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2/18/20

Dynamic Systems

Project 1

In this report a spring-mass-damper system modeled with discrete-time LMS AMESim models will be compared with closed form, continuous-time solutions solved either by hand, or with Mathematica depending on the problem. The AMESim models were ran, the data exported, then imported into MATLAB. while closed form solutions were generated directly in MATLAB with the MATLAB program attached in Appendix 2 of this report. Below in Figure 1 is the AMESim model that was solved for each problem with different parameters assigned to the damper and the initial displacement through the report, while the mass and spring constant remain constant shown in Table 1.

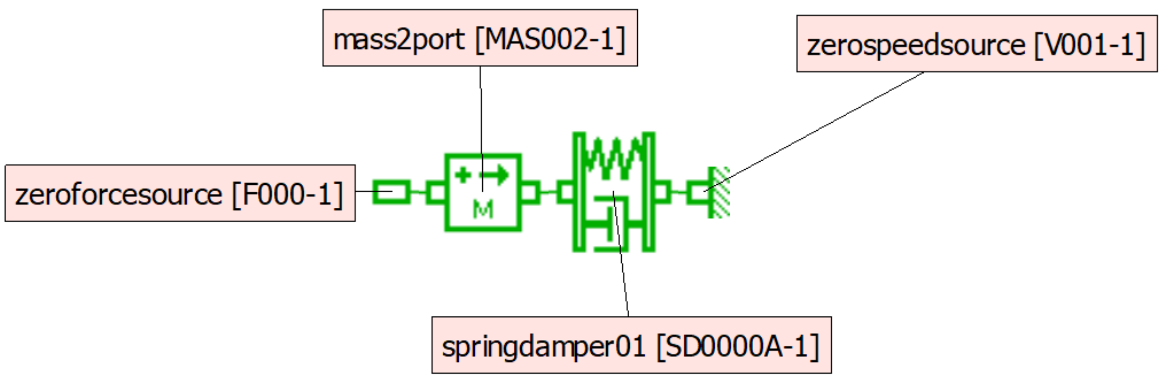


Figure 1: LMS AMESim model of the spring-mass-damper system

Table 1: System constants throughout this report

Parameter	**Value**	**Units**
Mass	23	kg
Spring constant	19990	N/m

This system can be modeled with the equation of motion defined below in Equation 1, which can then be transformed with the Laplace transform to simplify the equation into the transfer function shown in Equation 2. This equation of motion was defined by the free-body diagram that can be seen in Appendix 1, along with the steps to transform the equation of motion into the transfer function.

Equation 1: Equation of motion from free-body diagram of spring-mass-damper system

$$\sum F = M\ddot{x} = F_e - c\dot{x} - kx$$

Equation 2: Transfer function of spring-mass-damper system

$$\frac{X(s)}{F_e(s)} = \frac{1}{Ms^2 + cs + k} = \frac{1/M}{s^2 + 2\zeta\omega_n + {\omega_n}^2}$$

Where:

$$x = Displacement$$
$$F_e = External\ force$$
$$M = Mass\ of\ block$$
$$c = Damping\ coefficient$$
$$k = Spring\ coefficient$$
$$\omega_n = Natrual\ frequency = \sqrt{\frac{k}{m}}$$
$$\zeta = Damping\ ratio = \frac{c}{2M\omega_n}$$

First the undamped spring and mass system were modeled with an initial displacement of 10 millimeters. The plots generated are shown below in **Error! Reference source not found.**, and are identical by inspection of the two waveforms. The closed form solution calculated by taking the Laplace transform of the equation of motion in Equation 1, setting the damping coefficient to zero and the initial displacement to 0.01 meters, then taking the inverse Laplace transform of that. The closed form solution is shown below in Equation 3. This was plotted along with the AMESim simulation output in Figure *1*, and because they are so similar and the difference between them so small, it is almost hard to tell the difference between the two. By counting the zero-crosses of the solutions in a known timeframe, it is possible to calculate the frequency of the oscillations. By using this technique, the frequency can be found to be within 0.4 radians per second of the calculated natural frequency with the data collected, with the value measured from the plotted data being 29.48 radians per second, while the calculated value is 29.85 radians per second. It can also be clearly seen by the limits of the plot that the amplitude is what would be expected from an undamped oscillator, the magnitude of the initial displacement.

Equation 3: Closed form solution of undamped spring and mass system

$$x(t) = x_0 \cos(\omega_n t) = 0.01 \cos(29.48\ t)$$

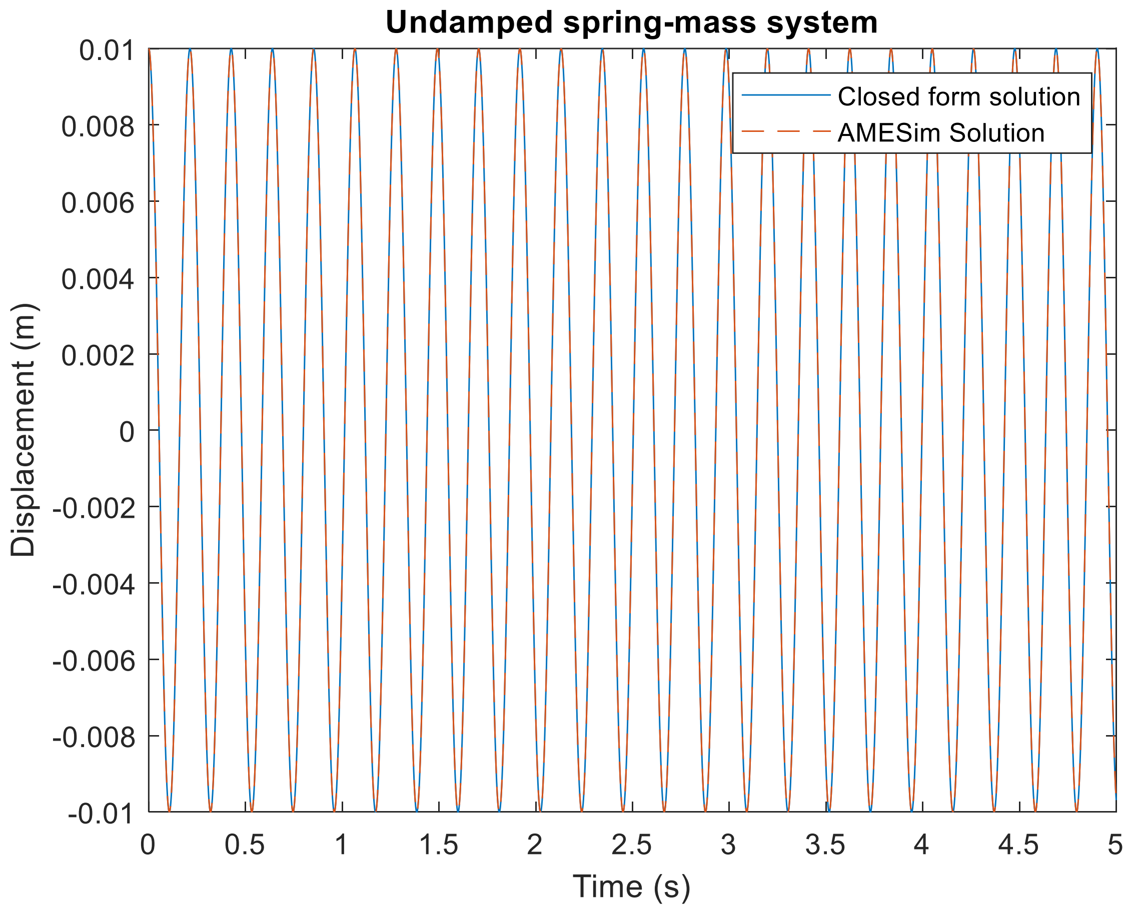


Figure 2: Undamped spring-mass system plot with the closed form in blue, and the AMESim simulation output in dotted orange.

The third part of the assignment was to analyze a mass-spring-damper system with 5% damping and an initial displacement of 10 millimeters. Using the equation described above as the definition of the damping ratio and setting the damping ratio to 0.05, the damping coefficient was found to be 67.8 N/m/s for 5% damping of this system. The two solutions calculated with AMESim and with the closed-form solution derived are shown below in Figure 3, and are very similar as with the previous system analyzed. The closed form solution for this system is shown below in Equation 4. By using the zero-cross counting technique as described in the previous section, the natural frequency of the calculated data was measured to be 29.86 radians per second. The damped natural frequency of this system can be calculated to be 29.44 radians per second, and so this value is close to the frequency measured from the data, within 2% error margin. By inspection of the calculated time constant on the plot, the time constant of the system can be assessed to be at least similar to the data's time constant, and most likely it is a similar case as with the frequency of being with a few percent of it.

Equation 4: Close form solution of 5% damped system with initial displacement of 10 millimeters

$$x(t) = e^{-0.05\omega_n t}\, 0.01\, (\cos(\omega_n t) + \frac{0.05\ *\ 0.01}{\sqrt{1 - 0.05}} \sin(\omega_n t)$$

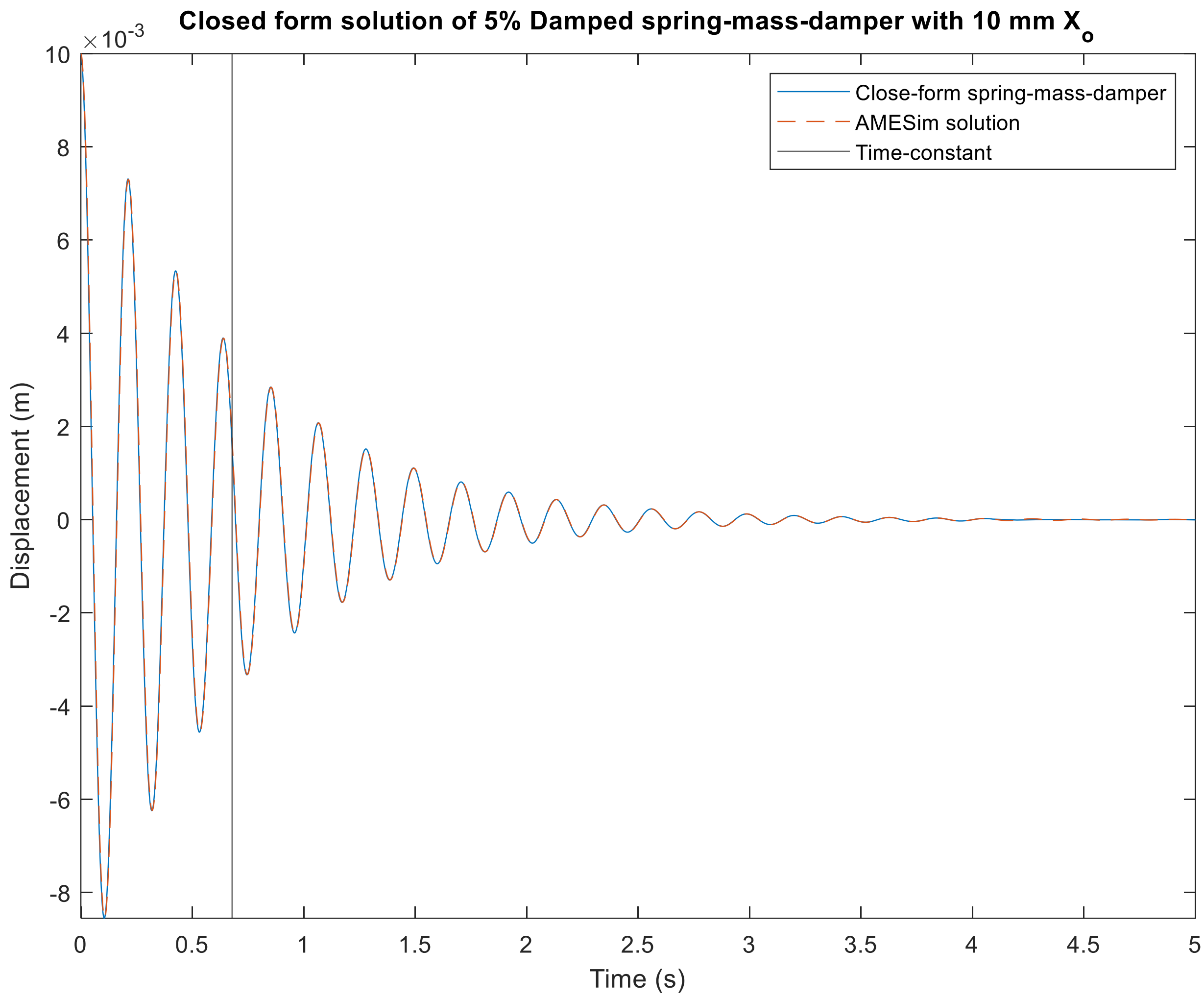


Figure 3: Closed-form solution and AMESim solution of a 5% damped, 10 millimeter initial displacement mass-spring-damper system plotted on same graph, with the time constant demarcated with a vertical line.

The fourth system analyzed for this report was a step of 10 Newtons to the 5% damped mass-spring-damper system. As before the two plots match up exactly as far as it can be ascertained. The new steady state for the system with a 10 Newton step is approximately 0.5 millimeters, and both can be seen to settle to this value. If the mean of the data is taken to find the actual steady-state value being settled toward, the value is calculated to be 5 millimeters. As previously, the zero-count technique shows that the frequency is also within a few percent of the expected value. The close-form solution is too long to write here, but it is enclosed in the source code of the MATLAB program in Appendix 2.

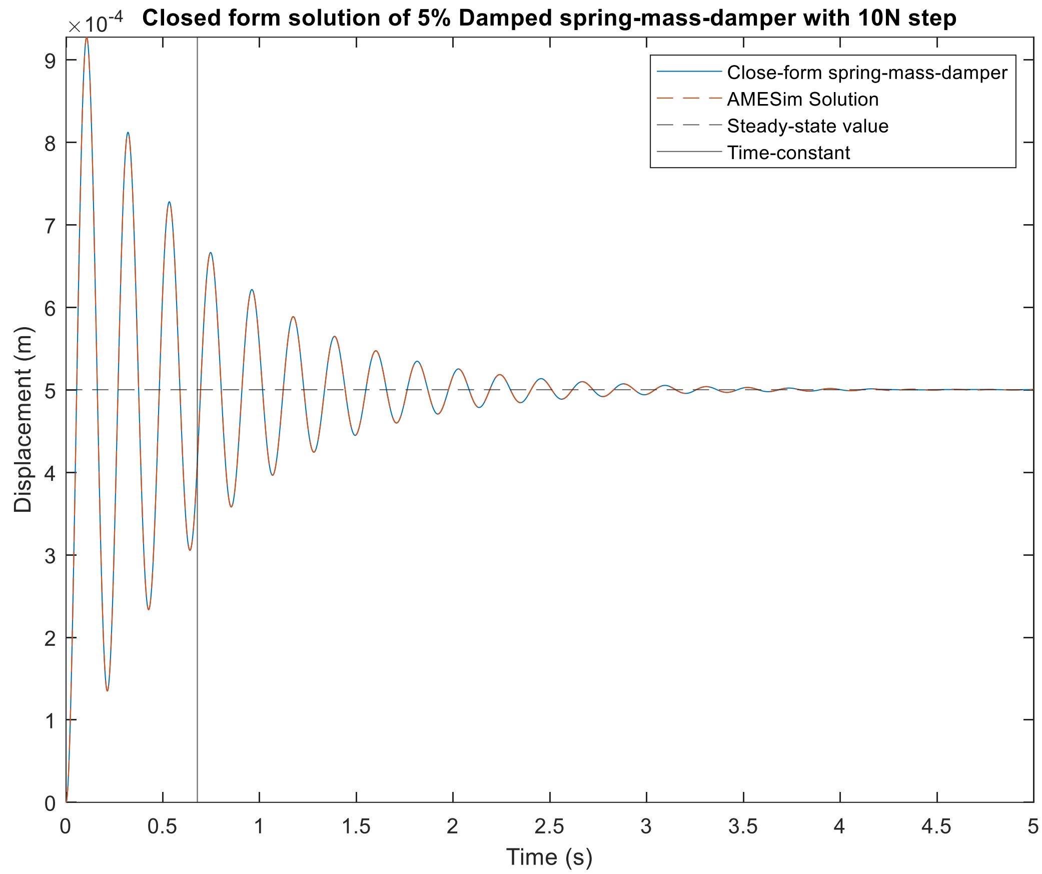


Figure 4: Closed-form solution and AMESim calculated values plotted, with the vertical line representing the calculated time constant and the horizontal dotted line representing the calculated steady-states value.

The final systems analyzed were three systems that had a sinusoidal force input with an amplitude of 5 Newtons, but with varying frequencies of the force applied to observe the beats between the force and the natural frequency. As with the previous systems analyzed, all the systems matched up between the closed-form solution and the AMESim simulation solutions. The closed-form solutions in this case were derived in Mathematica, with the code for that enclosed in Appendix 3. In this case as the frequency also changes slightly with time, the final 5 zero-crosses are used in this case instead of a mean over all the zero-crosses. The frequencies calculated from the data are tabulated below.

Force frequency percentage of natural frequency (frequency)	**Output Frequency (rad/s)**
100% (Natural frequency for reference)	29.48
50% (14.74)	14.75
90% (26.53)	26.40
130% (38.32)	38.31

Below the plot of all three systems are shown, with each system being plotted with both the closed form solution and the AMESim simulated solutions on the same plots. From inspection a difference between the AMESim data and the closed-form data cannot be seen and can be considered the same. The closed form solutions are too large to be included in line with this report but are also included in the supplied MATLAB source-code in Appendix 2.

The steady-state frequencies of all three-system line up almost exactly with the input force frequencies, even closer than some of the non-sinusoidal force input system analyzed previously. This steady-state makes sense as the natural frequency component would decrease as the damper would remove energy, but the input force would continue applying energy, so the system would eventually reach a steady-state frequency of matching the input force.

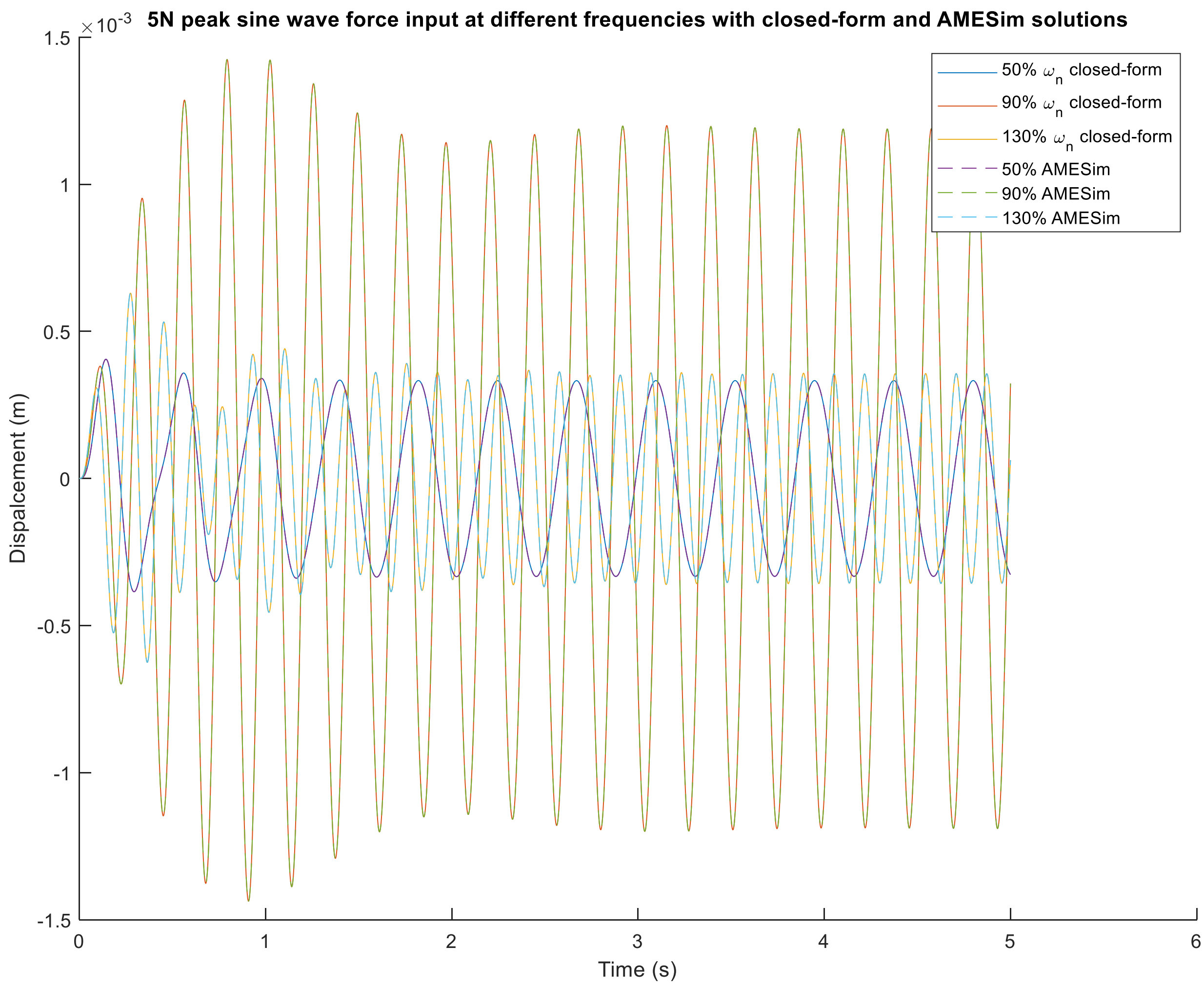
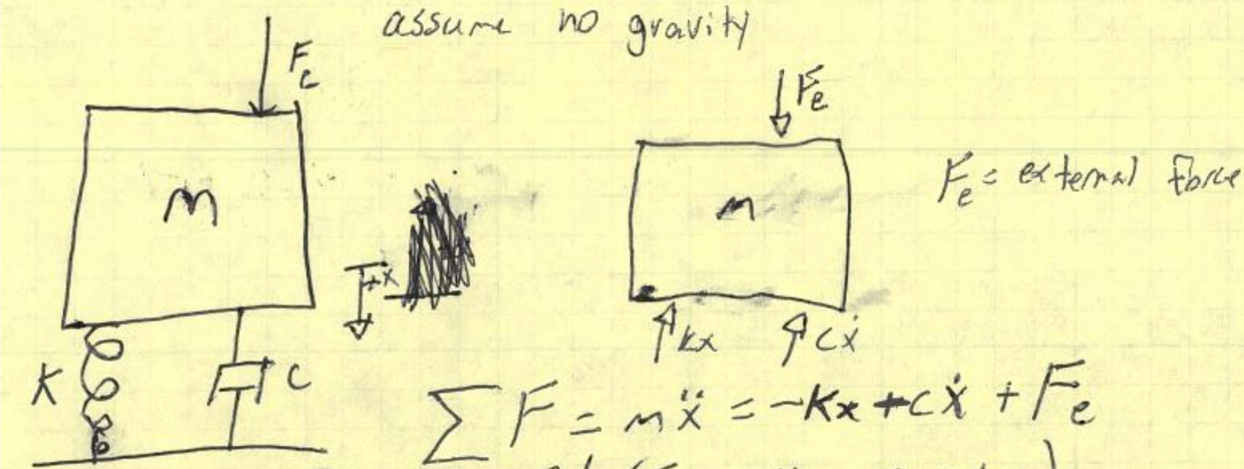


Figure 5: Three mass-spring-damper systems plotted with the closed-form and AMESim data overplayed on each other due to their near-perfect matches to each other.

Appendix 1 – Free-body diagram and transfer function



$$\sum F = m\ddot{x} = -kx + c\dot{x} + F_e$$

$$\mathcal{L}\left(F_e = m\ddot{x} + c\dot{x} + kx\right)$$

$$\boxed{\frac{X(s)}{F_e(s)} = \frac{1}{ms^2 + cs + k}}$$

If $x_0 = 10\,mm = 0.01\,m$ and $F_e = 0$ and $c = 0$

$$\mathcal{L}\left(0 = m\ddot{x} + c\dot{x} + kx\right)$$

$$0 = m\left(s^2 X(s) - s x_0 - \cancel{\dot{x}_0}\right) + c\left(s X(s) - x_0\right) + k X(s)$$

$$0 = ms^2 X(s) - ms x_0 + cs X(s) - c x_0 + k X(s)$$

$$0 = X(s)\left(ms^2 + cs + k\right) - s x_0 - m x_0$$

$$s x_0 + m x_0 = X(s)\left(ms^2 + cs + k\right)$$

$$X(s) = \frac{ms x_0 + \cancel{c x_0}}{ms^2 + \cancel{cs} + k} = \frac{s x_0 m}{ms^2 + k}$$

$$x(s) = \frac{x_0 \cdot m}{m} \frac{s}{s^2 + \frac{k}{m}} = x_0 \cdot \frac{s}{s^2 + \frac{k}{m}}$$

$$x(t) = x_0 \cdot \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

Appendix 2 – MATLAB Code

```
clear, clc, close all
% Ross Smyth
% 2/12/2019
% Dynamic System Project 1

% Some functions don't work on all systems (ilaplace) due to custom
```

```
% functions and different versions

k     = 19990; % N/m
m     = 23; % kg
nat_f = sqrt(k/m); % rad/s
syms t s
zero_cross = @(v) find(v(:).*circshift(v(:), [-1 0]) <= 0);
```

2. No damping

```
fun2 = 0.01 * cos(nat_f * t);

[t2, d2] = data_import('part_2.data');

figure
fplot(fun2, [0, 5])
title("Undamped spring-mass system")
xlabel("Time (s)")
ylabel("Displacement (m)")
hold on
plot(t2, d2, '--')
legend('Closed form solution', 'AMESim Solution')
hold off

mf2 = 2 * pi /(2 * mean(diff( t2(zero_cross(d2)))));
```

3. 5% Damping

```
c     = 2 * m * nat_f * 0.05;
nat_d = sqrt(1 - 0.05^2) * nat_f;
time3  = 1 / (nat_f * 0.05);

[t3, d3] = data_import('part_3.data');

fun3 = exp(-0.05 * nat_f * t)* 0.01 * (cos(nat_d * t) + 0.05 * 0.01 / sqrt(1-0.05) * sin(nat_f *
t));
figure
fplot(fun3, [0, 5])
hold on
plot(t3, d3, '--')
xline(time3)
hold off
title("Closed form solution of 5% Damped spring-mass-damper with 10 mm X_o")
xlabel("Time (s)")
ylabel("Displacement (m)")
legend('Close-form spring-mass-damper', 'AMESim solution', 'Time-constant')

mf3 = 2 * pi /(2 * mean(diff( t3(zero_cross(d3)))));
```

4 10N Step

```
fun4  = 10 / (m * nat_f^2) * (1 - 1 / sqrt(1 - 0.05^2) * exp(-0.05 * nat_f * t) * sin((nat_f *
sqrt(1 - 0.05^2) * t + atan(sqrt(1 - 0.05^2) / 0.05))));
time4 = 1/(0.05 * nat_f);

[t4, d4] = data_import('part_4.data');

figure
fplot(fun4, [0, 5])
hold on
plot(t4, d4, '--')
yline(10/k,'--')
xline(time4)
hold off
title("Closed form solution of 5% Damped spring-mass-damper with 10N step")
xlabel("Time (s)")
ylabel("Displacement (m)")
legend('Close-form spring-mass-damper', 'AMESim Solution', 'Steady-state value', 'Time-constant')

mf4 = 2 * pi /(2 * mean(diff( t4(zero_cross(d4)))));
```

5 5N sine wave, 50% natrual frequency

```
f5 = nat_f * 0.5;

[t5, d5] = data_import('part_5.data');

fun5 = (-5/225887).*(cos((9995/46).^(1/2).*t)+(-
15).*sin((9995/46).^(1/2).*t))+(5/225887).*399.^(-1/2).*exp(1).^((-
1/2).*(1999/230).^(1/2).*t).*(399.^(1/2).*cos((1/2).*(797601/230).^(1/2).*t)+(-
149).*sin((1/2).*(797601/230).^(1/2).*t));

mf5 = diff(t5(zero_cross(d5)));
mf5 = 2 * pi /(2 * mean(mf5(end-5)));
```

6 5N sine wave, 90% natrual frequency

```
f6 = nat_f * 0.9;

[t6, d6] = data_import('part_6.data');

fun6 = (-25/441779).*(9.*cos(9.*(1999/230).^(1/2).*t)+(-
19).*sin(9.*(1999/230).^(1/2).*t))+(75/441779).*(3/133).^(1/2).*exp(1).^((-
1/2).*(1999/230).^(1/2).*t).*(399.^(1/2).*cos((1/2).*(797601/230).^(1/2).*t)+(-
37).*sin((1/2).*(797601/230).^(1/2).*t));

mf6 = diff(t6(zero_cross(d6)));
mf6 = 2 * pi /(2 * mean(mf6(end-5)));
```

7 5N sine wave, 130% natrual frequency

```
f7 = 1.3 * nat_f;

[t7, d7] = data_import('part_7.data');

fun7 = (-
5/985507).*(13.*cos(13.*(1999/230).^(1/2).*t)+69.*sin(13.*(1999/230).^(1/2).*t))+(65/985507).*399
.^(-1/2).*exp(1).^((-
1/2).*(1999/230).^(1/2).*t).*(399.^(1/2).*cos((1/2).*(797601/230).^(1/2).*t)+139.*sin((1/2).*(797
601/230).^(1/2).*t));

mf7 = diff(t7(zero_cross(d7)));
mf7 = 2 * pi /(2 * mean(mf7(end-5)));

figure
hold on
fplot(fun5, [0, 5])
fplot(fun6, [0, 5])
fplot(fun7, [0, 5])

plot(t5, d5, '--', t6, d6, '--', t7, d7, '--')

legend('50% \omega_n closed-form', '90% \omega_n closed-form', '130% \omega_n closed-form', '50%
AMESim', '90% AMESim', '130% AMESim')
hold off
xlabel('Time (s)')
ylabel('Dispalcement (m)')
title('5N peak sine wave force input at different frequencies with closed-form and AMESim
solutions')
```

Appendix 3 – Mathematica Code

In[107]:=

```
m = 23
k = 19990
z = 5/100
nf = Sqrt[k/m]
fs = nf * (13/10)
```

Out[107]= 23

Out[108]= 19990

Out[109]= $\frac{1}{20}$

Out[110]= $\sqrt{\frac{19990}{23}}$

Out[111]= $13\sqrt{\frac{1999}{230}}$

In[112]:=

```
tf = (1/m) / (s^2 + 2 nf z s + nf^2)
```

Out[112]= $\frac{1}{23\left(\frac{19990}{23}+\sqrt{\frac{1999}{230}}\ s+s^2\right)}$

In[113]:=

```
xs = tf * (5 fs / (s^2 + fs^2))
```

Out[113]= $\frac{13\sqrt{\frac{9995}{46}}}{23\left(\frac{337831}{230}+s^2\right)\left(\frac{19990}{23}+\sqrt{\frac{1999}{230}}\ s+s^2\right)}$

In[114]:=

```
Apart[%]
```

Out[114]= $-\frac{65\sqrt{\frac{230}{1999}}\left(137931+\sqrt{459770}\ s\right)}{985507\left(337831+230\ s^2\right)}+\frac{65\sqrt{\frac{230}{1999}}\left(139930+\sqrt{459770}\ s\right)}{985507\left(199900+\sqrt{459770}\ s+230\ s^2\right)}$

In[115]:=

```
InverseLaplaceTransform[%, s, t]
```

Out[115]=

$$-\frac{5\left(13\,\mathrm{Cos}\left[13\sqrt{\frac{1999}{230}}\;t\right]+69\,\mathrm{Sin}\left[13\sqrt{\frac{1999}{230}}\;t\right]\right)}{985\,507}+\frac{65\,e^{-\frac{1}{2}\sqrt{\frac{1999}{230}}\;t}\left(\sqrt{399}\;\mathrm{Cos}\left[\frac{1}{2}\sqrt{\frac{797\,601}{230}}\;t\right]+139\,\mathrm{Sin}\left[\frac{1}{2}\sqrt{\frac{797\,601}{230}}\;t\right]\right)}{985\,507\sqrt{399}}$$