Prerequisites

- sets (notation)

Union, intersection, and complement of sets

Sets can be combined with each other in various ways. Sometimes this is used as yet another way to define sets, and sometimes it is an actual construction process where multiple sets serve as the input to some mechanism that returns a single set as the output. You'll see many examples of both usages throughout the course.

Union

Given two sets A and B, their **union** $A \cup B$ is the set that contains

- all elements of A, and
- all elements of B, and
- nothing else.

This means that the union of two sets is the result of taking everything that belongs to at least one set. So union builds bigger sets from smaller sets. This makes it something like the set counterpart of addition over numbers.

Example 1 The union of $\{1\}$ and $\{2, 3, 4\}$ is $\{1, 2, 3, 4\}$. The union of $\{1, 2\}$ and $\{3, 4\}$ is also $\{1, 2, 3, 4\}$.

Exercise 1 Compute the union of the following:

- $\{0, 1\} \cup \{2, 3\}$
- $\{0, 1\} \cup \{1, 2, 3\}$
- $\{0, 1\} \cup \{0, 1\}$
- $\{0, 1\} \cup \emptyset$
- $\{1, 2, 3\} \cup \{0, 1\}$

Union is **associative**, which means that $(A \cup B) \cup C = A \cup (B \cup C)$. This is just like $(5+4)+3 = 5+(4+3)$. It is also **commutative**. That is to say, the order of arguments does not matter: $A \cup B = B \cup A$. This again mirrors addition, where $5 + 3 = 3 + 5$. Also note that $A \cup \emptyset = \emptyset$ for any set A, just like $n + 0 = n$ for any number n.

Exercise 2 Compute the union of the following in a step-wise fashion:

- $\{0, 1\} \cup \{2, 3\} \cup \emptyset$
- $\{0, 1\} \cup \emptyset \cup \{2, 3\}$

Intersection

Intersection is the opposite of union in that it builds smaller sets rather than bigger ones. Given two sets A and B, their **intersection** $A \cap B$ is the set that contains only those elements that belong to A as well as B.

Example 2 The intersection of $\{1\}$ and $\{2,3,4\}$ is $\emptyset$, and so is the intersection of $\{1,2\}$ and $\{3,4\}$. But the intersection of $\{1,2\}$ and $\{2,3,4\}$ is $\{2\}$.

Exercise 3 Compute the intersection of the following:

- $\{0,1\} \cap \{2,3\}$
- $\{0,1\} \cap \{1,2,3\}$
- $\{0,1\} \cap \{0,1\}$
- $\{0,1\} \cap \emptyset$
- $\{1,2,3\} \cap \{0,1\}$

Note that $A \cap \emptyset = \emptyset$ no matter what the set A looks like. This is similar to how $n \cdot 0 = 0$ irrespective of the value of n. So intersection is akin to multiplication for sets. Like multiplication, intersection is associative, so that $(A \cap B) \cap C = A \cap (B \cap C)$. This mirrors the fact that $(5 \cdot 4) \cdot 2 = 5 \cdot (4 \cdot 2)$. Intersection is also commutative, again just like multiplication: $A \cap B = B \cap A$, and $m \cdot n = n \cdot m$.

The one difference between intersection and multiplication seems to be that the former produces something smaller and the latter something bigger. But as we will learn much later in the semester, this isn't really all that important, and the two operations are indeed very close counterparts in an abstract sense.

Relative complement

Given the set-counterparts for $+$ and $\cdot$, you probably expect one for subtraction, too. It exists, indeed, and is called **relative complement**. Given two sets A and B, their relative complement is written $A - B$ (sometimes $A \setminus B$). It contains all members of A that are not members of B.

Example 3 The complement of $\{2\}$ relative to $\{1,2,3\}$ is $\{1,2,3\} - \{2\} = \{1,3\}$. The complement of $\{3,4,5\}$ relative to $\{2,3\}$ is $\{2,3\} - \{3,4,5\} = \{2\}$.

Relative complement is **not** associative in the general case. For example, $(\{0,1\} - \{0\}) - \{1\} = \emptyset$, whereas $\{0,1\} - (\{0\} - \{1\}) = \{1\}$. Since associativity requires that the order of evaluation may never matter, this one example where it does matter is sufficient to show that associativity does not hold. That doesn't mean that there are never cases where one can't change the order of evaluation at all. For instance, $(\{0,1\} - \{0\}) - \{2\} = \{1\} = \{0,1\} - (\{0\} - \{2\})$ — but that is merely a coincidence. That relative complement is not associative mirrors subtraction for numbers, where $(5-2)-3 = 0 \neq 6 = 5-(2-3)$. Commutativity does not hold for relative complement either, as is shown by $\{5\} - \{5,4\} = \emptyset \neq \{4\} = \{5,4\} - \{4\}$.

Exercise 4 Give a concrete example where $A - B = B - A$. Then make a single change to A such that $A - B \neq B - A$.

Summary

Definition 1. Let A and B be arbitrary sets.

- The **union** of A and B is $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$.
- The **intersection** of A and B is $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$.

- The **relative complement** of A and B is $A - B := \{x \mid x \in A \text{ and } x \notin B\}$.
- If A is clear from context, we just write $\overline{B}$ for $A - B$ and call it the **complement** of B.

- Union and intersection are associative (order of evaluation doesn't matter) and commutative (order of arguments doesn't matter).
- Relative complement is neither associative nor commutative.