Switching between representations

The unit on universals argues that unrelated phenomena can be unified under the banner of monotonicity. The apparent differences between the phenomena are due to differences in the structures that monotonicity is evaluated over:

- the person hierarchy $(1 < 2 < 3)$,
- the adjectival gradation hierarchy (positive < comparative < superlative),
- the (partially ordered) case hierarchy,
- the person-person hierarchy for the PCC,
- the adjunct extension semilattices for the Adjunct Island Constraint.

What we have done, then, is given each empirical domain a specific **representation**. A representation is a structure that expresses crucial properties of the object is represents. Mathematically, we are free to choose our representations any way we want. In fact, there are **representation theorems** that establish how, sometimes, one representation can be transferred into another representation without losing any essential properties.

Example 1
Here is a very simple representation theorem. Every set S be represented as its **characteristic function** f_S such that $f_S(x) = 1$ if $x \in S$ and 0 otherwise. Note how this preserves all crucial aspects of a set. The set is still unordered because the characteristic function only expresses whether the element is in the set, not what position it occupies. Idempotency also holds: $S := \{a\}$ is the same set as $T := \{a, a\}$ because f_S and f_T are the same function. They map a to 1, and everything else to 0.
The functional view of sets is sometimes more convenient to work with. That's particularly the case for semantics, the linguistic study of meaning. Semanticists use a notational format that expresses everything in terms of functions (the **lambda calculus**).

Exercise 1
Suppose that the students in this class are *John*, *Sue*, and *Mary*. Write

- the characteristic function for the set of female students (assuming heteronormative naming conventions), and
- the characteristic function for the set of students whose name contains exactly four letters, and
- the characteristic function for the union of the two sets.

Define each function by explicitly writing down the values, e.g. John $\mapsto$ 1.

Exercise 2
Continuing the previous exercise, can you give a general definition for union that uses the values 0 and 1 assigned by characteristic functions? *Hint*: Check out the **max** function in the backgrounds section.

Exercise 3
Can you define the analog of a characteristic function for multisets?

Example 2
A more complicated representation theorem is the Kuratovsky definition of pairs. Rather than taking tuples as a primitive, Kuratovsky defined

$\langle a, b \rangle$ as a shorthand for $\{\{a\}, \{a, b\}\}$. Intuitively, the two don't seem to have anything in common. But think about the crucial property of pairs: $\langle a, b \rangle = \langle c, d \rangle$ iff $a = c$ and $b = d$. This equivalence also holds under Kuratovsky's definition of pairs.

Suppose that $\langle a, b \rangle = \langle c, d \rangle$. Then by Kuratowsky's definition, $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. If $\{a\} \neq \{c\}$, then $\{a\} = \{c, d\}$, which can only hold if $c = d$. But then $\{a\} = \{c, d\} = \{c, c\} = \{c\} \neq \{a\}$. As this is a contradiction, it must be the case that $\{a\} = \{c\}$. From this it follows that $a = c$, wherefore $\{a, b\} = \{c, d\}$ implies $b = d$, exactly as desired.

Exercise 4 Explain why the equivalence also holds in the other direction: if $a = c$ and $b = d$, then $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ (and thus $\langle a, b \rangle = \langle c, d \rangle$).

Exercise 5 Under Kuratovsky's definition, what set corresponds to the pair $\langle a, a \rangle$?

Exercise 6 In earlier units, we said that the crucial difference between sets and tuples is that the latter are ordered and not idempotent. So $\langle a, b \rangle \neq \langle b, a \rangle$ ("order matters") and $\langle a \rangle \neq \langle a, a \rangle$ ("tuples are not idempotent"). But now we only checked that $\langle a, b \rangle = \langle c, d \rangle$ iff $a = c$ and $b = d$. This biconditional actually entails that pairs are ordered and not idempotent, as desired. Explain why!

Exercise 7 The Kuratowski definition does not hold for arbitrary tuples. That is to say, $\langle a, b, c \rangle$ cannot always be represented as $\{\{a\}, \{a, b\}, \{a, b, c\}\}$. To see this, try to represent the following two triples in terms of these sets:

- $\langle a, a, b \rangle$
- $\langle a, b, b \rangle$

What is the problem?

Alright, so mathematically the sky is the limit when it comes to representations. That is certainly a good thing: using representation theorems, we can switch between representations in whatever way is convenient for the task at hand. The linguists among you might find that irritating. Linguists are used to talking about "*the* structure", and "*the* representation". Their implicit assumption is that there is one specific representation used by the human mind, and that is what linguists should use in their descriptions. But things simply aren't that straight-forward. Even with computers, there is no clear notion of representation. What exactly is the representation of, say, a set on your computer? In the programming language, it is a specific construct X, but that does not at all reflect how X is instantiated by your computer's hardware. And there's many intermediate representations between those two extremes. Computers, a man-made product, do not allow us to identify a unique representation even though we know exatly how they work. The human mind, by contrast, is still a giant black box to us. There is no reason to believe that there is something like "*the* uniquely correct representation" — and even if there is, odds are we will never get it quite right, so we might just as well switch between representations.

That said, linguists are right in insisting on the importance of representations in general. We want to have a good idea of what properties hold of linguistic

representations. For instance, why is the person hierarchy $1 < 2 < 3$? Why couldn't it simply be $1 < 2$, with 3 completely unordered with respect to 1 and 2? With such a hierarchy, we would expect a very different syncretism typology. Ideally, there is some important property that separates these two hierarchies, making one natural and the other unnatural. Both are weak partial orders, which we need for case syncretism and the PCC. So whatever is the relevant property, it can't be a limitation to weak partial orders.

The answer is still largely open - that's how it is with research, it takes a lot of effort to clear up the fog. But there are some simple restrictions that can be put in place. Like monotonicity, they are ubiquitous in language. In order to talk about them from a mathematical perspective, we need to bring in **graph theory**.