Prerequisites

- sets (comparison, powerset)
- multisets (operations)

Multiset relations

The (proper) subset and superset relations are readily extended to multisets.

Definition 1. Given two multisets A_M and B_M, A_M is a subset of B_M ($A_M \subseteq B_M$) iff it holds for all $a \in A_M$ that $A_M(a) \leq B_M(a)$. The subset relation is proper ($A_M \subsetneq B_M$) iff $A_M(a) < B_M(a)$ for all $a \in A_M$. Superset and proper superset are defined in an analogous fashion.

Note that this definition of subset entails that the powerset of a multiset can be much larger than that of a set.

Example 1 Returning once more to $A_M := \{a : 3, b : 2, c : 1\}$ and $B_M := \{a : 1, b : 1, c : 2, d : 1\}$, we can see immediately that neither is a subset of the other. First, $A_M \not\subseteq B_M$ because $A_M(a) = 3 \not\leq 1 = B_M(a)$. In the other direction, $B_M \not\subseteq A_M$ because $B_M(d) = 1 \not\leq 0 = A_M(d)$.
The powerset of B_M contains every multiset that is a subset of B_M. These are

- $\{a : 1, b : 1, c : 2, d : 1\}$
- $\{a : 1, b : 1, c : 2, d : 0\}$
- $\{a : 1, b : 1, c : 1, d : 1\}$
- $\{a : 1, b : 1, c : 1, d : 0\}$
- $\{a : 1, b : 1, c : 0, d : 1\}$
- $\{a : 1, b : 1, c : 0, d : 0\}$
- $\{a : 1, b : 0, c : 2, d : 1\}$
- $\{a : 1, b : 0, c : 2, d : 0\}$
- $\{a : 1, b : 0, c : 1, d : 1\}$
- $\{a : 1, b : 0, c : 1, d : 0\}$
- $\{a : 1, b : 0, c : 0, d : 1\}$
- $\{a : 1, b : 0, c : 0, d : 0\}$
- $\{a : 0, b : 1, c : 2, d : 1\}$
- $\{a : 0, b : 1, c : 2, d : 0\}$
- $\{a : 0, b : 1, c : 1, d : 1\}$
- $\{a : 0, b : 1, c : 1, d : 0\}$
- $\{a : 0, b : 1, c : 0, d : 1\}$
- $\{a : 0, b : 1, c : 0, d : 0\}$
- $\{a : 0, b : 0, c : 2, d : 1\}$
- $\{a : 0, b : 0, c : 2, d : 0\}$
- $\{a : 0, b : 0, c : 1, d : 1\}$
- $\{a : 0, b : 0, c : 1, d : 0\}$
- $\{a : 0, b : 0, c : 0, d : 1\}$
- $\{a : 0, b : 0, c : 0, d : 0\}$

The powerset of $B := \{a, b, c, d\}$, on the other hand, has only $2^{|B|} = 2^4 = 16$ members.

The contrast is even more pronounced when we consider a multiset like $\{a : 9\}$. While $\wp(\{a\})$ only consists of $\emptyset$ and $\{a\}$, $\wp(\{a : 9\})$ has 10 members.

The cardinality of the powerset of a multiset is computed by adding 1 to each count and then multiplying all counts. Again we can express this more precisely and succinctly with a formula:

$$|\wp(S_M)| := \prod_{s \in S_M} (S_M(s) + 1)$$

The operator $\prod$ behaves exactly like $\sum$, except that it is a shorthand for multiplication rather than addition. So the formula above expands to $(S_M(s_1)+1) \times (S_M(s_2)+1) \times \cdots \times (S_M(s_n)+1)$.

Example 2 We have already seen that $B_M := \{a : 1, b : 1, c : 2, d : 1\}$ has 24 subsets, so its powerset has cardinality 24. Our formula yields exactly the same value.

$$\begin{aligned}
|\wp(B_M)| &= \prod_{s \in B_M} (B_M(s) + 1) \\
&= (B_M(a) + 1) \times (B_M(b) + 1) \times (B_M(c) + 1) \times (B_M(d) + 1) \\
&= (1 + 1) \times (1 + 1) \times (2 + 1) \times (1 + 1) \\
&= 2 \times 2 \times 3 \times 2 \\
&= 24
\end{aligned}$$