**Prerequisites**

- sets (notation, operations)
- functions (basic notation, domain terminology)

# Cardinality

Sets can be compared based on how many elements they contain.

**Example 1**    The set $A := \{a, b, c\}$ contains exactly as many elements as the set $X := \{x, y, z\}$, namely 3. Each set has fewer members than $A \cup X$, which contains 6 elements, whereas $A \cap X$ has 0 members.

This is commonly called the **size** of a set, but the more accurate term is **cardinality**. The cardinality of a set $A$ is denoted $|A|$. We say that $|A| \leq B$ iff there is a function $f$ such that every element of $A$ is mapped to some element of $B$ and every element of $B$ has at most one element of $A$ mapped to it.

**Example 2**    Suppose that $A := \{a, b, c\}$ and $D := \{d, e, f, g\}$. Then $|A| \leq |D|$, as every element of $A$ can be mapped to some distinct element of $D$. For instance, we could have a function with $a \mapsto f$, $b \mapsto d$, $c \mapsto g$.
In the other direction, $|D| \not\leq |A|$. No matter how one maps the elements of $D$ to members of $A$, at least two members of $D$ will have to be mapped to the same element in $A$.

Clearly $|A| = |B|$ iff $|A| \leq |B|$ and $|B| \leq |A|$ are both true. But there is a more direct definition: $|A| = |B|$ iff there is a function $f$ such that $f$ maps every element of $A$ to some element of $B$ and every element of $B$ has exactly one element of $A$ mapped to it. We also say that there is a **bijection** between $A$ and $B$, which is a technical term for a 1-to-1 correspondence between the elements of $A$ and $B$.

**Example 3**    We already saw that $|A| = |X|$ in the previous example. A possible choice of $f$ would be $a \mapsto x$, $b \mapsto y$, $c \mapsto z$.

**Example 4**    The sets $A := \{0, 1, 2\}$ and $B := \{2, 3\}$ obviously have distinct cardinality. The set $A$ contains 3 elements, the set $B$ only 2. But let us see how we get the same result via our mathematical definition.
Suppose we have some arbitrary function $f : A \rightarrow B$. If $f$ is a bijection, then it must map every element of $A$ to some element of $B$. But since there are three elements in $A$ and only two in $B$, some element of $B$ must be the output for at least two elements of $A$. But then $f$ is not a bijection.
In the other direction, consider some arbitrary function $g : B \rightarrow A$. Since a function maps each input to at most one output, the two elements of $B$ are mapped to at most two elements of $A$. But $A$ has three elements, so one element of $A$ cannot be an output for any element of $B$. Again we find that $g$ cannot be bijection.
This exhausts all cases we need to consider, and we may conclude that no function from $A$ to $B$, or the other way round, can be a bijection. Hence $A$ and $B$ must have distinct cardinality.

**Exercise 1**   Show that $|\{0 \leq n < 10 \mid n \text{ is odd}\} = |\{0 \leq n < 10 \mid n \text{ is even}\}||$.

For finite sets, our intuitive notion of size closely matches the technical term of cardinality. However, size and cardinality diverge once we look at infinite sets.

**Example 5**   Consider the set $\mathbb{N} := \{0, 1, 2, ...\}$ of all natural numbers and the set $\mathbb{N}_+ := \{1, 2, ...\}$ of all positive natural numbers. Intuitively, $\mathbb{N}$ is larger than $\mathbb{N}_+$ because it contains all members of $\mathbb{N}_+$ as well as 0, which is not in $\mathbb{N}_+$. But the function $f : \mathbb{N} \to \mathbb{N}_+$ with $n \mapsto n + 1$ is a bijection. Hence $|\mathbb{N}| = |\mathbb{N}_+|$ even though intuitively the two sets have distinct size.

**Exercise 2**   Show that the set of natural numbers has the same cardinality as the set of all even natural numbers.

In a later unit, we will see that our definition of cardinality entails that there are different "sizes" of infinity, and that we want one specific infinity size to talk about language.