Magical Modality: The Logic of Possibility and Necessity

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A beautiful world we live in, when it IS possible, and when many other such things are possible, and not only possible, but done—done, see you!—under that sky there, every day. Long live the Devil. Let us go on.

Ernest Defarge, *A Tale of Two Cities*, Charles Dickens

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Chapter 0

Our first meeting all together

0.0 Introduction of myself

My name is Mei Rose. My background in Formal Logic consists mostly of undergraduate courses in Philosophical Logic, but I studied Mathematical Logic at UW and at the Universiteit van Amsterdam, for two and one semester respectively. I worked at a software development group in my undergraduate university and wrote a proof-checker in JavaScript. Modal logic has been a passion of mine since taking a course on the Philosophy of Mathematics and seeing its history and uses since the times of Leibniz and earlier to puzzle about statements about possible worlds. My personal logic and mathematics heroes of history are David Hilbert and Amalie Emmy Noether. In recent history, I have looked up to Saul Aaron Kripke, who died in 2022 and left a wave of controversy in his wake that we will hopefully have time to discuss during this project. And alive today, inspiring me to be a Logician, is J. F. A. K. (Johan) van Benthem, the only person I know of who holds permanent research positions at three different universities at the same time.

0.1 Introductions of students

I am interested to know how your journey through mathematics and logic has been going so far. All I know about you is that the committee recommended you into the project, what courses you are enrolled in, and which courses you have taken that are relevant to the reading group.

- Please tell me about your favorite mathematical experiences. What made them special or unique?
- What sorts of things do you like to do outside of your coursework? Does mathematics or logic ever factor into those activities?
- Who inspires you in history, or the present day?

0.2 Overview and goals

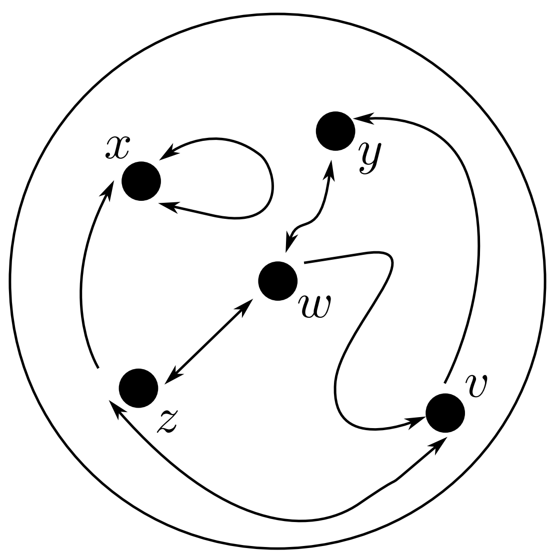
As stated in the project description, I aim for us to discuss many aspects of the so-called modal logic by the end of the semester. After establishing some groundwork in Kripke frames and models and a brief discussion of modal bisimulation, I plan to begin tackling the applications of modal logic. This is where your interests come in. I have the most experience working with modal logic applications in philosophy and linguistics, but there exist many more, such as information/preference theory, game theory, and more. Modal logic also touches more classical logic topics such as provability and model theory, and we can look into those if you are interested. (The "magical" [according to van Benthem] modal proof of Second Incompleteness is what gave this project its name.)

I would like to conclude the project with two connections between modal logic and first-order logic. These two connections are called van Benthem's Theorem and the Salqvist-van Benthem Algorithm. Each gives a slightly different viewpoint on how powerful modal logic is compared to first-order logic.

0.3 Propositional and First-order logic reminders

I will be assuming throughout this project that you are familiar with the following logical symbols from propositional logic: $\sim$ (logical negation, sometimes written as $\neg$), $\wedge$ (logical conjunction), $\vee$ (logical disjunction), $\rightarrow$ (material conditional), and $\leftrightarrow$ (biconditional).This means that you know their truth tables, and how to "read" them in a formula (e.g., $P \wedge Q$ gets read as "P and Q".)

As for the first-order logic I expect you to be familiar with, $\forall$ and $\exists$ are the starting point. Being able to read these symbols as "for all" and "there exists/there is at least one" is important, as is being able to write and interpret statements with relation or function symbols like $R(x, y)$ or $F(x, y, z)$. When I say that R is a two-place relation between a set and itself (as most of our relations will be in this project), I want you to be able to picture a diagram like the one below. The arrows show the relations. For example, we can say that $R(z, x)$ or that $(z, x) \in R$.



We all have preferences for what symbols and variables we use as Logicians. For example, I prefer to call propositions P, Q, etc. and use the $\sim$ symbol for the logical negation. I will try to accommodate your preferences, but know that I will fall back on mine if I forget. Van Benthem notably believes, or at least states, that you should be able to change notation for these sorts of things as one changes clothes.

Additionally, when we begin discussing things where the object language and the m etalanguage become important distinctions, my preferred metalanguage symbols for the above-referenced propositional logic connectives are: NOT for the logical negation, AND for the logical conjunction, OR for the logical disjunction, ONLY IF for the material conditional, and IFF for the biconditional. These last two are unlikely to be seen much, but I thought that I should establish them. For the metalanguage universal quantifier (equivalent to $\forall$) I like to use symbol $\bigwedge$, and for the metalanguage existential quantifier (equivalent to $\exists$) I like to use $\bigvee$. These should be distinguished from the object language $\wedge$ and $\vee$ that we will be using for the conjunction and disjunction, respectively.

0.4 A few new things

I mentioned above that there is a distinction between what is called the object language and the metalanguage. The object language is the (logical, though can be natural) language we are currently using. The majority of the time this will be the language of modal logic and its associated symbols. The metalanguage is the (same caveat) language we use to talk about the object language. A natural language example is the following, taken from A. Tarski: “Schnee ist weiss” if and only if snow is white. The object language, set off here in quotation marks, is German, and the metalanguage, left outside the quotation marks, is English. The philosophy people will notice that this is similar, but not quite the same, as the *de dicto*/*de re* distinction.

The distinction will come into play when we want to say things about modal logic, using the language of first-order logic. For example, when we define what it means for a formula of the form $\varphi \wedge \psi$ to be satisfied in a possible world α, we will make a statement like: “”$\alpha \Vdash \varphi \wedge \psi$ IFF $\alpha \Vdash \varphi$ AND $\alpha \Vdash \psi$.” The logical conjunction in the part of this sentence before (and not including) IFF is in the object language of modal logic. But the logical conjunction in the part of the sentence after (and including) IFF is in the metalanguage of first-order logic. Essentially, this allows us to talk about logic using logic, and keep clear when we are doing which. We need to be really careful about this distinction. Please prompt me if I am not being clear about when we are using the object language and when we are using the metalanguage.

0.5 Before the next time we meet

Please obtain the book Modal Logic for Open Minds by van Benthem. If you are able to get it before the next meeting, please read the first chapter, “A whirlwind history, and changes in perspective” and the second chapter, “Basic language and semantics”. Section 2.4 will be optional, but do feel free to read it and let me know if you want the project to look more closely at “game semantics” like this. I recommend an attempt of Exercises 1(a), 1(b), 2(a), 3(a), and 3(b) to cement your understanding of the reading. We will discuss these during the next meeting. If you choose to read Section 2.4, Exercises 2(b) and 2(c) may help your understanding.

Chapter 1

Sing a song of Kripke: Kripke frames and models

Two of the most fundamental objects of study in Modal Logic are Kripke frames and Kripke models, both named for Saul Aaron Kripke, (1940–2022). My lecturer in Amsterdam, who taught me the mathematical foundations of the field of Modal Logic, always said that there were two kinds of Modal Logicians: those who prefer to think in terms of Kripke frames, and those who prefer to think in terms of Kripke models. I suppose that I am more of a Kripke model person myself, despite not really being a model theorist myself. You will see that the two objects roughly correspond to a syntactic (proof theoretic) and semantic (model theoretic) position, respectively. I will leave that matter of taste to you.

1.0 The bare definitions

A *Kripke frame* (also known as a relational frame, among other names) is an ordered pair $\mathfrak{F} = (W, R)$ that consists of a set of possible worlds W and an accessibility relation between them denoted R. If we are discussing multiple different frames ($\mathfrak{F}$ and $\mathfrak{G}$ perhaps), we might choose to distinguish their sets of possible worlds as $W^{\mathfrak{F}}$ and $W^{\mathfrak{G}}$ respectively. And similarly, for their accessibility relations, we may denote the two relations as $R^{\mathfrak{F}}$ and $R^{\mathfrak{G}}$.

A *Kripke model* (also known as a relational model, possible–worlds model, among other names) is an ordered trirple $\mathfrak{M} = (W, R, V)$ that consists of a set of possible worlds W, an accessibility relation R, and a valuation function V. The valuation assigns propositions such as P truth values (either true or false) in worlds such as α. Another way of thinking about this is that V tells which possible worlds have which propositions true in them, therefore assigning propositions to worlds or sets of worlds. Another way to think about Kripke models is that they are Kripke frames that come with a truth function V. In fact, the defintion that I was taught was precisely that: a Kripke model is an ordered pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a Kripke frame and V is a valuation function.

Some quick notation notes

The accessibility relation R tells us how we can "get from" one possible world in the Kripke frame to another. If $(\alpha, \beta) \in R$, then we can say that "α sees β", or "we can reach β from α". The accessibility relation for a Kripke frame is usually written $R(\alpha, \beta)$ or as simply $R\alpha\beta$.

When a proposition P is true in a certain possible world α in a Kripke model, we say that "P is true at α", "α satisfies P", or, in certain texts, "α forces P". These all mean the same thing. They are written with the so–called *modal turnstile* symbol $\Vdash$. The symbol is an infix, so its correct usage is something like $\alpha \Vdash P$ in the case that α satisfies P. Some people like to use the *semantic turnstile* symbol $\models$ for this, reflecting the fact that we are asserting something about the semantics of a proposition. I think that the distinction between satisfaction in a Kripke model and satisfaction in general is important, so I will stay away from overloading the $\models$ symbol. Similarly to the "factions" that have arisen over whether the more primitive object is the Kripke frame or model, there are "factions" that "argue" about whether to use $\Vdash$ or $\models$ for modal satisfaction. Ultimately, this is a matter of taste.

1.1 A fast overview of the semantics of the symbols

You've already seen this in Chapter 2 of *Modal Logic for Open Minds* as Definition 2.2.2, but I will give it here anyway so that you don't have to go hunting for it the next time you need it. These are the truth–definitions of the operators as they behave in Kripke models. So here we go! For a Kripke model $\mathfrak{M}$ and a possible world α of that Kripke model, we say that . . .

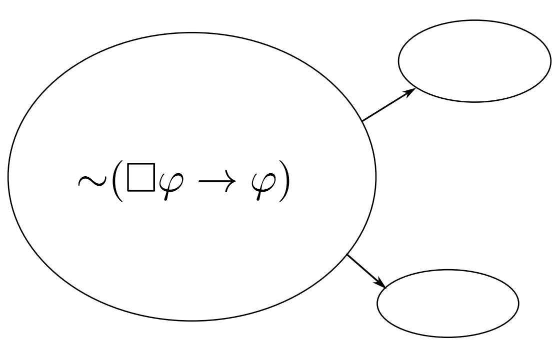
- $\alpha \Vdash P$ IFF the valuation operator V assigns True to P.
- $\alpha \Vdash \sim\varphi$ IFF NOT $(\alpha \Vdash \varphi)$.
- $\alpha \Vdash \varphi \wedge \psi$ IFF $\alpha \Vdash \varphi$ AND $\alpha \Vdash \psi$.
- $\alpha \Vdash \Diamond\varphi$ IFF $\bigvee \beta$ such that $R\alpha\beta$, $\beta \Vdash \varphi$.
- $\alpha \Vdash \Box\varphi$ IFF $\bigwedge \beta$ such that $R\alpha\beta$, $\beta \Vdash \varphi$.

The rest of the logical connectives' semantics can be deduced from these, as $\sim$ and $\wedge$ are a functionally complete set. I will leave it as an exercise for you to complete if you desire.

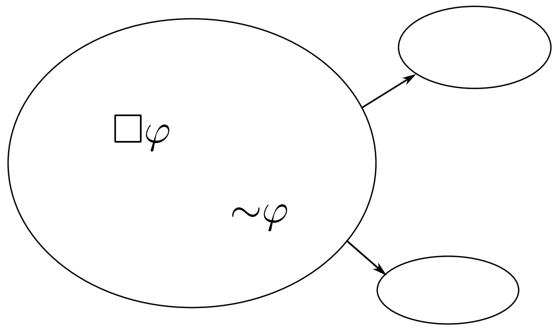
1.2 Modal axioms, what are they good for?

Modal axioms are formulae in the language of modal logic that are true for Kripke frames that have a particular structure to their accessibility relation. For example, in every Kripke frame that has a reflexive accessibility relation, it is the case that the Kripke model has the statement $\Box\varphi \to \varphi$ true. And the converse of this statement is true as well: Every Kripke frame for which the statement $\Box\varphi \to \varphi$ holds at every possible world has a reflexive accessibility relation. We can prove this using a picture.

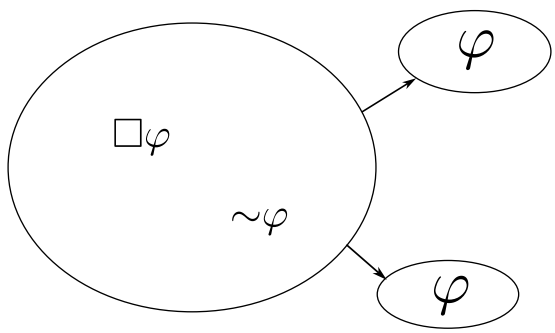
We will proceed by contradiction. Assume for the sake of contradiction that the axiom $\Box\varphi \to \varphi$ is false in some possible world α in our Kripke frame $\mathfrak{F}$. We will show this as the following (the large bubble is α):



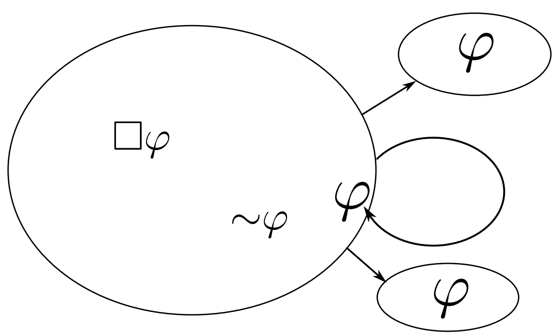
We will use the principle of classical logic that $\sim(\varphi \to \psi) \leftrightarrow \varphi \wedge \sim\psi$ to expand the assumption.



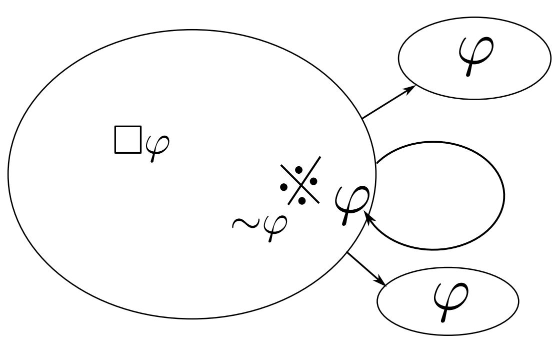
Separating the two conjuncts and applying the semantics of the box symbol means that every possible world that α can see satisfies φ.



But if our accessibility relation is reflexive, then every possible world can see itself. Then φ must be true in α for this reason.



A contradiction arises from the fact that we already asserted that $\sim\varphi$ is true in this world when we separated the two conjuncts. Thus we cannot have a Kripke frame that contains this statement and is not reflexive.



I will fully accept this kind of "picture proof" from you as a demonstration of these types of theorems, most notably because proving these things formally (in a natural deduction system perhaps), is not the goal of this project. I have written formal proofs of fairly simple correspondences of this type that involved 5 layers of nested subderivations, which I do not believe would give you more enlightenment into the axioms than the "picture proofs".

You may have noticed that we used the reasoning principle of *modus ponens* in this proof. Without this essential piece of reasoning, logic has a hard time getting started. So we build this into every Kripke frame through the following axiom: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi$. This axiom is so foundational, it was given the initial **K** in honor of Kripke. Nearly all modal axioms have initials. The one that we just discussed and proved a theorem about is called **T**. We will continue to explore modal axioms and their relation to Kripke frames and models as a theme in this project.

1.3 Looking into next week

For next week, I would like you to hear van Benthem's take on the "landscape of modal logics" by reading Chapter 8 in the book. Also, if you have gotten down to here, you should have read through my notes here thoroughly. I invite you to attempt his Exericse 2 in that chapter. And just for fun, becuase I think that you should do this to get acquainted with the way that modal axioms play with the accessibility relations on their Kripke frames, you should go to the Wikipedia page for the Kripke semantics and try to prove at least two of the correspondences shown in the table *Common modal axiom schemata*. Don't worry about reading the entire page and understanding it. The correspondences are what we will be building upon in the course of the project.