① What is a Kripke Frame?

A Kripke frame $\mathfrak{F}$ is an ordered pair $(W, \mathcal{R})$ where W is a non-empty set of possible worlds and $\mathcal{R}$ is a relation between worlds called the accessibility relation.

- Worlds contain propositions ($\varphi, \psi \ldots$) that can be joined by prop. connectives or modal connectives
- We write $\mathcal{R}(\alpha, \beta)$ when world α has access to world β

② What is a Kripke Model?

A Kripke model $\mathfrak{M}$ is an ordered pair $(\mathfrak{F}, \Vdash)$ where $\mathfrak{F}$ is a Kripke frame and $\Vdash$ is a valuation operation on propositions.

- We write $\alpha \Vdash \varphi$ when the proposition φ is true in the world α
  - Can also be read as "α satisfies φ" "α forces φ"
- $\Vdash$ essentially assigns a truth value to every proposition in the Kripke frame

③ Definitions of the Extensional Connectives

An extensional connective is a unary or binary relation between propositions with the property that its truth value is derived from the truth of the propositions alone.

	SYNTAX	SEMANTICS
$\sim$	$\alpha \Vdash \sim\varphi$	NOT$(\alpha \Vdash \varphi)$
$\wedge$	$\alpha \Vdash \varphi \wedge \psi$	$\alpha \Vdash \varphi$ AND $\alpha \Vdash \psi$
$\vee$	$\alpha \Vdash \varphi \vee \psi$	$\alpha \Vdash \varphi$ OR $\alpha \Vdash \psi$
$\rightarrow$	$\alpha \Vdash \varphi \rightarrow \psi$	(NOT$(\alpha \Vdash \varphi)$) OR $(\alpha \Vdash \psi)$

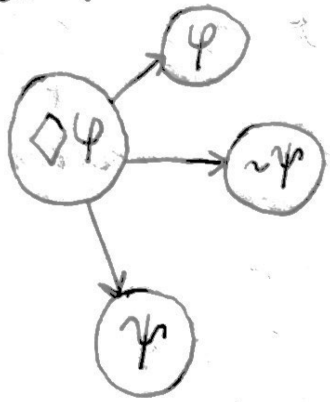
④ Definitions of the Modal Connectives

The modal connectives $\Box$ and $\Diamond$ are intensional connectives: their truth values depend on more than just the truth of the propositions involved.

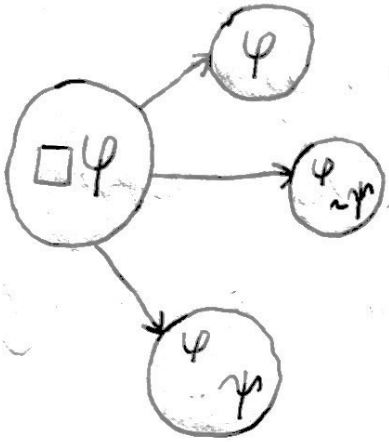
	SYNTAX	SEMANTICS
$\Diamond$	$\alpha \Vdash \Diamond\varphi$	$\exists \beta \in W (\mathcal{R}(\alpha,\beta)$ AND $\beta \Vdash \varphi)$
$\Box$	$\alpha \Vdash \Box\varphi$	$\forall \beta \in W (\mathcal{R}(\alpha,\beta) \Rightarrow \beta \Vdash \varphi)$

⑤ Pictures of Modal Connectives

COMPARE:



At least one accessible possible world has φ true



Every accessible possible world has φ true

⑥ Interpretations of $\Box$ and $\Diamond$

$\Diamond\varphi$	$\Box\varphi$
φ is possible	φ is neccesary
φ is permitted	φ is obligatory
φ is known by an agent A	φ is consistent with what an agent A knows
φ is consistent in a system $\mathcal{S}$	φ is provable in a system $\mathcal{S}$

$\longleftrightarrow$ swap

⑦ Modal Axioms

Modal Axioms are modal statements which we assume to be true in every world in a Kripke model so that $\Box$ and $\Diamond$ behave the ways that we expect

EXAMPLES

$\mathbf{4}: \Box\varphi \to \Box\Box\varphi$ for every φ

$\mathbf{T}: \Box\varphi \to \varphi$ for every φ

$\mathbf{B}: \varphi \to \Box\Diamond\varphi$ for every φ

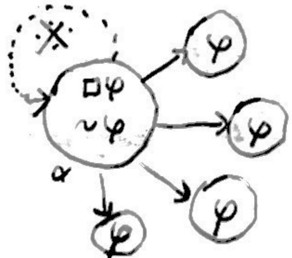
$\mathbf{K}: \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ for every φ, ψ

⑧ Theorem

<u>Theorem</u>: If and only if we impose the $\mathbf{T}$ axiom on our Kripke model, the accessibility relation $\mathcal{R}$ is reflexive in the associated Kripke frame.

<u>Proof</u>: Recall $\mathbf{T}: \Box\varphi \to \varphi$. Then the contrapositive of the theorem is $\sim\mathcal{R}$ refl. $\to \sim\mathbf{T}$.

$\sim\mathbf{T}: \Box\varphi \wedge \sim\varphi$. So let us construct a Kripke model:



But we cannot make φ true in α, because $\sim\varphi$ is true there. $\times$

9 The Kripke Semantics of Modal Logic

This correspondence is not the only such one. Others exist. These correspondences are called the Kripke Semantics of Modal Logic.

$\mathbf{4} \Leftrightarrow R$ is transitive

$\mathbf{B} \Leftrightarrow R$ is symmetric

Unfortunately, there is no axiom which gives us R to be a total relation on the possible worlds.

10 The $\mathbf{L}$ Axiom

Lest you think that all of the correspondences in the Kripke semantics give rise to first-order properties for R, here is an example:

$\Box(\Box\varphi \to \varphi) \to \Box\varphi \Longleftrightarrow$

R is transitive and converse-well-founded (R^{-1} is well-founded)

This axiom is the key to a beautiful proof of the First Incompleteness.

11 Going Beyond this Talk

Here are some topics to explore if you are interested in seeing more modal Logic than could fit in this talk.

- Bisimulations
  - van Benthem's Theorem
  - p-morphisms
- Salqvist-van Benthem Algorithm
- Soundness and Completeness of modal systems