overline{\alpha}\zeta)} \right] \right). \end{aligned}$$

From this we can see where and to what order the eigenlines coincide. The plus-minus sign produces two distinct lines unless one of the components of the homogeneous coordinates has a root, with the order of coincidence the same as the order of the root. As $ixe^{i\delta}$ is always in the left half of the complex plane, $1 - ixe^{i\delta}$ and its conjugate $1 + ixe^{-i\delta}$ never vanish. Hence the eigenlines coincide only over α and $\overline{\alpha}^{-1}$, and only to first order. The spectral curve is therefore $\eta^2 = (\zeta - \alpha)(1 - \overline{\alpha}\zeta)$, a genus zero hyperelliptic curve without singularities.

Let us explore how variation of the parameter α may alter the properties of the harmonic map f , and provide some non-rigorous intuition about the limit as α approaches the unit circle. From (2.10), we can see how the two continuous parameters δ and x have been incorporated into the definition of α . If we treat $xe^{i\delta}$ as a point in the upper half plane then (2.10) is the Cayley transform, a Möbius transformation of the upper half plane to the unit disc. One can write the inverse transformation as

$$xe^{i\delta} = i \frac{1 + \alpha}{1 - \alpha}.$$

Taking the magnitude of both sides shows that x is constant along arcs such that

$$\left| \frac{1 + \alpha}{1 - \alpha} \right|$$

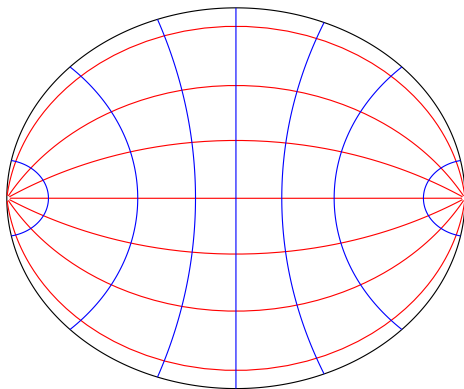


Figure 2.6
A plot of the unit disc with lines of constant δ shown in red and lines of constant x shown in blue. Notice that they are perpendicular. The lines of constant x are arcs of circles centred on the real axis.

is fixed. These are arcs of circles centred on the real axis with radii such that the circle is perpendicular to the unit circle. If x is constant, so too is τ , and along this arc the corresponding family of harmonic maps have the same domain but images changing as in Figures 2.2–2.5.

In the limit as α approaches the unit circle for fixed x , the parameter δ tends to 0 or π and the image of the harmonic map collapses into a circle, specifically a one-parameter subgroup of $SU(2)$. In Chapter 4, we will develop a process whereby we take the limit of a family of spectral curves as it tends towards a curve that has a double point on the unit circle, and that singular curve is normalised to produce a spectral curve of lower genus. The analogous process is not possible here, as the normalisation of $\eta^2 = e^{-i\varphi}(\zeta - e^{i\varphi})^2$ is the disjoint union of two spheres, which does not fit into our definition of spectral data (Definition 0.5). One can however appreciate the moral sentiment common to both these cases, that the development of a double point on the unit circle should be thought of as a family of harmonic maps degenerating to ‘simpler’ harmonic map.

Conversely if δ is fixed, say at $\delta = \pi/2$, but x is allowed to vary then α is given by

$$\alpha = \frac{x-1}{x+1},$$

and takes values along the real axis. Throughout this deformation, the image of the corresponding harmonic maps is fixed, it is only the domain that is changing. The two extremes, when $\alpha = -1$ or 1 , correspond to $x = 0$ and ∞ respectively. In these two limits, one of $\|X\|$ or $\|Y\|$ is dwarfing the other, which by (2.6) shows that in one direction the lattice of periods is becoming negligible. Another way to put this is that the limit of the lattice of periods will be only rank one, not rank two. Thus we should interpret this limit as corresponding to a harmonic map of the cylinder. Towards the end of this chapter, when we compute the

energy of these maps, we will see that as we approach the two points $\alpha = -1$ and 1 the torus tends to having infinite area, which supports this interpretation.

To summarise, the two parameters x and δ that constitute α control the conformal type of the domain and image of the harmonic map respectively, with the extreme cases corresponding to maps from a cylinder or maps to a circle. We shall see the features of these limits recur in the moduli space \mathcal{M}_1 , of spectral data with a genus one curve. The limit of a spectral curve when one of its branch points tends towards ± 1 is qualitatively different than when it tends to another point on the unit circle. At $\alpha \in \mathbb{S}^1 \setminus \{\pm 1\}$, the harmonic map degenerates to a simpler map, whereas at ± 1 the spectral data is not well defined in the limit.

Having found the spectral curve, we continue our quest to find the spectral data of the harmonic map f by computing the pair of differentials. The pair of differentials Θ^1 and Θ^2 arise as the derivatives of the logarithms of the eigenvalues μ^1 and μ^2 of the holonomy matrices H^1 and H^2 respectively. The two eigenvalues of H^2 are $(\mu^2)^{\pm 1} = \exp(\pm \nu)$, so in this example $\Theta^2 = d \log \mu^2 = d\nu$. To compute ν , an eigenvalue of $B(\zeta)$, we note that as B is a traceless matrix

$$\begin{aligned} \nu^2 &= -\det(-G\bar{\tau} + G^*\tau - \zeta^{-1}F\tau + \zeta F^*\bar{\tau}) \\ &= -\zeta^{-2}(\kappa\tau + \bar{\kappa}\bar{\tau}\zeta)^2 \|X\|^2 |1 - ixe^{i\delta}|^2 (\zeta - \alpha)(1 - \bar{\alpha}\zeta). \end{aligned} \quad (2.11)$$

Therefore the differential Θ^2 corresponding to the eigenvalue of $H^2(\zeta)$ is

$$\Theta^2 = d \log \mu^2 = d \left[\zeta^{-1}(\kappa\tau + \bar{\kappa}\bar{\tau}\zeta)i\|X\| |1 - ixe^{i\delta}| \eta \right].$$

Let us pause for a moment to make a small calculation to simplify the coefficients appearing in this equation. First note that by the definition of α , equation (2.10),

$$|1 - \alpha| = \frac{2}{|1 - ixe^{i\delta}|}, \text{ and } |1 + \alpha| = \frac{2x}{|1 - ixe^{i\delta}|}.$$

It follows then

$$\begin{aligned} r^1 &:= i\kappa_1\|X\| |1 - ixe^{i\delta}| = \frac{\pi}{2} \left(\frac{1}{|1 + \alpha|} + i \frac{1}{|1 - \alpha|} \right) \\ r^2 &:= i\kappa_2\|X\| |1 - ixe^{i\delta}| = \frac{\pi}{2} \left(\frac{1}{|1 + \alpha|} - i \frac{1}{|1 - \alpha|} \right), \end{aligned}$$

and so finally from (2.7) that

$$\kappa i\|X\| |1 - ixe^{i\delta}| = (n^1\kappa_1 + m^1\kappa_2)i\|X\| |1 - ixe^{i\delta}| = n^1r^1 + m^1r^2.$$

Therefore the differentials Θ^1 and Θ^2 may be written as

$$\begin{aligned} \Theta^1 &= d \left\{ \zeta^{-1} \left[(n^1r^1 + m^1r^2) + \overline{(n^1r^1 + m^1r^2)}\zeta \right] \eta \right\} \\ \Theta^2 &= d \left\{ \zeta^{-1} \left[(n^2r^1 + m^2r^2) + \overline{(n^2r^1 + m^2r^2)}\zeta \right] \eta \right\}, \end{aligned} \quad (2.12)$$

which both lie in a lattice spanned by the basis

$$\Psi^1 := d \left\{ \zeta^{-1}(r^1 + \overline{r^1} \zeta) \eta \right\} \text{ and } \Psi^2 := d \left\{ \zeta^{-1}(r^2 + \overline{r^2} \zeta) \eta \right\}. \quad (2.13)$$

Conversely every differential on Σ that satisfies conditions (P.1)–(P.11) belongs to this lattice. Hence we may identify this lattice of differentials with the lattice of periods of the map f , lending weight to the interpretation that the pair of differentials in the spectral data determine the winding of the torus onto its image. This same interpretation holds for the general construction of a harmonic map from spectral data, where the domain of the map is constructed as the parallelogram spanned by the pair of differentials.

The final piece of the spectral data, though one that we do not treat extensively in this thesis, is the eigenline bundle E on Σ . As Σ is a sphere, up to isomorphism there is only one line bundle for each degree. By condition (P.10), the line bundle E must be degree -1 , so there is a unique choice.

We have computed the spectral data for all maps f of the form (2.3) and noted that they were all and exactly the spectral data with a genus zero spectral curve. Let us turn then to describing the moduli space \mathcal{M}_0 of triples $(\Sigma, \Theta^1, \Theta^2)$ as a whole. We have seen that the spectral curve Σ is completely determined by its sole branch point α in the unit disc D . And for every α we may choose Θ^1 and Θ^2 from a rank two lattice. However by condition (P.9) they must be real linearly independent. Given a basis of the lattice of differentials, such as Ψ^1 and Ψ^2 in (2.13) above, we may represent our choice of lattice points in terms of two pairs of integers. As a matrix equation, this takes the form

$$\begin{pmatrix} \Theta^1 \\ \Theta^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Psi^1 \\ \Psi^2 \end{pmatrix}.$$

Then linear independence is equivalent to the integer matrix having non-zero determinant. This provides a concise way to refer to the choice of differentials, as a matrix in $\text{Mat}_2^* \mathbb{Z} = \{ M \in \text{Mat}_2 \mathbb{Z} \mid \det M \neq 0 \}$. The moduli space \mathcal{M}_0 can be described succinctly as the product $D \times \text{Mat}_2^* \mathbb{Z}$. One should bear in mind that this is not a canonical identification, as it is dependent on the choice of basis.

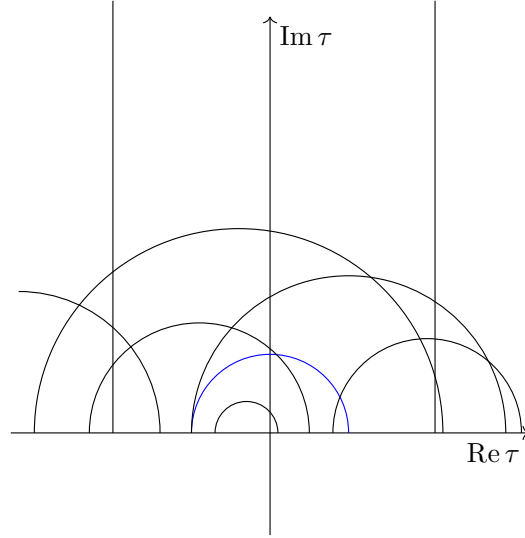
This matrix formulation is also well suited to talk about changes of the conformal parameter. Recall that the conformal parameter of the domain of the harmonic map may be computed by taking the ratio of the principal parts of the two differentials of the corresponding spectral data. If $(\Sigma, \Theta^1, \Theta^2)$ is a triple of spectral data, with conformal parameter τ , then so too is $(\Sigma, c\Theta^1 + d\Theta^2, a\Theta^1 + b\Theta^2)$, for integers $a, b, c, d \in \mathbb{Z}$, and the new conformal parameter is

$$\frac{a + b\tau}{c + d\tau}. \quad (2.14)$$

But this is just the action of $\text{Mat}_2^* \mathbb{Z}$ on a point τ in the upper half plane by Möbius transformations. The conformal type of the domain of the harmonic

Figure 2.7

The upper half plane. The conformal parameter τ may vary along a semicircle with rational endpoints. The blue semicircle is the conformal type of the domain of the harmonic map corresponding to (Σ, Ψ^1, Ψ^2) as one moves the branch point $\alpha \in D$.



map corresponding to (Σ, Ψ^1, Ψ^2) , where Ψ^1 and Ψ^2 is the basis (2.13) given above, is equivalent to

$$\tau = \frac{1 - x^2 + 2ix}{1 + x^2},$$

so as α moves in the unit disc, τ sweeps out the upper half of the unit circle. The range of the conformal parameter for any other genus zero harmonic map is the image of this semicircle under some element of $\text{Mat}_2^* \mathbb{Z}$. Hence the possible ranges of the conformal parameter τ under deformation of the harmonic map are semicircles in the upper half plane centred on the real axis with endpoints in \mathbb{Q} (or vertical rays with a rational endpoint, which are a special case), as shown in Figure 2.7.

The previous chapter developed a method for computing the deformations of spectral data. In the genus zero case, it is clear that given a triple of spectral data one can freely move the sole branch point α in the unit disc, and the path of α uniquely determines the deformation. An infinitesimal deformation therefore is determined by the value of the derivative of the branch point, $\dot{\alpha}$. However, let us work through the description of infinitesimal deformations from the previous chapter, via the definitions of the functions \dot{q}^i and Q (equations (1.3) and (1.9)), to gain some familiarity with its workings.

Because the spectral curve Σ is simply connected in the present situation, all differentials on it are exact. Thus we shall not need to solve (1.4) to find the polynomials \hat{c}^i . Instead we may directly compute the functions q^i and differentiate along a deformation to obtain $\dot{q}^i = \zeta^{-1} \eta^{-1} \hat{c}^i$. Starting from equation (2.12), the integration is immediate:

$$q^i = \zeta^{-1} (s^i - \bar{s}^i \zeta) \eta + C,$$

for some constant C and $s^i \in \mathbb{Z}\langle r^1, r^2 \rangle$. If the deformation is given by a parametrised path $\alpha(t)$, we may differentiate with respect to a deformation parameter t to deduce

$$\begin{aligned} \dot{q}^i = \frac{1}{\zeta\eta} & \left[- \left(\alpha \dot{s}^i + \frac{1}{2} \dot{\alpha} \bar{s}^i \right) + \left((1 + \alpha \bar{\alpha}) \dot{s}^i + \alpha \dot{s}^i + \frac{1}{2} \dot{\alpha} \bar{s}^i + \frac{1}{2} (\dot{\alpha} \bar{\alpha} + \alpha \dot{\bar{\alpha}}) s^i \right) \zeta \right. \\ & \left. - \left((1 + \alpha \bar{\alpha}) \dot{\bar{s}}^i + \bar{\alpha} \dot{s}^i + \frac{1}{2} \dot{\bar{\alpha}} s^i + \frac{1}{2} (\dot{\alpha} \bar{\alpha} + \alpha \dot{\bar{\alpha}}) \bar{s}^i \right) \zeta^2 + \left(\bar{\alpha} \dot{s}^i + \frac{1}{2} \dot{\bar{\alpha}} s^i \right) \zeta^3 \right]. \end{aligned}$$

Note that this is a cubic polynomial that is imaginary with respect to the involution ρ , as expected. We wish to factorise it, but to do so we first need to simplify the expressions appearing in the coefficients. By labourious calculation, using the specific form of s^i , the coefficient of ζ^1 may be shown to be equal to the conjugate of the coefficient of ζ^0 .

(If one wishes to verify this calculation for oneself, it is recommended to first check for s^i equal to $|1 + \alpha|^{-1}$ and then $i|1 - \alpha|^{-1}$. As s^i is in fact an integral combination of the functions r^1 and r^2 , r^i is a real combination of the two suggested functions, and because the expressions in the coefficients are real linear in s^i , this check is sufficient.)

We have then

$$q^i = \frac{1}{\zeta\eta} (\zeta^2 - 1) \left[\left(\alpha \dot{s}^i + \frac{1}{2} \dot{\alpha} \bar{s}^i \right) + \left(\bar{\alpha} \dot{s}^i + \frac{1}{2} \dot{\bar{\alpha}} s^i \right) \zeta \right]$$

The factor of $\zeta^2 - 1$ is a consequence of the closing conditions (P.11) being preserved throughout the deformation. Also, the remainder after factoring $\zeta^2 - 1$ is a linear real polynomial, $c^i(\zeta) \in \mathcal{P}_{\mathbb{R}}^1$. Next we take the exterior derivative of q^i to find the polynomials $b^i(\zeta)$,

$$b^i(\zeta) \frac{d\zeta}{\zeta^2\eta} := dq^i = \frac{d\zeta}{\zeta^2\eta} \left[\alpha \dot{s}^i - \frac{1}{2} (1 + \alpha \bar{\alpha}) \dot{s}^i \zeta - \frac{1}{2} (1 + \alpha \bar{\alpha}) \dot{\bar{s}}^i \zeta^2 + \bar{\alpha} \dot{s}^i \zeta^3 \right]. \quad (2.15)$$

These polynomials are also cubic and are real with respect to ρ . They also satisfy the residue condition (0.9). If we take b^i and c^i and substitute them into $b^1 c^2 - b^2 c^1$, this is the left hand side of equation (1.9). Dividing by P leaves the polynomial Q ,

$$\begin{aligned} Q = \frac{\pi^2}{4} \frac{1}{|1 - \alpha^2|} (n^1 m^2 - n^2 m^1) & \left[-2i\alpha \left(\operatorname{Re} \frac{\dot{\alpha}}{1 - \alpha^2} \right) \right. \\ & \left. + (1 - \alpha \bar{\alpha}) \left(\operatorname{Im} \frac{\dot{\alpha}}{1 - \alpha^2} \right) \zeta + 2i\bar{\alpha} \left(\operatorname{Re} \frac{\dot{\alpha}}{1 - \alpha^2} \right) \zeta^2 \right]. \end{aligned}$$

This is a real quadratic polynomial, but the form of its coefficients hold some further information. At a fixed α , we see that Q_0 can only take values on a real line, whereas we may have expected that it may take any complex value. This is

in accordance with the remark that Q_0 determines the change in the conformal parameter (see (1.33)) and the observation above that during a deformation the conformal parameter τ moves along an arc in the upper half plane.

When $\alpha \neq 0$, the conditions of Lemma 1.23 are met. In this case, the infinitesimal deformation should be completely specified by Q . We see that this is indeed the case, as Q_0 and Q_1 determine the real and imaginary parts of $\dot{\alpha}(1 - \alpha^2)^{-1}$ respectively, which exactly determines the value of $\dot{\alpha}$. It is somewhat suggestive to note that if α moves along an arc of constant x then $\operatorname{Re} \dot{\alpha}(1 - \alpha^2)^{-1}$ is zero, while if it moves along an arc of constant δ then it is $\operatorname{Im} \dot{\alpha}(1 - \alpha^2)^{-1}$ that vanishes. Thus the coefficients of Q are not only determining an infinitesimal deformation, they are doing so in a manner that aligns with the geometric properties of the harmonic map.

In the conformal case, when $\alpha = 0$, we must refer instead to Lemma 1.29. Here we see that $Q = Q_1\zeta$ as required. Now however, the deformation is not determined solely by Q , as $Q_1 = \operatorname{Im} \dot{\alpha}$ only gives part of the information of $\dot{\alpha}$. However, the infinitesimal deformation is still determined by the polynomials c^i ,

$$c^i(\zeta) = \frac{1}{2}\bar{s}^i\dot{\alpha} - \frac{1}{2}s^i\dot{\bar{\alpha}}\zeta.$$

The deficit of information in Q is accounted for by a degree of freedom in the solutions to (1.17). Observe that the polynomials b^i in (2.15) factor as $b^i = \zeta\tilde{b}^i$, for $\tilde{b}^i \in \mathcal{P}_{\mathbb{R}}^1$. Explicitly,

$$\tilde{b}^i(\zeta) = \frac{1}{2}s^i + \frac{1}{2}\bar{s}^i\zeta.$$

In Lemma 1.29, given a solution to (1.17) we are free to add any real multiple of \tilde{b}^i . This is exactly the freedom to choose a value of $\operatorname{Re} \dot{\alpha}$, which was not determined by Q .

Finally, there is a formula to compute the energy of harmonic map from its spectral data, given in [Hit90, Theorem 12.17]. In the nonconformal case

$$E = \frac{4i}{P_0}(b_2^1b_0^2 - b_2^2b_0^1),$$

where the lower indices refer to the coefficients of the polynomial. For example $b^1 = b_0^1 + b_2^1\zeta + \dots$. In particular, when the genus of the spectral curve is zero the coefficients of the polynomials b^1 and b^2 are entirely determined by the choice of the four integers n^1, m^1, n^2, m^2 and a point α in the unit disc, as in (2.12). After substitution and simplification, one arrives at

$$E = \pi^2(1 + \alpha\bar{\alpha})\frac{m^1n^2 - n^1m^2}{|1 - \alpha^2|}.$$

One can interpret the fraction as giving the area of the domain of the map, this is the ‘area’ of the parallelogram spanned by the differentials. The factor $1 + \alpha\bar{\alpha}$

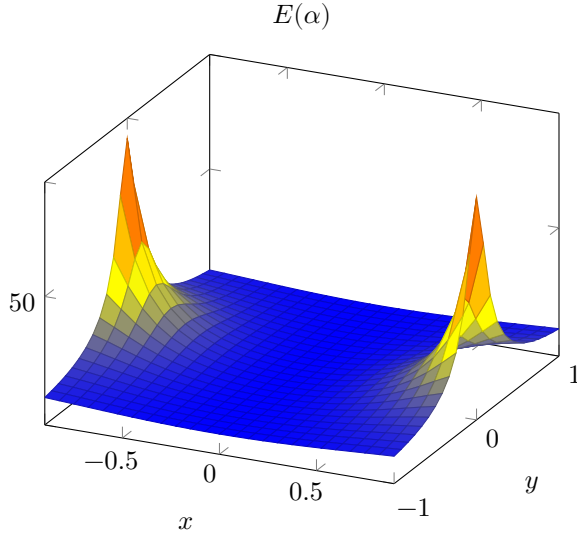


Figure 2.8

A plot of the energy as a function over $\alpha = x + iy$. There are singularities at $\alpha = 1, -1$, where the domain becomes a cylinder.

may be seen as a measure of how far the map is from being conformal. If we compute the derivative of this expression, we observe that

$$\dot{E} = 0 \Rightarrow (\operatorname{Re} \alpha) \left(\operatorname{Re} \frac{\dot{\alpha}}{1 - \alpha^2} \right) = 0.$$

The left factor corresponds to the imaginary axis. The right factor corresponds to a circle centred on the real axis that cuts the unit circle perpendicularly. In other words, \dot{E} is zero precisely when α moves on an arc that preserves τ . This was to be expected because, for a given conformal class, harmonic maps are minimisers for the energy.

The Genus One Moduli Space

The aim of this chapter is to describe the topology of the moduli space of spectral curves \mathcal{S}_1 , which we consider as a subspace of the space of marked curves \mathcal{E}_1 . We will construct the universal cover $\tilde{\mathcal{E}}_1$ of \mathcal{E}_1 and recover \mathcal{S}_1 as the quotient of a certain subspace $\tilde{\mathcal{S}}_1 \subset \tilde{\mathcal{E}}_1$ by the group of covering transformations. Recall that a deformation of spectral data is defined to be a path in the moduli space. We will prove that the path connected components of \mathcal{S}_1 are indexed by two rational numbers $p > 0$ and q . For $p \neq 1$ the components are ribbons $(0, 1) \times \mathbb{R}$, whereas for $p = 1$ the components are annuli.

The key to describing the topology of \mathcal{S}_1 lies in the construction of coordinate charts for $\tilde{\mathcal{E}}_1$ with the dual properties that the group of covering transformations acts by translation and also that the components of the preimage $\tilde{\mathcal{S}}_1$ of \mathcal{S}_1 in $\tilde{\mathcal{E}}_1$ are the subsets where two of the coordinates take rational values. To accomplish the first goal, we will construct a global coordinate chart on $\tilde{\mathcal{E}}_1$,

$$\tilde{\mathcal{E}}_1 = \left\{ (p, k, \tilde{U}, \tilde{V}) \in \mathbb{R}^+ \times (0, 1) \times \mathbb{R} \times \mathbb{R} \mid \tilde{U} < \tilde{V} < \tilde{U} + 2\pi \right\}, \quad (3.1)$$

in which the covering transformations act as translations. The group \mathcal{G} of covering transformations is proved to be $\mathbb{Z}\langle\tilde{\lambda}\rangle$, where the transformation $\tilde{\lambda}$ acts on $\tilde{\mathcal{E}}_1$ by

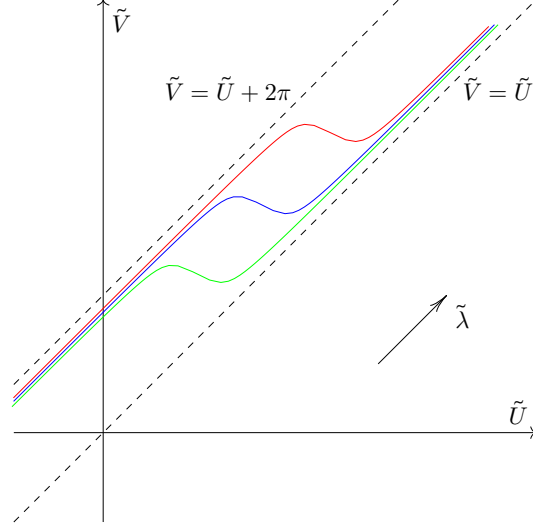
$$\tilde{\lambda} : (p, k, \tilde{U}, \tilde{V}) \mapsto (p, k, \tilde{U} + \pi, \tilde{V} + \pi).$$

Therefore we see that $\mathcal{E}_1 = \tilde{\mathcal{E}}_1 / \mathcal{G}$ is the product of $\mathbb{R}^+ \times (0, 1)$ and a cylinder.

Within $\tilde{\mathcal{E}}_1$ we must characterise $\tilde{\mathcal{S}}_1$. We will produce two functions S and \tilde{T} on $\tilde{\mathcal{E}}_1$, with ranges \mathbb{R}^+ and \mathbb{R} respectively, such that a marked curve belongs to $\tilde{\mathcal{S}}_1$ precisely when both S and \tilde{T} are rationally valued. In fact, the first coordinate introduced by (3.1) is already given by the function S . If $\tilde{\mathcal{E}}_1(p, q)$ is the level set $S = p$ and $\tilde{T} = q$ in $\tilde{\mathcal{E}}_1$, the space $\tilde{\mathcal{S}}_1$ is the union

$$\tilde{\mathcal{S}}_1 = \coprod_{p \in \mathbb{Q}^+, q \in \mathbb{Q}} \tilde{\mathcal{E}}_1(p, q).$$

Figure 3.1
 Sketch of some level sets of \tilde{T} on a cross-section of $\tilde{\mathcal{C}}_1$. The transformation $\tilde{\lambda}$ acts as a translation in the given direction. It maps level sets to level sets. Note that in this figure that the \tilde{V} -derivative of \tilde{T} does not vanish.



Crucially, the value of S is fixed by the action of $\tilde{\lambda}$ and precomposing \tilde{T} with $\tilde{\lambda}$ increases its value by $S-1$ (Lemma 3.53). Thus $\tilde{\lambda}$ maps $\tilde{\mathcal{C}}_1(p, q)$ to $\tilde{\mathcal{C}}_1(p, q+p-1)$. If p and q are rational, so too is $q+(p-1)$ and hence the group $\mathcal{G} = \mathbb{Z}\langle\tilde{\lambda}\rangle$ of covering transformations restricts to give a group action on $\tilde{\mathcal{S}}_1$. We can therefore conclude that $\mathcal{S}_1 \cong \tilde{\mathcal{S}}_1/\mathcal{G}$.

As p is rational it is constant along any path in $\tilde{\mathcal{S}}_1$. When working with the path connected components of $\tilde{\mathcal{S}}_1$ we may therefore restrict ourselves to the submanifolds $\tilde{\mathcal{C}}_1(p)$ of $\tilde{\mathcal{C}}_1$ where p is fixed. Further, the covering transformations do not change p so these submanifolds are \mathcal{G} -invariant. The three remaining coordinates $(k, \tilde{U}, \tilde{V})$ form a global coordinate chart for each $\tilde{\mathcal{C}}_1(p)$. However, in order to produce a coordinate chart where both goals are met, that is $\tilde{\lambda}$ acts as translation and $\tilde{\mathcal{S}}_1(p) := \tilde{\mathcal{S}}_1 \cap \tilde{\mathcal{C}}_1(p)$ is the union of coordinate planes, in Lemma 3.48 we show that for each p there is a coordinate chart where $q = \tilde{T}$ may be used as a coordinate in place of either \tilde{U} or \tilde{V} . Specifically for $p \leq 1$,

$$\{(q, k, \tilde{U}) \in \mathbb{R} \times (0, 1) \times \mathbb{R}\}$$

is a coordinate chart covering $\tilde{\mathcal{C}}_1(p)$, whereas for $p \geq 1$ a coordinate chart covering $\tilde{\mathcal{C}}_1(p)$ is

$$\{(q, k, \tilde{V}) \in \mathbb{R} \times (0, 1) \times \mathbb{R}\}.$$

This shows that the above union of $\tilde{\mathcal{C}}_1(p, q)$ was a decomposition of $\tilde{\mathcal{S}}_1$ into path connected components. If we let \tilde{X} stand for either \tilde{U} or \tilde{V} depending on the magnitude of p , then the action of $\tilde{\lambda}$ in each coordinate chart reads

$$\tilde{\lambda} : (p, q, k, \tilde{X}) \mapsto (p, q + (p-1), k, \tilde{X} + \pi).$$

The main result of the chapter takes the quotient of each of these submanifolds $\tilde{\mathcal{C}}_1(p)$ by the group of covering transformations \mathcal{G} . For $p \neq 1$, in Theorem 3.55 we have that

$$\mathcal{C}_1(p) \cong \tilde{\mathcal{C}}_1(p)/\mathcal{G} = \left\{ ([q], k, \tilde{X}) \in \mathbb{R}/(p-1)\mathbb{Z} \times (0, 1) \times \mathbb{R} \right\}.$$

If further $p \neq 1$ is rational, the moduli space $\mathcal{S}_1(p)$ is the subset where $[q] \in \mathbb{Q}/(p-1)\mathbb{Z}$. For $p = 1$, by contrast, Theorem 3.56 shows that

$$\mathcal{C}_1(1) \cong \tilde{\mathcal{C}}_1(1)/\mathcal{G} = \left\{ (q, k, [\tilde{X}]) \in \mathbb{R} \times (0, 1) \times \mathbb{R}/\pi\mathbb{Z} \right\},$$

where again we recover $\mathcal{S}_1(1)$ as the subset where q is rational.

In either case, we have a foliation of $\mathcal{C}_1(p)$ such that $\mathcal{S}_1(p)$ is a dense collection of leaves. Though the topology of the components changes from ribbons to annuli at $p = 1$, we can understand each foliation as belonging to a family parametrised by p . As k is unaffected by the quotient, we may consider only $(q, \tilde{X}) \in \mathbb{R}^2$, as shown in Figure 3.2. For each p , the translation $\tilde{\lambda}$ takes (q, \tilde{X}) to $(q + (p-1), \tilde{X} + \pi)$. Let us rotate each plane so that these translations are in the vertical direction. The lines of constant q now make an angle $\arctan \frac{p-1}{\pi}$ to the vertical. We may therefore take the quotient of this space, a cylinder, and for $p \neq 1$ the lines of constant q become helices wrapped around this cylinder. The parameter p changes the slope of helices. When $p = 1$ however the lines of constant q are in the direction of translation and close up into circles under the group action. This exceptional case $p = 1$ is the transition from left-handed to right-handed helices.

Now that we have given a picture of the results of this chapter, let us run through the stages of the proof. The intermediate way-point is the construction of the functions S and \tilde{T} , which derive from the closing conditions, Conditions (P.11). The difference between a marked curve of genus one, a point of \mathcal{C}_1 , and a spectral curve, a point of \mathcal{S}_1 , is that the latter possesses a pair of linearly independent differentials that satisfy the closing conditions. For any genus g , every marked curve of \mathcal{C}_g meets Conditions (P.1)–(P.3) and admits a plane \mathcal{B}_Σ of differentials satisfying Conditions (P.4)–(P.7). Further, in the case of marked curves of genus one, we will show that every curve has differentials with integral periods, Condition (P.8). But not every curve admits differentials that meet the closing conditions.

We therefore proceed by the construction of coordinates adapted to the closing conditions and the action of the covering transformations. In the initial three sections of this chapter we work with the parameter space \mathcal{A}_1 , which we recall from equation (0.12) to be the space $\{(\alpha, \beta) \in D^2 \mid \alpha \neq \beta\}$, where D is the open unit disc. We take the quotient space $\mathcal{A}_1/\mathbb{Z}_2$ as a model for the space of marked curves \mathcal{C}_1 , where \mathbb{Z}_2 acts by permutation of (α, β) . Using Σ to denote the covering map, the curve

$$\Sigma(\alpha, \beta) = \{(\zeta, \eta) \mid \eta^2 = P(\zeta) = (\zeta - \alpha)(1 - \bar{\alpha}\zeta)(\zeta - \beta)(1 - \bar{\beta}\zeta)\} \in \mathcal{C}_1,$$

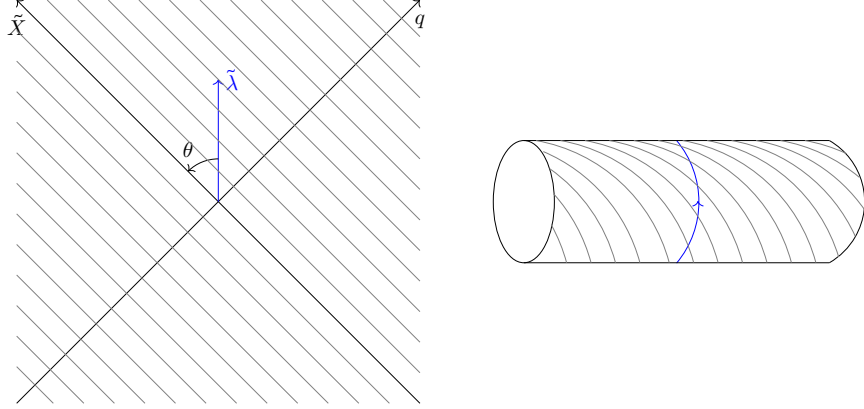


Figure 3.2

On the left, a cross-section of $\tilde{\mathcal{E}}_1(p)$ for fixed k , with the q and \tilde{X} axes in black and direction of the translation $\tilde{\lambda}$ in blue. The grey lines are the lines of constant q . The angle θ is given by $\text{atan} \frac{p-1}{\pi}$. In this figure, $p \approx 1 + \pi > 1$.

On the right, the result of taking the quotient by $\mathcal{G} = \mathbb{Z}\langle\tilde{\lambda}\rangle$. The plane has been rolled into a cylinder, and the level sets are helices. As p is changed, the slope of the helices changes also. When $p = 1$, they close-up into circles on the cylinder.

is the marked curve with branch points $\{\alpha, \bar{\alpha}^{-1}, \beta, \bar{\beta}^{-1}\}$ and the standard scaling given by (0.7).

For each point of \mathcal{A}_1 , in Section 3.1 we give a transformation of the corresponding marked curve to an elliptic curve in Jacobi normal form. This allows us to find on each marked curve the differentials that satisfy (P.4)–(P.7) and have integral periods. Having found these differentials, in Section 3.2 we reformulate the closing conditions (P.11) to produce a function S and a multi-valued function T on \mathcal{A}_1 , such that they are valued in \mathbb{Q} exactly when a given marked curve admits differentials that further satisfy the closing conditions. A marked curve with a pair of such differentials constitute a triple of spectral data, and so we call such a curve a spectral curve. The first function, S (defined by (3.17)), is strictly positive and has an explicit algebraic formula; its level sets foliate \mathcal{A}_1 into solid tori. The second function, T (defined by (3.20)), is however multi-valued and transcendental; it is dependent on paths of integration on the curve, and its formula contains elliptic integrals.

The functions S and T may be analysed by introducing a new set of coordinates on \mathcal{A}_1 that are suited to calculation (Section 3.3). These new coordinates allows us to reasonably compute the derivatives of T . They also dovetail with the results of Section B.5, allowing us to lift T to a single valued function \tilde{T} on $\tilde{\mathcal{E}}_1$, the universal cover of \mathcal{A}_1 . We then use the implicit function theorem to

prove that the level sets $\tilde{\mathcal{C}}_1(p, q)$ of S and \tilde{T} are graphs and foliate the space $\tilde{\mathcal{C}}_1$ (Lemma 3.48). This allows for an explicit description of $\tilde{\mathcal{S}}_1$, the preimage of \mathcal{S}_1 in $\tilde{\mathcal{C}}_1$, as a union as above. Finally, we push this foliation back down to \mathcal{C}_1 and recover \mathcal{S}_1 as a quotient by the group of covering transformations \mathcal{G} (Section 3.4).

To close the chapter, in Section 3.5 we show that the moduli space of spectral data \mathcal{M}_1 is not a trivial bundle over \mathcal{S}_1 , as \mathcal{M}_0 was over \mathcal{S}_0 . We show however that the total space of the bundle is simply connected. We also consider some well-known examples of harmonic maps, namely the Gauss maps of Delaunay surfaces, and identify them with a specific path connected component of \mathcal{S}_1 . Building on this, we identify a symmetry of \mathcal{S}_g , for any genus g , and provide a geometrical interpretation. We illustrate this interpretation for the analogous symmetry of genus zero spectral curves using the explicit equations for harmonic maps derived in the previous chapter.

We summarise the spaces that we have introduced and their relationships to one another in the diagram below. One starts out with the space of marked curves \mathcal{C}_1 in the top right corner. It is covered by the parameter space \mathcal{A}_1 , which is in turn covered by the universal cover $\tilde{\mathcal{C}}_1$. On the next line we consider within each of these spaces the subspaces on which the function S has the value $p \in \mathbb{Q}^+$, denoted by p in parentheses. On the bottom line we have the statement that $\tilde{\mathcal{S}}_1$, the preimage of the space of spectral curves \mathcal{S}_1 , is a union of level sets $\tilde{\mathcal{C}}_1(p, q)$, the subsets of $\tilde{\mathcal{C}}_1$ on which $S = p$ and $\tilde{T} = q$. The horizontal arrows represent covering maps, labelled with the group of covering transformations, whereas the vertical arrows represent inclusions.

$$\begin{array}{ccccc}
\tilde{\mathcal{C}}_1 & \xrightarrow{2\pi\mathbb{Z}} \gg & \mathcal{A}_1 & \xrightarrow{\mathbb{Z}_2} \gg & \mathcal{C}_1 \\
\uparrow & & \uparrow & & \uparrow \\
\tilde{\mathcal{C}}_1(p) & \xrightarrow{2\pi\mathbb{Z}} \gg & \mathcal{A}_1(p) & \xrightarrow{\mathbb{Z}_2} \gg & \mathcal{C}_1(p) \\
\uparrow & & & & \uparrow \\
\tilde{\mathcal{S}}_1(p) = \coprod_{q \in \mathbb{Q}} \tilde{\mathcal{C}}_1(p, q) & \xrightarrow{\quad\quad\quad} \gg & \mathcal{S}_1(p) & &
\end{array}$$

3.1 The Differentials of an Elliptic Marked Curve

It will be necessary to have explicit formulae for the differentials on a marked curve Σ that have integral periods. In this section, we construct such differen-

tials in four stages. First, we give a coordinate transformation f of the marked curve Σ to the Jacobi normal form of an elliptic curve. On any marked curve it is possible to find differentials that meet conditions (P.4)–(P.6). These conditions are:

(P.4) Θ has double poles and no residues over $\zeta = 0$ and ∞ ,

(P.5) that with respect to the hyperelliptic involution σ , Θ obeys $\sigma^*\Theta = -\Theta$, and

(P.6) that Θ is real with respect to ρ , which is to say $\rho^*\Theta = -\bar{\Theta}$.

The second step is to therefore write down set of differentials meeting these three conditions; they form a three-dimensional real vector space W .

Next, we use the transformation f to compute the periods of these differentials in terms of the Jacobi elliptic integrals. Finally, we leverage the standard relations among the Jacobi elliptic integrals to describe explicitly the differentials with integral periods, those which meet condition (P.8). Any differential that meets all the conditions (P.4)–(P.8) may be written as a linear combination of two distinguished differentials, Θ^E and Θ^P . The choice of these two differentials on $\Sigma(\alpha, \beta)$ varies smoothly in the parameters (α, β) . The space of differentials with integral periods will be shown to be trivial bundles over \mathcal{A}_1 and \mathcal{C}_1 .

Let us get to work. With the standard scaling (0.7), any marked curve of genus one may be written as

$$\eta^2 = P(\zeta) := (\zeta - \alpha)(1 - \bar{\alpha}\zeta)(\zeta - \beta)(1 - \bar{\beta}\zeta), \quad (3.2)$$

with roots $\alpha, \bar{\alpha}^{-1}, \beta$ and $\bar{\beta}^{-1}$. On the other hand, every elliptic curve may be transformed into the Jacobi normal form

$$w^2 = (1 - z^2)(1 - k^2 z^2)$$

where k is a complex number called the elliptic modulus. For a given marked curve, how is one to compute the modulus and therefore determine the appropriate Jacobi form into which to transform? The answer lies in the cross ratio of the roots

$$[\alpha, \bar{\alpha}^{-1}; \beta, \bar{\beta}^{-1}] = \frac{|\alpha - \beta|^2}{|1 - \bar{\alpha}\beta|^2}. \quad (3.3)$$

This is a real quantity, and so the four roots lie on a circle (or a line). Thus the roots of the Jacobi form do also, which forces k to be real. Any transformation between the curves must take branch points to branch points, so we must decide on a correspondence for the roots. There are twenty-four possible choices, but we choose the correspondence

$$\begin{array}{c|c|c|c} \zeta & \alpha & \bar{\alpha}^{-1} & \beta \\ \hline z & 1 & -1 & k^{-1} \end{array} \quad \begin{array}{c|c} \bar{\beta}^{-1} \\ \hline -k^{-1} \end{array} \quad (3.4)$$

This correspondence has three properties that distinguish it from the others. By convention, $k \in (0, 1)$, which rules out sixteen of the choices. Second, consider the behaviour of the curve as $k \rightarrow 1$. The Jacobi form of the curve develops two nodes at $z = \pm 1$. This corresponds to forming nodes $\alpha = \beta$ and $\bar{\alpha}^{-1} = \bar{\beta}^{-1}$. But the value of

$$[1, k^{-1}; -1, -k^{-1}] = \frac{4k}{(k+1)^2}$$

disagrees with eqn (3.3) in this limit, which rules out this correspondence and the three others with the same cross ratio. Finally, our choice of correspondences takes the interior of the unit disc to the right half plane, which will be our convention throughout this chapter.

There is only one other correspondence that has all three of these properties, namely

$$\frac{\zeta}{z} \parallel \begin{array}{c|c|c|c} \beta & \bar{\beta}^{-1} & \alpha & \bar{\alpha}^{-1} \\ \hline 1 & -1 & k^{-1} & -k^{-1} \end{array}$$

The difference between this correspondence and our preferred correspondence is the choice of which root, α or β , is mapped to 1. This is the reason that we must work with \mathcal{A}_1 and not \mathcal{E}_1 . On the latter space, there would be no way to consistently make the choice. For example, take the path $t \mapsto \Sigma(0.5e^{it}, -0.5e^{it})$ in \mathcal{E}_1 . This is a closed loop where the branch points are interchanged as t varies from 0 to π .

From our correspondence, equating the cross ratios $[\alpha, \bar{\alpha}^{-1}; \beta, \bar{\beta}^{-1}]$ and $[1, -1; k^{-1}, -k^{-1}]$ gives

$$k = \frac{|1 - \bar{\alpha}\beta| - |\alpha - \beta|}{|1 - \bar{\alpha}\beta| + |\alpha - \beta|}, \quad (3.5)$$

and the map, which we shall call f , can be computed from the relation

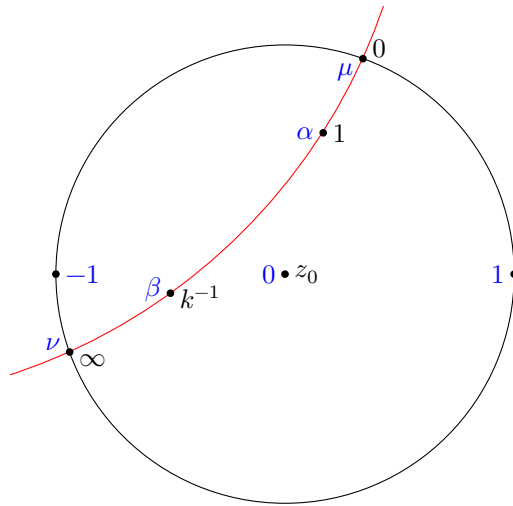
$$[\alpha, \bar{\alpha}^{-1}; \beta, \zeta] = [1, -1; k^{-1}, f(\zeta)].$$

Instead of solving this relation for f immediately, we turn to understanding the geometry of the transformation so that we might produce a meaningful expression.

The unit circle in the ζ -plane plays an important role in the definition of a spectral curve, so it is natural to ponder its image under f . The involution $\rho(\zeta)$ fixes the unit circle, and exchanges the pairs of branch points $\alpha, \bar{\alpha}^{-1}$ and $\beta, \bar{\beta}^{-1}$. The corresponding antiholomorphic involution $\tilde{\rho}(z)$ in the z -plane that exchanges $1, -1$ and $k^{-1}, -k^{-1}$ is $z \mapsto -\bar{z}$. Its fixed point set is the imaginary axis, which therefore must be the image of the unit circle under f .

As already mentioned, the four roots of the spectral curve lie on a circle (or a line), which we shall call the branch circle. Let the two points at the intersection of the branch circle with the unit circle be μ and ν , with μ lying between α and

The ζ -plane, with points marked in blue. The black labels are their images under f . The red line is the circle through the branch points.



A Möbius transformation, such as f , is determined up to scaling by the points it sends to 0 and ∞ , in this case μ and ν . One other point is therefore needed to determine this scaling. We write $z_0 := f(0)$. Using the reality structure $\tilde{\rho}(z)$, we have $f(\infty) = -\overline{z_0}$. These points allow us to write concise formulae for f and f^{-1}

$$z = f(\zeta) = -\bar{z}_0 \frac{\zeta - \mu}{\zeta - \nu}, \quad (3.6)$$

$$\zeta = f^{-1}(z) = \nu \frac{z - z_0}{z + \bar{z}_0}. \quad (3.7)$$

Proof. Recall that μ is the point such that $f(\mu) = 0$. We may find a formula for μ in terms of α and β using the cross-ratio relation $[\alpha, \bar{\alpha}^{-1}; \beta, \mu] = [1, -1; k^{-1}, 0]$. Rearranging gives

$$\mu = \frac{(\alpha - \beta) |1 - \bar{\alpha}\beta| + \alpha(1 - \bar{\alpha}\beta) |\alpha - \beta|}{\bar{\alpha}(\alpha - \beta) |1 - \bar{\alpha}\beta| + (1 - \bar{\alpha}\beta) |\alpha - \beta|}.$$

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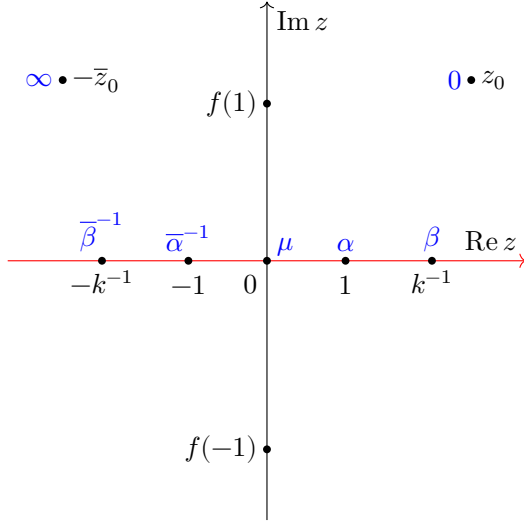


Figure 3.4
The z -plane, with points marked in black. The blue labels are their images under f^{-1} .

factor $1 - \bar{\alpha}\beta$ is zero if and only $\bar{\alpha}^{-1} = \beta$ and, as both α and β are inside the unit disc, this is impossible. Finally, the denominator is zero exactly when

$$\bar{\alpha} = -\frac{1 - \bar{\alpha}\beta}{|1 - \bar{\alpha}\beta|} \frac{|\alpha - \beta|}{\alpha - \beta}.$$

But the right hand side is an element of the unit circle, so again this possibility is eliminated. The proof of smoothness is entirely similar for

$$\begin{aligned} \nu &= \frac{(\alpha - \beta) |1 - \bar{\alpha}\beta| - \alpha(1 - \bar{\alpha}\beta) |\alpha - \beta|}{\bar{\alpha}(\alpha - \beta) |1 - \bar{\alpha}\beta| - (1 - \bar{\alpha}\beta) |\alpha - \beta|}, \\ z_0 &= \frac{\alpha(\bar{\alpha} - \bar{\beta}) |1 - \bar{\alpha}\beta| + (1 - \alpha\bar{\beta}) |\alpha - \beta|}{\alpha(\bar{\alpha} - \bar{\beta}) |1 - \bar{\alpha}\beta| - (1 - \alpha\bar{\beta}) |\alpha - \beta|}, \\ (z_0)^{-1} &= \frac{\alpha(\bar{\alpha} - \bar{\beta}) |1 - \bar{\alpha}\beta| - (1 - \alpha\bar{\beta}) |\alpha - \beta|}{\alpha(\bar{\alpha} - \bar{\beta}) |1 - \bar{\alpha}\beta| + (1 - \alpha\bar{\beta}) |\alpha - \beta|}, \text{ and} \\ k &= \frac{|1 - \bar{\alpha}\beta| - |\alpha - \beta|}{|1 - \bar{\alpha}\beta| + |\alpha - \beta|}. \end{aligned} \tag{3.9}$$

By the formula (3.6), we also conclude that f is smooth so long as the denominator is nonzero.

The final claim is that $\mu - \nu$ is never zero. This is clear from the geometry, as the branch circle is a circle that passes through points both inside and outside the unit circle and so must intersect the unit circle at distinct two points. Algebraically, the difference vanishes only if

$$2(|\alpha|^2 - 1)(\alpha - \beta)(1 - \bar{\alpha}\beta) = 0,$$

and this has already shown not to occur on \mathcal{A}_1 . \square

Because of the holomorphic involution $\sigma : \eta \rightarrow -\eta$, equations (3.6) and (3.7) almost but not quite specify a relation between η and w : there is a free sign choice to make. On the marked curve $\Sigma = \{(\zeta, \eta) | \eta^2 = P(\zeta)\}$ there are two disjoint circles in Σ lying over the unit circle in \mathbb{CP}^1 . At a point (ζ, η) over the unit circle in Σ , we have that the value of η is $\pm \zeta |\zeta - \alpha| |\zeta - \beta|$. Thus there is a notion of the ‘positive’ unit circle, the one on which η is positive over $\zeta = 1$. We define the function η^+ to be $\zeta |\zeta - \alpha| |\zeta - \beta|$.

Under the transformation f the unit circle is mapped to the imaginary axis. At the points over the imaginary axis in the Jacobi elliptic curve, those points (z, w) for which $z = iu$, we have that $w = \pm \sqrt{1 + u^2} \sqrt{1 + k^2 u^2}$. Again, it is possible to make a consistent choice of sign along these two disjoint circles. For the sake of being concrete, we choose the transformation between elliptic curves to map the positive unit circle to points over the imaginary axis where w is positive. In a slight abuse of notation, we shall also use f to denote the map between elliptic curves.

Having found the map f that transforms a genus one marked curve Σ into Jacobi normal form, we may now turn our attention to the other part of the spectral data, the differentials that satisfy conditions (P.4)–(P.8). Let us first find the vector space of differentials that satisfy just conditions (P.4)–(P.6), and write a basis for this space. Following the notation of (0.10), recall that all differentials on a genus one marked curve Σ that have (at worst) a double pole over $\zeta = 0$ and $\zeta = \infty$ may be written the form

$$\Theta = b(\zeta) \frac{d\zeta}{\zeta^2 \eta},$$

for a polynomial $b(\zeta)$ of degree 4. Condition (P.6) forces $b \in \mathcal{P}_{\mathbb{R}}^4$, the space of polynomials that are real with respect to ρ . Practically, if we write $b(\zeta) = b_0 + \dots + b_4 \zeta^4$, this condition forces $b_i = b_{4-i}$. Lastly, if we write out the equation of Σ as $\eta^2 = P(\zeta) = P_0 \zeta + \dots + P_4 \zeta^4$, Θ has no residues exactly when $P_1 b_0 - 2P_0 b_1 = 0$ (equation 0.9). If we count the degrees of freedom remaining, the last equation shows that we may choose (b_0, b_1) from a complex line, whereas b_2 may be any real number. Hence for each marked curve Σ there is a real three-dimensional vector space W of differentials that meet conditions (P.4)–(P.6).

This presents an obvious choice of basis, but one that we will not choose. Instead, we shall choose a basis that it suited to computing the periods of the differentials so that we may be able to satisfy condition (P.8), that the periods of the differentials lie in $2\pi i$. For this we turn to the the classical theory of elliptic curves, which has long studied differentials and their periods. It is standard to refer to differentials with double poles and no residues as differentials of the second kind. Condition (P.4) may therefore be rephrased as the differential must be of the second kind and have poles at the points of Σ over $\zeta = 0$ and ∞ . The standard Jacobi differential of the second kind is defined to be

$$e := (1 - k^2 z^2) \frac{dz}{w}. \quad (3.10)$$

Every differential of the second kind may be written as the linear combination of e , the holomorphic differential ω ,

$$\omega := \frac{dz}{w},$$

and an exact differential [Han10, Art. 167].

Note that the holomorphic differential ω lies in the space W and so makes for an obvious first basis vector. It accounts for every possible choice of b_2 . In genus one, there is a real and exact differential with double poles over $\zeta = 0$ and ∞ , namely

$$\Theta^E := i d \left(\frac{\eta}{\zeta} \right).$$

We take it as the second basis vector. The superscript E is a mnemonic for exact. Given that we already have taken ω as a basis vector, we seek to complete the basis of W with the sum of e and an exact differential.

Recall equation (3.10), the definition of e . It is real with respect to $\tilde{\rho}(z) = -\bar{z}$, but it has a pole at $z = \infty$ ($\zeta = \nu$), which not allowed under condition (P.4). This pole can be moved to $\zeta = \infty$ by adding an exact differential. We assert that

$$e + d \left[\frac{w}{z + \bar{z}_0} \right]$$

has no pole at $z = \infty$. To check this let $z' = z^{-1}$, and expand e about $z' = 0$.

$$\begin{aligned} w &= kz'^{-2} (1 + O(z'^2)) \\ e &= -k^2 z'^{-2} (1 - k^{-2} z'^2) \times -z'^{-2} dz' \times k^{-1} z'^2 (1 + O(z'^2)) \\ &= kz'^{-2} dz' (1 + O(z'^2)), \end{aligned}$$

whereas the exact differential has the following expansion

$$\begin{aligned} d \left[\frac{w}{z + \bar{z}_0} \right] &= d \left[kz'^{-1} (1 + O(z'^2)) (1 - \bar{z}_0 z' + \bar{z}_0^2 z'^2 + \dots) \right] \\ &= d \left[kz'^{-1} (1 + O(z')) \right] \\ &= -kz'^{-2} dz' (1 + O(z'^2)), \end{aligned}$$

which shows that their sum is holomorphic at $z = \infty$. Unfortunately, this differential is not real with respect to $\tilde{\rho}$. To correct this deficiency we shall have to add another exact differential. The set of exact differentials (not necessarily real) with the double poles over $\zeta = 0, \infty$ is

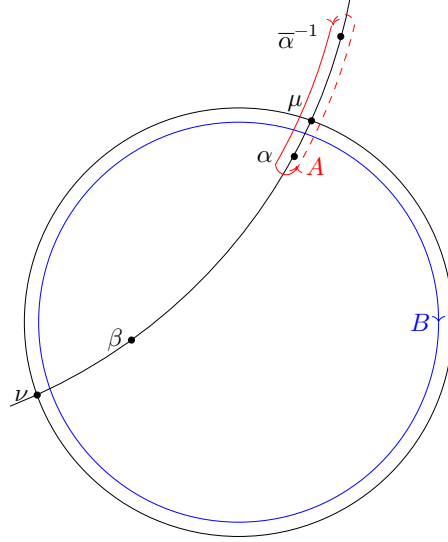
$$\left\{ C d \left[\frac{w}{(z - z_0)(z + \bar{z}_0)} \right] \mid C \in \mathbb{C} \right\},$$

so we add a differential of this form to restore the reality. The differential

$$e + d \left[\frac{w}{z + \bar{z}_0} \right] + C d \left[\frac{w}{(z - z_0)(z + \bar{z}_0)} \right] = e + d \left[\frac{(z - z_0 + C)w}{(z - z_0)(z + \bar{z}_0)} \right]$$

Figure 3.5

The anticlockwise, real period A in red and the clockwise, imaginary period B in blue.



is real when $z_0 - C$ is an imaginary number. Thus we should choose $C \in \text{Re } z_0 + i\mathbb{R}$. The apparent freedom to choose the imaginary part of C is exactly adding a scalar of Θ^E and so will not change the span of the resulting basis. There are two natural choices; taking $z_0 - C$ to be zero, or taking C to be purely real. Both have their merits, but the latter choice ends up being superior as it makes the principal part of this differential perpendicular to the principal part of Θ^E , which introduces a symmetry that we will use later. Hence we take as our third basis differential

$$\varepsilon := e + d \left[\frac{w}{z + \bar{z}_0} \right] + \text{Re } z_0 \, d \left[\frac{w}{(z - z_0)(z + \bar{z}_0)} \right] = e + d \left[\frac{(z - i \, \text{Im } z_0)w}{(z - z_0)(z + \bar{z}_0)} \right]. \quad (3.11)$$

We are now in a position to compute the periods of our basis $\{\omega, \Theta^E, \varepsilon\}$. We choose a basis of homology of the marked curve Σ following way. First, take the branch cuts to be along the branch circle, between α and β and between $\bar{\alpha}^{-1}$ and $\bar{\beta}^{-1}$, in particular so that they do not cross the unit circle. Under f , this corresponds to the standard choice of $[1, k^{-1}]$ and $[-k^{-1}, -1]$. We can speak of points of Σ as being on the positive or negative sheet, where as before the positive sheet is where η is positive over $\zeta = 1$.

For the loop A , start on the positive unit circle at μ , traverse in and around α anticlockwise (crossing a branch cut), then cross the negative unit circle and continue anticlockwise around $\bar{\alpha}^{-1}$ before returning to the starting point. For the loop B , start at the same point we began A and follow the unit circle clockwise. These loops are shown in Figure 3.5.

The image of A under f is the anticlockwise loop around -1 and 1 with the

left to right part of the path on the positive sheet. The loop $f(B)$ is simply a traversal of the imaginary axis from bottom to top on the positive sheet. But this is homologous to a clockwise loop around 1 and k^{-1} . Thus we have chosen the basis of homology A, B such that their images under f are the standard choice of homology on an elliptic curve in Jacobi normal form, as depicted in Figure B.1.

The loop A is a real period, which means that the integral of a real differential over A is real. To prove this, recall that a differential Θ is real with respect to ρ when $\rho^*\Theta = -\bar{\Theta}$. By construction, $\rho_*A = -A$. Together,

$$\overline{\int_A \Theta} = \int_A \bar{\Theta} = - \int_A \rho^*\Theta = - \int_{\rho_*A} \Theta = - \int_{-A} \Theta = \int_A \Theta,$$

which shows the A -period of Θ to be real. Likewise, the integral of a real differential over B is always an imaginary number.

We wish to be able to describe the differentials of W that have both imaginary and integral periods.

Lemma 3.12. *On a genus one marked curve Σ , suppose there is a differential Θ^P satisfying Conditions (P.4)–(P.8) such that*

$$\int_B \Theta^P = 2\pi i.$$

Then for any other differential Θ satisfying Conditions (P.4)–(P.8), Θ lies in $\mathbb{R}\Theta^E + \mathbb{Z}\Theta^P \subset W$.

Proof. It is always the case that the real periods of a holomorphic differential on a compact Riemann surface are non-zero [Mir95, Cor VIII.4.3]. Therefore $\{\omega, \Theta^E, \Theta^P\}$ is a linearly independent set of differentials in W , and thus a basis. By satisfying (P.4)–(P.6), Θ is forced to lie in W and so is a combination of this basis.

If Θ were exact, which is to say that its integrals over A and B were zero, it follows immediately that Θ would have to be a real multiple of Θ^E . If Θ were non-exact, then by (P.8) it would have an imaginary period of $2\pi il$, for some $l \in \mathbb{Z}$. By subtracting $l\Theta^P$ the result would be an exact differential, and therefore would have to be a real multiple of Θ^E . \square

Thus we have reduced the problem of describing differentials on Σ meeting Conditions (P.4)–(P.8) to that of finding a differential $\Theta^P \in W$ with vanishing A -period and a B -period of $2\pi i$. The superscript P is a mnemonic for period. Clearly, we may add a real multiple of Θ^E to Θ^P without altering affecting this requirement. Therefore, for some real constants a and b , let $\Theta^P = a\omega + b\varepsilon$ be a combination of the other two basis vectors of $W = \mathbb{R}\langle\omega, \Theta^E, \varepsilon\rangle$.

It is useful at this point to summarise the standard elliptic integrals, the periods of the differentials ω and e ,

$$\begin{aligned} \int_{f(A)} \omega &= 4K(k) & \int_{f(B)} \omega &= 2iK' \\ \int_{f(A)} e &= 4E(k) & \int_{f(B)} e &= 2i(K' - E') \end{aligned}$$

where K and E are the complete elliptic integrals of the first and second kind, and the prime denotes not the derivative but instead the elliptic complement. The elliptic complement is by definition $k' = \sqrt{1 - k^2}$, and $K'(k) = K(k')$ and $E'(k) = E(k')$. Further properties of elliptic integrals may be found in Appendix B. Note that ε is the sum of e and an exact differential and so has the same periods as e .

We require that

$$\begin{aligned} \int_A \Theta^P &= 4Ka + 4Eb = 0 \\ \int_B \Theta^P &= 2iK'a + 2i(K' - E')b = 2\pi i. \end{aligned}$$

From the first equation, we can write $a = cE$ and $b = -cK$ for some $c \in \mathbb{R}$. Substituting this into the second equation gives

$$\begin{aligned} \pi &= cK'E - c(K' - E')K \\ &= c(K'E + KE' - KK') \\ &= \frac{\pi}{2}c \\ c &= 2 \end{aligned} \tag{3.13}$$

where (3.13) uses Legendre's relation (see Lemma B.16). Thus $\Theta^P = 2E\omega - 2K\varepsilon$, or if we unwind the definition of ε , (3.11),

$$\Theta^P = 2E\omega - 2Ke - 2Kd \left[\frac{(z - i \operatorname{Im} z_0)w}{(z - z_0)(z + \bar{z}_0)} \right]. \tag{3.14}$$

This equation shows a nice division, with the first two terms providing the desired periods and the last term giving the required poles. Though the choices up to this point may seem arbitrary and contrived, they are not, as we can characterise the differential Θ^P in the following way.

Lemma 3.15. *The differential Θ^P is the unique real differential on the marked curve Σ with double poles and no residues over $\zeta = 0$ and ∞ , with periods 0 and $2\pi i$ over A and B respectively, and with its principal part over $\zeta = 0$ satisfying*

$$\operatorname{pp}.\Theta^P \in i\mathbb{R} \operatorname{pp}.\Theta^E.$$

Proof. We first verify that Θ^P has such properties, then verify uniqueness. The only property not yet demonstrated is the third one, concerning the principal part. We note that $\zeta = 0$ corresponds to $z = z_0$, so for some real scalar r

$$\text{pp. } \Theta^E = i \text{ pp. } d \left(\frac{\eta}{\zeta} \right) = i \text{ pp. } d \left(\frac{rw}{(z - z_0)(z + \bar{z}_0)} \right) = -ir \frac{w(z_0)}{z_0 + \bar{z}_0} \frac{dz}{(z - z_0)^2}.$$

And on other side we have

$$\begin{aligned} \text{pp. } \Theta^P &= -2K \text{ pp. } d \left[\frac{(z - i \text{Im } z_0)w}{(z - z_0)(z + \bar{z}_0)} \right] \\ &= +2K \frac{(z_0 - i \text{Im } z_0)w(z_0)}{z_0 + \bar{z}_0} \frac{dz}{(z - z_0)^2} \\ &= 2K(\text{Re } z_0) \frac{w(z_0)}{z_0 + \bar{z}_0} \frac{dz}{(z - z_0)^2}. \end{aligned}$$

To establish uniqueness, suppose that Θ was another such differential. Then $\Theta - \Theta^P$ would be exact, real and have double poles with no residues. So for some real s

$$\Theta = \Theta^P + s\Theta^E.$$

As taking principal part is a linear operation, the third condition forces $s = 0$. \square

As we have already remarked, having found this pair of differentials any other differential satisfying (P.4)–(P.8) may be written in the form $a\Theta^E + n\Theta^P$ for some $a \in \mathbb{R}$ and $n \in \mathbb{Z}$. We may think of the differentials satisfying (P.4)–(P.7) as forming a rank 2 real vector bundle over \mathcal{A}_1 spanned by Θ^E and Θ^P , and those differentials satisfying (P.4)–(P.8) as forming a $\mathbb{Z} \times \mathbb{R}$ -subbundle. This pair of differentials Θ^E and Θ^P vary smoothly with respect to (α, β) and are always linearly independent so they trivialises that bundle.

Recall that \mathcal{A}_1 double covers \mathcal{C}_1 ; the points (α, β) and (β, α) in \mathcal{A}_1 correspond to the same marked curve $\Sigma \in \mathcal{C}_1$. Suppose we have a section of differentials $\Theta : (\alpha, \beta) \mapsto \Omega^1(\Sigma(\alpha, \beta))$. As $\Sigma(\alpha, \beta)$ and $\Sigma(\beta, \alpha)$ are the same curve, this raises the question of what the difference $\Theta(\alpha, \beta) - \Theta(\beta, \alpha)$ is. Asking this for $\Theta^E(\alpha, \beta)$ and $\Theta^P(\alpha, \beta)$, we observe directly that $\Theta^E = id(\eta/\zeta)$ is invariant under the interchange of α and β . Now we may use the characterisation given by Lemma 3.15 to conclude the same for Θ^P , because

$$\text{pp. } \Theta^P(\beta, \alpha) \in i\mathbb{R} \text{ pp. } \Theta^E(\beta, \alpha) = i\mathbb{R} \text{ pp. } \Theta^E(\alpha, \beta)$$

and so uniqueness forces $\Theta^P(\beta, \alpha) = \Theta^P(\alpha, \beta)$.

The consequence of this is that $\langle \Theta^E, \Theta^P \rangle$ pushes forward to a well-defined basis of the differentials over \mathcal{C}_1 as well, and just as for \mathcal{A}_1 they trivialise the bundle $\mathcal{B}_1 \rightarrow \mathcal{C}_1$. Though we mainly concentrate on the subspace of \mathcal{C}_1 of marked

curves that admit spectral data, at the end of this chapter we will consider the space of spectral data within the total space of pairs of differentials

$$\mathcal{B}_1 \times \mathcal{B}_1 = (\mathbb{R}\langle \Theta^E, \Theta^P \rangle)^2.$$

3.2 The Closing Conditions

The closing conditions, Conditions (P.11), are the conditions that spectral data must meet in order that they correspond to a harmonic map of the torus, rather than a harmonic map of the plane (of finite type). If $(\Sigma, \Theta, \tilde{\Theta})$ is a triple of spectral data, Θ satisfies the closing condition at $\zeta = 1$ if

$$\int_{\gamma_+} \Theta \in 2\pi i\mathbb{Z},$$

where γ_+ is a path that begins at $(1, -\eta^+(1))$ and ends at $(1, \eta^+(1))$, the two points on the spectral curve lying over $\zeta = 1$. Recall that we have defined $\eta^+(\zeta) = \zeta |\zeta - \alpha| |\zeta - \beta|$ to be the value of η on the positive unit circle in Σ . However, the value of the integral is dependent on the particular path chosen. To see that this condition is none-the-less well defined, suppose that γ and γ' are two paths between the two points over $\zeta = 1$. Their difference $\gamma - \gamma'$ is a closed loop and homologous to an integral combination $aA + bB$ of the periods A and B . The difference in the values of the integrals is therefore

$$\int_{\gamma} \Theta - \int_{\gamma'} \Theta = \int_{\gamma - \gamma'} \Theta = a \int_A \Theta + b \int_B \Theta = 2\pi i n b,$$

because by the period condition (P.8) the real period of Θ is zero and its imaginary period is a multiple of $2\pi i$. So although the value of the integral is dependent on the path, the condition that the value must lie in $2\pi i\mathbb{Z}$ is not. Likewise, the closing condition at $\zeta = -1$ is defined by taking γ_- , a path from $(-1, -\eta^+(-1))$ to $(-1, \eta^+(-1))$, and requiring the integral of Θ over this path to lie in $2\pi i\mathbb{Z}$ also.

We have seen that every marked curve $\Sigma \in \mathcal{E}_1$ admits differentials that meet Conditions (P.4)–(P.8), but it is not possible to find differentials on every curve that further satisfy Condition (P.11). It will be our ongoing aim to find all such curves in \mathcal{E}_1 .

Let us begin by formulating a condition on (α, β) which will determine when $\Sigma(\alpha, \beta)$ admits an exact differential that meets the closing conditions. For an exact differential, such as Θ^E , the particular path of integration is irrelevant and the value of the integral is

$$\int_{\gamma_+} \Theta^E = i \, d\left(\frac{\eta}{\zeta}\right) \Big|_{(1, -\eta^+(1))}^{(1, \eta^+(1))} = 2i\eta^+(1) = 2i|1 - \alpha| |1 - \beta|$$

And similarly over the other marked point

$$\int_{\gamma_-} \Theta^E = -2i\eta^+(-1) = 2i|1+\alpha||1+\beta|.$$

Any other exact differential with the properties (P.4)–(P.8) must be a real multiple of Θ^E . For a real scalar $a \in \mathbb{R}$, the two closing conditions applied to $a\Theta^E$ are

$$\left. \begin{aligned} 2i\eta^+(1)a &\in 2\pi i\mathbb{Z}, \\ -2i\eta^+(-1)a &\in 2\pi i\mathbb{Z}. \end{aligned} \right\} \quad (3.16)$$

Eliminating a from the two equations, there is a common solution for a if and only if

$$S(\alpha, \beta) := \frac{2i\eta^+(1)}{-2i\eta^+(-1)} = \frac{|1-\alpha||1-\beta|}{|1+\alpha||1+\beta|} \in \mathbb{Q}^+. \quad (3.17)$$

This gives the flavour of what we are aiming to achieve. We will produce two explicit functions such that a marked curve admits spectral data exactly when these functions take rational values. The two functions may be interpreted as defining equations for the subspace of spectral curves \mathcal{S}_1 within the space of all marked curves \mathcal{C}_1 .

Before we plough ahead to differentials with periods, there is a simplification we can make. Suppose that we have a triple of spectral data $(\Sigma, \Theta^1, \Theta^2)$ such that the differentials Θ^1 and Θ^2 have imaginary periods $2\pi il_1$ and $2\pi il_2$ respectively. Let $l > 0$ be the greatest common denominator of l_1 and l_2 , and by Bézout's identity let x and y be integers that satisfy

$$xl_1 + yl_2 = l.$$

Then consider the differentials Ψ^E and Ψ^P defined by the following integer combination

$$\begin{pmatrix} \Psi^E \\ \Psi^P \end{pmatrix} = \begin{pmatrix} \frac{l_2}{l} & -\frac{l_1}{l} \\ x & y \end{pmatrix} \begin{pmatrix} \Theta^1 \\ \Theta^2 \end{pmatrix}$$

The new pair of differentials are simpler in the sense that their imaginary periods are 0 and $2\pi il$ respectively. They also meet the closing condition, because they are an integer combination of differentials that do. And the integer matrix has determinant one, so is invertible over the integers. Further, the two differentials are linearly dependent exactly when Ψ^E is zero. Hence,

Lemma 3.18. *A marked curve admits spectral data if and only if it admits a pair of nonzero differentials with imaginary periods 0 and $2\pi il$, for some positive integer l , that also satisfy the closing conditions (P.11). \square*

The condition above, $S \in \mathbb{Q}^+$, is a necessary condition for a spectral curve to admit spectral data. To find a second necessary condition, concerning the existence of a differential with imaginary period $2\pi il$, we follow the same line of

reasoning. For some real number b , we may write $\Psi^P = b\Theta^E + l\Theta^P$. Fix two paths γ_+, γ_- . The two closing conditions applied to Ψ^P are then

$$\left. \begin{aligned} 2i\eta^+(1)b + l \int_{\gamma_+} \Theta^P &= 2\pi i \Gamma_+^P \in 2\pi i \mathbb{Z}, \\ -2i\eta^+(-1)b + l \int_{\gamma_-} \Theta^P &= 2\pi i \Gamma_-^P \in 2\pi i \mathbb{Z}. \end{aligned} \right\} \quad (3.19)$$

Again elimination of b yields the condition for a common solution to exist. This condition can be written as

$$2\pi i T(\alpha, \beta, \gamma_+, \gamma_-) := S(\alpha, \beta) \int_{\gamma_-} \Theta^P - \int_{\gamma_+} \Theta^P \in 2\pi i \mathbb{Q}, \quad (3.20)$$

using the definition of $S(\alpha, \beta)$ to substitute for $-\eta^+(1)/\eta^+(-1)$. As was shown at the beginning of this section, whether the integral of a differential over γ_+ or γ_- lies in $2\pi i \mathbb{Z}$ is independent on the particular path chosen between the marked points. In the same manner, if $S(\alpha, \beta)$ is rational then the condition $T(\alpha, \beta, \gamma_+, \gamma_-) \in \mathbb{Q}$ is independent of the choice of paths because a different path γ_+ or γ_- will change the corresponding integral of Θ^P by a multiple of $2\pi i$. Therefore $T(\alpha, \beta)$ is defined up to an element of $\mathbb{Z}\langle 1, S(\alpha, \beta) \rangle \subset \mathbb{Q}$. We may therefore consider $T(\alpha, \beta)$, without a particular choice of path, as a multi-valued function on \mathcal{A}_1 .

We shall prove that these two conditions are sufficient.

Lemma 3.21. *A marked curve admits a pair of nonzero differentials, one exact and one inexact, which both satisfying conditions (P.4)–(P.8) and also the closing conditions (P.11) if and only if $S \in \mathbb{Q}^+$ and $T \in \mathbb{Q}$ for any paths γ_+, γ_- .*

Proof. From the above discussion, these are a necessary conditions.

For the converse suppose that the two functions are both rational, say $S = n/m$ and $T = n'/m'$. Further assume that n and m are coprime, as we can always take them to be. First we will attempt to solve (3.16) to produce an exact differential Ψ^E . The rationality of S directly ensures the consistent solution of an a . Namely, we may define

$$a := \frac{2\pi i n}{2i\eta^+(1)}.$$

Then observe that

$$a = \frac{2\pi i n}{2i\eta^+(1)} = \frac{2\pi i n}{-2i\eta^+(-1)} \frac{-2i\eta^+(-1)}{2i\eta^+(1)} = \frac{2\pi i n}{-2i\eta^+(-1)} \frac{m}{n} = \frac{2\pi i m}{-2i\eta^+(-1)},$$

and hence for $\Psi^E := a\Theta^E$ its integrals over γ_+ and γ_- are

$$\begin{aligned}\int_{\gamma_+} \Psi^E &= a \int_{\gamma_+} \Theta^E = 2i\eta^+(1)a = 2\pi in \\ \int_{\gamma_-} \Psi^E &= a \int_{\gamma_-} \Theta^E = -2i\eta^+(-1)a = 2i\pi m.\end{aligned}$$

This demonstrates the existence of an exact differential Ψ^E that satisfies the closing conditions (P.11).

To find a non-exact differential Ψ^P that also satisfies the closing conditions, we must solve a similar equation. For some real number b and integer l , suppose that $\Psi^P = b\Theta^E + l\Theta^P$. We must find such b and l so that (3.19) holds for some integers Γ_+^P and Γ_-^P . Recall that T is a rational number n'/m' and so eliminating b from (3.19) gives

$$\begin{aligned}\frac{1}{2i\eta^+(1)} \left(2\pi i\Gamma_+^P - l \int_{\gamma_+} \Theta^P \right) &= \frac{1}{-2i\eta^+(-1)} \left(2\pi i\Gamma_-^P - l \int_{\gamma_-} \Theta^P \right) \\ 2\pi i\Gamma_+^P - l \int_{\gamma_+} \Theta^P &= \frac{2i\eta^+(1)}{-2i\eta^+(-1)} \left(2\pi i\Gamma_-^P - l \int_{\gamma_-} \Theta^P \right) \\ 2\pi i(S\Gamma_-^P - \Gamma_+^P) &= l \left(S \int_{\gamma_-} \Theta^P - \int_{\gamma_+} \Theta^P \right) \\ \frac{n\Gamma_-^P - m\Gamma_+^P}{m} &= l \frac{n'}{m'} \\ m'(n\Gamma_-^P - m\Gamma_+^P) &= lmn'.\end{aligned}\tag{3.22}$$

If one were simply interested in getting a solution to this equation, one could take $l = m'$. However in Lemma 3.57 we will need the minimal solution to this equation. To avoid repeating ourselves, let us do the necessary extra work now. By considering the integer factorisation of each side, we see that a solution is only possible if m' divides the right hand side. Therefore, l must be at least

$$l = \frac{m'}{\gcd(m', mn')}.$$

We may then divide through by m' . To then solve this equation for Γ_+^P and Γ_-^P , as n and m are coprime let x and y be integers such that $nx - my = 1$. By Bézout's Identity, the solution set is

$$\left\{ (\Gamma_+^P, \Gamma_-^P) = \left(\frac{mn'}{\gcd(m', mn')}y + mr, \frac{mn'}{\gcd(m', mn')}x + nr \right) \mid r \in \mathbb{Z} \right\}.$$

Therefore we may take

$$\Gamma_+^P = \frac{mn'}{\gcd(m', mn')}y, \quad \Gamma_-^P = \frac{mn'}{\gcd(m', mn')}x,$$

to obtain equality in (3.22). Hence, as for the exact differential, we may define

$$b := \frac{1}{2i\eta^+(1)} \left(2\pi i \frac{mn'}{\gcd(m', mn')} y - \frac{m'}{\gcd(m', mn')} \int_{\gamma_+} \Theta^P \right). \quad (3.23)$$

Observe,

$$\begin{aligned} b &= \frac{1}{2i\eta^+(1)} \left(2\pi i \frac{mn'}{\gcd(m', mn')} y - \frac{m'}{\gcd(m', mn')} \int_{\gamma_+} \Theta^P \right) \\ &= \frac{1}{-2i\eta^+(-1)} \frac{m}{n} \left(2\pi i \frac{mn'}{\gcd(m', mn')} \frac{nx-1}{m} \right. \\ &\quad \left. - \frac{m'}{\gcd(m', mn')} \left(\frac{n}{m} \int_{\gamma_-} \Theta^P - 2\pi i \frac{n'}{m'} \right) \right) \\ &= \frac{1}{-2i\eta^+(-1)} \left(2\pi i \frac{mn'}{\gcd(m', mn')} x - \frac{m'}{\gcd(m', mn')} \int_{\gamma_-} \Theta^P \right. \\ &\quad \left. - 2\pi i \frac{mn'}{n \gcd(m', mn')} + 2\pi i \frac{mn'}{n \gcd(m', mn')} \right) \\ &= \frac{1}{-2i\eta^+(-1)} \left(2\pi i \frac{mn'}{\gcd(m', mn')} x - \frac{m'}{\gcd(m', mn')} \int_{\gamma_-} \Theta^P \right). \end{aligned}$$

With these definitions of b and l , it follows that $\Psi^P = b\Theta^E + l\Theta^P$ satisfy the closing conditions:

$$\begin{aligned} \int_{\gamma_+} \Psi^P &= 2i\eta^+(1)b + l \int_{\gamma_+} \Theta^P \\ &= 2\pi i \frac{mn'}{\gcd(m', mn')} y - \frac{m'}{\gcd(m', mn')} \int_{\gamma_+} \Theta^P + \frac{m'}{\gcd(m', mn')} \int_{\gamma_+} \Theta^P \\ &= 2\pi i \frac{mn'}{\gcd(m', mn')} y \in 2\pi i\mathbb{Z} \\ \int_{\gamma_-} \Psi^P &= -2i\eta^+(-1)b + l \int_{\gamma_-} \Theta^P = 2\pi i \frac{mn'}{\gcd(m', mn')} x \in 2\pi i\mathbb{Z}. \end{aligned}$$

In summary, we have found real constants a and b and an integer l such that $\Psi^E = a\Theta^E$ and $\Psi^P = b\Theta^E + l\Theta^P$ satisfy the closing conditions. \square

Thus the space of genus one spectral curves that admit closing spectral data is defined by the equations $S(\alpha, \beta) \in \mathbb{Q}^+$ and $T(\alpha, \beta) \in \mathbb{Q}$ inside \mathcal{A}_1 . For higher genus spectral curves, the closing conditions are difficult to work with, harder than even the period conditions. In the genus one case the integrals of Θ^E lead to an algebraic expression, as we have just seen in (3.17), but the integrals of Θ^P will lead to a transcendental conditions involving incomplete elliptic integrals.

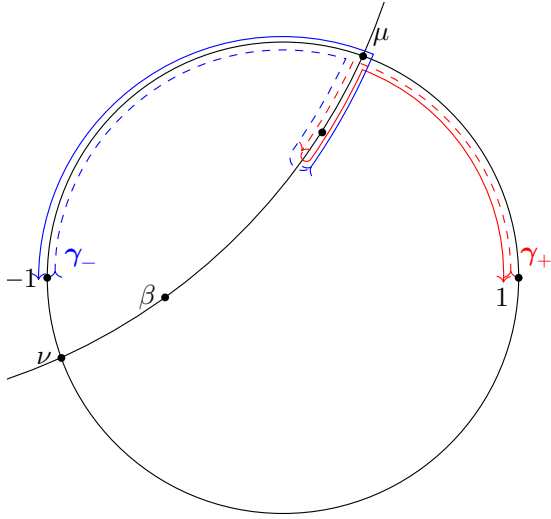


Figure 3.6

The principal path γ_+ , in red, starts at 1 on the lower sheet, traverses around α and then returns to 1 on the upper sheet. There is a branch cut between α and β . Likewise, the principal path γ_- is marked in blue.

Immediately before this lemma we observed that although the condition $T \in \mathbb{Q}$ is well defined, $T(\alpha, \beta)$ is a multi-valued function on \mathcal{A}_1 which is dependent on the paths of integration. So that we may work with a well-defined function, we will make some branch cuts on \mathcal{A}_1 and choose a principal branch of T . On each curve we will have to make a choice of paths γ_+ and γ_- , and we shall refer to these choices as principal paths.

Consider the open dense subset $\mathcal{A}_1 \setminus \{\nu = \pm 1\}$ of \mathcal{A}_1 . On any marked curve $\Sigma(\alpha, \beta)$ corresponding to a point of $\mathcal{A}_1 \setminus \{\nu = \pm 1\}$, let $\gamma_+ = \gamma_+(\alpha, \beta)$ be the path that begins at $(1, -\eta^+(1))$, traverses the unit circle to the point μ without crossing ν , follows the branch circle to α , circles this branch point anticlockwise, goes back along the arc to the unit circle (though on a different sheet now), and back to $(1, \eta^+(1))$. This path is illustrated in Figure 3.6.

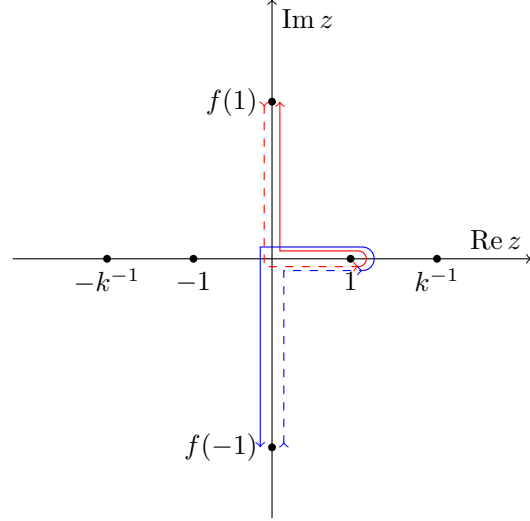
Likewise choose $\gamma_- = \gamma_-(\alpha, \beta)$ to be the path from $(-1, -\eta^+(-1))$ to $(-1, \eta^+(-1))$ along the unit and branch circles that does not cross ν . In the case $\nu = \pm 1$, it would be impossible to ‘avoid’ ν , so this case had to be excluded. Note that these paths depend on the order of the branch points $(\alpha, \beta) \in \mathcal{A}_1$, since ν is defined to be the intersection of the unit and branch circles which lies between β and $\bar{\beta}^{-1}$.

Definition 3.24. *The principal branch cut T_0 of T is defined on $\mathcal{A}_1 \setminus \{\nu = \pm 1\}$ to be*

$$T_0(\alpha, \beta) := T(\alpha, \beta, \gamma_+, \gamma_-) = \frac{1}{2\pi i} \left(S(\alpha, \beta) \int_{\gamma_-} \Theta^P - \int_{\gamma_+} \Theta^P \right). \quad (3.25)$$

As $\nu \neq 1$, we know that $f(1) \neq \infty$ and so $f(\gamma_+)$ and $f(\gamma_-)$ lie in the plane (they

Figure 3.7
The paths $f(\gamma_+)$ in red and $f(\gamma_-)$ in blue.



do not pass through $z = \infty$). By design it is easy to describe these paths in terms of the (z, w) coordinates. For example, we may describe the path $f(\gamma_+)$ as follows. Start from the point $f(1)$ on the imaginary axis and go to the origin. Go out along the real axis, around $z = 1$ (which corresponds to $\zeta = \alpha$) and back again to the origin. Return along the imaginary axis to $f(1)$. Both paths are illustrated in Figure 3.7.

Having fixed a choice of paths, we can express the integrals of the differentials ω and e along these particular paths in terms of Legendre elliptic integrals,

$$\begin{aligned} \int_{\gamma_+} \omega &= \left(2 \int_0^{f(1)} - 2 \int_0^1 \right) \omega = 2F(f(1); k) - 2K(k) \\ \int_{\gamma_+} e &= 2\tilde{E}(f(1); k) - 2E(k), \end{aligned}$$

and so integrating (3.14) gives a formula for Θ^P over γ_+ ,

$$\begin{aligned} \int_{\gamma_+} \Theta^P &= \int_{\gamma_+} (2E\omega - 2Ke) - 2K \int_{\gamma_+} d \left[\frac{(z - i \operatorname{Im} z_0)w}{(z - z_0)(z + \bar{z}_0)} \right] \\ &= 4E(k)F(f(1); k) - 4K(k)E(f(1); k) - 4K \frac{(f(1) - i \operatorname{Im} z_0)w(f(1))}{(f(1) - z_0)(f(1) + \bar{z}_0)}. \end{aligned} \quad (3.26)$$

For the integral of Θ^P over the path γ_- for $\zeta = -1$, we have similarly

$$\begin{aligned} \int_{\gamma_-} \Theta^P &= 4E(k)F(f(-1); k) - 4K(k)E(f(-1); k) \\ &\quad - 4K \frac{(f(-1) - i \operatorname{Im} z_0) w(f(-1))}{(f(-1) - z_0)(f(-1) + \bar{z}_0)}. \end{aligned} \quad (3.27)$$

These two formulae may be substituted into (3.25) to compute T_0 . In the next section however, we will make a change of coordinates that simplifies these formulae.

Previously, the comment was made that the particular algorithm to chose a path is not valid when $\nu = \pm 1$. Indeed, the result of this can be seen directly in the formulae we have derived. When ν takes either of these values, then one of $f(1)$ or $f(-1)$ will be infinite. We also note that these integrals are purely imaginary, as we expected on theoretical grounds, because $f(1)$ and $f(-1)$ are purely imaginary, $F(z; k)$ and $E(z; k)$ take the imaginary axis to itself and

$$(f(1) - z_0)(f(1) + \bar{z}_0) = -(f(1) - z_0)\overline{(f(1) - z_0)} = -|f(1) - z_0|^2.$$

The function T_0 is therefore real valued.

To summarise our calculations up to this point, we determined that every differential on a marked curve Σ that satisfies conditions (P.4)–(P.6) must lie in a real three-dimensional vector space W . We found a basis $\{\omega, \Theta^E, \}$ of W , gave a basis A, B for the homology of Σ and computed the periods of the basis differentials. This allowed us to find a differential $\Theta^P \in \mathbb{R}\{\omega, \lambda\} \subset W$ that satisfied (P.8). We observed that every differential on Σ that meet the conditions (P.4)–(P.8) was the sum of a real multiple of Θ^E and an integer multiple of Θ^P .

With these two differentials we then attempted to further satisfy (P.11). This was not possible for an arbitrary marked curve Σ , which lead us to define the functions $S(\alpha, \beta)$ by (3.17) and $T(\alpha, \beta)$ by (3.20). Lemmata 3.18 and 3.21 taken together imply that a marked curve admits spectral data exactly when the functions S and T simultaneously take rational values. The last part of this section noted that T is a multi-valued function, and so took a principal branch of it and derived the corresponding explicit formulae.

3.3 Coordinates for \mathcal{A}_1

In the previous section, we found a condition for a point of \mathcal{A}_1 to correspond a spectral curve, namely the functions S and T must be rationally valued. Thus on \mathcal{S}_1 , the space of spectral curves, these functions must be (locally) constant. To understand \mathcal{S}_1 we should therefore be examining the joint level sets of S

and T . This will require invoking the implicit function theorem, for which the necessary computation will be the differentiation of the two functions.

It is therefore prudent to adopt a parametrisation of the space of marked curves $\mathcal{A}_1 \subset D \times D$ that is suited to the task of differentiating $T(\alpha, \beta)$. Elliptic integrals are the most difficult part of (3.26) to differentiate, so to minimise our labour we choose three coordinates to be k , $iu = f(1)$ and $iv = f(-1)$. For the final coordinate we shall take $p = S(\alpha, \beta)$ itself, as then we can enforce the condition $S \in \mathbb{Q}^+$ simply by holding this coordinate constant.

However, these coordinates (p, k, u, v) only cover part of \mathcal{A}_1 , since f is a Möbius function $\mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ and so, for example, there will be points of \mathcal{A}_1 where $iu = f(1)$ is infinite. To cover these cases we must introduce the additional coordinates $u' = u^{-1}$ and $v' = v^{-1}$. The purpose of Lemma 3.28 is that verify that these are in fact coordinates for \mathcal{A}_1 and that together they cover it.

After having established this coordinate change, the remainder of the section is devoted to calculations. First we rewrite the formulae derived thus far in terms of our new coordinates: equations (3.32) and (3.33) are rewrites of (3.26) and (3.27) respectively. From these it is feasible to compute the u -derivative of T_0 , (3.35), and in thereby in Lemma 3.38 we essentially prove that the derivative does not vanish.

Lemma 3.28. *The following functions are diffeomorphisms:*

$$\begin{aligned} \varphi_0 : \mathcal{A}_1 \setminus \{\nu = \pm 1\} &\rightarrow \mathbb{R}^+ \times (0, 1) \times \mathbb{R} \times \mathbb{R} \setminus \{u = v\} \\ (\alpha, \beta) &\mapsto (p, k, u, v) := (S(\alpha, \beta), k(\alpha, \beta), -if(1), -if(-1)), \end{aligned}$$

$$\begin{aligned} \varphi_1 : \mathcal{A}_1 \setminus \{\mu = 1 \text{ or } \nu = -1\} &\rightarrow \mathbb{R}^+ \times (0, 1) \times \mathbb{R} \times \mathbb{R} \setminus \{u'v = 1\} \\ (\alpha, \beta) &\mapsto (p, k, u', v) := (S(\alpha, \beta), k(\alpha, \beta), if(1)^{-1}, -if(-1)), \end{aligned}$$

$$\begin{aligned} \varphi_2 : \mathcal{A}_1 \setminus \{\mu = -1 \text{ or } \nu = 1\} &\rightarrow \mathbb{R}^+ \times (0, 1) \times \mathbb{R} \times \mathbb{R} \setminus \{uv' = 1\} \\ (\alpha, \beta) &\mapsto (p, k, u, v') := (S(\alpha, \beta), k(\alpha, \beta), -if(1), if(-1)^{-1}), \end{aligned}$$

for the functions S given by (3.17), k given by (3.5) and f given by (3.6). Also, the union of the domains covers \mathcal{A}_1 .

Proof. First note that the exclusions from the codomains are correct. Were, for example, u and v to be equal then

$$f(1) = iu = iv = f(-1),$$

but f is an invertible transformation and this thus this would be a contradiction. Likewise for the codomains of other two functions.

Next, $S(\alpha, \beta)$ is smooth since α and β are inside the unit disc. The other functions are smooth by Lemma 3.8. Hence the function φ_0 is smooth. The

other two functions are necessary because the map f is a Möbius transformation, and so takes the value infinity. Indeed, we have seen that $f(\nu) = \infty$, and so one of u or v is infinite when $\nu = \pm 1$. Using (3.6), on the subset of \mathcal{A}_1 where $\mu \neq 1$

$$-iu' = -\frac{1}{\bar{z}_0} \frac{1-\nu}{1-\mu},$$

and where $\mu \neq -1$

$$-iv' = -\frac{1}{\bar{z}_0} \frac{1+\nu}{1+\mu},$$

demonstrating that φ_1 and φ_2 are smooth functions on their respective domains of definition.

It remains to show that these functions have smooth inverses. As the parameters α and β are points in the ζ -plane, one method to derive the inverse functions is to express the transformation $f^{-1}(z)$ in term of our new parameters (p, k, u, v) . Then $\alpha = f^{-1}(1)$ and $\beta = f^{-1}(k^{-1})$, entirely analogous to how the transformation $f(\zeta)$ is determined by (α, β) and the coordinate u is defined by $iu = f(1)$. As a Möbius transformation is described, up to a scalar, by the points sent to 0 and ∞ , $f^{-1}(z)$ is a scalar multiple of

$$\frac{z - z_0}{z + \bar{z}_0},$$

(cf. (3.7)). Thus the construction of f^{-1} proceeds in two steps; first find z_0 , then determine the correct scaling factor. To find z_0 , we will identify it as the intersection of two circles: one arising from S and one arising from the geometry of the Möbius transformation f . In (3.17), the definition of S , there are two ratios. We may use the following trick using the cross ratio to write each ratio in terms of the new coordinates and z_0 . Observe

$$\left| \frac{\alpha - 1}{\alpha + 1} \right| = \left| \frac{\alpha - 1}{\alpha + 1} \right| \left| \frac{0 + 1}{0 - 1} \right| = |[\alpha, 0; 1, -1]| = |[1, z_0; iu, iv]| = \left| \frac{1 - iu}{1 - iv} \right| \left| \frac{z_0 - iv}{z_0 - iu} \right|$$

The same trick gives a similar formula for β .

$$\left| \frac{\beta - 1}{\beta + 1} \right| = \left| \frac{1 - kiu}{1 - kiv} \right| \left| \frac{z_0 - iv}{z_0 - iu} \right|$$

We will show that together these imply that z_0 lies on a particular circle determined by the parameters (p, k, u, v) . We have that

$$\begin{aligned} p = S(\alpha, \beta) &= \left| \frac{\alpha - 1}{\alpha + 1} \right| \left| \frac{\beta - 1}{\beta + 1} \right| = \left| \frac{1 - iu}{1 - iv} \right| \left| \frac{1 - kiu}{1 - kiv} \right| \left| \frac{z_0 - iv}{z_0 - iu} \right|^2 \\ |z_0 - iv|^2 &= p \frac{\sqrt{1+v^2}}{\sqrt{1+u^2}} \frac{\sqrt{1+k^2v^2}}{\sqrt{1+k^2u^2}} |z_0 - iu|^2 = p \frac{w(iv)}{w(iu)} |z_0 - iu|^2, \end{aligned}$$

where we recall that w is defined by the relation $w^2 = (1 - z^2)(1 - k^2 z^2)$. This equation for z_0 defines a circle. Explicitly, if we decompose $z_0 = x + iy$ then

$$x^2 + y^2 + 2y \frac{puw(iv) - vw(iu)}{w(iu) - pw(iv)} + \frac{v^2 w(iu) - pu^2 w(iv)}{w(iu) - pw(iv)} = 0,$$

which is centred on the imaginary axis.

On the other hand, in the ζ -plane the points -1 , 0 , and 1 all lie on a straight line that is perpendicular to the unit circle at both -1 and 1 , and that is invariant under the real involution ρ . Applying the Möbius transformation f we can therefore say that iv , z_0 , and iu all lie on a circle that is perpendicular to the imaginary axis and symmetric under reflection in the imaginary axis. Therefore z_0 lies on the circle

$$x^2 + \left(y - \frac{u+v}{2}\right)^2 = \frac{(u-v)^2}{4},$$

which simplifies to the relation

$$x^2 + y^2 = y(u+v) - uv. \quad (3.29)$$

Thus we have determined two circles that z_0 lies on. As these two circles are both centred on the imaginary axis, they intersect in two points: z_0 and $-\overline{z_0}$. We may solve for z_0 , giving

$$x = \frac{\sqrt{pw(iu)w(iv)}}{pw(iv) + w(iu)} |u - v|, \quad y = \frac{puw(iv) + vw(iu)}{pw(iv) + w(iu)}, \quad (3.30)$$

where the sign of x is chosen to make z_0 lie in the right half of the z -plane. This choice amounts to choosing the branch points α and β inside the unit circle. Note that these are smooth functions of (p, k, u, v) , because the term under the square root and the denominators are strictly positive functions, and $u - v$ is not zero by the definition of the codomain of φ_0 .

Having found z_0 in terms of (p, k, u, v) it remains to find the correct scaling of f^{-1} . We use the fact that $f^{-1}(iu) = 1$ and $f^{-1}(iv) = -1$ to conclude

$$f^{-1}(z) = \frac{iu + \overline{z_0} z - z_0}{iu - z_0 z + \overline{z_0}} = -\frac{iv + \overline{z_0} z - z_0}{iv - z_0 z + \overline{z_0}}.$$

As was previously presented, one can simply take $\alpha = f^{-1}(1)$ and $\beta = f^{-1}(k^{-1})$ to give formulae for the branch points in terms of the new parameters. A problem could potentially occur if z_0 were to equal iu or iv , in which case the scaling factor would be $0/0$. This could only occur if $\text{Re } z_0 = 0$, which itself only occurs if $u = v$. But we have already noted that this is impossible. Likewise the formula would be ill-defined if $z_0 = -1$ or $-k^{-1}$ (for then α or β would be infinite), but again this occurs only if $\text{Re } z_0 < 0$, which is excluded by our decision to take z_0 in the right half-plane.

Now that we have constructed an inverse for φ_0 , we must also construct inverses for φ_1 and φ_2 . But we may do so by modifying the formula for x and y , given in equation (3.30), to define them for the primed coordinates u' or v' . Such a definition extends smoothly to the points where u' or v' is zero. Using the notation $w'(it)^2 = (1+t^2)(k^2+t^2)$ we have

$$\begin{aligned} x &= \frac{\sqrt{pw(iu)w(iv)}}{pw(iv) + w(iu)} |u - v| = \frac{\sqrt{u^2 \times pw'(iu')w(iv)}}{pw(iv) + u^2 w'(iu')} |u| |1 - u'v| \\ &= \frac{\sqrt{pw'(iu')w(iv)}}{pu'^2 w(iv) + w'(iu')} |1 - u'v|. \end{aligned}$$

Likewise

$$x = \frac{\sqrt{pw(iu)w(iv')}}{pw'(iv') + v'^2 w'(iu')} |uv' - 1|,$$

and

$$y = \frac{pu'w(iv) + vw'(iu')}{pu'^2 w(iv) + w'(iu')} = \frac{puw'(iv') + v'w(iu)}{pw'(iv') + v'^2 w(iu)}. \quad (3.31)$$

Having made these changes in formula for z_0 , the same formula for f^{-1} applies without modification.

It is interesting to see that it was necessary to exclude the plane where $u = v$ (or $u'v = 1$ or $uv' = 1$), for otherwise x would be zero, z_0 would be equal to $-\bar{z}_0$ and $f^{-1}(z)$ would be a constant function. We shall see later that these points correspond to the diagonal $\{\alpha = \beta\} \subset D \times D$ and represent a degeneration of marked curves.

Finally, it remains to be demonstrated that the three domains cover \mathcal{A}_1 . If there was some point $(\alpha, \beta) \in \mathcal{A}_1$ that was not covered, then one could compute $\mu(\alpha, \beta)$ and $\nu(\alpha, \beta)$. But the intersection

$$\{\nu = \pm 1\} \cap \{\mu = 1 \text{ or } \nu = -1\} \cap \{\mu = -1 \text{ or } \nu = 1\} \subset \mathcal{A}_1$$

consists of only those points where $\mu = \nu = 1$ or $\mu = \nu = -1$, and Lemma 3.8 proves that μ and ν are never equal. \square

Though standard, it is perhaps still of some interest to consider the above geometrical argument in the limit $u \rightarrow \infty$ to assure ourselves that nothing singular is happening. Suppose that $u' = 0$, which is to say geometrically that 1 is mapped to infinity by f . Then the transformation f takes the line through 1, 0, and -1 to a line perpendicular to the imaginary axis, cutting at $f(-1)$. This line is therefore horizontal and so z_0 and iv have the same imaginary parts. This gives $y = v$ directly, as can be observed by setting $u' = 0$ in (3.31).

Now that we have establish that these are valid changes of coordinates, we can put them to work. First we will show how to compute T_0 in these coordinates, and then develop related coordinates for the universal cover $\tilde{\mathcal{C}}_1$. Recall that we

defined a principal branch cut T_0 of T on $\mathcal{A}_1 \setminus \{\nu = \pm 1\}$. This is exactly the domain of φ_0 , and in equation (3.30) we have a formula for the inverse.

To rewrite T_0 in terms of (p, k, u, v) , we must compute the factors in the last terms of equations (3.26) and (3.27). Note that in the previous lemma we have denoted $\text{Im } z_0$ by y , and that by definition $iu = f(1)$ and $iv = f(-1)$. Therefore, by direct computation

$$\begin{aligned} u - y &= \frac{w(iu)(u - v)}{pw(iv) + w(iu)} & |iu - z_0|^2 &= \frac{w(iu)(u - v)^2}{pw(iv) + w(iu)} \\ v - y &= -\frac{w(iv)(u - v)}{pw(iv) + w(iu)} & |iv - z_0|^2 &= \frac{w(iv)(u - v)^2}{pw(iv) + w(iu)}, \end{aligned}$$

and thus

$$\begin{aligned} -4K \frac{w(f(1))(f(1) - i \text{Im } z_0)}{(f(1) - z_0)(f(1) + \bar{z}_0)} &= 4iK \frac{w(iu)(u - y)}{|iu - z_0|^2} = 4iK \frac{w(iu)}{u - v}, \\ -4K \frac{w(f(-1))(f(-1) - i \text{Im } z_0)}{(f(-1) - z_0)(f(-1) + \bar{z}_0)} &= 4iK \frac{w(iv)(v - y)}{|iv - z_0|^2} = -4iK \frac{w(iv)}{u - v}. \end{aligned}$$

We may make these replacements in (3.26) and (3.27) to yield

$$\int_{\gamma_+} \Theta^P = 4E(k)F(iu; k) - 4K(k)E(iu; k) + 4iK \frac{w(iu)}{u - v}, \quad (3.32)$$

$$\int_{\gamma_-} \Theta^P = 4E(k)F(iv; k) - 4K(k)E(iv; k) - 4iK \frac{w(iv)}{u - v}, \quad (3.33)$$

and hence by the equation (3.20) we finally arrive at

$$\begin{aligned} &2\pi i T_0(p, k, u, v) \\ &= 4p \left[EF(iv; k) - KE(iv; k) - iK \frac{w(iv)}{u - v} \right] - 4 \left[EF(iu; k) - KE(iu; k) + iK \frac{w(iu)}{u - v} \right] \\ &= 4p [EF(iv; k) - KE(iv; k)] - 4 [EF(iu; k) - KE(iu; k)] - 4iK \frac{pw(iv) + w(iu)}{u - v}. \end{aligned} \quad (3.34)$$

Be aware that we have omitted the elliptic modulus k in the complete elliptic integrals, as is standard, so that $K = K(k)$ and $E = E(k)$.

The purpose for constructing these new coordinates was to make it feasible to compute the derivatives of T . While T may be a multi-valued function on \mathcal{A}_1 , its derivative is not. For a fixed value of p the ambiguity present in T is locally a constant, which is removed by differentiation. In particular then let us compute the u -derivative of T_0 , a principal branch cut of T , from the explicit formula in (3.34). The v -derivative is similar, but later we will employ a symmetry in T_0

to avoid the need to compute it directly. As $F(z; k)$ and $E(z; k)$ are parameter integrals in z , we have that

$$\frac{\partial}{\partial u} F(iu; k) = \frac{i}{w(iu)}, \quad \frac{\partial}{\partial u} E(iu; k) = i \frac{1 + k^2 u^2}{w(iu)},$$

and we recall the definition $w(iu) = \sqrt{(1 + u^2)(1 + k^2 u^2)}$, so it follows elementarily that

$$\frac{\partial}{\partial u} w(iu) = \frac{\partial}{\partial u} \sqrt{1 + u^2} \sqrt{1 + k^2 u^2} = \frac{(1 + k^2)u + 2k^2 u^3}{w(iu)}.$$

Equipped with the derivatives of these factors, the calculation of the derivative of T_0 is mechanical if tedious.

$$\frac{\pi}{2} \frac{\partial T_0}{\partial u} = -\frac{E}{w(iu)} + \frac{pKw(iv)}{(u-v)^2} + \frac{K}{w(iu)(u-v)^2} [1 + u^2 - uv + v^2 + k^2 uv + k^2 u^2 v^2] \quad (3.35)$$

From this formula, we factor out the common denominator $[w(iu)(u-v)^2]^{-1}$ to define a function L ,

$$L(p, k, u, v) := -(u-v)^2 E + pKw(iu)w(iv) + K [1 + u^2 - uv + k^2 uv + v^2 + k^2 u^2 v^2]. \quad (3.36)$$

We shall need the value of the derivative ‘at infinity’ too, and so let us also define

$$\begin{aligned} L'(p, k, v) &:= \lim_{u \rightarrow \infty} u^{-2} L(p, k, u, v) \\ &= \lim_{u \rightarrow \infty} \left(-(1 - u^{-1}v)^2 E + pKw(iv) \cdot u^{-2} w(iu) \right. \\ &\quad \left. + K [u^{-2} + 1 - u^{-1}v + k^2 u^{-1}v + u^{-2}v^2 + k^2 v^2] \right) \\ &= -E + pKw(iv) + K [1 + k^2 v^2]. \end{aligned} \quad (3.37)$$

The final lemma of this section shows these functions L and L' are non-vanishing, which will be used to later prove that certain derivatives of T are non-vanishing also. The lemma below is not completely sufficient to establish this latter fact by itself, because T_0 is only defined in the (p, k, u, v) coordinate patch.

Lemma 3.38. *The functions L and L' , defined by (3.36) and (3.37) respectively are strictly positive for $p \geq 1$, $k \in (0, 1)$, and $u, v \in \mathbb{R}$.*

Proof. For this proof, we shall draw upon several inequalities that are explained in Section B.2. The first step is to eliminate E . We apply the crude estimate that $K > E$, from (B.3), and also the assumption that $p \geq 1$ to simplify

$$\begin{aligned} L(p, k, u, v) &= -(u-v)^2 E + pKw(iu)w(iv) + K [1 + u^2 - uv + k^2 uv + v^2 + k^2 u^2 v^2] \\ &> -(u-v)^2 K + Kw(iu)w(iv) + K [1 + u^2 - uv + k^2 uv + v^2 + k^2 u^2 v^2] \\ &= K [w(iu)w(iv) + 1 + (1 + k^2)uv + k^2 u^2 v^2] \end{aligned}$$

This formula is almost sufficient. The only term that could be negative is the one featuring uv . However, a lower bound for the square root terms is

$$w(iu) = \sqrt{1 + (1 + k^2)u^2 + k^2u^4} > \sqrt{(1 + k^2)u^2} = \sqrt{(1 + k^2)} |u|,$$

so applying this to both $w(iu)$ and $w(iv)$ gives

$$\begin{aligned} L(p, k, u, v) &> K [(1 + k^2) |uv| + 1 + (1 + k^2)uv + k^2u^2v^2] \\ &\geq K [1 + k^2u^2v^2]. \end{aligned}$$

As $K > \pi/2$, this is strictly positive, as required. The proof of the positivity of the second function, L' , is almost immediate. Again using $K > E$,

$$L'(p, k, v) > K [-1 + pkw(iv) + 1 + k^2v^2].$$

This establishes that L' is positive too. □

3.4 The Topology of the Moduli Space

The purpose of this chapter overall is to describe the set of genus one spectral curves \mathcal{S}_1 . The coordinates constructed in the previous chapter are apt for computation, but because T is a multi-valued function any work on its level sets in \mathcal{A}_1 will be limited to local results. The way forward is to transition to the universal cover $\tilde{\mathcal{E}}_1$ of \mathcal{A}_1 , which is covered by coordinates $(p, k, \tilde{u}, \tilde{v})$. The universal cover allows us to pull back the multi-valued function T to a single valued function \tilde{T} . This global function \tilde{T} will allow us to gain global results about \mathcal{S}_1 .

More precisely, in Lemma 3.48 we demonstrate that for any values $p \in \mathbb{R}^+$, $q \in \mathbb{R}$ the level set $\tilde{\mathcal{E}}_1(p, q)$ defined by $p = S$ and $q = \tilde{T}$ is a graph over two of the coordinates. This follows by an application of the implicit function theorem and relies on the non-vanishing result from Lemma 3.38. Hence each level set is diffeomorphic to a ribbon $(0, 1) \times \mathbb{R}$.

It will follow from the observation that \tilde{T} is rational exactly when T is that the preimage $\tilde{\mathcal{S}}_1$ of \mathcal{S}_1 in the universal cover $\tilde{\mathcal{E}}_1$ may be written as the disjoint union of level sets,

$$\tilde{\mathcal{S}}_1 = \coprod_{p \in \mathbb{Q}^+, q \in \mathbb{Q}} \tilde{\mathcal{E}}_1(p, q).$$

The second half of this section seeks to recover \mathcal{S}_1 from $\tilde{\mathcal{S}}_1$. To do so, we investigate the action of the group \mathcal{G} , the covering transformations of $\tilde{\mathcal{E}}_1$ over \mathcal{E}_1 , on these level sets. This culminates in Theorems 3.55 and 3.56, wherein we take the quotient of $\tilde{\mathcal{S}}_1$ by this group and thereby enumerate the path connected components of the moduli space \mathcal{S}_1 of genus one spectral curves and describe the topology of each component.

To motivate the definition of the universal cover in Lemma 3.39, and to provide context for Figures 3.11–3.13, we will deduce the topology of \mathcal{A}_1 from the coordinates (p, k, u, v) introduced in the previous section. These coordinates embed \mathcal{A}_1 into $\mathbb{R}^+ \times (0, 1) \times \mathbb{T}^2$ in the following way. The first two factors $\mathbb{R}^+ \times (0, 1)$ are obvious, they come from $p \in \mathbb{R}^+$ and $k \in (0, 1)$. To see the torus part, consider

$$\{(u, v) \in \mathbb{R} \times \mathbb{R}\} \cup \{(u', v) \in \mathbb{R} \times \mathbb{R}\} \cup \{(u, v') \in \mathbb{R} \times \mathbb{R}\} \cup \{(u', v') \in \mathbb{R} \times \mathbb{R}\}$$

with the identifications $u' = u^{-1}$ and $v' = v^{-1}$. This is the product of two circles, which is to say a torus \mathbb{T}^2 . Specifically \mathcal{A}_1 is the subset of this torus where $u \neq v$. The line $u = v$ can be represented as the line where the toroidal and poloidal angles are equal, and removing this line leaves an annulus. Moreover, consider the subsets $\mathcal{A}_1(p)$ of the parameter space \mathcal{A}_1 for which the coordinate p is fixed. This shows that topologically it is the product of an interval and an annulus, a feature not as easily seen from the (α, β) description.

A more instructive way of visualising $\mathcal{A}_1(p)$ is to think of it as a solid cylinder with a line along the central axis removed. One should think of the ‘radius’ of point being given by $1 - k$, so that the central axis is identified with the value $k = 1$. To motivate this, consider formula (3.5) for k .

$$k = \frac{|1 - \bar{\alpha}\beta| - |\alpha - \beta|}{|1 - \bar{\alpha}\beta| + |\alpha - \beta|}.$$

In the limit as $\alpha \rightarrow \beta$, this formula says that $k \rightarrow 1$. From the equation of S , the subspace of $D \times D$ where $\alpha = \beta$ and $S(\alpha, \beta) = p$ is an arc. In this visualisation we are imagining this arc as the central axis of the cylinder. In Section 4.1, the interesting structure of the moduli space in this limit will be investigated.

The fact that the parameter space is not simply connected is fundamentally tied to the fact that T is not a single valued function. We have defined a principal branch cut T_0 of T and given a formula, but the more natural way to correct this deficiency is to move to the universal cover. By constructing a lift \tilde{T} of T which is single valued, we will be able to treat the level sets $T \in \mathbb{Q}$ globally and thereby acquire complete description of the topology of the space \mathcal{S}_1 , significantly more than Theorem 1.32 which is a local result showing it to be a surface.

Lemma 3.39. *The universal cover of \mathcal{A}_1 is*

$$\tilde{\mathcal{E}}_1 = \{(p, k, \tilde{u}, \tilde{v}) \in \mathbb{R}^+ \times (0, 1) \times \mathbb{R} \times \mathbb{R} \mid \tilde{u} < \tilde{v} < \tilde{u} + 2\pi\},$$

with the projection map $\tilde{\pi} : \tilde{\mathcal{E}}_1 \rightarrow \mathcal{A}_1$ is given by

$$\begin{aligned} p &= p, \\ k &= k, \\ u &= \tan \frac{\tilde{u}}{2}, & u' &= \cot \frac{\tilde{u}}{2}, \\ v &= \tan \frac{\tilde{v}}{2}, & v' &= \cot \frac{\tilde{v}}{2}. \end{aligned}$$

Proof. The justification of this definition of $\tilde{\mathcal{C}}_1$ precedes in two steps. First, we have already observed that \mathcal{A}_1 embeds into $\mathbb{R}^+ \times (0, 1) \times \mathbb{T}^2$. The universal cover of this larger space is $\mathbb{R}^+ \times (0, 1) \times \mathbb{R}^2$, with the covering map given above using the standard 2π -periodic tan mapping of \mathbb{R} to \mathbb{S}^1 . The second step is to recall that \mathcal{A}_1 is the complement of the hyperplane $u - v = 0$. When pulled back to the universal cover, this hyperplane becomes a collection of hyperplanes $\tilde{u} - \tilde{v} \in 2\pi\mathbb{Z}$. Thus we may take a simply connected region of the complement $\tilde{u} - \tilde{v} \notin 2\pi\mathbb{Z}$ to cover \mathcal{A}_1 and this is $\tilde{\mathcal{C}}_1$ above. \square

It is straightforward to lift T_0 , defined on $\mathcal{A}_1 \setminus \{\nu = \pm 1\} \subset \mathcal{A}_1$, to a single valued function \tilde{T} on $\tilde{\mathcal{C}}_1$. Recall the definitions of F_0 and E_0 from B.6:

$$F_0(x; k) = \text{Im } F(ix; k), \quad E_0(x; k) = \text{Im } F(ix; k) - kx.$$

Using these, we rewrite T_0 in the following way.

$$\begin{aligned} 2\pi T_0(p, k, u, v) &= 4p [EF_0(v; k) - KE_0(v; k)] - 4 [EF_0(u; k) - KE_0(u; k)] \\ &\quad - 4K \left[p \left(\frac{w(iv)}{u - v} + kv \right) + \left(\frac{w(iu)}{u - v} - ku \right) \right]. \end{aligned} \quad (3.40)$$

In Section B.5, analytic extensions of $F_0(x; k)$ and $E_0(x; k)$ are constructed. They are denoted respectively as \tilde{F} and \tilde{E} . Thus the first two brackets of T_0 in (3.40) can be lifted to the universal cover by replacing F_0 and E_0 with their extensions. The following lemma will establish that the third bracket is analytic, and so lifts to the universal cover without the need for modification at all. Therefore we define

$$\begin{aligned} 2\pi \tilde{T}(p, k, \tilde{u}, \tilde{v}) &:= 4p [E\tilde{F}(\tilde{v}; k) - K\tilde{E}(\tilde{v}; k)] - 4 [E\tilde{F}(\tilde{u}; k) - K\tilde{E}(\tilde{u}; k)] \\ &\quad - 4K \left[p \left(\frac{w(iv)}{u - v} + kv \right) + \left(\frac{w(iu)}{u - v} - ku \right) \right]. \end{aligned}$$

Lemma 3.41. *The function*

$$\frac{w(iu)}{u - v} - ku$$

defined on $\mathcal{A}_1 \setminus \{\nu = \pm 1\}$ extends to an analytic function on \mathcal{A}_1 .

Proof. As $u - v \neq 0$ on $\mathcal{A}_1 \setminus \{\nu = \pm 1\}$, this is an analytic function of the coordinates (p, k, u, v) . It remains to show that it is similarly analytic in the other coordinates required to cover \mathcal{A}_1 . Firstly, examining this function when using the coordinate v' gives

$$\frac{w(iu)}{u - v} - ku = \frac{w(iu)v'}{uv' - 1} - ku,$$

which is analytic. Next, when using the coordinate u' we have

$$\frac{w(iu)}{u-v} - ku = \frac{w(iu) - ku^2}{u-v} + \frac{ kuv }{u-v} \quad (3.42)$$

$$= \frac{1 + (1+k^2)u^2}{(u-v)(w(iu) + ku^2)} + \frac{kv}{1-u'v} \quad (3.43)$$

$$= \frac{u'((u')^2 + (1+k^2))}{(1-u'v)(w'(iu') + k)} + \frac{kv}{1-u'v}, \quad (3.44)$$

where we have again used the auxiliary function $w'(it)^2 = (1+t^2)(k^2+t^2)$. This is a sum of analytic functions of u' . As these three coordinate patches cover all of \mathcal{A}_1 we are done. \square

To actually compute the value of the function \tilde{T} , it is simply a matter of substituting the correct expression for \tilde{F} or \tilde{E} . If we define the winding number $\text{Wind} : \mathbb{R} \rightarrow \mathbb{Z}$ of a number x to be the integer $\text{Wind}(x)$ such that $-\pi < x - 2\pi \text{Wind}(x) < \pi$, then recall (B.17) and (B.18),

$$\begin{aligned} \tilde{F}(\tilde{x}; k) &= 2 \text{Wind}(\tilde{x})K' + F_0 \left(\tan \frac{\tilde{x}}{2}; k \right), \\ \tilde{E}(\tilde{x}; k) &= 2 \text{Wind}(\tilde{x})(K' - E') + E_0 \left(\tan \frac{\tilde{x}}{2}; k \right). \end{aligned}$$

Then

$$\begin{aligned} 2\pi\tilde{T}(p, k, \tilde{u}, \tilde{v}) &= 2\pi T_0(p, k, u, v) + 4p[2EK' - 2K(K' - E')] \text{Wind}(\tilde{v}) \\ &\quad - 4[2EK' - 2K(K' - E')] \text{Wind}(\tilde{u}) \\ \tilde{T}(p, k, \tilde{u}, \tilde{v}) &= T_0(p, k, u, v) + 2(p \text{Wind}(\tilde{v}) - \text{Wind}(\tilde{u})), \end{aligned} \quad (3.45)$$

using Legendre's relation (see Section B.4). Thus to do computations with the function \tilde{T} , for the most part one can continue to work with the function T_0 downstairs on \mathcal{A}_1 and keep track of the winding numbers. For example, the following lemma uses this fact to motivate us looking at the level sets of \tilde{T} as a proxy for looking at the level sets of T .

Lemma 3.46. *If p is rational, $T \circ \tilde{\pi} \in \mathbb{Q}$ if and only if $\tilde{T} \in \mathbb{Q}$.* \square

This relationship between \tilde{T} and T_0 is also used in the next lemma to show that the range of \tilde{T} is \mathbb{R} , which is used in Lemma 3.48.

Lemma 3.47. *The range of \tilde{T} on $\tilde{\mathcal{C}}_1$ is \mathbb{R} .*

Proof. We shall prove below that the range of T_0 on $\tilde{\mathcal{C}}_1 \setminus \{\nu = \pm 1\}$ is \mathbb{R} . Because on each component of the preimage of $\tilde{\mathcal{C}}_1 \setminus \{\nu = \pm 1\}$ in $\tilde{\mathcal{C}}_1$ equation (3.45) tells

us that \tilde{T} and $T_0 \circ \tilde{\pi}$ differ by a constant, it follows that the range of \tilde{T} is also \mathbb{R} .

Fix p , but also fix any value for k . By (B.8), the magnitude $|F_0(x; k)|$ is bounded by K' and $|E_0(x; k)|$ is bounded by $K' - E'$. Thus there is some constant, dependent on both p and k but independent of u and v , such that

$$-C \leq 4p[EF_0(v) - KE_0(v)] - 4[EF_0(u) - KE_0(u)] \leq C.$$

Substituting this into (3.40), the definition of T_0 ,

$$\begin{aligned} & -4K \left[\frac{pw(iv) + w(iu)}{u - v} + k(pv - u) \right] - C \\ & \leq 2\pi T_0(p, k, u, v) \leq \\ & -4K \left[\frac{pw(iv) + w(iu)}{u - v} + k(pv - u) \right] + C, \end{aligned}$$

and so it is sufficient to show that for any fixed p and k the bracketed expression has range equal to the real line. But this is easy to show. Consider the limit as $u \rightarrow v^+$,

$$\lim_{u \rightarrow v^+} \frac{pw(iv) + w(iu)}{u - v} + k(pv - u) = k(p - 1)v + (p + 1)w(iv) \lim_{u \rightarrow v^+} \frac{1}{u - v} = +\infty.$$

From the other side,

$$\lim_{u \rightarrow v^-} \frac{pw(iv) + w(iu)}{u - v} + k(pv - u) = k(p - 1)v + (p + 1)w(iv) \lim_{u \rightarrow v^-} \frac{1}{u - v} = -\infty.$$

By continuity T_0 obtains every value. \square

Let us make two definitions. Define $\tilde{\mathcal{E}}_1(p)$ to be the subspace of $\tilde{\mathcal{E}}_1$ on which $S \circ \tilde{\pi} = p$ is a fixed constant. We further denote the subset of $\tilde{\mathcal{E}}_1(p)$ on which the value of \tilde{T} is q by $\tilde{\mathcal{E}}_1(p, q)$.

The spaces $\tilde{\mathcal{E}}_1(p)$ are simple to understand. They are just the points of $\tilde{\mathcal{E}}_1$ where the first coordinate is fixed, and so they are simply covered by the three remaining coordinates $(k, \tilde{u}, \tilde{v}) \in (0, 1) \times \mathbb{R} \times \mathbb{R}$. As demonstrated by the following lemma, the level sets $\tilde{\mathcal{E}}_1(p, q)$ are similarly well behaved.

Lemma 3.48. *If $p \leq 1$ then there is a diffeomorphism between $\tilde{\mathcal{A}}_1(p)$ and*

$$\{(q, k, \tilde{u}) \in \mathbb{R} \times (0, 1) \times \mathbb{R}\},$$

such that fixing a value q gives the level set $\tilde{\mathcal{E}}_1(p, q)$, on which $S \circ \tilde{\pi} = p$ and $\tilde{T} = q$. Likewise, if $p \geq 1$ then there is a diffeomorphism between $\tilde{\mathcal{A}}_1(p)$ and

$$\{(q, k, \tilde{v}) \in \mathbb{R} \times (0, 1) \times \mathbb{R}\},$$

such that again fixing a value q gives the level set $\tilde{\mathcal{E}}_1(p, q)$. In either case, the level sets are ribbons $(0, 1) \times \mathbb{R}$.

Proof. Fix a value of p and consider the function $G(q, k, \tilde{u}, \tilde{v}) = \tilde{T}(p, k, \tilde{u}, \tilde{v}) - q$ on $\tilde{\mathcal{E}}_1(p) \times \mathbb{R}$. $G^{-1}(0)$ is a graph over $\tilde{\mathcal{E}}_1(p)$ given by $q = \tilde{T}$, so they are diffeomorphic. We will apply the implicit function theorem to show that $G^{-1}(0)$ can also be written as a graph over either (q, k, \tilde{u}) or (q, k, \tilde{v}) , depending on the magnitude of p .

Suppose first that the fixed value of p is greater than or equal to one. We compute the following formula for the derivative of G with respect to \tilde{u} .

$$\begin{aligned} \frac{\partial G}{\partial \tilde{u}} &= \frac{\partial \tilde{T}}{\partial \tilde{u}} = \frac{du}{d\tilde{u}} \frac{\partial T_0}{\partial u} \\ &= \frac{1}{2} \sec^2 \left(\frac{\tilde{u}}{2} \right) \frac{\partial T_0}{\partial u} \\ &= \frac{1}{2} \sec^2 \left(\frac{\tilde{u}}{2} \right) \times \frac{2}{\pi} \times \frac{1}{w(iu)(u-v)^2} L(p, k, u, v) \\ &= \frac{1}{\pi} \frac{1+u^2}{w(iu)(u-v)^2} L(p, k, u, v), \end{aligned}$$

which holds for $\tilde{u} \notin \pi + 2\pi\mathbb{Z}$, using (3.36). As witnessed in Lemma 3.38, L is never zero, and neither are the other three factors present. Hence $\partial \tilde{G} / \partial \tilde{u}$ is never zero on this open set.

It remains to check it does not vanish when $\tilde{u} \in \pi + 2\pi\mathbb{Z}$. Recall the definition of L' from (3.37),

$$L'(p, k, v) := \lim_{u \rightarrow \infty} u^{-2} L(p, k, u, v).$$

As \tilde{T} is an analytic function its derivatives are continuous and so we may compute their value at these points by taking a limit. Therefore

$$\begin{aligned} \lim_{\tilde{u} \rightarrow \pi + 2\pi\mathbb{Z}} \frac{\partial G}{\partial \tilde{u}} &= \frac{1}{\pi} \lim_{u \rightarrow \infty} \frac{1+u^2}{w(iu)(u-v)^2} L(p, k, u, v) \\ &= \frac{1}{\pi} \lim_{u \rightarrow \infty} \frac{(1+u^2)u^2}{w(iu)(u-v)^2} \times u^{-2} L(p, k, u, v) \\ &= \frac{1}{\pi} \frac{1}{k} L'(p, k, v), \end{aligned}$$

which is also nonzero by Lemma 3.38. The implicit function theorem states that there is a function h such that $G^{-1}(0)$ is a graph of the form $\{(q, k, \tilde{u}, h(q, k, \tilde{u})) \mid q \in \text{Range } \tilde{T}, k \in (0, 1), \tilde{u} \in \mathbb{R}\}$. By Lemma 3.47, the range of \tilde{T} is \mathbb{R} . Finally, if we hold q fixed, then the level set $\tilde{\mathcal{E}}_1(p, q)$ is parametrised by the remaining two coordinates (k, \tilde{u}) .

When $p \leq 1$, we employ the following symmetry.

$$\begin{aligned}
T_0(p, k, u, v) &= 4p [EF(iv; k) - KE(iv; k)] - 4 [EF(iu; k) - KE(iu; k)] \\
&\quad - 4iK \frac{pw(iv) + w(iu)}{u - v} \\
&= -p \left(-4 [EF(iv; k) - KE(iv; k)] + \frac{4}{p} [EF(iu; k) - KE(iu; k)] \right. \\
&\quad \left. + 4iK \frac{w(iv) + \frac{1}{p}w(iu)}{u - v} \right) \\
&= -pT_0\left(\frac{1}{p}, k, v, u\right), \tag{3.49}
\end{aligned}$$

so that it is now the \tilde{v} derivative of \tilde{T} that is non-vanishing. Again the implicit function theorem gives the result. \square

With this lemma in hand, we are within striking distance of results about \mathcal{S}_1 . Recall that $\mathcal{S}_1 \subset \mathcal{C}_1$ is the set of spectral curves, those marked curves that admit spectral data. We will recover \mathcal{S}_1 as the quotient of the level sets of \tilde{T} by the group \mathcal{G} of covering transformations $\tilde{\mathcal{C}}_1$ over \mathcal{C}_1 .

Define $\tilde{\mathcal{S}}_1 \subset \tilde{\mathcal{C}}_1$ to be the preimage of \mathcal{S}_1 in the universal cover. As we have shown that \mathcal{S}_1 is the subspace of \mathcal{C}_1 on which S is a positive rational and T is any rational (Lemma 3.21) and that a point in $\tilde{\mathcal{C}}_1$ is the preimage of a point in \mathcal{C}_1 exactly when p is a positive rational and $\tilde{T} \in \mathbb{Q}$ (Lemma 3.46), it follows that

$$\tilde{\mathcal{S}}_1 = \coprod_{p \in \mathbb{Q}^+, q \in \mathbb{Q}} \tilde{\mathcal{C}}_1(p, q). \tag{3.50}$$

There are several topological implications of the previous lemma. First, each level set $\tilde{\mathcal{C}}_1(p, q)$ is connected. More precisely each of these is diffeomorphic to a product of $(0, 1)$ and \mathbb{R} . Therefore the above disjoint union is the decomposition of $\tilde{\mathcal{S}}_1$ into its path connected components. By varying the value of q , we see that the $\tilde{\mathcal{C}}_1(p, q)$ foliate $\tilde{\mathcal{C}}_1(p)$, so $\tilde{\mathcal{S}}_1$ is arranged densely in $\tilde{\mathcal{C}}_1(p)$, analogously to how \mathbb{Q} is arranged densely in \mathbb{R} .

To understand the topology of \mathcal{S}_1 , we must understand the action of the covering transformations of $\tilde{\mathcal{C}}_1 \rightarrow \mathcal{A}_1 \rightarrow \mathcal{C}_1$ as restricted to $\tilde{\mathcal{S}}_1$, and so we must first describe the group of covering transformations \mathcal{G} . The covering transformations of \mathcal{A}_1 over \mathcal{C}_1 are easy to understand; besides the identity there is only

$$\lambda : (\alpha, \beta) \mapsto (\beta, \alpha),$$

which swaps the labelling of the branch points inside the unit disc. We will use λ as a stepping stone to \mathcal{G} .

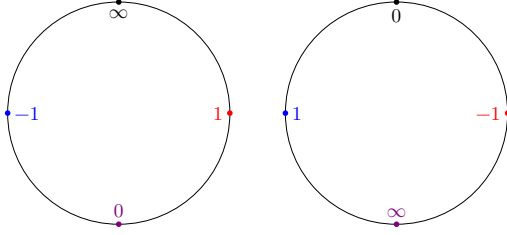


Figure 3.8
The circle on the left is \mathbb{RP}^1 with four points marked. The map $U \mapsto -U^{-1}$ is applied to produce the circle on the right, with corresponding points coloured the same. We can see that the circle on the right is rotated by half.

Let us describe λ in terms of the (p, k, u, v) coordinates. By inspection of the definitions of $p = S(\alpha, \beta)$ and $k = k(\alpha, \beta)$, equations (3.17) and (3.5), they are unchanged if the labelling of the branch points is exchanged. We therefore must say what happens to u and v . Recall that $iu = f(1)$ and $iv = f(-1)$. Our construction thus far relies on a definition of f that sends α to 1 and β to k^{-1} . Let f_s be the Möbius transformation which instead standardises the branch points of the marked curve in the other way, taking β to 1 and α to k^{-1} . Equivalently, f_s is the result of first swapping the order of branch points (α, β) and then applying f . Consider the composition of $f_s \circ f^{-1}$. It is a Möbius transformation that exchanges 1 and k^{-1} and also -1 and $-k^{-1}$. It therefore must be the map

$$z \mapsto \frac{1}{kz}.$$

Under the map $f_s \circ f^{-1}$ the point iu is taken to $-i(ku)^{-1}$ and iv is taken to $-i(kv)^{-1}$. Thus, under the label-swapping involution λ ,

$$\lambda : (p, k, u, v) \mapsto (p, k, -(ku)^{-1}, -(kv)^{-1}). \quad (3.51)$$

To gain a geometric understanding of λ , we imagine should again consider u and $u' = u^{-1}$ as coordinates on \mathbb{RP}^1 . The action of λ above on u is then a distorted rotation of the circle. To see this clearly, we may rescale u in the following manner. Let $U = \sqrt{k}u$ and $U' = \frac{1}{\sqrt{k}}u'$. Then

$$U \mapsto \sqrt{k}\lambda\left(\frac{U}{\sqrt{k}}\right) = -\frac{1}{U} = -U',$$

which is half-rotation of the circle (as shown in Figure 3.8).

We shall use this rescaled coordinate U , and likewise $V = \sqrt{k}v$, to construct coordinates on $\tilde{\mathcal{C}}_1$ such that the covering transformations are simply translations. This is motivated by the fact that translating the line \mathbb{R} rotates the quotient \mathbb{R}/\mathbb{Z} .

In analogy to Lemma 3.39, we define coordinates \tilde{U} and \tilde{V} on $\tilde{\mathcal{E}}_1$ to be

$$\begin{aligned} U &= \tan \frac{\tilde{U}}{2}, & U' &= \cot \frac{\tilde{U}}{2}, \\ V &= \tan \frac{\tilde{V}}{2}, & V' &= \cot \frac{\tilde{V}}{2}. \end{aligned}$$

Using $U = \sqrt{k}u$, the change of coordinates from \tilde{u} to \tilde{U} is

$$\begin{aligned} \tan \frac{\tilde{U}}{2} &= U = \sqrt{k}u = \sqrt{k} \tan \frac{\tilde{u}}{2} \\ \tilde{U} &= 2\pi \text{Wind}(\tilde{u}) + 2 \operatorname{atan} \left[\sqrt{k} \tan \frac{\tilde{u}}{2} \right], \end{aligned}$$

and similarly

$$\tilde{V} = 2\pi \text{Wind}(\tilde{v}) + 2 \operatorname{atan} \left[\sqrt{k} \tan \frac{\tilde{v}}{2} \right].$$

Observe that these formula fix certain points, namely multiples of \mathbb{Z} . Hence \tilde{u} and \tilde{U} have the same winding number. Recall also from Lemma 3.39 that the range of the coordinates \tilde{u} and \tilde{v} is restricted to $\tilde{u} < \tilde{v} < \tilde{u} + 2\pi$. Since \tan and atan are increasing functions, it follows that the transformation is order preserving, so we may derive that $\tilde{U} < \tilde{V} < \tilde{V} + 2\pi$ as well.

These coordinates \tilde{U} and \tilde{V} can be used interchangeably with \tilde{u} and \tilde{v} . Consider for example the coordinates $(p, k, \tilde{u}, \tilde{v})$ on $\tilde{\mathcal{E}}_1$ defined in Lemma 3.39. The determinant of the Jacobian of the change of coordinates to $(p, k, \tilde{U}, \tilde{V})$ is

$$\det \text{Jac} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{\partial \tilde{U}}{\partial k} & \frac{\partial \tilde{U}}{\partial \tilde{u}} & 0 \\ 0 & \frac{\partial \tilde{V}}{\partial k} & 0 & \frac{\partial \tilde{V}}{\partial \tilde{v}} \end{vmatrix} = \frac{\sqrt{k}}{\cos^2 \frac{\tilde{u}}{2} + k \sin^2 \frac{\tilde{u}}{2}} \times \frac{\sqrt{k}}{\cos^2 \frac{\tilde{v}}{2} + k \sin^2 \frac{\tilde{v}}{2}} \neq 0,$$

so this is indeed a valid change of coordinates. The same is true for the coordinates on $\tilde{\mathcal{E}}_1(p)$ provided by Lemma 3.39. For $p \leq 1$, we may change coordinates from (p, q, k, \tilde{u}) to (p, q, k, \tilde{U}) , and for $p \geq 1$ from (p, q, k, \tilde{v}) to (p, q, k, \tilde{V}) .

First let us use the coordinate $(p, k, \tilde{U}, \tilde{V})$ on $\tilde{\mathcal{E}}_1$ to find the group \mathcal{G} of covering transformations $\tilde{\mathcal{E}}_1 \rightarrow \mathcal{E}_1$. We can divide the covering transformations into two types: those that push forward to \mathcal{A}_1 to give the identity and those that push forward to give λ . Recall that the projection of the universal cover $\tilde{\mathcal{E}}_1$ to \mathcal{A}_1 in part reads $U = \tan \tilde{U}/2$. This is the standard covering of the circle by \mathbb{R} , so the covering transformations that push forward to the identity are simply

$$\left\{ (p, k, \tilde{U}, \tilde{V}) \mapsto (p, k, \tilde{U} + 2\pi n, \tilde{V} + 2\pi n) \mid n \in \mathbb{Z} \right\}.$$

To find those covering transformations of \mathcal{G} that push forward to λ , take a point $(\alpha, \beta) \in \mathcal{A}_1$ with some U coordinate. Then the \tilde{U} coordinate of any point of $\tilde{\mathcal{E}}_1$

that lies over (β, α) must lie in

$$2 \operatorname{atan} \left(-\frac{1}{U} \right) + 2\pi\mathbb{Z} = 2 \left(\pm \frac{\pi}{2} + \operatorname{atan} U \right) + 2\pi\mathbb{Z}$$

and so is $2\pi m + \pi + \tilde{U}$ for some integer m . Therefore, these covering transformations lie within the set

$$\left\{ (p, k, \tilde{U}, \tilde{V}) \mapsto (p, k, 2\pi m + \pi + \tilde{U}, 2\pi n + \pi + \tilde{V}) \mid m, n \in \mathbb{Z} \right\}.$$

Not every element of this set is a well defined map on $\tilde{\mathcal{E}}_1$ however, because we require that $\tilde{U} < \tilde{V} < \tilde{U} + 2\pi$. Hence above we must choose $m = n$. The group of covering transformations \mathcal{G} is generated by

$$\tilde{\lambda} : (p, k, \tilde{U}, \tilde{V}) \mapsto (p, k, \tilde{U} + \pi, \tilde{V} + \pi), \quad (3.52)$$

and we may write $\mathcal{G} = \mathbb{Z}\langle \tilde{\lambda} \rangle$. If we apply this transformation twice, we see that $\tilde{\lambda}^2$ generates the subgroup of transformations that pushforward to the identity transformation on \mathcal{A}_1 . It is unsurprising that this subgroup is index two in \mathcal{G} because the group of covering transformations of \mathcal{A}_1 over \mathcal{E}_1 has just two elements.

If we wish to see the effect of the covering transformations as restricted to $\tilde{\mathcal{S}}_1$, we must determine how the value of \tilde{T} changes when it is precomposed with $\tilde{\lambda}$, since $\tilde{\mathcal{S}}_1$ is a collection of its level sets.

Lemma 3.53. *The effect of precomposing \tilde{T} with $\tilde{\lambda}$ is to increase its value by $S - 1$. That is,*

$$\tilde{T} \circ \tilde{\lambda} - \tilde{T} = S - 1.$$

Proof. It is fruitful to consider first the effect of λ on T_0 in the ζ -plane. Suppose that μ and ν are chosen such that $\nu, \mu, 1$ and -1 are arranged clockwise as shown in Figure 3.9. The principal choice of path $\gamma_+ = \gamma_+(\alpha, \beta)$ is shown in red, whereas the principal choice after swapping the labels of the roots is shown in blue. Let it be denoted $\gamma'_+ = \gamma_+ \circ \lambda = \gamma_+(\beta, \alpha)$.

The difference between these two paths is homologous to a loop anticlockwise around the upper unit circle. So by the construction of Θ^P ,

$$\int_{\gamma'_+} \Theta^P - \int_{\gamma_+} \Theta^P = \int_{\mathbb{S}^1} \Theta^P = -2\pi i.$$

Likewise if we consider the difference between the principal path γ_- and the path $\gamma'_- = \gamma_- \circ \lambda$ we again have a anticlockwise loop of the upper unit circle, shown in Figure 3.10. Hence

$$\int_{\gamma'_-} \Theta^P - \int_{\gamma_-} \Theta^P = \int_{\mathbb{S}^1} \Theta^P = -2\pi i.$$

Figure 3.9
The principal path γ_+ in red
and path $\gamma'_+ = \gamma_+ \circ \lambda$ in blue.

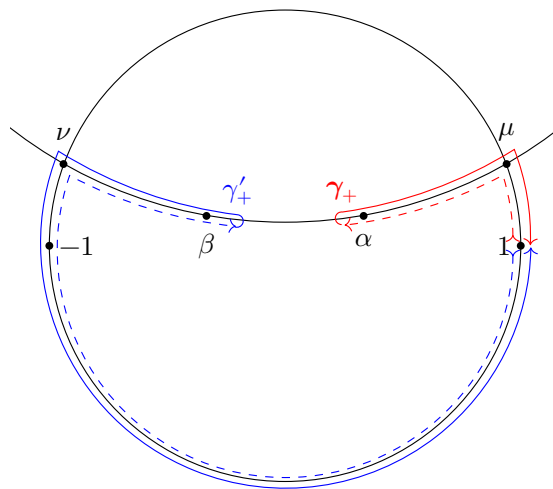
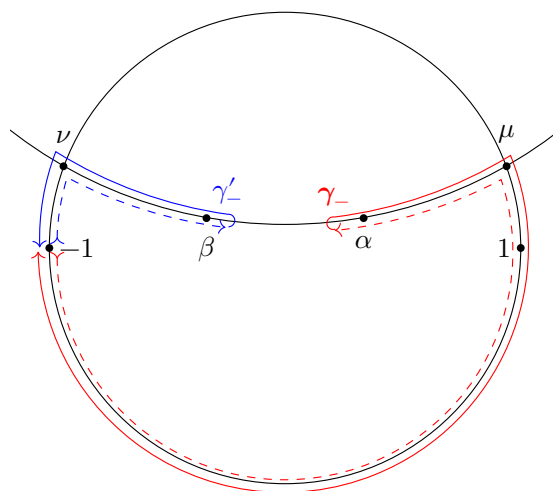


Figure 3.10
The principal path γ_- in red
and $\gamma'_- = \gamma_- \circ \lambda$ in blue.



Putting these together we conclude that the value of T_0 changes by $1 - S$ under this transformation at these points:

$$\begin{aligned} 2\pi i(T_0 \circ \lambda - T_0) &= \left(S \int_{\gamma'_-} \Theta^P - \int_{\gamma'_+} \Theta^P \right) - \left(S \int_{\gamma_-} \Theta^P - \int_{\gamma_+} \Theta^P \right) \\ &= S(-2\pi i) - (-2\pi i) \\ &= 2\pi i(1 - S). \end{aligned}$$

To infer the effect of the transformation on \tilde{T} however, we also must take into account how the coordinates \tilde{u} and \tilde{v} may have changed, and consequently any alteration to their winding numbers. If the points μ and ν have been arranged as described, then this restricts the arrangement of iu and iv . By definition,

$$0 = f(\mu), \quad iu = f(1), \quad iv = f(-1), \quad \infty = f(\nu),$$

and as one traverses the unit circle clockwise in the ζ -plane, one traverses the imaginary axis in the z -plane upwards. The clockwise arrangement of μ , 1 , -1 , and ν therefore corresponds to $0 < u < v < \infty$.

As u and v are both positive, it must be that $\tilde{u} \in (2\pi n, 2\pi n + \pi)$ and $\tilde{v} \in (2\pi m, 2\pi m + \pi)$ for some integers n and m . Hence \tilde{U} and \tilde{V} also lie in those two intervals respectively. Under applying the transformation $\tilde{\lambda}$, the coordinate \tilde{U} will be translated by π and so lie in $(2\pi(n+1) - \pi, 2\pi(n+1))$. In other words its winding number has increased by 1. The same can be said for \tilde{V} . As \tilde{u} and \tilde{v} have the same winding numbers as \tilde{U} and \tilde{V} , combining the effect of λ on T_0 with the change of winding number in (3.45) shows

$$\begin{aligned} \tilde{T} \circ \tilde{\lambda} - \tilde{T} &= (T_0 \circ \lambda + 2(S(m+1) - (n+1))) - (T_0 + 2(Sm - n)) \\ &= 1 - S + 2(S - 1) \\ &= S - 1. \end{aligned}$$

This relation a between analytic functions on an open set, so by continuation it applies everywhere. \square

Thus the effect of the covering transformation $\tilde{\lambda}$ on points in the level set $\tilde{\mathcal{E}}_1(p, q)$, where $\tilde{T} = q$, is to move them into the $\tilde{\mathcal{E}}_1(p, q + p - 1)$ level set. To be precise, fix a value for p and take a point $(p, k, \tilde{U}, \tilde{V}) \in \tilde{\mathcal{E}}_1(p)$. If $p \leq 1$ let \tilde{X} be \tilde{U} and otherwise take it to be \tilde{V} . We know by Lemma 3.48 and the above change of coordinates that (q, k, \tilde{X}) , where $q = \tilde{T}$, are coordinates for $\tilde{\mathcal{E}}_1(p)$. In particular, $\tilde{\mathcal{E}}_1(p)$ is foliated by the level sets $\tilde{\mathcal{E}}_1(p, q)$ of \tilde{T} . Under the covering transformation $\tilde{\lambda}$,

$$(p, q, k, \tilde{X}) \mapsto (p, q + (p - 1), k, \tilde{X} + \pi). \quad (3.54)$$

Viewing the cosets of the group of covering transformations as an equivalence relation, for $p \neq 1$ they provide an identification between the level set $\tilde{\mathcal{E}}_1(p, q)$ and the level set $\tilde{\mathcal{E}}_1(p, q + l(p - 1))$ for any integer l .

Theorem 3.55. *For $p \neq 1$, the space of marked curves $\mathcal{C}_1(p)$ is diffeomorphic to*

$$\left\{ ([q], k, \tilde{X}) \in (\mathbb{R}/(p-1)\mathbb{Z}) \times (0, 1) \times \mathbb{R} \right\},$$

such that the subspace of spectral curves $\mathcal{S}_1(p)$ is

$$\mathcal{S}_1(p) = \{\mathcal{C}_1(p) \mid [q] \in \mathbb{Q}/(p-1)\mathbb{Z}\}.$$

Proof. Fix $p \neq 1$ and consider $\mathcal{C}_1(p)$. It is the quotient of $\tilde{\mathcal{C}}_1(p)$ by the group of covering transformations $\mathcal{G} = \mathbb{Z}\langle\tilde{\lambda}\rangle$. By Lemma 3.48, $\tilde{\mathcal{C}}_1(p)$ is foliated by $\tilde{\mathcal{C}}_1(p, q)$ and we have just shown in (3.54) how different $\tilde{\mathcal{C}}_1(p, q)$ can be identified if their values of q differ by a multiple of $p-1$. Hence it is sufficient to take one representative from each element of $\mathbb{R}/(p-1)\mathbb{Z}$ to cover the image.

Lemma 3.46 also demonstrates that $\tilde{\mathcal{S}}_1(p)$ is the union of those level sets $\tilde{\mathcal{C}}_1(p, q)$ with $q \in \mathbb{Q}$. $\mathcal{S}_1(p)$ is the image of $\tilde{\mathcal{S}}_1(p)$ under the covering map, so it is the subset of $\mathcal{C}_1(p)$ where q is in the image of \mathbb{Q} , that is $q \in \mathbb{Q}/(p-1)\mathbb{Z}$. \square

This leaves just one special case, where $p = 1$. We see that the action of $\tilde{\lambda}$ on $\tilde{\mathcal{C}}_1(1)$ fixes the value of \tilde{T} and so does not identify different level sets $q = \tilde{T}$. Instead, the group action creates an equivalence relation on each level set. Specifically, the action of $\tilde{\lambda}$ on $\tilde{\mathcal{C}}_1(1, q)$ given by (3.54) reads

$$(1, q, k, \tilde{X}) \mapsto (1, q, k, \tilde{X} + \pi).$$

and so in particular only the fourth coordinates is changed. Using these coordinates $(1, q, k, \tilde{X})$ on $\tilde{\mathcal{C}}_1(1)$, it is trivial to deduce the quotient by \mathcal{G} .

Theorem 3.56. *The space of marked curves $\mathcal{C}_1(1)$ is diffeomorphic to*

$$\left\{ (q, k, [\tilde{X}]) \in \mathbb{R} \times (0, 1) \times \mathbb{R}/\pi\mathbb{Z} \right\},$$

the product of \mathbb{R} and an annulus, such that the subspace of spectral curves $\mathcal{S}_1(1)$ is the restriction of the first component of the product to \mathbb{Q} .

Proof. The orbit of a point $(1, q, k, \tilde{X})$ of $\tilde{\mathcal{C}}_1(1)$ is

$$(1, q, k, \tilde{X} + \pi\mathbb{Z}),$$

so the quotient sends $\tilde{X} \in \mathbb{R}$ to $[\tilde{X}] \in \mathbb{R}/\pi\mathbb{Z}$.

As in the previous theorem, by construction of the coordinates (p, q, k, \tilde{X}) and Lemma 3.46, a point is in $\mathcal{S}_1 \subset \mathcal{C}_1$ exactly when it is the image of a point where p and q are rational. As $p = 1$ in this case, the points of $\mathcal{S}_1(1)$ are those where $q \in \mathbb{Q}$. \square

These two theorems complete our quest to gain an understanding of the subspace of spectral curves \mathcal{S}_1 within \mathcal{E}_1 . First we characterised when a marked curve is a spectral curve by way of a necessary and sufficient condition: that the two real valued functions $S(\alpha, \beta)$ and $T(\alpha, \beta)$ take rational values. The first condition, $S \in \mathbb{Q}^+$, lead immediately to a dense disjoint collection of subspaces $\mathcal{E}_1(p)$ on which the condition was met, where p was the value of S .

Then we moved to the universal cover $\tilde{\mathcal{E}}_1$, so that we could construct a well defined pullback \tilde{T} of the multi-valued function T . We then considered the subspaces of $\tilde{\mathcal{E}}_1(p)$ where \tilde{T} was constant. These were shown to be ribbons in Lemma 3.48. The function \tilde{T} is rational exactly when T is, so the preimage of \mathcal{S}_1 in its universal cover is the union of the level sets $\tilde{\mathcal{E}}_1(p, q)$ of $p = S$ and $q = \tilde{T}$ where p and q are rational.

The final step was to push these level sets back down to \mathcal{E}_1 . To do this, one must quotient out by the action of the group of covering transformations. This group \mathcal{G} was shown to be $\mathbb{Z}\langle\tilde{\lambda}\rangle$ and its action, described by (3.54), was that of translations. This allowed us to describe the quotients in the two Theorems 3.55 and 3.56. For p not equal to one, $\mathcal{S}_1(p)$ is a dense collection of ribbons within \mathcal{E}_1 . As can be seen in Figures 3.11–3.13 below, they should be thought of as being intertwined around a central axis, similar to a family of helicoids. But for p equal to one, instead we have a dense collection of annuli $\mathcal{S}_1(1)$.

Unlike helicoids however, cross-sections perpendicular to the central axis do not meet the axis. Instead they spiral infinitely closer. In this respect, they behave like the cone of a spiral. This aspect is especially prominent in Figure 3.12, which one could think of as a family of cones with a common vertex (though the plotting software has difficulty for $k \approx 1$, so the ‘cones’ appear to be heading towards the white hole in the middle of the figure but are truncated. The centre of this white hole is the vertex). This vertex structure is proved in Section 4.1.

We saw in the introduction to this chapter that the parameter p can be thought of as controlling the slope of the level sets as they wind around the central axis. From this point of view, $p = 1$ is the intermediate case between right and left handed spirals where the slope is ‘flat’ and the level sets ‘close up’.

3.5 Corollaries

In the last section, we deduced the topology of the path components of the moduli space of spectral curves. Most components are ribbons, but the components of $\mathcal{S}_1(1)$ are annuli. In this section we will first investigate the moduli space \mathcal{M}_1 of spectral data $(\Sigma, \Theta^1, \Theta^2)$. For $p \neq 1$, over each component of $\mathcal{S}_1(p)$ it is a trivial bundle. Its bundle structure over the components of $\mathcal{S}_1(1)$ is more complicated, but we will prove that the total space of the bundle is simply connected.

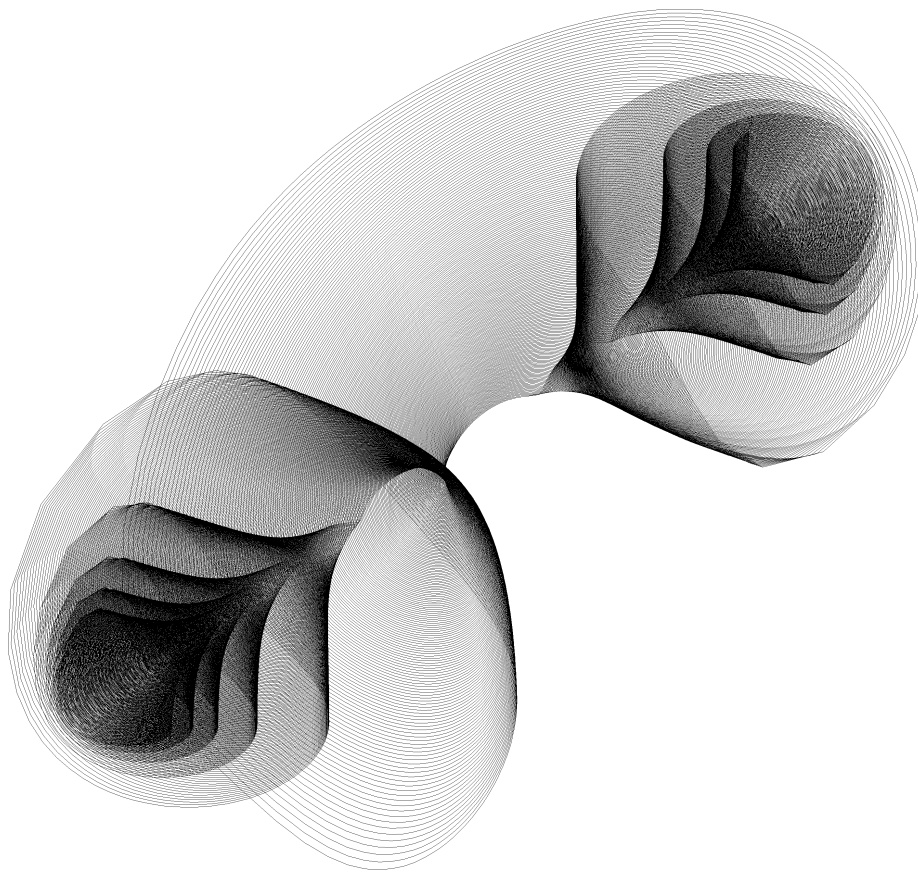


Figure 3.11
 One component of the moduli space $\mathcal{S}_1(0.5)$. Notice that the ribbon is wrapped in a left handed direction around a central axis.

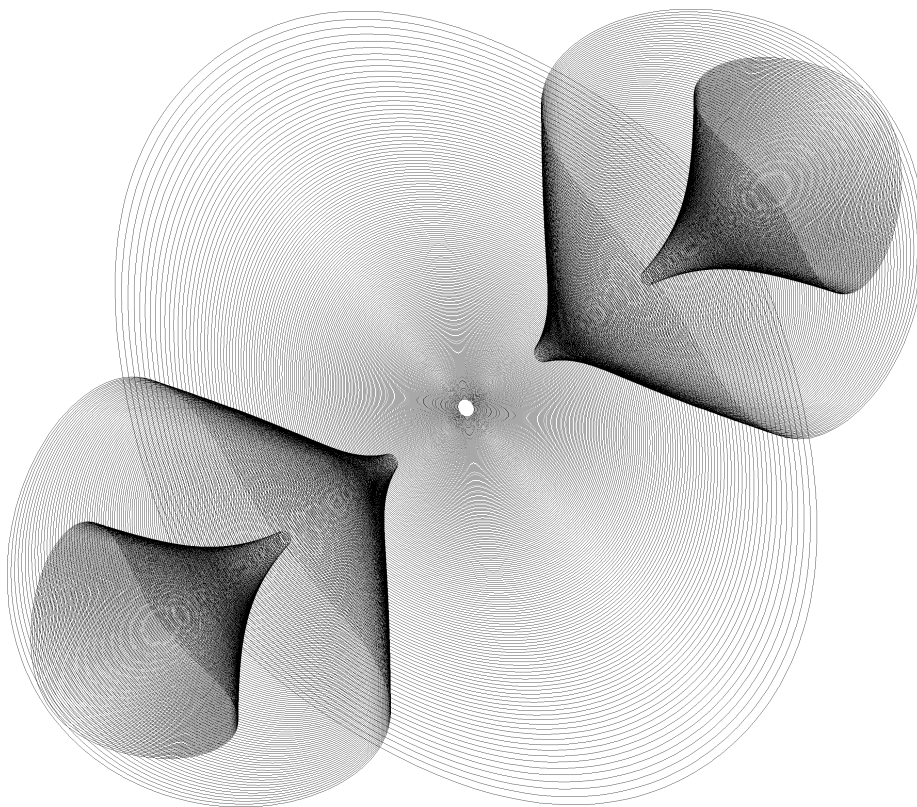


Figure 3.12

Select components of the moduli space $\mathcal{S}_1(1)$, namely those components on which T is -3 , -1 , 0 , 1 , or 3 . The component on which T is 0 is the disc in the middle.

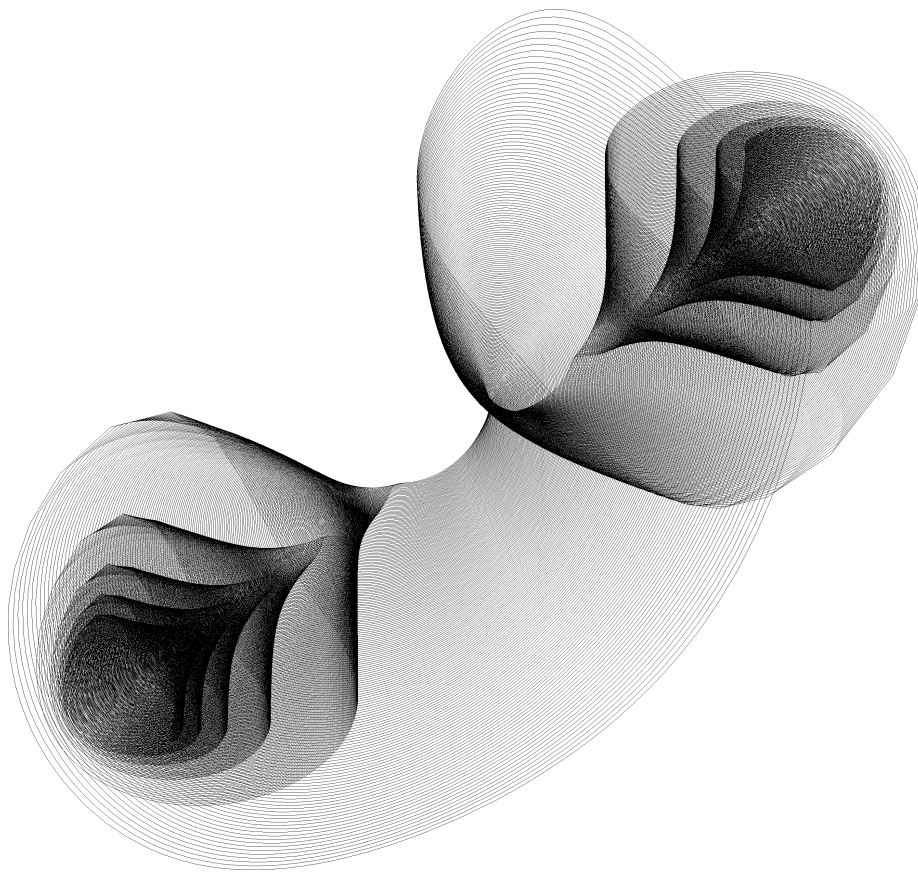


Figure 3.13
 One component of the moduli space $\mathcal{S}_1(2)$. Notice, contra Figure 3.11, that the ribbon is wrapped in a right handed direction around a central axis.

Second, we shall extend the symmetry exhibited in (3.49) to a general transformation χ on the space of spectral curves of a fixed genus \mathcal{S}_g and give a geometric interpretation. We will illustrate this interpretation by observing how it applies to the space \mathcal{S}_0 of spectral curves with genus zero. Finally, we will examine the special case of harmonic maps to a 2-sphere and show that those with a genus one spectral curve can be identified with a particular path component of $\mathcal{S}_1(1)$.

Before we can prove results about the moduli space of spectral data, we must revisit and improve on Lemma 3.21. In that lemma, we saw that the rationality of the values of the functions S and T are necessary and sufficient conditions for the existence of spectral data. The proof was constructive, in that it finds a set of spectral data on any marked curve that meets both conditions. However, to examine the moduli of spectral data, we must find a ‘minimal’ set of spectral data from which all other on that curve may generated.

Lemma 3.57. *On any curve $\Sigma \in \mathcal{S}_1$, there is a pair of differentials Ψ^E and Ψ^P such that every differential that satisfies conditions (P.4)–(P.11) is an integer combination of that pair.*

Proof. The method of proof is similar to that of Lemma 3.21. As Σ is a spectral curve, if we fix paths γ_+ and γ_- we know that the functions S and T take rational values. Let $S = n/m$ and $T = n'/m'$, where n and m are coprime. Recall the definitions of Ψ^E and Ψ^P as derived in that lemma, namely $\Psi^E = a\Theta^E$ and $\Psi^P = b\Theta^E + l\Theta^P$, for constants

$$\begin{aligned} a &:= \frac{2\pi i n}{2i\eta^+(1)}, \\ l &:= \frac{m'}{\gcd(m', mn')}, \\ b &:= \frac{1}{2i\eta^+(1)} \left(2\pi i \frac{mn'}{\gcd(m', mn')} y - \frac{m'}{\gcd(m', mn')} \int_{\gamma_+} \Theta^P \right). \end{aligned} \quad (3.58)$$

We shall prove this lemma first for exact differentials. Suppose that Θ is an exact differential meeting conditions (P.4)–(P.11). We assert that it must be an integer multiple of Ψ^E . To see this, note that it is a real scalar of Θ^E and compute its integrals over γ_+ and γ_-

$$\begin{aligned} \int_{\gamma_+} \Theta &= a' \int_{\gamma_+} \Theta^E = 2i\eta^+(1)a' =: 2\pi i\Gamma_+, \\ \int_{\gamma_-} \Theta &= a' \int_{\gamma_-} \Theta^E = -2i\eta^+(-1)a' =: 2\pi i\Gamma_-, \end{aligned}$$

for some $a' \in \mathbb{R}$ and $\Gamma_+, \Gamma_- \in \mathbb{Z}$. By the definition of $S(\alpha, \beta)$

$$S = \frac{n}{m} = -\frac{\eta^+(1)}{\eta^+(-1)} = \frac{\Gamma_+}{\Gamma_-},$$

so since n/m is a simplified fraction we must have that $\Gamma_+ = cn$ and $\Gamma_- = cm$ for an integer c . It follows that $a' = ca$ and hence $\Theta = c\Psi^E$.

Now if Θ is any differential that satisfies the closing conditions, we may write it as $\Theta = b'\Theta^E + l'\Theta^P$ for $b' \in \mathbb{R}$ and $l' \in \mathbb{Z}$. Its imaginary period is $2\pi il'$ and let its integrals over γ_+ and γ_- be $2\pi i\Gamma_+$ and $2\pi i\Gamma_-$. Similar to Lemma 3.21, eliminating b' from (3.19) leads to

$$\begin{aligned} l'mn' &= m'(n\Gamma_- - m\Gamma_+) \\ l' \frac{mn'}{\gcd(m', mn')} &= \frac{m'}{\gcd(m', mn')} (n\Gamma_- - m\Gamma_+) \\ &= l(n\Gamma_- - m\Gamma_+) \end{aligned}$$

Now, by construction l is coprime to mn' . Hence we see that l must divide l' . Finally, consider the differential

$$\Theta - \frac{l'}{l}\Psi^P.$$

It meets conditions (P.4)–(P.11) and is exact, so must be an integer multiple of Ψ^E by the first part of this proof. Rearranging, we have written Θ as an integer combination of Ψ^E and Ψ^P . \square

Morally, the differentials Ψ^E and Ψ^P should be used locally as a frame for the space of differentials meeting (P.4)–(P.11). These differentials form a lattice, so each fibre of the bundle \mathcal{M}_1 over \mathcal{S}_1 is a discrete space. The global structure of the bundle is therefore given by the monodromy action on the lattice; which determines fibres connect to one another.

Before we can make this precise however there is a technical point that must be addressed, namely that \mathcal{S}_1 is only an immersed submanifold. This causes issues if one tries to describe bundles over \mathcal{S}_1 as subspaces of bundles over \mathcal{E}_1 . For example, we demonstrated at the end of Section 3.1, using Lemma 3.15, that the differentials meeting conditions (P.4)–(P.7) form a rank two vector bundle \mathcal{B}_1 over \mathcal{E}_1 framed by $\langle \Theta^E, \Theta^P \rangle$. If we try to consider Ψ^E as a section of this bundle restricted to \mathcal{S}_1 then it is not even a continuous function! Indeed, consider a sequence of points

$$\Sigma_j \in \mathcal{S}_1(1 + j^{-1}),$$

with a limit $\Sigma_\infty \in \mathcal{S}_1(1)$. The integers j and $j+1$ are always coprime, so in the above notation $S = n/m = (j+1)/j$ and the differential Ψ_j^E on Σ_j is defined by

$$\Psi_j^E = \frac{2\pi i(j+1)}{2i\eta^+(1)} \Theta^E,$$

which does not have a well defined limit, whereas on the other hand

$$\Psi_\infty^E = \frac{2\pi i}{2i\eta^+(1)} \Theta^E.$$

Instead, we must consider bundles over each path connected component of \mathcal{S}_1 separately, each of which is an embedded submanifold of \mathcal{E}_1 . This is the more natural choice anyway if one recalls that deformations have been defined to be paths in the moduli space. Let us label the path connected components of \mathcal{S}_1 . Following Theorem 3.55, for each $p \in \mathbb{Q}^+ \setminus \{1\}$ and $[q] \in \mathbb{Q}/(p-1)\mathbb{Z}$, there is a ribbon-like component $\mathcal{S}_1(p, [q]) \cong (0, 1) \times \mathbb{R}$. By Theorem 3.56, for $p = 1$ there is an annuli component $\mathcal{S}_1(1, q)$ for each rational number q .

To examine the local triviality of the bundle \mathcal{M}_1 over each component, fix a path connected component \mathcal{X} of \mathcal{S}_1 . For any fixed value of $p \in \mathbb{Q}^+$, the differential Ψ^E is a well-defined smooth function on $\mathcal{E}_1(p)$, and as \mathcal{X} is entirely contained within some $\mathcal{E}_1(p)$ it follows that Ψ^E is well-defined and smooth on \mathcal{X} . The same is not necessarily true for Ψ^P . Locally though, in a simply connected neighbourhood $\mathcal{V} \subset \mathcal{X}$ it is possible to choose paths γ_+ and γ_- on each spectral curve that vary smoothly with changes of the branch points. Hence by Lemma 3.57, the moduli space of differentials satisfying (P.4)–(P.11) is the trivial \mathbb{Z}^2 -bundle over \mathcal{V} framed by Ψ^E and Ψ^P .

From this we may establish the local structure of \mathcal{M}_1 . We recall from Chapter 2 the definition of the integer matrices $\text{Mat}_2^* \mathbb{Z} = \{ M \in \text{Mat}_2 \mathbb{Z} \mid \det M \neq 0 \}$. There the moduli space of spectral data with a genus zero spectral curve was described as the product $\mathcal{M}_0 = D \times \text{Mat}_2^* \mathbb{Z}$. Similarly, we may describe the moduli space of spectral data whose spectral curve lies in \mathcal{V} as $\mathcal{V} \times \text{Mat}_2^* \mathbb{Z}$, where the differentials (Θ^1, Θ^2) on any curve may be described by a matrix via

$$\begin{pmatrix} \Theta^1 \\ \Theta^2 \end{pmatrix} = \begin{pmatrix} b_1 & l_1 \\ b_2 & l_2 \end{pmatrix} \begin{pmatrix} \Psi^E \\ \Psi^P \end{pmatrix},$$

using the frame (Ψ^E, Ψ^P) . The non-vanishing of the determinant of this matrix is equivalent to Condition (P.9), that the pair of differentials (Θ^1, Θ^2) is linearly independent.

With the local structure established, we can now turn our attention to the global structure of \mathcal{M}_1 . Just as we decomposed \mathcal{S}_1 into path components $\mathcal{S}_1(p, [q])$ and $\mathcal{S}_1(1, q)$, we may similarly decompose \mathcal{M}_1 according to which component its spectral curve belongs. As the path connected components $\mathcal{X} = \mathcal{S}_1(p, [q])$ are simply connected, immediately we have that

$$\mathcal{M}_1(p, [q]) = \mathcal{S}_1(p, [q]) \times \text{Mat}_2^* \mathbb{Z}.$$

For $p = 1$, let us fix a path component $\mathcal{S}_1(1, q)$ of $\mathcal{S}_1(1)$. This component is an annulus and not simply connected. The differential Ψ^P is not well-defined on all of $\mathcal{S}_1(1, q)$ because its definition depends on a choice of paths γ_+ and γ_- , and it is not possible to make a choice of paths that varies continuously as the branch points of the spectral curve are moved. For example, recall the family of curves $t \mapsto \Sigma(0.5e^{it}, -0.5e^{it})$ in \mathcal{E}_1 . If one takes γ_+ to be a path that wraps around $\zeta = 0.5$ on $\Sigma(0)$ (as the principal choice of path γ_+ does), then as t is

increased to π , the path now passes around the other branch point $\zeta = -0.5$ on the same spectral curve $\Sigma(\pi) = \Sigma(0)$.

Consider therefore a nontrivial loop $\ell : [0, 1] \rightarrow \mathcal{S}_1(1, q)$ that wraps around the annulus once. If we make a choice of $\gamma_+(t)$ and $\gamma_-(t)$ on each $\Sigma(\ell(t))$ such that the paths vary smoothly in t then note in particular that $\gamma_+(0)$ and $\gamma_+(1)$ will be different paths on the same curve. One may ask: if we construct Ψ^P using $\gamma_+(t)$ and $\gamma_-(t)$ then what will be the change in Ψ^P when we return to $\ell(1) = \ell(0)$?

From (3.54) it follows that $T = n'/m'$ is well defined and constant on the whole annulus. We may simplify so that it is a reduced fraction. Writing $S = 1 = n/m$ as a reduced fraction implies that $m = n = 1$. The period l of Ψ^P is therefore $m'/\gcd(m', mn') = m'$. Let $\Psi^P(0) = b(0)\Theta^E + m'\Theta^P$ be the differential on $\Sigma(\ell(0))$. As the periods of the differential are integral, they cannot change along the path ℓ and so

$$\Psi^P(t) = b(t)\Theta^E + m'\Theta^P.$$

As we move from the start of ℓ to the end, from our previous computation in Lemma 3.53, each of

$$\int_{\gamma_+(t)} \Theta^P \quad \text{and} \quad \int_{\gamma_-(t)} \Theta^P$$

will be incremented or decremented (depending on the orientation of ℓ) by $2\pi i$. The difference $\Psi^P(1) - \Psi^P(0) = (b(1) - b(0))\Theta^E$ can be explicitly computed from (3.58) as

$$\begin{aligned} b(1) &= \frac{1}{2i\eta^+(1)} \left(2\pi i \eta n' m - m' \left(\int_{\gamma_+(1)} \Theta^P \right) \right) \\ &= \frac{1}{2i\eta^+(1)} \left(2\pi i \eta n' m - m' \left(\int_{\gamma_+(0)} \Theta^P + 2\pi i \right) \right) \\ &= b(0) - \frac{2\pi i}{2i\eta^+(1)} m' \\ &= b(0) - am', \end{aligned}$$

so that $\Psi^P(1) - \Psi^P(0) = -am'\Theta^E = -m'\Psi^E$. This equations shows that every time you loop around the annulus, the non-exact differential Ψ^P shifts by $m'\Psi^E$.

Translating the shift of Ψ^P into a statement about $\text{Mat}_2^*\mathbb{Z}$, given a tuple of spectral data $(\Sigma, \Theta^1, \Theta^2)$ in $\mathcal{M}_1(1, q)$, we may write $\Theta^i = b_i\Psi^E + l_i\Psi^P$ for integers b_i and l_i . One is free to vary Σ within $\mathcal{S}_1(1, q)$, but the effect of looping

around on the differentials is

$$\begin{aligned} \begin{pmatrix} \Theta^1 \\ \Theta^2 \end{pmatrix} &= \begin{pmatrix} b_1 & l_1 \\ b_2 & l_2 \end{pmatrix} \begin{pmatrix} \Psi^E \\ \Psi^P \end{pmatrix} \\ &\mapsto \begin{pmatrix} b_1 & l_1 \\ b_2 & l_2 \end{pmatrix} \begin{pmatrix} \Psi^E \\ \Psi^P - m' \Psi^E \end{pmatrix} \\ &= \begin{pmatrix} b_1 & l_1 \\ b_2 & l_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -m' & 1 \end{pmatrix} \begin{pmatrix} \Psi^E \\ \Psi^P \end{pmatrix}. \end{aligned}$$

In short, if B_q is the subgroup of $\text{Mat}_2^* \mathbb{Z}$ of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ m' \mathbb{Z} & 1 \end{pmatrix},$$

then the connected components of $\mathcal{M}_1(1, q)$ are enumerated by the right B_q -orbits of $\text{Mat}_2^* \mathbb{Z}$.

Finally, we can prove the space $\mathcal{M}_1(1, q)$ is simply connected. Given any closed path ℓ in it, we may project this loop down to $\mathcal{S}_1(1, q)$. If the projection of the loop is null-homotopic, then it is contained in a simply connected neighbourhood \mathcal{V} of $\mathcal{S}_1(1, q)$ and we may use the frame $\langle \Psi^E, \Psi^P \rangle$ to lift to a null-homotopy of ℓ in $\mathcal{M}_1(1, q)$. If the projection is non-trivial, then it winds a certain number of times around the annulus. The above calculation shows that if the differentials are unchanged from the beginning to end of the path then either $m' = 0$ or $l_1 = l_2 = 0$. The former is excluded because by definition $n'/m' = q$ and so m' is never zero. The latter implies that the differentials are both multiples of Θ^E , which by (P.9) contradicts their linear independence. This demonstrates that every closed path in $\mathcal{M}_1(1, q)$ is null-homotopic, and so it is simply connected.

It is therefore the case that each connected component of $\mathcal{M}_1(1, q)$ is diffeomorphic to the universal cover of the annulus $\mathcal{S}_1(1, q)$, which is to say that each of them is a ribbon $(0, 1) \times \mathbb{R}$. The connected components of $\mathcal{M}_1(p, [q])$ were also ribbons. In summary, \mathcal{M}_1 is the disjoint union

$$\begin{aligned} \mathcal{M}_1 &= \coprod_{q \in \mathbb{Q}} \mathcal{M}_1(1, q) \quad \amalg \quad \coprod_{\substack{p \in \mathbb{Q}^+, p \neq 1 \\ [q] \in \mathbb{Q}/(p-1)\mathbb{Z}}} \mathcal{M}_1(p, [q]) \\ &= \coprod_{q \in \mathbb{Q}} \left[(0, 1) \times \mathbb{R} \times (\text{Mat}_2^* \mathbb{Z} / B_q) \right] \quad \amalg \quad \coprod_{\substack{p \in \mathbb{Q}^+, p \neq 1 \\ [q] \in \mathbb{Q}/(p-1)\mathbb{Z}}} \left[(0, 1) \times \mathbb{R} \times \text{Mat}_2^* \mathbb{Z} \right]. \end{aligned}$$

There are some symmetries and special cases that also bear mention. First, we have already remarked upon, and made use of, the symmetry in (3.49),

$$T_0(p, k, u, v) = -p T_0\left(\frac{1}{p}, k, v, u\right),$$

but how should one interpret it geometrically? Looking at the transformation

$$p = \frac{|1 - \alpha| |1 - \beta|}{|1 + \alpha| |1 + \beta|} \mapsto \frac{1}{p} = \frac{|1 + \alpha| |1 + \beta|}{|1 - \alpha| |1 - \beta|},$$

the natural guess would be that it is induced by

$$\begin{aligned}\chi : \mathcal{A}_1 &\rightarrow \mathcal{A}_1 \\ (\alpha, \beta) &\mapsto (-\alpha, -\beta).\end{aligned}\tag{3.59}$$

Indeed this can be seen to be the case, as k is invariant under such a transformation, and the associated map between the spectral curves

$$\begin{aligned}\chi_{(\alpha, \beta)} : \Sigma(\alpha, \beta) &\rightarrow \Sigma(-\alpha, -\beta) \\ (\zeta, \eta) &\mapsto (-\zeta, -\eta)\end{aligned}$$

interchanges 1 and -1 , in effect swapping the roles of $u = -if(1)$ and $v = -if(-1)$. The pullback of the differentials under the map $\chi_{(\alpha, \beta)}$ preserves the integrality of the periods and the integrals over γ_+ and γ_- and so spectral data on $\Sigma(\alpha, \beta)$ is transformed into spectral data on $\Sigma(-\alpha, -\beta)$. The exact same reasoning applies in general to marked curves of any genus,

$$\begin{aligned}\chi : \mathcal{A}_g &\rightarrow \mathcal{A}_g \\ (\alpha_1, \alpha_2, \dots, \alpha_g) &\mapsto (-\alpha_1, -\alpha_2, \dots, -\alpha_g), \\ \chi_{(\alpha_1, \alpha_2, \dots, \alpha_g)} : \Sigma(\alpha_1, \alpha_2, \dots, \alpha_g) &\rightarrow \Sigma(-\alpha_1, -\alpha_2, \dots, -\alpha_g) \\ (\zeta, \eta) &\mapsto (-\zeta, i^g \eta).\end{aligned}$$

The harmonic map $g(z) : \mathbb{T}^2 \rightarrow \mathbb{S}^3$ arises from the spectral data as the gauge transformation between the connections corresponding to $\zeta = 1$ and $\zeta = -1$ in (0.3), so exchanging these points with $\chi_{(\alpha_1, \alpha_2, \dots, \alpha_g)}$ gives the inverted map $g(z)^{-1}$. It is also harmonic [Uhl89, Prop 8.2].

This can be seen directly in the genus zero case, which was treated in Chapter 2. Recall that the equation of any harmonic map corresponding to spectral data with a genus zero spectral curve may, as in (2.3), be written

$$g(w_R + iw_I) = \exp(-4w_R X) \exp(4w_I Y),$$

for

$$X = \|X\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = \|Y\| \begin{pmatrix} 0 & e^{i\delta} \\ -e^{-i\delta} & 0 \end{pmatrix}.$$

The norms, $\|X\|$ and $\|Y\|$, come from the identification of \mathfrak{su}_2 with \mathbb{R}^3 using the standard basis $\{\sigma_1, \sigma_2, \sigma_3\}$ defined in (2.4). Geometrically, δ is the angle between X and Y . Under inversion,

$$g(w)^{-1} = \exp(-4w_I Y) \exp(4w_R X).$$

To bring this back into the form of (2.3), we must perform two operations. First, we must change coordinates on the domain so that the real part is in the first factor. The multiplication $\tilde{w} = iw$ accomplishes this:

$$g(\tilde{w})^{-1} = \exp(4\tilde{w}_R Y) \exp(4\tilde{w}_I X).$$

Second we must rotate the image so that $-Y$ is aligned with σ_2 and X lies in the plane spanned by σ_2 and σ_3 . This may be achieved by $SU(2)$ conjugation:

$$\begin{aligned} \begin{pmatrix} ie^{i\delta/2} & 0 \\ 0 & -ie^{-i\delta/2} \end{pmatrix}^{-1} \|Y\| \begin{pmatrix} 0 & -e^{i\delta} \\ e^{-i\delta} & 0 \end{pmatrix} \begin{pmatrix} ie^{i\delta/2} & 0 \\ 0 & -ie^{-i\delta/2} \end{pmatrix} &= \|Y\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} ie^{i\delta/2} & 0 \\ 0 & -ie^{-i\delta/2} \end{pmatrix}^{-1} \|X\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} ie^{i\delta/2} & 0 \\ 0 & -ie^{-i\delta/2} \end{pmatrix} &= \|X\| \begin{pmatrix} 0 & e^{i(\pi-\delta)} \\ -e^{i(\pi-\delta)} & 0 \end{pmatrix}. \end{aligned}$$

Thus the angle parameter has become $\pi - \delta$. As an aside, as seen in Figure 2.5 this angle parameter determines the image of the harmonic map up to an $SO(4)$ rotation of \mathbb{S}^3 . In particular, these maps g and g^{-1} have congruent images.

Recall that the parameter x of g was defined by (2.8) to be $\|Y\|/\|X\|$. The value of this ratio is inverted for g^{-1} . Now using (2.10) to determine the branch point of the spectral curve associated to g^{-1} we have

$$\frac{\frac{1}{x}e^{i(\pi-\delta)} - i}{\frac{1}{x}e^{i(\pi-\delta)} + i} = \frac{-e^{-i\delta} - ix}{-e^{-i\delta} + ix} = \frac{-1 - ixe^{i\delta}}{-1 + ixe^{i\delta}} = -\frac{xe^{i\delta} - i}{xe^{i\delta} + i} = -\alpha,$$

which is to say the branch point of the spectral curve associated to g^{-1} is the negative of the branch point of the spectral curve associated to g , as asserted above.

Returning to genus one spectral curves, of special interest are the spectral curves that are a fixed point of this transformation $\chi : \Sigma(\alpha, \beta) \mapsto \Sigma(-\alpha, -\beta)$. For these spectral curves $\chi_{(\alpha, \beta)}$ is an extra involution and hence $\beta = -\alpha$. These are exactly the genus one marked curves that would meet the conditions, if they admit spectral data, for the associated harmonic map to have a totally geodesic two-sphere as its image (see discussion at end of Section 0.2). Hitchin [Hit90, p693] identifies a particular one parameter family of these maps as the Gauss maps of Delaunay surfaces. We shall show that all such marked curves in fact admit spectral data and further identify in which component of \mathcal{S}_1 they reside.

Suppose that $\Sigma(\alpha, -\alpha)$ is a curve that is fixed by χ . As χ sends $p \mapsto p^{-1}$, it follows that p is one. To show that $\Sigma(\alpha, -\alpha)$ admits spectral data it remains to show that T is rationally valued at this point $(\alpha, -\alpha) \in \mathcal{A}_1$. Computing the value of T from its principal branch cut T_0 requires us to know which coordinates $(1, k, u, v)$ correspond to $(\alpha, -\alpha)$. The annulus $\{(\alpha, -\alpha) \in \mathcal{A}_1\}$ is two-dimensional, but there are three parameters (k, u, v) , so there must be some relation between them. As the four branch points lie on a line it follows that $\mu = \hat{\alpha}$ and $\nu = -\hat{\alpha}$, where $\hat{\alpha}$ is the unit vector of α . Directly from (3.9) and (3.5),

$$z_0 = \frac{1 + |\alpha|}{1 - |\alpha|}, \quad k = \left(\frac{1 - |\alpha|}{1 + |\alpha|} \right)^2.$$

It follows from (3.6) that

$$iv = f(-1) = \left(\frac{1 + |\alpha|}{1 - |\alpha|} \right)^2 \left(\frac{1 + |\alpha|}{1 - |\alpha|} \frac{1 + \hat{\alpha}}{1 - \hat{\alpha}} \right)^{-1} = \frac{1}{k} f(1)^{-1} = \frac{1}{k} (iu)^{-1},$$

or concisely that $v = -(ku)^{-1}$. But $u \mapsto -(ku)^{-1}$ is exactly the formula for the change of u under the label swapping involution λ , described by (3.51). Thus we see that χ and λ act in the same way on $\{(\alpha, -\alpha) \in \mathcal{A}_1\}$, as one would expect because they are both swapping α and $-\alpha$, the two branch points inside the unit disc. Precomposing T_0 with λ shifts its value by $1 - p$, which in this case is zero. Hence using this together with the fact that precomposing T_0 with χ negates the function shows that at a point of $\{(\alpha, -\alpha) = (1, k, u, -(ku)^{-1}) \in \mathcal{A}_1\}$,

$$\begin{aligned} T_0(1, k, u, -(ku)^{-1}) &= T_0(1, k, -(ku)^{-1}, u) + 1 - p \\ &= -T_0(1, k, u, -(ku)^{-1}), \end{aligned}$$

from which we deduce that T_0 is zero. Conversely, the disjoint annuli that constitute $\mathcal{S}_1(1)$ are determined uniquely by the value T_0 takes on them. Therefore we have shown that $\{\Sigma(\alpha, -\alpha) \in \mathcal{C}_1\}$ is the only annulus of $\mathcal{S}_1(1)$ where T_0 is zero, and these are exactly the harmonic maps to the sphere with a genus one spectral curve.

The Boundary of the Moduli Space

Throughout this thesis we have restricted ourselves to the study of nonsingular spectral curves. In this chapter we examine two different limiting cases of \mathcal{S}_1 in which the spectral curves develop nodal singularities. In the previous chapter, we proved that \mathcal{S}_1 was composed of ribbons $(0, 1) \times \mathbb{R}$ and annuli $(0, 1) \times \mathbb{S}^1$ where the $(0, 1)$ factor is given by the coordinate k . We will consider the limits as $k \rightarrow 0$ and $k \rightarrow 1$ as the two boundaries of the moduli space \mathcal{S}_1 . This is the appropriate notion of boundary for this space for two reasons. Firstly, \mathcal{S}_1 is dense in the space of marked curves so the usual topological definition of boundary is trivial, and secondly we are interested in deformations which are paths in the moduli space so it is natural to study limits along a path in \mathcal{S}_1 .

We will show that in the case corresponding to $k \rightarrow 0$, which we shall call the exterior boundary, that we can identify the limit of a path in \mathcal{S}_1 with a genus zero spectral curve in \mathcal{S}_0 by normalisation of the singular curve. Further, over these paths in \mathcal{S}_1 the limits of spectral data $(\Sigma, \Theta^1, \Theta^2) \in \mathcal{M}_1$ are well-defined, and we may pull back the differentials to the normalisation to obtain spectral data in \mathcal{M}_0 . On the other hand, in the case corresponding to $k \rightarrow 1$ which we shall call the interior boundary the spectral curves of each path connected component may only tend to a particular nodal curve. This curve is determined by the value of the function S , defined by (3.17), which is constant on each component. This nodal curve corresponds to the vertex of the conical spiral, which is especially visible at the centre of Figure 3.12. The limit of spectral data is not well-defined as $k \rightarrow 1$, so there is no analogous interpretation of these limits in \mathcal{M}_1 .

Recall the methodology of Chapter 3 where we expended our efforts primarily in studying the topology of the space \mathcal{S}_1 of spectral curves and then secondarily showed how to construct all spectral data on a curve from a lattice of differentials. We adopt a parallel approach in this chapter, with our focus on determining the behaviour of the space \mathcal{S}_1 of spectral curves as a subset of the space \mathcal{C}_1 of genus one marked curves at the boundary of \mathcal{C}_1 . We recall that

\mathcal{S}_1 is the subset of \mathcal{E}_1 where the function S and the multi-valued function T take rational values. An essential part of this chapter is to compute the limits of these functions. We also compute the limits of the basis Θ^E and Θ^P of the real vector bundle \mathcal{B}_1 of differentials with imaginary periods, from which the results about spectral data will follow.

Recall that in order to work with the space of marked curve we introduced a covering parameter space $\mathcal{A}_1 = \{(\alpha, \beta) \in D \times D \mid \alpha \neq \beta\}$. The two points α and β correspond to the branch points of the marked curve that lie within the unit disc. Let $\overline{\mathcal{A}}_1$ be the closure of \mathcal{A}_1 in $\overline{D} \times \overline{D}$, the product of two closed discs. Consider the elliptic modulus k of a marked curve as a function of its branch points, as given by equation (3.5)

$$k = \frac{|1 - \bar{\alpha}\beta| - |\alpha - \beta|}{|1 - \bar{\alpha}\beta| + |\alpha - \beta|}.$$

As $k \rightarrow 1$ we must have $|\alpha - \beta|$ tending to zero, and hence α tending to β . We call this the interior boundary,

$$\mathcal{D} := \{(\alpha, \alpha) \in D \times D\}.$$

These points correspond to curves with a node inside the unit disc (and by the reality condition there is a node outside the unit disc too). As $k \rightarrow 0$, we have that

$$|1 - \bar{\alpha}\beta|^2 = |\alpha - \beta|^2 \Leftrightarrow (1 - |\alpha|^2)(1 - |\beta|^2) = 0,$$

so this extreme of k occurs as a branch point tends to the unit circle. We will refer to this case as the exterior boundary. These points correspond to curves with one or two nodal singularities over the unit circle. We will focus on the case where just a single branch point tends to the unit circle because these are the ones that correspond to spectral data of genus zero spectral curves. We may identify the limit with two branch points in the unit circle with a harmonic map to a circle. This was seen explicitly in Chapter 2, where as the branch point $\alpha \in D$ of the genus zero spectral curve tended to the unit circle, the parameter δ tended to zero and the image of the harmonic map degenerated to a circle.

There is one further feature of the exterior boundary which bears consideration. On \mathcal{A}_1 we defined a function S ,

$$S(\alpha, \beta) := \frac{|1 - \alpha||1 - \beta|}{|1 + \alpha||1 + \beta|}, \quad (3.17)$$

such that a marked curve $\Sigma(\alpha, \beta)$ could only be a spectral curve if $S(\alpha, \beta)$ was rational. It follows that the value of S must be fixed during a deformation. If we consider the limit as $\alpha \rightarrow 1$, in order to hold S constant we would have to send β to -1 . Similarly, if we send α to -1 this forces β to 1 . Therefore there are two points $(1, -1)$ and $(-1, 1)$ in the closure $\overline{\mathcal{A}}_1$ that are common to the the closure of every $\mathcal{A}_1(p)$. As one may expect, spectral data is not well-behaved

close to these points and indeed in Lemma 4.4 we will have to exclude taking limits of branch points to ± 1 in order to preserve the closing conditions. This relates back to the similar limit $\alpha \rightarrow \pm 1$ in the genus case, where we observed that this limit corresponded to a map from a cylinder.

4.1 Interior Boundary

We shall examine first the moduli space on the interior boundary \mathcal{D} . On the interior boundary \mathcal{D} of $\overline{\mathcal{A}}_1$ the branch points are equal: $\alpha = \beta$. At such point the function S reduces to

$$S = \frac{|1 - \alpha|^2}{|1 + \alpha|^2}. \quad (4.1)$$

For each value of $S = p$ this is the equation of an arc in the disc. Therefore the subsets $\mathcal{D}(p)$ of \mathcal{D} on which $S = p$ are lines in $\overline{\mathcal{A}}_1$.

We would like to be able to reuse the formula for T_0 from the previous chapter so that we may compute the limit of its level sets by computing a suitable limit of the function itself. However, the coordinates (p, k, u, v) on \mathcal{E}_1 provided by Lemma 3.28 do not extend to coordinates on its closure. Recall that these coordinates embed \mathcal{E}_1 into $\mathbb{R}^+ \times (0, 1) \times \mathbb{T}^2$. We would be inclined to identify $\mathcal{D}(p)$ with the subset of $\mathbb{R}^+ \times [0, 1] \times \mathbb{T}^2$ where p is fixed and $k = 1$, which would imply that $\mathcal{D}(p)$ is parametrised by the two coordinates (u, v) and so would be two-dimensional. However $\mathcal{D}(p)$ is just one-dimensional.

Instead, these coordinates are describing a covering space of $\mathcal{D}(p)$ that we may think of as a blowup. The geometric motivation for this point of view is provided by working through the construction of the (p, k, u, v) coordinates. The branch circle was defined to be the circle that passes through $\alpha, \beta, \overline{\alpha}^{-1}$ and $\overline{\beta}^{-1}$. As $\alpha \rightarrow \beta$, the branch circle straightens into a line, which is entirely determined by its intersections μ and ν with the unit circle. The point $\alpha = \beta$ in $\mathcal{D}(p)$ is therefore the intersection of the arc given by (4.1) with the line joining μ and ν . There are many such lines that give the same intersection, and each one encodes a different direction that α may approach β .

Recall that under the map f which transforms the marked curve into Jacobi normal form, the unit circle is mapped to the imaginary axis. The coordinates u and v are defined by $iu = f(1)$ and $iv = f(-1)$. Similarly, the points μ and ν are mapped to 0 and ∞ respectively. By changing the positions of μ and ν on the unit circle one changes the positions of iu and iv on the imaginary axis. Conversely therefore u and v encode the positions of μ and ν , and hence also the direction of approach of $\alpha \rightarrow \beta$.

To find $\mathcal{S}_1(p)$ we must look at the level sets of the multi-valued function T . We

shall work with a principal branch cut T_0 . Recall from (3.25), (3.32) and (3.33)

$$\begin{aligned} T_0(\alpha, \beta) &:= T(\alpha, \beta, \gamma_+, \gamma_-) = \frac{1}{2\pi i} \left(S(\alpha, \beta) \int_{\gamma_-} \Theta^P - \int_{\gamma_+} \Theta^P \right), \\ \int_{\gamma_+} \Theta^P &= 4E(k)F(iu; k) - 4K(k)E(iu; k) + 4iK \frac{w(iu)}{u-v}, \text{ and} \\ \int_{\gamma_-} \Theta^P &= 4E(k)F(iv; k) - 4K(k)E(iv; k) - 4iK \frac{w(iv)}{u-v}, \end{aligned}$$

where the paths γ_+ and γ_- have been chosen according to an algorithm described on page 77. Let us compute the limit of T_0 as $k \rightarrow 1$. This is not trivial, because as $k \rightarrow 1$, $K(k) \rightarrow \infty$. We shall see however that the integrals remain finite, at least in some places.

Lemma 4.2. *The limit as $k \rightarrow 1$ of the integral of Θ^P over γ_+ is given by*

$$\lim_{k \rightarrow 1} \int_{\gamma_+} \Theta^P = 4i \operatorname{atan} u + 4i \lim_{k \rightarrow 1} K \frac{1+uv}{u-v}.$$

Proof. As $k \rightarrow 1$ the complete elliptic integral of the first kind $K(k)$ has a logarithmic singularity. Specifically we have the inequality

$$\ln 4 \leq K + \frac{1}{2} \ln(1-k) + \frac{1}{2} \ln(1+k) \leq \frac{\pi}{2}. \quad (\text{B.4})$$

Our method for evaluating the above limit is to rewrite (3.32) as a sum of terms, most of which vanish. Observe that

$$\begin{aligned} 4KE(iu; k) - 4iK \frac{w(iu)}{u-v} &= 4iK \left[\operatorname{Im} E(iu; k) - \frac{w(iu)}{u-v} \right] \\ &= 4iK \left[(\operatorname{Im} E(iu; k) - ku) + (ku - u) + \left(u - \frac{1+u^2}{u-v} \right) + \frac{1+u^2-w(iu)}{u-v} \right]. \end{aligned}$$

We can now compute the limit of each grouping. In the first grouping we recognise from Definition B.6 that $\operatorname{Im} E(iu; k) - ku$ is the function $E_0(u; k)$. This function is bounded independent of u by

$$-(K' - E') \leq E_0(u; k) \leq K' - E', \quad (\text{B.8})$$

where recall $K' = K(\sqrt{1-k^2})$ and $E' = E(\sqrt{1-k^2})$ are elliptic integrals of the complementary modulus. It therefore follows

$$\begin{aligned} \lim_{k \rightarrow 1} |K(k) (\operatorname{Im} E(iu; k) - ku)| &\leq \lim_{k \rightarrow 1} K(K' - E') \\ &= \lim_{k \rightarrow 1} -\frac{\pi}{2} + K'E \\ &= 0, \end{aligned}$$

where we used Legendre's relation $KE' + K'E - KK' = \frac{1}{2}\pi$ in the second step. By applying inequality (B.4) the second grouping is also zero,

$$\begin{aligned} (\ln 4)(k-1)u &\leq (k-1)uK + \frac{1}{2}(k-1)u \ln(1-k) + \frac{1}{2}(k-1)u \ln(1+k) \leq \frac{\pi}{2}(k-1)u \\ 0 &\leq u \lim_{k \rightarrow 1} (k-1)K + 0 + 0 \leq 0, \end{aligned}$$

using the elementary limit $x \ln x \rightarrow 0$ as $x \rightarrow 0^+$. Finally, the fourth grouping is dispatched by the following manipulation:

$$\begin{aligned} \lim_{k \rightarrow 1} K \frac{1+u^2-w(iu)}{u-v} &= \lim_{k \rightarrow 1} 4iK \frac{(1+u^2)((1+u^2)-(1+k^2u^2))}{(u-v)(1+u^2+w(iu))} \\ &= \lim_{k \rightarrow 1} 4iK(1-k^2) \frac{(1+u^2)u^2}{(u-v)(1+u^2+w(iu))} \\ &= 0. \end{aligned}$$

The reuses the fact that $(k-1)K$ vanishes as k approaches 1, which was established above.

Combining these three parts shows that

$$\begin{aligned} \lim_{k \rightarrow 1} \int_{\gamma_+} \Theta^P &= \lim_{k \rightarrow 1} 4E(k)F(iu; k) - \lim_{k \rightarrow 1} 4iK(k) \left[\operatorname{Im} E(iu; k) - \frac{w(iu)}{u-v} \right] \\ &= \lim_{k \rightarrow 1} 4E(k)F(iu; k) - \lim_{k \rightarrow 1} 4iK(k) \left[u - \frac{1+u^2}{u-v} \right]. \end{aligned}$$

It remains to compute the first limit. But $E(k)$ is well defined at $k = 1$, where it takes the value 1, so by Lemma B.9 we have

$$\lim_{k \rightarrow 1} 4E(k)F(iu; k) = 4 \times 1 \times i \operatorname{atan} u$$

After rearranging the term on the right into a single fraction, we have demonstrated that

$$\lim_{k \rightarrow 1} \int_{\gamma_+} \Theta^P = 4i \operatorname{atan} u + 4i \lim_{k \rightarrow 1} K \frac{1+uv}{u-v}.$$

□

In the same way, we have that

$$\lim_{k \rightarrow 1} \int_{\gamma_-} \Theta^P = 4i \operatorname{atan} v - 4i \lim_{k \rightarrow 1} K \frac{1+uv}{u-v}.$$

Together they show that the limit of the principal cut T_0 as k tends to one is

$$\begin{aligned} \pi \lim_{k \rightarrow 1} T_0(p, k, u, v) &= p \left(2 \operatorname{atan} v - 2 \lim_{k \rightarrow 1} K \frac{1+uv}{u-v} \right) - \left(2 \operatorname{atan} u + 2 \lim_{k \rightarrow 1} K \frac{1+uv}{u-v} \right) \\ &= 2p \operatorname{atan} v - 2 \operatorname{atan} u - 2(p+1) \lim_{k \rightarrow 1} K \frac{1+uv}{u-v} \end{aligned}$$

For most values of u and v this limit will diverge. This means that we can choose neighbourhoods of $(p, 1, u, v)$ where the function T_0 is arbitrarily large. Therefore there are no paths in the space $\mathcal{S}_1(p)$ with such a point as their limit. Colloquially, it is not possible to deform spectral data into such a singular curve.

However, if the fractional term were to tend to zero as k went to one then it may still be possible to tend towards a finite value. As K has a logarithmic singularity, let us rescale k so that k tends to 1 very rapidly. Set $k = 1 - \exp(-1/\varepsilon)$ for $\varepsilon \rightarrow 0$. Then treating K as a function of ε , the inequality (B.4) reads

$$\ln 4 \leq K(1 - \exp(-1/\varepsilon)) - \frac{1}{2} \frac{1}{\varepsilon} + \frac{1}{2} \ln(2 - \exp(-1/\varepsilon)) \leq \frac{\pi}{2},$$

so that $K \sim \frac{1}{2}\varepsilon^{-1}$ to first order. Then if we choose u and v to be functions of ε such that for some constant C

$$\frac{1 + uv}{u - v} = C\varepsilon,$$

then

$$\lim_{k \rightarrow 1} K \frac{1 + uv}{u - v} = C \lim_{\varepsilon \rightarrow 0} K\varepsilon = \frac{C}{2}.$$

So

$$\pi \lim_{\varepsilon \rightarrow 0} T_0(p, k(\varepsilon), u(\varepsilon), v(\varepsilon)) = 2p \operatorname{atan} v(0) - 2 \operatorname{atan} u(0) - (p + 1)C.$$

Note that any such choice of u and v as functions of ε forces $1 + uv \rightarrow 0$, so that in the limit we have that $v(0) = -u(0)^{-1}$. One naturally asks to which points $(\alpha, \alpha) \in \mathcal{D}(p)$ these points $(p, 1, u, -u^{-1})$ correspond. Recall that the point z_0 was essential to computing the transition from (p, k, u, v) to (α, β) . Under the map f that took a marked curve to its Jacobi normal form, z_0 was the image of 0. Looking back at the equations that determine $z_0 = x + iy$, equation (3.29) states

$$x^2 + y^2 = y(u + v) - uv.$$

Knowing that $uv = -1$ implies that $\pm 1 \in \mathbb{C}$ lies on this circle, pictured in Figure 4.1. We also know that this is the circle through iu , iv , z_0 , and $-\bar{z}_0$. Applying the transformation f^{-1} sends iu to 1, iv to -1 , and z_0 to 0. Therefore the image of this circle is the real line. Further, it sends 1 to α and thus $\alpha \in \mathbb{R}$.

Using the condition that the value of S is fixed, we may solve (4.1) explicitly to derive

$$\alpha = \frac{1 - \sqrt{p}}{1 + \sqrt{p}}.$$

We see then that the points $\{(p, 1, u, -u^{-1}) \mid u \in \mathbb{R} \cup \{\infty\}\}$ blow-down to a single point in $\mathcal{D}(p)$. This should be a surprise if we consider the following heuristic. The subset $\mathcal{S}_1(p)$ is a dense collection of surfaces in the 3-space

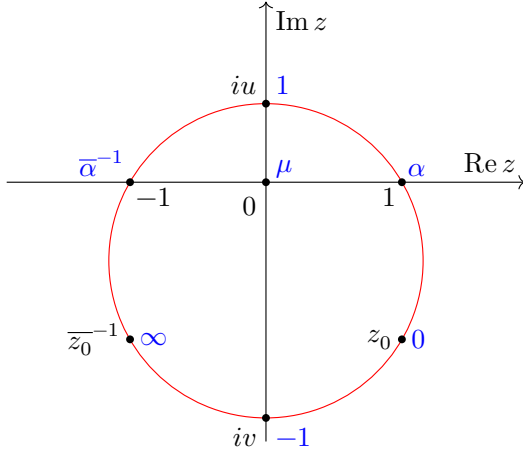


Figure 4.1
The z -plane, with points marked in black. The blue labels are their images under f^{-1} . The red circle is mapped to the real line.

$\mathcal{C}_1(p)$. Generically, each surface should intersect the line $\mathcal{D}(p)$ at discrete points. One may have anticipated then that limit of paths in $\mathcal{S}_1(p)$ would be a dense collection of points in the line $\mathcal{D}(p)$. Instead, for each p there is only a single point on the interior boundary that deformations may tend towards.

4.2 Exterior Boundary

In this section we establish a correspondence between the limit of genus one spectral data and genus zero spectral data in the case that one pair of branch points of the spectral curve accumulates on the unit circle. The limit of the genus one marked curve is singular, and after normalising we have a genus zero marked curve. We obtain differentials on the genus zero curve by pulling back those of the singular curve via the normalisation map. By computing explicitly the limits of differentials, we will see that this operation preserves Conditions (P.4)–(P.6). As there are no periods on a genus zero curve, Condition (P.8) is automatic. It will then remain to show that the closing conditions (P.11) are preserved, which is proved in Lemma 4.4.

Recall that a marked curve of genus one is described by two distinct points α and β inside the unit disc. As has been remarked upon several times now, there is no way to consistently assign labels to these branch points on \mathcal{C}_1 . Instead one works with its double cover \mathcal{A}_1 , the space of pairs (α, β) . In the limit that one of these points tends to the unit circle, the elliptic modulus of the marked curve tends to zero. For example, for $\beta \rightarrow \nu \in \mathbb{S}^1$,

$$k = \frac{|1 - \bar{\alpha}\beta| - |\alpha - \beta|}{|1 - \bar{\alpha}\beta| + |\alpha - \beta|} \rightarrow \frac{|\nu| |\bar{\nu} - \bar{\alpha}| - |\alpha - \nu|}{|1 - \bar{\alpha}\nu| + |\alpha - \nu|} = 0.$$

Many of the formulae in the preceding chapters rely on the map f which brings

the marked curve from the form (0.7) to Jacobi normal form. In Jacobi normal form, the branch points of the elliptic curve are ± 1 and $\pm k^{-1}$. In particular, in the limit that $k \rightarrow 0$ we see that the two points $\pm k^{-1}$ move together at infinity. However, the map f always takes the branch point label α to 1 and $\bar{\alpha}^{-1}$ to -1 . Thus as α tends to the unit circle where it will meet $\bar{\alpha}^{-1}$, the corresponding branch points in the Jacobi normal form do not form a double point. This is an example of the well-known phenomenon where the configuration of points in a limit can depend on the coordinates used.

It for this reason that we will consider only the limit as β tends to the unit circle. This is sufficient however, because we will be proving results about the differentials Θ^E and Θ^P . In Section 3.1 for each point of $(\alpha, \beta) \in \mathcal{A}_1$ we constructed an exact differential Θ^E and a differential Θ^P with imaginary period $2\pi i$ on the marked curve $\Sigma(\alpha, \beta)$ such that the plane of differentials \mathcal{B}_Σ satisfying conditions (P.4)–(P.7) is spanned by the differentials Θ^E and Θ^P . The content of Lemma 3.15 was that this construction was actually independent of the labelling of branch points. Therefore, any result we obtain in the limit as $\beta \rightarrow \mathbb{S}^1$ immediately applies to α in the same limit. One may alternatively recall from (3.4) that there was a freedom in the choice of f of which point α or β to send to 1. The choice f_s that instead sends β to 1 is just as valid, and it would be possible to carry out the entire analysis of this thesis using f_s in place of f . One would then have a set of formulae that they could use to compute limits as $\alpha \rightarrow \mathbb{S}^1$.

On a point of notation, recall that $\nu \in \mathbb{S}^1$ was defined to be the point of intersection between the unit circle and the branch circle on which $\alpha, \bar{\alpha}^{-1}, \beta$ and $\bar{\beta}^{-1}$ lie, more specifically the point of intersection that lies between β and $\bar{\beta}^{-1}$. Thus as β tends to the unit circle it and $\bar{\beta}^{-1}$ necessarily tend towards the point ν . We use ν to represent the limiting value of β .

In this limit as $\beta \rightarrow \nu$, the curve $\Sigma(\alpha, \beta)$ becomes a nodal curve

$$\Sigma(\alpha, \nu) = \{(\zeta, \eta) \mid \eta^2 = P(\zeta) = (-\bar{\nu})(\zeta - \nu)^2(\zeta - \alpha)(1 - \bar{\alpha}\zeta)\},$$

which has a genus zero normalisation

$$\Sigma(\alpha) = \{(\zeta, \eta) \mid \eta^2 = P(\zeta) = (\zeta - \alpha)(1 - \bar{\alpha}\zeta)\},$$

where the normalisation map is given by

$$\begin{aligned} N_{(\alpha, \nu)} : \Sigma(\alpha) &\rightarrow \Sigma(\alpha, \nu) \\ (\zeta, \eta) &\mapsto (\zeta, \sqrt{-\bar{\nu}}(\zeta - \nu)\eta). \end{aligned}$$

The presence of the square root represents an unavoidable sign issue inherent in this normalisation map. On both curves η takes real values at the two points of lying over $\zeta = 1$. By the hyperelliptic involution σ , these two values necessarily have opposite signs. One may try to choose the sign of $\sqrt{-\bar{\nu}}$ in such a way that

the normalisation map sends the point over $\zeta = 1$ on $\Sigma(\alpha)$ where η is positive to the point on $\Sigma(\alpha, \nu)$ where η is positive. However, there is no way to smoothly make this choice as ν is allowed to vary, as after ν traverses the unit circle once the sign of $\sqrt{-\bar{\nu}}$ will have been reversed. Further, this kind of identification does not even make sense when $\nu = 1$, as there is only one point over $\zeta = 1$ on $\Sigma(\alpha, 1)$. The consequence of this is that the above formula actually defines two different normalisation maps that differ by composition with the hyperelliptic involution σ . Our results will not depend on the choice of sign though, so either map is as good as the other.

Under this map $N_{(\alpha, \nu)}$, the exact differential Θ^E on $\Sigma(\alpha, \nu)$ pulls back to

$$N_{(\alpha, \nu)}^* \Theta^E = N_{(\alpha, \nu)}^* i d \left(\frac{\eta}{\zeta} \right) = i d \left(\zeta^{-1} \eta \sqrt{-\bar{\nu}} (\zeta - \nu) \right),$$

which is of the form (2.12). That is to say, it is a genus zero differential that meets conditions (P.4)–(P.7). To compute the limit and pullback of Θ^P is more involved. Recall from above that $k \rightarrow 0$ as β approaches the unit circle. This enables us to compute the limit of the first two terms of (3.14). In particular, Definition (3.10) of the differential e reads

$$e := (1 - k^2 z^2) \frac{dz}{w} \rightarrow \frac{dz}{w} = \omega,$$

where ω is a holomorphic differential. Also recall that the complete elliptic integrals $K(k)$ and $E(k)$ take the value $\pi/2$ when $k = 0$, see (B.3). Thus from (3.14),

$$\begin{aligned} \lim_{\beta \rightarrow \nu} \Theta^P &= \lim_{\beta \rightarrow \nu} \left(2E\omega - 2Ke - 2Kd \left[\frac{(z - i \operatorname{Im} z_0)w}{(z - z_0)(z + \bar{z}_0)} \right] \right) \\ &= 2\frac{\pi}{2}\omega - 2\frac{\pi}{2} \left(\lim_{k \rightarrow 0} e \right) - 2\frac{\pi}{2} \lim_{\beta \rightarrow \nu} d \left[\frac{(z - i \operatorname{Im} z_0)w}{(z - z_0)(z + \bar{z}_0)} \right] \\ &= -\pi \lim_{\beta \rightarrow \nu} d \left[\frac{(z - i \operatorname{Im} z_0)w}{(z - z_0)(z + \bar{z}_0)} \right], \end{aligned}$$

which disposes of the transcendental part of Θ^P . We wish to convert this remaining term into an expression in terms of ζ and η , so that we may normalise as above. Because $z_0 = f(0)$ and $-\bar{z}_0 = f(\infty)$, by consideration of their respective poles we may deduce that

$$\frac{w}{(z - z_0)(z + \bar{z}_0)} = r \frac{\eta}{\zeta},$$

for some scalar r which may depend on α and β . Further, they are both real functions with respect to the antiholomorphic involution so r must also be real.

By direct substitution of (3.6), the other factor becomes

$$\begin{aligned} z - i \operatorname{Im} z_0 &= \frac{1}{\zeta - \nu} (-\zeta(\bar{z}_0 + i \operatorname{Im} z_0) + (\mu\bar{z}_0 + i\nu \operatorname{Im} z_0)) \\ &= -\frac{\operatorname{Re} z_0}{\zeta - \nu} \left(\zeta - \nu \frac{i \operatorname{Im} z_0 + \mu\nu^{-1}\bar{z}_0}{i \operatorname{Im} z_0 + \bar{z}_0} \right). \end{aligned} \quad (4.3)$$

From the definition $z_0 = f(0)$ and (3.6) we see that $\mu\nu^{-1}\bar{z}_0 = -z_0$. Making this substitution, we recognise from (3.7) that the second term inside the bracket above is $f^{-1}(i \operatorname{Im} z_0)$.

We may evaluate this expression by means of a geometric argument, illustrated by Figure 4.2. Consider the z -plane, and the horizontal line that passes through $i \operatorname{Im} z_0$. On the following points of this line, we know that f^{-1} acts by

$$\frac{z}{\zeta = f^{-1}(z)} \parallel \begin{array}{c|c|c|c|c} -\bar{z}_0 & i \operatorname{Im} z_0 & z_0 & \infty \\ \hline \infty & f^{-1}(i \operatorname{Im} z_0) & 0 & \nu \end{array}$$

Hence $f^{-1}(i \operatorname{Im} z_0)$ lies on a line in the ζ -plane through the origin and ν . On the other hand, $i \operatorname{Im} z_0$ lies on the imaginary axis of the z -plane, and the imaginary axis is taken to the unit circle by f^{-1} . Therefore it must be that $f^{-1}(i \operatorname{Im} z_0) = -\nu$, the intersection of these two lines. Using this expression, we may simplify (4.3) to

$$z - i \operatorname{Im} z_0 = -\frac{\operatorname{Re} z_0}{\zeta - \nu} (\zeta + \nu).$$

Returning to our calculation of the limit of Θ^P , we have that

$$\lim_{\beta \rightarrow \nu} \Theta^P = r\pi(\operatorname{Re} z_0) d \left[\frac{\eta}{\zeta} \times \frac{\zeta + \nu}{\zeta - \nu} \right].$$

The pull back of this by the normalisation map N is then

$$N_{(\alpha, \nu)}^* \lim_{\beta \rightarrow \nu} \Theta^P = r\pi(\operatorname{Re} z_0) d \left[\zeta^{-1} \eta \sqrt{-\bar{\nu}} (\zeta + \nu) \right],$$

which also lies in the plane of differentials on the genus zero curve $\Sigma(\alpha)$ meeting conditions (P.4)–(P.6). What we have shown is that if we move along a path in the space \mathcal{M}_1 of spectral data where one of the branch points β tends to the unit circle, then after normalising the double point that develops on $\Sigma(\alpha, \nu)$ the limit of this spectral data is potentially spectral data on the genus zero marked curve $\Sigma(\alpha)$ because it meets conditions (P.4)–(P.6). There are no periods on a curve of genus zero, so (P.8) is automatic. The only conditions that we have not yet shown to be satisfied in the limit is (P.11), the closing conditions. In fact, we will prove a more general proposition.

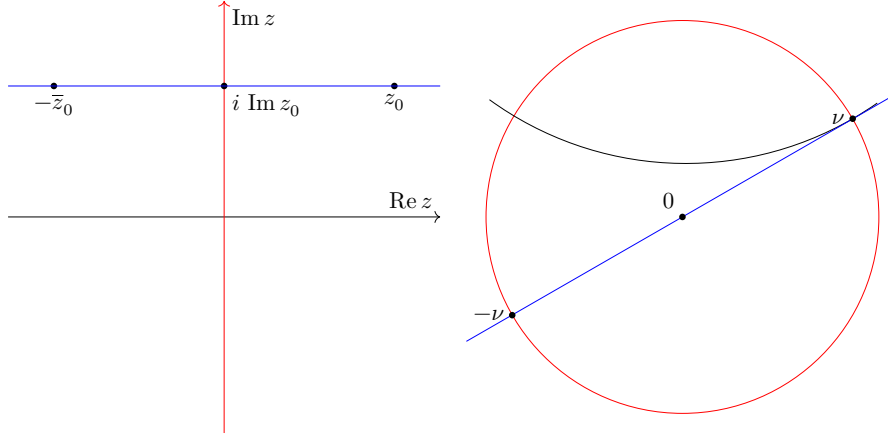


Figure 4.2

On the left the z -plane and on the right the ζ -plane. The transformation f^{-1} takes points of the z -plane to the ζ -plane. The imaginary axis is mapped to the unit circle, the real axis is mapped to the branch circle, and the horizontal line through $i \operatorname{Im} z_0$ is mapped to the line through the origin and ν .

Lemma 4.4. *Suppose that Θ is a smooth section $\Theta : (\alpha, \beta) \mapsto \Omega^1(\Sigma(\alpha, \beta))$ of differentials satisfying (P.4)–(P.8). Then for $\nu \neq \pm 1$,*

$$\int_{\gamma_+(\alpha)} N_{(\alpha, \nu)}^* \lim_{\beta \rightarrow \nu} \Theta = \pm \lim_{\beta \rightarrow \nu} \int_{\gamma_+(\alpha, \beta)} \Theta, \quad (4.5)$$

where $\gamma_+(\alpha)$ is a smooth family of paths between the two points over $\zeta = 1$ on $\Sigma(\alpha)$ and $\gamma_+(\alpha, \beta)$ are the principal paths between the two points over $\zeta = 1$ on $\Sigma(\alpha, \beta)$, defined on page 3.2. The ambiguity in sign is exactly the ambiguity in the choice of sign of the square root discussed above.

The same result holds for paths $\gamma_-(\alpha)$ between the two points over $\zeta = 1$ on $\Sigma(\alpha)$ and principal paths $\gamma_-(\alpha, \beta)$ between the two points over $\zeta = 1$ on $\Sigma(\alpha, \beta)$.

Proof. The method of proof will be to show this for Θ^E and Θ^P and then because any Θ meeting (P.4)–(P.8) may be expressed as a smooth combination of these, the general result will follow. For exact differentials, such as Θ^E , one may compute both sides explicitly and compare. Consider the exact differential $d[g(\alpha, \beta)(\zeta) + \eta h(\alpha, \beta)(\zeta)]$ on $\Sigma(\alpha, \beta)$, for smooth sections $g, h : (\alpha, \beta) \rightarrow M^0(\Sigma(\alpha, \beta))$ of meromorphic functions. On the left hand side

we have

$$\begin{aligned}
& \int_{\gamma_+(\alpha)} N_{(\alpha,\nu)}^* \lim_{\beta \rightarrow \nu} d[g(\alpha, \beta)(\zeta) + \eta h(\alpha, \beta)(\zeta)] \\
&= \int_{\gamma_+(\alpha)} N_{(\alpha,\nu)}^* d[g(\alpha, \nu)(\zeta) + \eta h(\alpha, \nu)(\zeta)] \\
&= \int_{\gamma_+(\alpha)} d[g(\alpha, \nu)(\zeta) + \eta \sqrt{-\bar{\nu}}(\zeta - \nu) h(\alpha, \nu)(\zeta)] \\
&= [g(\alpha, \nu)(\zeta) + \eta \sqrt{-\bar{\nu}}(\zeta - \nu) h(\alpha, \nu)(\zeta)]_{(1, -\eta(1))}^{(1, \eta(1))} \\
&= 2\eta(1) \sqrt{-\bar{\nu}}(1 - \nu) h(\alpha, \nu)(1), \\
&= 2|1 - \alpha| \sqrt{-\bar{\nu}}(1 - \nu) h(\alpha, \nu)(1),
\end{aligned}$$

and on the right hand side

$$\begin{aligned}
\lim_{\beta \rightarrow \nu} \int_{\gamma_+(\alpha, \beta)} d[g(\alpha, \beta)(\zeta) + \eta h(\alpha, \beta)(\zeta)] &= \lim_{\beta \rightarrow \nu} 2\eta(1) h(\alpha, \beta)(1) \\
&= \lim_{\beta \rightarrow \nu} 2|1 - \alpha| |1 - \beta| h(\alpha, \beta)(1) \\
&= 2|1 - \alpha| |1 - \nu| h(\alpha, \nu)(1),
\end{aligned}$$

which are the same up to a sign as $\sqrt{-\bar{\nu}}(1 - \nu) = \pm |1 - \nu|$. A similar calculation holds for γ_- . This establishes the relationship for Θ^E .

For Θ^P , we must deal with its elliptic integral terms. But from (3.26) we have that

$$\int_{\gamma_+} 2E\tilde{\omega} - 2Ke = 4E(k)F(f(1); k) - 4K(k)\tilde{E}(f(1); k).$$

We refer to Lemma B.9, which computes the limits of incomplete elliptic integrals as $k \rightarrow 0$. That lemma proves that so long as $f(1)$ is contained in a compact set, that we may pass the limit through to the integrand and that the incomplete elliptic integrals degenerate to asinh. In this situation that implies,

$$\lim_{k \rightarrow 0} \int_{\gamma_+} 2E\tilde{\omega} - 2Ke = 2\pi [\operatorname{asinh}(-if(1)) - \operatorname{asinh}(-if(1))] = 0.$$

For the range of $f(1) \in i\mathbb{R}$ to be bounded, we must forbid $f(1) \rightarrow \infty$. Translating this into the ζ -plane by applying f^{-1} , we have that ν must not approach 1. From (3.27), by the same reasoning,

$$\lim_{k \rightarrow 0} \int_{\gamma_-} 2E\tilde{\omega} - 2Ke = 2\pi [\operatorname{asinh}(-if(-1)) - \operatorname{asinh}(-if(-1))] = 0,$$

as long as ν does not approach -1 . Combining these two facts, for $\nu \neq \pm 1$,

$$\begin{aligned}
\int_{\gamma_+(\alpha)} N_{(\alpha,\nu)}^* \lim_{\beta \rightarrow \nu} \Theta^P &= \int_{\gamma_+(\alpha)} 0 - N_{(\alpha,\nu)}^* \lim_{\beta \rightarrow \nu} 2Kd \left[\frac{(z - i \operatorname{Im} z_0)w}{(z - z_0)(z + \bar{z}_0)} \right] \\
&= \pm \lim_{\beta \rightarrow \nu} \int_{\gamma_+(\alpha,\beta)} 0 - 2Kd \left[\frac{(z - i \operatorname{Im} z_0)w}{(z - z_0)(z + \bar{z}_0)} \right] \\
&= \pm \lim_{\beta \rightarrow \nu} \int_{\gamma_+(\alpha,\beta)} 2E\omega - 2Ke - 2Kd \left[\frac{(z - i \operatorname{Im} z_0)w}{(z - z_0)(z + \bar{z}_0)} \right] \\
&= \pm \lim_{\beta \rightarrow \nu} \int_{\gamma_+(\alpha,\beta)} \Theta^P.
\end{aligned}$$

Thus the relationship (4.5) holds for Θ^P as well, and by linear extension all differentials satisfying conditions (P.4)–(P.8). \square

This lemma is sufficient to finalise the correspondence. The only conditions that we had left to satisfy were the closing conditions (P.11), that the integrals of the differentials over paths γ_+ and γ_- must lie in $2\pi i\mathbb{Z}$. In particular, take a path $\ell(t)$ of spectral data $(\Sigma(\alpha(t), \beta(t)), \Theta^1(t), \Theta^2(t))$ in \mathcal{M}_1 , such that as $t \rightarrow t_0$ we have $\beta(t) \rightarrow \nu(t_0) \neq \pm 1$. Along the path these integrals over γ_+ and γ_- are constant and so are unchanged in the limit. The above lemma has proved that the integrals of the pulled-back differentials are equal, up to a sign, to the limit of the integrals on the genus one curves. They therefore lie in $2\pi i\mathbb{Z}$ also. In other words, the corresponding differentials on the normalised marked curve $\Sigma(\alpha)$ satisfy (P.4)–(P.11) if the differentials on $\Sigma(\alpha, \beta)$ do. The corresponding spectral data in \mathcal{M}_0 is

$$(\Sigma(\alpha(t_0)), N^*\Theta^1(t_0), N^*\Theta^2(t_0)).$$

Glossary

A The standard real period of a elliptic marked curve. See Figure B.1.

\mathcal{A}_g A parameter space that double covers the space \mathcal{E}_g of marked curves of genus g . It is defined by (0.12) to be the space of ordered tuples of distinct branch points inside the unit disc.

b^i Every differential on a marked curve of genus g that satisfies (P.4)–(P.7) may be written in the form

$$\Theta = b(\zeta) \frac{d\zeta}{\zeta^2 \eta},$$

for a polynomial $b(\zeta) \in \mathcal{P}_{\mathbb{R}}^{g+3}$. The superscript refers to one of the pair of differentials in the spectral data $(\Sigma, \Theta^1, \Theta^2)$.

\dot{b}^i The derivative of b^i with respect to t along a path $\ell(t)$ in \mathcal{M}_g .

\tilde{b}^i A factor of b^i . $b^1 = FF^1G\tilde{b}^1$.

B The standard imaginary period of a elliptic marked curve. See Figure B.1.

\mathcal{B}_g The rank two bundle of differentials satisfying Conditions (P.4)–(P.7) over \mathcal{E}_g .

c^i A real polynomial of degree $g+1$ defined by factoring \hat{c}^i . $c^i = (\zeta^2 - 1)\hat{c}^i$.

\hat{c}^i An infinitesimal deformation of differential $\Theta^i = dq$ satisfying (P.4)–(P.8) may be encoded as a meromorphic function with certain poles,

$$\dot{q}^i = \frac{1}{\zeta \eta} \hat{c}^i(\zeta),$$

for a polynomial $\hat{c}^i(\zeta) \in \mathcal{P}_{\mathbb{R}}^{g+3}$.

\mathcal{E}_g The space of marked curves of genus g .

$\tilde{\mathcal{E}}_1$ The universal cover of \mathcal{E}_1 .

\mathcal{D} The interior boundary of \mathcal{A}_1 , defined to be $\{(\alpha, \alpha) \in D \times D\}$.

e The Jacobi differential of the second kind.

$$e = (1 - k^2 z^2) \frac{dz}{w}.$$

$E(k)$ The complete elliptic integral of the second kind. $E(k) = E(1; k)$.

$E(z; k)$ The incomplete elliptic integral of the second kind. See Section B.1.

$$E(z; k) = \int_0^z \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt.$$

$E_0(x; k)$ A variant of $E(z; k)$ focused on imaginary arguments and which is well-behaved at infinity. $E_0(x; k) := \text{Im } E(ix; k) - kx$

$f(\zeta)$ In genus one, the Möbius transformation that takes the standard form of the spectral curve (3.2) to the Jacobi form. See (3.6).

$f_s(\zeta)$ In genus one, another the Möbius transformation that takes the standard form of the spectral curve (3.2) to the Jacobi form. See page 92.

F The common factor of P , b^1 and b^2 .

F^i The common factor of P/F and b^i/F .

$F(z; k)$ The incomplete elliptic integral of the first kind. See Section B.1.

$$F(z; k) = \int_0^z \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}.$$

$F_0(x; k)$ A variant of $F(z; k)$ focused on imaginary arguments. $F_0(x; k) := \text{Im } F(ix; k)$

G The common factor of b^1/FF^1 and b^2/FF^2 .

\mathcal{G} The group of covering transformations of $\tilde{\mathcal{C}}_1$ over \mathcal{C}_1 .

k The elliptic modulus. The elliptic modulus of a genus one marked curve is given by (3.5).

$K(k)$ The complete elliptic integral of the first kind. $K(k) = F(1; k)$.

l The imaginary period of the differential Ψ^P is $2\pi il$, for some positive integer $l \in \mathbb{Z}^+$.

ℓ A path in the moduli space \mathcal{M}_g of spectral data.

L A function related to the derivatives of T . The subject of Lemma 3.38.

L' A function derived from L related to the derivatives of T over the points where $u = \infty$. The subject of Lemma 3.38.

\mathcal{M}_g The moduli space of spectral data $(\Sigma, \Theta^1, \Theta^2)$ consisting of a marked curve Σ and two differentials satisfying conditions (P.4)–(P.11).

$\text{Mat}_2^* \mathbb{Z}$ The set of two-by-two integer matrices with nonzero determinant.

$N_{(\alpha, \nu)}$ The normalisation map of a singular marked curve $\Sigma(\alpha, \nu)$ for $\nu \in \mathbb{S}^1$.

p Represents a value of the function S . A coordinate on \mathcal{A}_1 .

$P(\zeta)$ The equation of the spectral curve is $\eta^2 = P(\zeta)$. In standard form (0.7), where the roots are given by conjugate inverse pairs $\alpha_j, \bar{\alpha}_j^{-1}$,

$$P(\zeta) = \prod (\zeta - \alpha_j)(1 - \bar{\alpha}_j \zeta).$$

P_k The coefficient of ζ^k in $P(\zeta)$.

\dot{P} The derivative of P with respect to t along a path $\ell(t)$ in \mathcal{M}_g .

\mathcal{P}^k The space of polynomials of degree at most k . See Definition 0.6.

$\mathcal{P}_{\mathbb{R}}^k$ The space of polynomials of degree at most k that are real with respect to ρ . See Definition 0.6.

q Represents a value of the function \tilde{T} . A coordinate on $\tilde{\mathcal{E}}_1$. See Lemma 3.48.

Q A real quadratic polynomial that describes an infinitesimal deformation of the moduli space of spectral data. See (1.9).

\hat{Q} An imaginary quartic polynomial that describes an infinitesimal deformation of a pair of differentials that satisfy (P.4)–(P.8). See (1.8).

R A function $R : \mathcal{U}^{(i)} \times \mathbb{R}^3 \rightarrow \mathbb{C}$ given by Definition 1.20. This function must be zero at a point of $\mathcal{M}_g \cap \mathcal{U}^{(i)}$ in order for deformations to exist.

S A positive function on \mathcal{A}_1 such that a genus one marked curve $\Sigma(\alpha, \beta)$ admits an exact differential meeting the closing conditions (P.11) only if $S(\alpha, \beta)$ is rational. See (3.17).

\mathcal{S}_g The moduli space of spectral curves of genus g . These are marked curves for which there are a pair of differentials which satisfy (P.4)–(P.11).

$\tilde{\mathcal{S}}_1$ The universal cover of \mathcal{S}_1 . It is identified with a subspace of $\tilde{\mathcal{E}}_1$. See (3.50).

\mathfrak{su}_2 The Lie algebra of SU_2 .

SU_2 The group of two-by-two special unitary matrices, $A\bar{A}^{-T} = I$.

T A multi-valued function on \mathcal{A}_1 such that a genus one marked curve $\Sigma(\alpha, \beta)$ admits differentials meeting the closing conditions (P.11) only if $T(\alpha, \beta)$ is rational. See (3.20).

- T_0 A principal branch cut of T . See Definition 3.24.
- \tilde{T} A lift of T_0 from \mathcal{A}_1 to the universal cover $\tilde{\mathcal{C}}_1$. It may be computed from T_0 via the formula (3.45).
- \mathbb{T}^2 The torus.
- u A coordinate on \mathcal{A}_1 , defined by $iu = f(1)$. See Lemma 3.28.
- u' A coordinate on \mathcal{A}_1 , defined by $u' = u^{-1}$. See Lemma 3.28.
- \tilde{u} A coordinate on $\tilde{\mathcal{C}}_1$. See Lemma 3.39.
- U A coordinate on \mathcal{A}_1 on which λ acts by a rotation by π radians. Defined by $U = \sqrt{k}u$.
- \tilde{U} A coordinate on $\tilde{\mathcal{C}}_1$ defined in relation to U in the same way that \tilde{u} is related to u (via a half-tan covering map).
- \mathcal{U} An open subset of the affine space $\mathcal{P}_{\mathbb{R}}^{2g+2} \times \mathcal{P}_{\mathbb{R}}^{g+3} \times \mathcal{P}_{\mathbb{R}}^{g+3}$ on which the moduli space \mathcal{M}_g of spectral data is embedded via the reduction of a triple of spectral data $(\Sigma, \Theta^1, \Theta^2)$ to polynomials (P, b^1, b^2) . See Definition 1.1.
- \mathcal{U}' An open subset of \mathcal{U} . See Definition 1.13.
- \mathcal{U}'' A subset of \mathcal{U} corresponding to conformal harmonic maps. See Definition 1.13
- $\mathcal{U}^{(i)}$ An open dense subset of \mathcal{U}' where the roots of the polynomials P , b^1 and b^2 associated to spectral data have distinct roots. See Definition 1.13
- v A coordinate on \mathcal{A}_1 , defined by $iv = f(-1)$.
- v' A coordinate on \mathcal{A}_1 , defined by $v' = v^{-1}$.
- \tilde{v} A coordinate on $\tilde{\mathcal{C}}_1$. See Lemma 3.39.
- V A coordinate on \mathcal{A}_1 on which λ acts by a rotation by π radians. Defined by $V = \sqrt{k}v$.
- \tilde{V} A coordinate on $\tilde{\mathcal{C}}_1$ defined in relation to V in the same way that \tilde{v} is related to v (via a half-tan covering map).
- $w(z)$ The equation of an elliptic curve in Jacobi form. $w(z)^2 = (1 - z^2)(1 - k^2 z^2)$.
- $w'(z')$ Variant of $w(z)$ used to concisely write equations in the coordinates ‘at infinity’, such as u' and v' .
- $$w'(z')^2 = (z')^2 w((z')^{-1}) = (1 - (z')^2)(k^2 - (z')^2).$$
- W The vector 3-space of differentials on a marked curve satisfying conditions (P.4)–(P.6). It has bases $\{\omega, \Theta^E, \varepsilon\}$ and $\{\omega, \Theta^E, \Theta^P\}$. See Section 3.1.

- x A parameter that determines the conformal type of the domain of a harmonic map with a genus zero spectral curve. See (2.8).
- X A vector in \mathfrak{su}_2 . A harmonic map g corresponding to a genus zero spectral curve may be brought into the form $g(w_R + iw_I) = \exp(-4w_R X) \exp(4w_I Y)$, as in (2.3).
- Y A vector in \mathfrak{su}_2 . See the glossary entry for X .
- z A coordinate on \mathbb{CP}^1 , over which the Jacobi elliptic curve is written.
- z_0 The image of 0 under f . It is crucial to the construction of the map f^{-1} in terms of the parameters (p, k, u, v) , and given by (3.30).
- α A branch point of a marked curve, in the unit disc.
- β Another branch point of a marked curve, in the unit disc.
- γ_+ A path on a marked curve Σ between the two points lying over $\zeta = 1$.
- γ_+ A principal choice of the path γ_+ . See page 77.
- γ_- A path on a marked curve Σ between the two points lying over $\zeta = -1$.
- γ_- A principal choice of the path γ_- . See page 77.
- δ A parameter which determines the image of a harmonic map corresponding to a spectral curve of genus zero. It is defined in (2.9) to be the angle between X and Y .
- ε A differential on a genus one marked curve used to construct a basis of W . Defined by (3.11).
- ζ A coordinate on \mathbb{CP}^1 , over which the marked curve is defined.
- η A marked curve is given in standard form by $\eta^2 = P(\zeta)$, for P given by (0.7).
- $\eta^+(\zeta)$ The value of the fibre coordinate over $\zeta \in \mathbb{S}^1$. For a point ζ on the unit circle and a spectral curve in standard form, for odd genus g ,

$$\eta^+(e^{i\theta}) = \zeta^{\frac{g+1}{2}} \prod_j |\zeta - \alpha_j|.$$

- Θ^E An exact differential on a genus one marked curve which satisfies conditions (P.4)–(P.8).

$$\Theta^E = i d \left(\frac{\eta}{\zeta} \right).$$

- Θ^P A differential on a genus one marked curve which satisfies conditions (P.4)–(P.8) and has an imaginary period of $2\pi i$. See (3.14).

- κ A parameter for translation invariant solutions to (0.1) that determines the angle of the lattice of periods to the z -axis. See (2.2).
- λ The involution on \mathcal{A}_1 that swaps the labelling of the branch points. See (3.51).
- $\tilde{\lambda}$ The translation on $\tilde{\mathcal{C}}_1$ that generates the group of covering transformation of $\tilde{\mathcal{C}}_1$ over \mathcal{C}_1 . See (3.52).
- μ In the genus one case, the branch points lie on a circle. The point μ is the intersection of the branch circle with the unit circle that lies between α and $\bar{\alpha}^{-1}$. See Figure 3.3.
- ν The intersection of the branch circle with the unit circle that lies between β and $\bar{\beta}^{-1}$. See Figure 3.3.
- ξ A point on a marked curve Σ .
- $\tilde{\pi}$ The projection of $\tilde{\mathcal{C}}_1$ to \mathcal{A}_1 . See Lemma 3.39.
- ρ The real involution on a marked curve. See Definition 0.5.
- $\tilde{\rho}$ The real involution ρ when written in the z -coordinate. $\tilde{\rho}(z) = -\bar{z}$.
- σ The hyperelliptic involution. See Definition 0.5.
- $\sigma_1, \sigma_2, \sigma_3$ The standard basis of \mathfrak{su}_2 . See (2.4).
- Σ A marked curve. See Definition 0.5.
- $\Sigma(\alpha, \beta)$ The genus one marked curve with branch points $\{\alpha, \bar{\alpha}^{-1}, \beta, \bar{\beta}^{-1}\}$. This can also be viewed as the covering map \mathcal{A}_1 over \mathcal{C}_1 .
- τ The conformal parameter of the torus. See (1.33) or (2.8).
- Φ The Higgs field, a $(1, 0)$ section of the vector bundle $\text{ad } P$ for principal G -bundle over M . Every harmonic map corresponds to a pair (A, Φ) satisfying (0.1).
- χ An involution on \mathcal{A}_g that take branch points to their negatives. See (3.59).
- Ψ^E An exact differential on a genus one marked curve that satisfies Conditions (P.4)–(P.11). See Lemma 3.21.
- Ψ^P A differential on a genus one marked curve that satisfies Conditions (P.4)–(P.11) and has imaginary period $2\pi i$. See Lemma 3.21.
- ω A holomorphic differential on a genus one marked curve given by dz/w .

Elliptic Integrals

The purpose of this appendix is to provide background information about elliptic integrals. It contains their definitions and basic properties, as well as some results that are used in the body of the thesis. It is not a comprehensive treatment, specifically elliptic integrals of the third kind are not treated, and incomplete integrals are only treated for an imaginary argument.

B.1 Definitions and Periods

Elliptic integrals are the integrals of differentials on elliptic curves (curves of genus one). The descriptor ‘elliptic’ comes from the problem of finding the arc length of an ellipse, and elliptic integrals are in turn the origin of the term elliptic curve. This topic has a long history, at least two hundred years, and consequently there are several competing conventions. In this thesis we shall use the Jacobi elliptic integrals, sometimes also called Legendre elliptic integrals, of a modulus k . A history of the development of elliptic integral, first by Legendre and taken up by Jacobi and Abel, may be found in [BG13]. Every elliptic integral may be reduced to a combination of Jacobi elliptic integrals [AE06, Han10].

Definition B.1. *The incomplete elliptic integral of the first kind is defined to be*

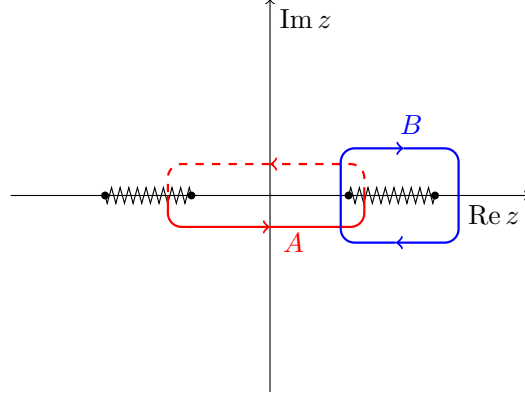
$$F(z; k) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Of particular interest are *complete* elliptic integrals, where $z = 1$. The first complete integral is denoted $K(k)$. Often the modulus is understood and it is written simply as K . The complete integrals are used to compute the periods of the elliptic curve

$$w^2 = (1-z^2)(1-k^2z^2).$$

Figure B.1

The z -plane with branch cuts along $[1, k^{-1}]$ and $[-1, -k^{-1}]$. The A period is in red and B period is in blue.



Every elliptic curve may be brought to this highly symmetric form, known as *Jacobi's form*. If the curve carries a reality structure (if it admits an antiholomorphic involution) then k must be real and by a further Möbius transformation one can arrange for $0 < k < 1$. We may choose to take branch cuts along $[1, k^{-1}]$ and $[-1, -k^{-1}]$.

Consider the A -period, the anticlockwise loop around the branch points -1 and 1 . By the symmetry $w \mapsto -w$, the integral of the holomorphic differential dz/w around this loop is twice the integral from -1 to 1 , which itself is twice the integral from 0 to 1 . Thus this period of the holomorphic differential is $4K$, which earns K the nickname of ‘quarter-period’.

To complete a basis of homology, choose the other period, B , to be a clockwise loop around 1 and k^{-1} . To compute the integral around this loop, one employs a trick called *Jacobi's imaginary transformation* that makes the following quadratic substitution [WW15]. Let $z^{-2} = 1 - k'^2 s^2$, where $k'^2 = 1 - k^2$. Then

$$dz = \frac{k'^2 s}{(1 - k'^2 s^2)^{3/2}} ds,$$

$$w = \pm \frac{ik'^2 s \sqrt{1 - s^2}}{1 - k'^2 s^2}$$

so that

$$\begin{aligned} \int_B \frac{dz}{w} &= \int_1^0 \frac{ds}{(+i)\sqrt{(1-s^2)(1-k'^2 s^2)}} + \int_0^1 \frac{ds}{(-i)\sqrt{(1-s^2)(1-k'^2 s^2)}} \\ &= 2iK(k') \end{aligned}$$

Often $K(k')$ is abbreviated to simply K' , and this should not be confused with the derivative of K . This auxillary parameter k' is called the complementary modulus.

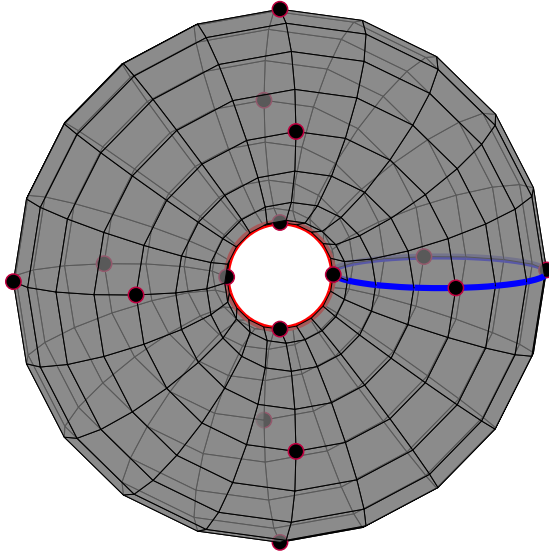


Figure B.2
The torus $w^2 = (1 - z^2)(1 - k^2 z^2)$ with A -period in red (in the centre) and B -period in blue. The upper and lower halves of the torus correspond to the two sheets of \mathbb{C} .

We need also in this thesis to consider periods of differentials of the second kind. The standard differential of the second kind is characterised by having a double pole at infinity with no residue. Analogously to the integrals of the first kind,

Definition B.2. *The incomplete elliptic integral of the second kind is*

$$E(z; k) = \int_0^z \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt = \int_0^z \frac{1 - k^2 t^2}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} dt,$$

and $E(k) = E(1; k)$ is the complete integral.

Unfortunately, it is standard usage to denote both complete and incomplete integrals by E , so to avoid confusion we will always show incomplete integrals with two arguments, while we may omit the argument for complete integrals.

The A -period of this differential is $4E$. To compute the B -period requires another clever substitution. Let this time $k^2 z^2 = 1 - k'^2 s^2$. Then

$$dz = -\frac{k'^2}{k} \frac{s}{\sqrt{1 - k'^2 s^2}} ds,$$

$$w = \pm i \frac{k'^2}{k} s \sqrt{1 - s^2}$$

so that

$$\begin{aligned}
\int_B (1 - k^2 z^2) \frac{dz}{w} &= 2i \int_0^1 \frac{k'^2 s^2}{\sqrt{(1 - s^2)(1 - k'^2 s^2)}} ds \\
&= 2i \int_0^1 \frac{ds}{\sqrt{(1 - s^2)(1 - k'^2 s^2)}} - 2i \int_0^1 \sqrt{\frac{1 - k'^2 s^2}{1 - s^2}} ds \\
&= 2i [K(k') - 2iE(k')].
\end{aligned}$$

Again, we introduce the notation that $E' = E(k')$.

B.2 Inequalities and Limits

Throughout this thesis, we use inequalities to bound the behaviour of elliptic integrals. One can obtain a crude bound from the fact that K is an increasing function and E is a decreasing one. The exact values

$$K(0) = E(0) = \frac{\pi}{2}, \quad E(1) = 1, \quad (\text{B.3})$$

therefore give a lower bound for K and a narrow range of values for E . More precise inequalities may be found in [AV85]. Their inequality (2) contains only K and confines its behaviour to within a strip-like region of known width.

$$\ln 4 \leq K + \frac{1}{2} \ln(1 - k^2) \leq \frac{\pi}{2}. \quad (\text{B.4})$$

In particular, as $k \rightarrow 1$, the complete elliptic integral of the first kind has a logarithmic singularity. A stronger and more precise statement is

$$\lim_{k \rightarrow 1} \left[K + \frac{1}{2} \ln(1 - k^2) \right] = \frac{3}{2} \ln 2.$$

A similar result, inequality (1) in [AV85], ties the integral of the second kind to that of the first kind,

$$\frac{\pi}{4} k^2 \leq E - (1 - k^2)K \leq k^2. \quad (\text{B.5})$$

For the incomplete integrals, in this thesis we need only investigate their properties on the imaginary axis. Both $F(z; k)$ and $E(z; k)$ take purely imaginary values on the imaginary axis. We shall see that $E(z; k)$ has a pole at infinity, so we shall concentrate instead on $E(z; k) - kz$ which is well behaved everywhere. We make the following definitions.

Definition B.6. *Let*

$$F_0(x) = F_0(x; k) := \text{Im } F(ix; k) = \int_0^x \frac{dt}{\sqrt{(1 + t^2)(1 + k^2 t^2)}},$$

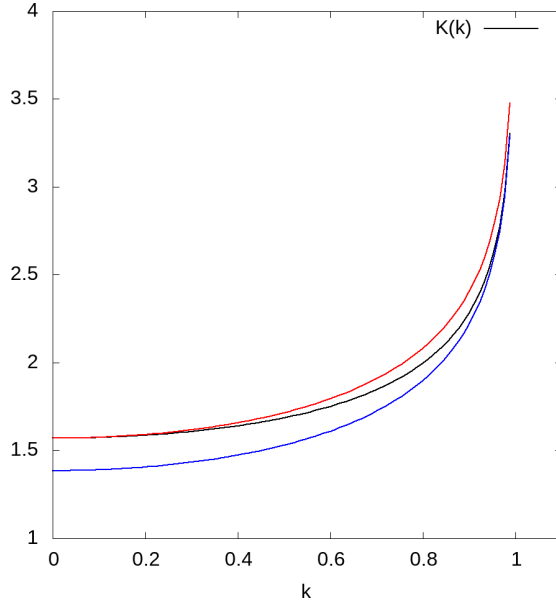


Figure B.3

Plot of K (black) and its upper and lower bounding approximates. The blue lower bound is

$$\ln 4 - \frac{1}{2} \ln(1 - k^2),$$

and the red upper bound is

$$\frac{\pi}{2} - \frac{1}{2} \ln(1 - k^2).$$

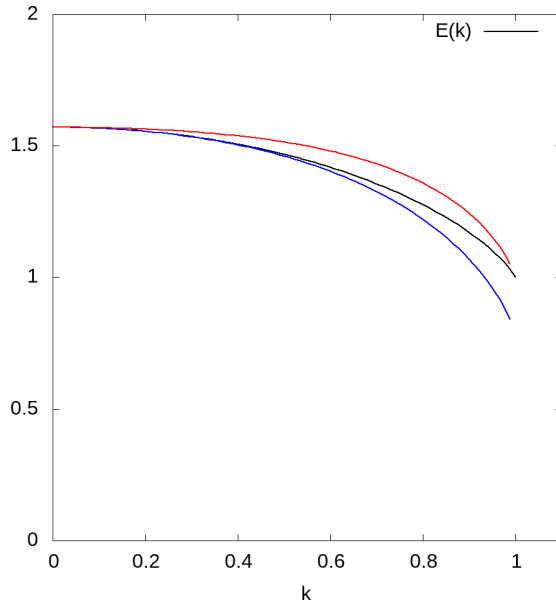


Figure B.4

Plot of E (black) and its upper and lower bounding approximates. The blue lower bound is

$$\frac{\pi}{4} k^2 + (1 - k^2) K(k),$$

and the red upper bound is

$$k^2 + (1 - k^2) K(k).$$

and

$$E_0(x) = E_0(x; k) := \operatorname{Im} E(ix; k) - kx = \int_0^x \sqrt{\frac{1 + k^2 t^2}{1 + t^2}} - k \, dt,$$

where the k is implicit if it is omitted.

Both are real valued functions of $x \in \mathbb{R}$ and $k \in (0, 1)$. We begin with the simple observation that both F_0 and E_0 are odd functions of x . Next note that both functions are increasing functions of x . In the case of F_0 this is obvious as the integrand is positive. For the other function, $k\sqrt{1 + t^2} < \sqrt{1 + k^2 t^2}$ from which it follows that

$$E_0(x) = \int_0^x \frac{\sqrt{1 + k^2 t^2} - k\sqrt{1 + t^2}}{\sqrt{1 + t^2}} \, dt$$

is increasing as well. For increasing functions it is natural to wonder whether they increase towards a limit. In our case they do, and we shall compute the value of the limit by a standard technique of complex analysis: integration around a closed semicircular contour.

Lemma B.7. *The functions F_0 and E_0 are increasing functions, with the following limits as x tends to infinity.*

$$\lim_{x \rightarrow +\infty} F_0(x; k) = K', \quad \lim_{x \rightarrow +\infty} E_0(x; k) = K' - E'.$$

Proof. Consider once again the standard incomplete elliptic integrals. Let the aforementioned semicircular contour be composed of an interval along the imaginary axis from $-iR$ to iR , and let C_R be the semicircular arc in the right half plane from iR back down to $-iR$. The contour is homologous to a standard B period around $[1, k^{-1}]$.

We treat first the integral of the first kind. The contribution coming from the semicircular arc is negligible as R tends to infinity, as can be seen below.

$$\begin{aligned} \left| \int_{C_R} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} \right| &= \left| \int_{\pi/2}^{-\pi/2} \frac{Re^{i\theta} d\theta}{\sqrt{(1 - R^2 e^{2i\theta})(1 - k^2 R^2 e^{2i\theta})}} \right| \\ &\leq \frac{R}{\sqrt{(R^2 - 1)(k^2 R^2 - 1)}} \times \pi \\ &\rightarrow 0. \end{aligned}$$

And since F_0 is an odd function of x , we can conclude that

$$\begin{aligned} 2iK' &= \lim_{R \rightarrow \infty} \left(\int_{-iR}^{iR} + \int_{C_R} \right) \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} \\ &= 2i \lim_{R \rightarrow \infty} \int_0^R \frac{dt}{\sqrt{(1 + t^2)(1 + k^2 t^2)}} + 0 \\ &= 2i \lim_{R \rightarrow \infty} F_0(R; k), \end{aligned}$$

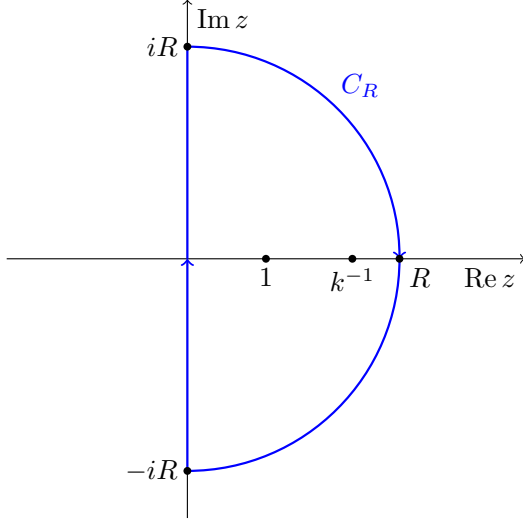


Figure B.5
The contour C_R

which establishes the first half of the result.

The analysis of the second limit proceeds in the same manner, however there is an extra step arising from the cancellation of the pole at infinity. Using the same contour as before, we again show that the contribution from the semicircular arc is vanishing.

$$\begin{aligned}
 \left| \int_{C_R} \sqrt{\frac{1-k^2 z^2}{1-z^2}} - k \, dz \right| &= \left| \int_{\pi/2}^{-\pi/2} \frac{\sqrt{1-k^2 R^2 e^{2i\theta}} - k\sqrt{1-R^2 e^{2i\theta}}}{\sqrt{1-R^2 e^{2i\theta}}} R e^{i\theta} d\theta \right| \\
 &\leq \frac{\sqrt{1+k^2 R^2} - k\sqrt{R^2-1}}{\sqrt{R^2-1}} R \times \pi \\
 &= \pi \frac{R}{\sqrt{R^2-1}} \frac{1+k^2}{\sqrt{1+k^2 R^2} + k\sqrt{R^2-1}} \\
 &\rightarrow 0.
 \end{aligned}$$

As before, we are dealing with an odd function of x and hence

$$\lim_{x \rightarrow \infty} E_0(x; k) = K' - E'.$$

□

It immediately follows from this lemma that we have the following bounds independent of x .

$$\begin{aligned}
 -K' &\leq F_0(x; k) \leq K' \\
 -(K' - E') &\leq E_0(x; k) \leq K' - E'.
 \end{aligned} \tag{B.8}$$

It can also be useful to bound the growth of the two functions independently of k . As the integrands are monotonically decreasing functions of k , for $x > 0$ so too are F_0 and E_0 . In the extreme cases that k is 0 or 1, the following lemma applies.

Lemma B.9. *The elliptic integrals degenerate to the following elementary functions as k approaches 0 or 1. Uniformly in x ,*

$$\begin{aligned}\lim_{k \rightarrow 1} F_0(x; k) &= \operatorname{atan} x \\ \lim_{k \rightarrow 1} E_0(x; k) &= 0.\end{aligned}$$

And uniformly for x in a compact set,

$$\begin{aligned}\lim_{k \rightarrow 0} F_0(x; k) &= \operatorname{asinh} x \\ \lim_{k \rightarrow 0} E_0(x; k) &= \operatorname{asinh} x.\end{aligned}$$

Proof. As these are decreasing functions, for all $k > k_0$ we have

$$\begin{aligned}|F_0(x; k)| &< |F_0(x; k_0)| < K'(k_0), \\ |E_0(x; k)| &< |E_0(x; k_0)| < K'(k_0) - E'(k_0),\end{aligned}$$

using the previous lemma. This shows that the integrals are dominated, and we may therefore pass the limit $k \rightarrow 1$ under the integral sign.

$$\begin{aligned}\lim_{k \rightarrow 1} F_0(x; k) &= \int_0^x \frac{dt}{1+t^2} = \operatorname{atan} x, \\ \lim_{k \rightarrow 1} E_0(x; k) &= \int_0^x \sqrt{\frac{1+t^2}{1+t^2}} - 1 \, dt = 0.\end{aligned}$$

For the limit as $k \rightarrow 0$, observe that for positive x

$$\begin{aligned}F_0(x; k) &\leq \int_0^x \frac{dt}{\sqrt{1+t^2}} = \operatorname{asinh} x, \\ E_0(x; k) &\leq \int_0^x \frac{1}{\sqrt{1+t^2}} - 0 \, dt = \operatorname{asinh} x.\end{aligned}$$

These inequalities serve to show that the integrals are dominated by functions independent of k , but not uniformly in x as asinh is an unbounded function. Thus for only values of x in a compact set does the dominated convergence theorem hold and give the result. \square

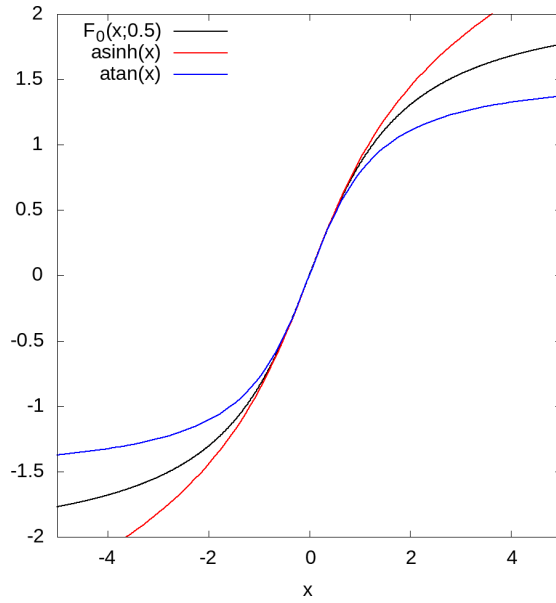


Figure B.6
Plot of $F_0(x; 0.5)$ (black) and its upper and lower bounds. For each value of k the function $F_0(x; k)$ is bounded, but as $k \rightarrow 0$ it tends to the unbounded $\text{asinh } x$.

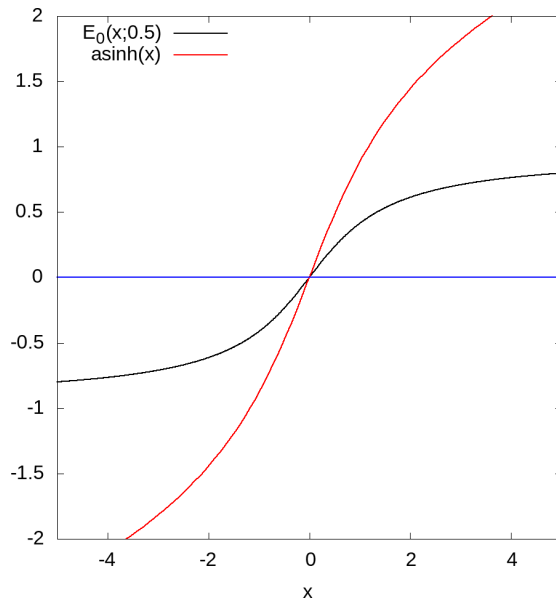


Figure B.7
Plot of $E_0(x; 0.5)$ (black) and its upper and lower bounds. Like F_0 , it is bounded for each value of k , but unbounded in the limit $k \rightarrow 0$.

B.3 Derivatives

The purpose of this section is to compute the derivatives of the elliptic integrals. The derivatives of the incomplete integrals with respect to the variable z are trivial because they are simply parameter integrals

$$\frac{\partial}{\partial z} F(z; k) = \frac{1}{\sqrt{(1-z^2)(1-k^2 z^2)}}, \quad (\text{B.10})$$

$$\frac{\partial}{\partial z} E(z; k) = \sqrt{\frac{1-k^2 z^2}{1-z^2}}. \quad (\text{B.11})$$

The derivatives of elliptic integrals with respect to the modulus are again elliptic integrals. In the interest of being concise, the correct combinations are presented *ex nihilo*. One may do the computation from scratch by differentiating the integrand and then subtracting terms to cancel off the poles and zeroes until only an exact differential remains. Differentiation under the integral sign is permitted because the integrals are dominated, as established in Lemma B.9. We compute the difference between the k derivative of dz/w and a certain combination of elliptic integrand terms.

$$\begin{aligned} \frac{\partial}{\partial k} \left(\frac{1}{\sqrt{(1-z^2)(1-k^2 z^2)}} \right) - \frac{1}{k(1-k^2)} \sqrt{\frac{1-k^2 z^2}{1-z^2}} + \frac{1}{k} \frac{1}{\sqrt{(1-z^2)(1-k^2 z^2)}} \\ = \frac{-k}{1-k^2} \frac{\partial}{\partial z} z \sqrt{\frac{1-z^2}{1-k^2 z^2}}. \end{aligned}$$

Integrating and rearranging gives

$$\frac{\partial}{\partial k} F(z; k) = \frac{1}{k(1-k^2)} E(z; k) - \frac{1}{k} F(z; k) - \frac{k}{1-k^2} z \sqrt{\frac{1-z^2}{1-k^2 z^2}}. \quad (\text{B.12})$$

In a similar way, consider that

$$k \frac{\partial}{\partial k} \left(\sqrt{\frac{1-k^2 t^2}{1-t^2}} \right) - \sqrt{\frac{1-k^2 t^2}{1-t^2}} + \frac{1}{\sqrt{(1-t^2)(1-k^2 t^2)}} = 0.$$

Thus we can integrate to obtain

$$\frac{\partial}{\partial k} E(z; k) = \frac{1}{k} E(z; k) - \frac{1}{k} F(z; k). \quad (\text{B.13})$$

We can recover the well known formulae for the derivatives of the complete elliptic integrals by making the substitution $z = 1$.

$$\frac{d}{dk} K = \frac{1}{k(1-k^2)} E - \frac{1}{k} K, \quad (\text{B.14})$$

$$\frac{d}{dk} E = \frac{1}{k} E - \frac{1}{k} K. \quad (\text{B.15})$$

B.4 Legendre's Relation

There is a useful relation between the complete elliptic integrals. It can be used to construct a basis of differentials with normalised periods. It also links the behaviour of the elliptic integrals at $k = 0$ and $k = 1$. In the computation of certain limits one could directly apply bounds, but it can often be more expedient to use Legendre's relation to transform the limiting term into a well behaved function. The standard proof of Legendre's relation is reproduced below.

Lemma B.16 (Legendre's relation).

$$KE' + K'E - KK' = \frac{\pi}{2},$$

Proof. We shall prove Legendre's relation in two stages. First we shall differentiate to show that it is constant. Then we will compute the constant by taking the limit as k tends to 0. Recall that the primes refer to the complementary modulus $k' = \sqrt{1 - k^2}$. Its derivative is

$$\frac{dk'}{dk} = -\frac{k}{k'},$$

so

$$\begin{aligned} & \frac{d}{dk} (KE' + K'E - KK') \\ &= \left(\frac{1}{k(1-k^2)} E - \frac{1}{k} K \right) E' - \frac{k}{k'} K \left(\frac{1}{k'} E' - \frac{1}{k'} K' \right) \\ & \quad - \frac{k}{k'} \left(\frac{1}{k'(1-k'^2)} E' - \frac{1}{k'} K' \right) E + K' \left(\frac{1}{k} E - \frac{1}{k} K \right) \\ & \quad - \left(\frac{1}{k(1-k^2)} E - \frac{1}{k} K \right) K' + \frac{k}{k'} K \left(\frac{1}{k'(1-k'^2)} E' - \frac{1}{k'} K' \right) \\ &= 0. \end{aligned}$$

Thus we have shown that the relation is constant. Determining the value of the constant is somewhat delicate. One could naïvely attempt to set $k = 0$, but then $k' = 1$ and K' is infinite. And conversely, if we were to set $k = 1$, then K would be infinite. Instead, let us take the limit as $k \rightarrow 0$,

$$\lim_{k \rightarrow 0} KE' + K'E - KK' = \frac{\pi}{2} + \lim_{k \rightarrow 0} (E - K)K'.$$

It remains to show this latter limit is zero. We will show this using the inequalities for K and E . From B.5 we have

$$\left(\frac{\pi}{4} - K \right) k^2 \leq E - K \leq (1 - K) k^2.$$

And substituting the complementary modulus, B.4 becomes

$$\ln 4 \leq K' + \ln k \leq \frac{\pi}{2}.$$

Since $k^2 \ln k$ goes to 0 as k does, the limit is established. Hence the constant in Legendre's relation is $\pi/2$.

□

B.5 Analytic Continuation

Above, we computed the limits of $F_0(x; k)$ and $E_0(x; k)$ as $x \rightarrow \infty$, but because both functions are odd, there is no way to extend them to single valued functions on $\mathbb{R} \cup \{\infty\}$. Instead we must extend these functions analytically. It is a standard fact that the elliptic integrals are analytic functions on the plane. To extend, we will use the fact that parameter integrals of analytic functions are analytic. Let $\tilde{x} \in \mathbb{R}$ and consider the covering map $\mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ given by

$$\tilde{x} \mapsto \tan \frac{\tilde{x}}{2}.$$

We identify $x \in \mathbb{R}$ with $\tilde{x} \in (-\pi, \pi)$.

Let \tilde{F} and \tilde{E} be the analytic continuations of F_0 and E_0 respectively. We will show that these continuations exist for all \tilde{x} by explicit construction. Take \tilde{x} in the range $(-\pi, \pi)$ and pull back the covering map

$$\begin{aligned} \tilde{F}(\tilde{x}; k) &= \int_0^{2 \operatorname{atan} \tilde{x}} \frac{dt}{\sqrt{(1+t^2)(1+k^2 t^2)}} \\ &= \int_0^{\tilde{x}} \frac{1}{\sqrt{(1+(\tan s/2)^2)(1+k^2(\tan s/2)^2)}} \times \frac{1}{2} \sec^2 \frac{s}{2} ds \\ &= \frac{1}{2} \int_0^{\tilde{x}} \frac{1}{\sqrt{\cos^2 s/2 + k^2 \sin^2 s/2}} ds. \end{aligned}$$

The integrand is analytic, and so \tilde{F} is an analytic function for all values of

$\tilde{x} \in \mathbb{R}$. The same approach works for \tilde{E} , but requires a little more work.

$$\begin{aligned}
\tilde{E}(\tilde{x}; k) &= \int_0^{2 \operatorname{atan} \tilde{x}} \frac{\sqrt{1+k^2 t^2} - k\sqrt{1+t^2}}{\sqrt{1+t^2}} dt \\
&= \frac{1}{2} \int_0^{\tilde{x}} \frac{\sqrt{\cos^2 s/2 + k^2 \sin^2 s/2} - k}{\cos^2 s/2} ds \\
&= \frac{1}{2} \int_0^{\tilde{x}} \frac{(\cos^2 s/2 + k^2 \sin^2 s/2) - k^2}{\cos^2 s/2} \frac{1}{\sqrt{\cos^2 s/2 + k^2 \sin^2 s/2 + k}} ds \\
&= \frac{1}{2} (1 - k^2) \int_0^{\tilde{x}} \frac{1}{\sqrt{\cos^2 s/2 + k^2 \sin^2 s/2 + k}} ds.
\end{aligned}$$

Manipulating the integrand into a form that is plainly analytic demonstrates that \tilde{E} is as well. One could reasonably ask how to compute the values of these extended functions. This is answered by noticing the integrands are $2\pi i$ periodic. Write $\tilde{x} = 2\pi n + y$ for an integer n and $y \in (-\pi, \pi)$. Then

$$\begin{aligned}
\tilde{F}(\tilde{x}; k) &= \frac{1}{2} \left(\int_0^{2\pi n} + \int_{2\pi n}^{2\pi n+y} \right) \frac{1}{\sqrt{\cos^2 s/2 + k^2 \sin^2 s/2}} ds \\
&= \frac{1}{2} \left(n \int_0^{2\pi} + \int_0^y \right) \frac{1}{\sqrt{\cos^2 s/2 + k^2 \sin^2 s/2}} ds \\
&= \left(n \oint_{\mathbb{R} \cup \{\infty\}} + \int_0^x \right) \frac{dt}{\sqrt{(1+t^2)(1+k^2 t^2)}} \\
&= 2nK' + F_0(x; k),
\end{aligned} \tag{B.17}$$

where $x = \tan \tilde{x}/2 = \tan y/2$ and recalling the period of F_0 from earlier. Likewise

$$\tilde{E}(\tilde{x}; k) = 2n(K' - E') + E_0(x; k). \tag{B.18}$$

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