Prerequisite

- functions (basics)

Relations

Relations are similar to functions in that they establish connections between objects. But whereas a function associates only one output with every input, a relation is more flexible and allows connections to arbitrarily many elements.

Example 1 The question *Is x a biological child of y* is a function because it maps any two x and y to either true or false, but never both. But if we slightly change the question to just *Name a biological child of y*, we are no longer dealing with a function because multiple answers are possible if y has more than one child. Instead, we can talk about the *biological child* relation R such that x is related to y via R iff x is a biological child of y.

Basic notation

Given some relation R, we write $x \; R \; y$ to indicate that R relates x to y. Note that $x \; R \; y$ does not imply that $y \; R \; x$ also holds.

Example 2 One relation you know very well is the "less than" relation $<$ over numbers. When we write $5 < 7$, we are saying that the relation $<$ relates 5 to 7. However, it is not the case that 7 is related to 5 via $<$, since $7 < 5$ does not hold.

Example 3 Suppose *John* has exactly two siblings, *Mary* and *Sue*. Then the sibling relation establishes two connections for John: *John-Mary*, and *John-Sue*. Using S for the sibling relation, we have John S Mary and John S Sue.
In contrast to $<$, the sibling relation is symmetric. That is to say, if Mary is a sibling of John, then John is also a sibling of Mary. Therefore we also have Mary S John and Sue S John.
The relation is also transitive. If John is a sibling of Mary, and Sue is a sibling of John, then Sue is a sibling of Mary. So is also holds that Mary S Sue, and via symmetry we also get Sue S Mary.
Overall, we have $x \; S \; y$ where $x, y \in \{\text{John}, \text{Mary}, \text{Sue}\}$ and $x \neq y$.

Example 4 The **substring relation** $\sqsubseteq$ holds between two strings u and v iff u is a substring of v. That is to say, $u \sqsubseteq v$ iff there are $w, w' \in \Sigma^*$ such that $w \cdot u \cdot w' = v$. Note that for any given string u, there are infinitely many v that u is a substring of. Even if the alphabet contains only a, the string aa is a substring of aaa, $aaaa$, $aaaaa$, and so on, ad infinitum. It is also a substring of itself (in this case, $w = w' = \varepsilon$).

Example 5 Relations can be defined over more complex objects like sets. An example of this is the subset relation $\subseteq$.

Example 6 Just like a function can take multiple arguments to return a single output, a relation can connect multiple elements. In the real world, the "jointly conceived" relation J would connect two individuals to their offspring. So the expression

$$\text{John}, \text{Mary} \ R \ \text{Sue}$$

encodes that John and Mary are the biological parents of Sue (let's just hope that those are not the same people as in the first example).

Example 7 Here is an example of a very abstract relation. Consider the space of all possible functions from real numbers to real numbers — that's a lot of functions. Now let's define a boundedness relation B which relates function f to function g iff $f(x) \leq g(x)$ for every natural number x. Suppose, for instance, that $f(x) = x$ and $g(x) = x^2$. Then $f \ B \ g$, but not $g \ B \ f$.
Many functions aren't related via B at all. One example of this is $f(x) = -x + 1$ and $g(x) = x^2$. It is the case that $f(x) \leq g(x)$ for all $x \geq 1$, but $f(0) = 1 > 0 = g(0)$.

We will mostly be dealing with the special case of **binary relations** where exactly one element is related to some other element.

Example 8 Almost all relations above are binary relations. The only exception is the "jointly conceived" relation J, which is a ternary relation as it relates three elements.

Given a binary relation R, $a \ R$ is the set of objects that a is related to. Similarly, $R \ b$ is the set of objects that are related to b.

$$a \ R := \{b \mid a \ R \ b\}$$

$$R \ b := \{a \mid a \ R \ b\}$$

Example 9 Suppose that the parent-of relation P establishes the following relations between elements: John P Sue and Mary P Sue. Then John $P = \{\text{Sue}\}$ and P Sue $= \{\text{John}, \text{Mary}\}$.

Exercise 1 Let R be the relation that connects words to their parts of speech (N for nouns, V for verbs, A for adjectives, P for prepositions, D for determiners, and so on). List the following for English:

- export R
- apple R
- R P

Relations versus functions

Every function can be regarded as a relation.

Example 10 Consider the function $f : \mathbb{N} \to \mathbb{N}$ with $x \mapsto 2x$. It is identical to the relation R with $x \ R \ y$ iff $y = 2x$.

The crucial difference is that functions are **right-unique relations**. That's a fancy way of saying that a function cannot provide more than one output, whereas a relation can unless it is right-unique. When we view a function f as a relation R, then it must hold for every a that $a\ R$ is either empty or contains exactly one element. Hence the term right-unique: if we look at the expression $a\ R\ b$, a is the left side and b the right side. If there cannot be more than one choice for b, then the right side of $a\ R\ b$ is uniquely determined.

The bottom line: every function is a relation, but not every relation is a function. If a is related to two elements or more (i.e. $|a\ R| \geq 2$), then R cannot be a function.

Exercise 2 For each one of the following, say whether it is a function or just a relation.

- the parent-of relation (e.g. $j\ P\ m$ for "John is a parent of Mary")
- the parent-of relation in a world where the one-child policy is enforced globally
- the relation between a car's license plate and its owners
- the prefix relation, where u is a prefix of v iff there is some $w \in \Sigma^*$ such that $v = u \cdot w$

Exercise 3 Is the following statement true or false? Justify your answer.
Every relation R can be regarded as a function that maps x to $x\ R$.