Prerequisites

- strings (string parts)

Finite-state automata

Prefix trees, albeit being limited to tree structures rather than arbitrary graphs, generalize our standard notion of graphs in that they have both vertex labels (which we called *colors*) and edge labels (the actual characters of the strings). We briefly entertained the notion of generalizing prefix trees to prefix DAGs, but that did not turn out to be particularly useful for our intended application, namely a more efficient encoding of word lists. But when we take one more step and generalize prefix trees from trees to arbitrary graphs, we do get a very useful kind of object: *finite-state automata*.

Automata as graphs

A finite-state automaton (FSA) is a finite graph that has both edge labels and vertex labels. The edges are usually called **arcs**, and the vertices are called **states** (by now you're hopefully accustomed to one and the same thing having many different names). We will freely switch between these terms depending on how much we want to emphasize the graph-theoretic nature of FSAs.

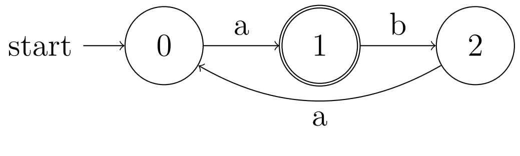
As was just said, FSAs have both edge labels and vertex labels. The edge labels are drawn from some arbitrary alphabet. The vertex labels are used to distinguish between four types of vertices, two of which are already familiar from prefix trees:

1. normal vertices,
2. **final** vertices,
3. **initial** vertices,
4. vertices that are both initial and final.

We already had normal and final vertices for prefix trees (they were color-coded as red and blue, respectively). Initial vertices are a new type. For prefix trees, it was obvious that we always wanted to start at the source, i.e. the root of the tree. An arbitrary graph may have multiple sources, however, or none at all, so instead the possible starting points have to be indicated explicitly by marking them as initial.

Any graph that satisfies the requirements above is a finite-state automaton. As for prefix trees, we can look at the strings that are associated with paths from an initial to a final vertex and thus compute a (possibly infinite) set of strings.

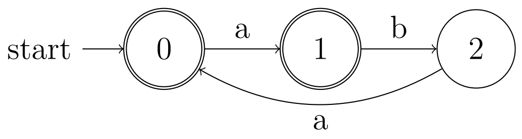
Example. Consider the FSA below, where initial states are marked by an edge labeled *start* and final nodes are doubly circled.



The shortest path from an initial to a final state goes from 0 to 1, or simply $\langle 0, 1\rangle$. This path contains only an a along the way. So the string associated with this path is a.

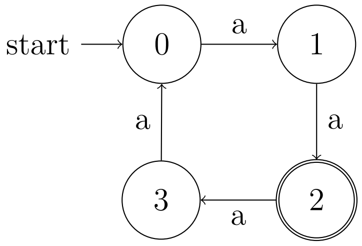
The next longer path is $\langle 0, 1, 2, 0, 1\rangle$, yielding *abaa*. After that, one can follow the path $\langle 0, 1, 2, 0, 1, 2, 0, 1\rangle$ and obtain the string *abaabaa*. In sum, all the associated strings start with an a, followed by 0 or more instances of *baa*.

Example. In the minor variant below, the initial state is also a final state.



As a result, the empty path is a valid path from an initial state to a final state. The empty path is associated with the empty string ε. In addition, for every valid path ending in 1 there is now a valid truncated version missing the final step from 0 to 1. This also allows for the following strings: *a*, *aba*, *ababa*, and so on.

Example. The automaton below produces strings over $\{a\}$ of length l such that $l \mod 4 = 2$.

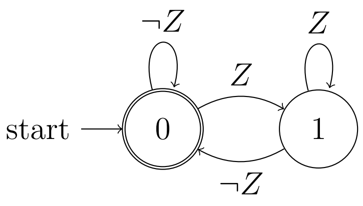


FSAs are incredibly useful for modeling natural language. For example, the n-gram grammars we have seen are all special cases of FSAs.

Example. Consider the SL grammar that bans word-final voicing for German:

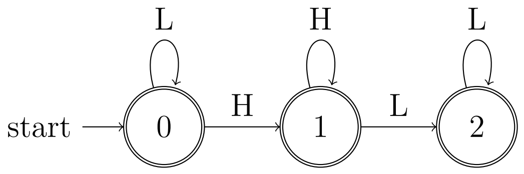
$$\{\mathrm{b}\ltimes, \mathrm{d}\ltimes, \mathrm{v}\ltimes, \mathrm{z}\ltimes\}$$

We can convert this to an FSA that will move us into a non-final vertex whenever a voiced consonant is encountered. Since we only consider paths that end in a final vertex, it is impossible for a word to end in a voiced consonant. For the sake of succinctness, we denote all voiced consonants by Z and all other sounds by $\neg Z$.



Example. An SP grammar is similar to an SL grammar except that each n-gram is a forbidden subsequence, rather than a forbidden substring. For example, the phenomenon of unbounded tone plateauing forbids low tones (L) from occurring between high tones (H) no matter how far apart the two high tones are. So LHLLLLL and LLLLLHL are well-formed, but not LHLLLHL. An SP-grammar can captured this by forbidding the subsequence HLH.

Equivalently, one can construct an FSA where seeing an L after an H moves us into a special part of the graph where all edges are labeled L. This way it becomes impossible to continue a string like LHLLL with an H.



The last example illustrates how vertices in an FSA serve as a limited kind of memory. The fact that we are in a specific vertex implicitly encodes that a certain symbol was encountered along the path to this vertex, and by carefully placing edges from this vertex we can regulate how the computation proceeds from here. This connection between vertices and "memory states" is why the term is finite **state** automata.

Definition and terminology

The canonical definition of FSAs looks very different from the graph-theoretic one. This is because FSAs were invented independently, and none of the important theorems about them build on the insights of graph theory. I will first define FSAs in graph-theoretic terms, and then contrast those definitions with the canonical one from formal language theory.

Definition 1. A **finite-state automaton** (FSA) is a vertex- and edge-labeled graph $A := \langle V, E, c, \ell \rangle$ such that

1. V is finite, and
2. ℓ maps edges to members of $\wp(\Sigma)$ for some fixed set Σ, and
3. c maps vertices to members of $\wp(\{I, F\})$.

A vertex v is called **initial** iff $I \in c(v)$ and **final** iff $F \in c(v)$. We also call vertices **states**.

The definition above is mostly straight-forward, except that the edge labeling function has $\wp(\Sigma)$ as its co-domain instead of Σ. This is done to accommodate cases like the devoicing automaton above, where several symbols can take us from state u to another state v. Since our definition of graphs only allows for at most one edge between two states, cases with several symbols must be handled by tying all those symbols to that edge. The easiest way of doing so is to label edges with subsets of Σ.

Every path through the graph is also associated with a set of strings. Intuitively, these are all the strings that can be built by following along the path.

Definition 2. With every path $p = \langle v_1, v_2, v_3 \dots, v_{n-1}, v_n \rangle$ we associate a string set $L(p) := \ell(v_1, v_2) \times \ell(v_2, v_3) \cdots \times \ell(v_{n-1}, v_n)$. If $p = \langle \rangle$, $L(p) = \emptyset$. If $p = \langle v_1 \rangle$, then $L(p) = \{\varepsilon\}$ if v_1 is final, and $\emptyset$ otherwise. Let P be the set of all paths from an initial state to a final state. Then the language **recognized** by A is $\bigcup_{p \in P} L(p)$. For every stringset $L \subseteq \Sigma^*$, L is **regular** iff L is recognized by some FSA.

The canonical definition of FSAs can avoid the complication of set-labeled edges by directly treating edges as triples of the form *start*, *label*, *end*.

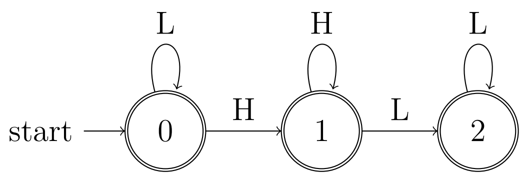
Definition 3. A **finite-state automaton** (FSA) is a 5-tuple $A := \langle \Sigma, Q, I, F, \Delta \rangle$ such that

1. the alphabet Σ is a finite, non-empty set,
2. Q is a finite set of **states**,
3. $I \subseteq Q$ is the set of **initial** states,
4. $F \subseteq Q$ is the set of **final** states,
5. $\Delta \subseteq Q \times \Sigma \times Q$ is the **transition relation**.

Given a string $s := \sigma_1 \cdots \sigma_n \in \Sigma^n$ ($n \geq 0$), a **run** of A over s is a tuple $r := \langle q_0, q_1, \dots, q_n \rangle$ such that $q_0 \in I$ and for all $0 < i \leq n$, $\langle q_{i-1}, \sigma_i, q_i \rangle \in \Delta$. A run is **accepting** iff its last component is a final state. A string s is **recognized** or **generated** by A iff there is some accepting run of A over s. The string language $L(A)$ recognized/generated by A is the smallest set containing all strings recognized by A.

A run is just a record of which states an automaton passes through when processing a string. The run is accepting if it starts in an initial state and ends in a final state. Note that one string may allow for multiple runs. A string is recognized by the automaton iff there is at least one accepting run.

Example. Consider once more the automaton for unbounded tone plateauing.

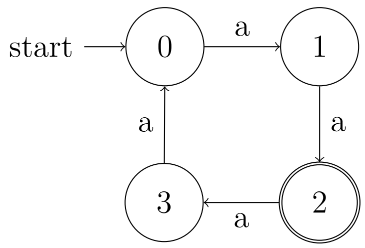


In this automaton, the string LLLHH has only one run, which is 000011. Note how the run is one symbol longer than the string. That's because we start in 0, and then the first symbol (i.e. L) moves us from 0 to 0. In more detail:

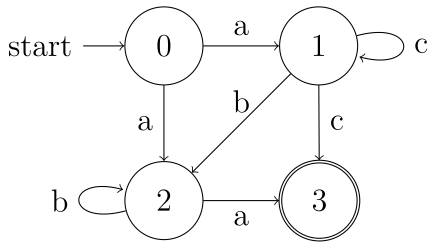
1. start: 0
2. L: 0

3. L: 0
4. L: 0
5. H: 1
6. H: 1
7. done

Example. In the automaton below, the string *aaaaaaa* receives the run 01230123.



Example. Now consider the automaton below.

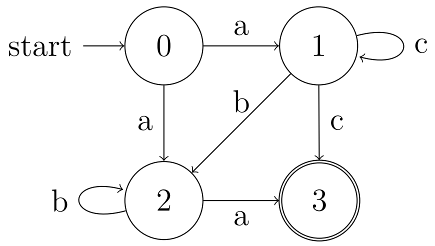


The string *aba* has two distinct runs. One is 0123, the other one is 0223.

[Exercise 1] Draw an FSA that recognizes the language a^*b^+, where a^* denotes "0 or more *a*s" and b^+ is short for "1 or more *b*s".

[Exercise 2] Draw an FSA that recognizes the language $a^+b^+a^*$.

[Exercise 3] Consider once more the following automaton:



For each one of the following strings, list all accepting runs with respect to this automaton. If there is no such run, say so.

1. *aa*

2. $acbba$
3. $abba$
4. $abab$