Prerequisites

- sets (notation, operations)
- functions (basic notation)

Cardinality

Sets can be compared based on how many elements they contain.

Example 1 The set $A := \{a,b,c\}$ contains exactly as many elements as the set $X := \{x,y,z\}$, namely 3. Each set has fewer members than $A \cup X$, which contains 6 elements, whereas $A \cap X$ has 0 members.

This is commonly called the **size** of a set, but the more accurate term is **cardinality**. The cardinality of a set A is denoted $|A|$. We say that $|A| \leq B$ iff there is a function f such that every element of A is mapped to some element of B and every element of B has at most one element of A mapped to it.

Example 2 Suppose that $A := \{a,b,c\}$ and $D := \{d,e,f,g\}$. Then $|A| \leq |D|$, as every element of A can be mapped to some distinct element of D. For instance, we could have a function with $a \mapsto f$, $b \mapsto d$, $c \mapsto g$.
In the other direction, $|D| \not\leq |A|$. No matter how one maps the elements of D to members of A, at least two members of D will have to be mapped to the same element in A.

Clearly $|A| = |B|$ iff $|A| \leq |B|$ and $|B| \leq |A|$ are both true. But there is a more direct definition: $|A| = |B|$ iff there is a function f such that f maps every element of A to some element of B and every element of B has exactly one element of A mapped to it. We also say that there is a **bijection** between A and B, which is a technical term for a 1-to-1 correspondence between the elements of A and B.

Example 3 We already saw that $|A| = |X|$ in the previous example. A possible choice of f would be $a \mapsto x$, $b \mapsto y$, $c \mapsto z$.

Exercise 1 Show that $|\{0 \leq n < 10 \mid n \text{ is odd}\} = |\{0 \leq n < 10 \mid n \text{ is even}\}||$.

For finite sets, our intuitive notion of size closely matches the technical term of cardinality. However, size and cardinality diverge once we look at infinite sets.

Example 4 Consider the set $\mathbb{N} := \{0,1,2,\ldots\}$ of all natural numbers and the set $\mathbb{N}_+ := \{1,2,\ldots\}$ of all positive natural numbers. Intuitively, $\mathbb{N}$ is larger than $\mathbb{N}_+$ because it contains all members of $\mathbb{N}_+$ as well as 0, which is not in $\mathbb{N}_+$. But the function $f : \mathbb{N} \to \mathbb{N}_+$ with $n \mapsto n+1$ is a bijection. Hence $|\mathbb{N}| = |\mathbb{N}_+|$ even though intuitvely the two sets have distinct size.

Exercise 2 Show that the set of natural numbers has the same cardinality as the set of all even natural numbers.