Prerequisites

- sets (operations)
- multisets (basics)

Operations on multisets

Standard set operations

The set operations union, intersection and relative complement can be generalized to multisets.

Definition 1. Given two natural numbers m and n with $m \leq n$, let $\max(m, n) = n$ and $\min(m, n) = m$. Then for any two multisets A_M and B_M

- the union $A_M \cup B_M$ maps every a to $\max(A_M(a), B_M(a))$,
- the intersection $A_M \cap B_M$ maps every a to $\min(A_M(a), B_M(a))$
- the relative complement $A_M - B_M$ maps every a to $A_M(a) - B_M(a)$ (or 0 if the value would be negative)

Example 1 Let $A_M := \{a : 3, b : 2, c : 1\}$ and $B_M := \{a : 1, b : 1, c : 2, d : 1\}$. Then

- $A_M \cup B_M = B_M \cup A_M = \{a : 3, b : 2, c : 2, d : 1\}$
- $A_M \cap B_M = B_M \cap A_M = \{a : 1, b : 1, c : 1\}$
- $A_M - B_M = \{a : 2, b : 1, c : 0\}$
- $B_M - A_M = \{c : 1, d : 1\}$

Exercise 1 Fill each gap with a matching multiset or operator.

- $\{a : 3, b : 2, c : 1\} \cup \{a : 5, b : 1, d : 8\} = \_$
- $\{c : 17\} \_ \{a : 5, b : 1, d : 8\} = \{c : 17\}$
- $\{a : 3, b : 3\} \cup \_ = \{a : 5, b : 3, c : 5, d : 6\}$
- $\_ \_ \{a : 5, b : 1, d : 8\} = \{a : 3, b : 1\}$

Special operations for multisets

Since multisets are a generalization of sets, they allow for certain operations that would not make much sense with sets. These are *multiset sum* ($\uplus$) and *scalar multiplication* ($\otimes$).

Definition 2. Let A_M and B_M be two multisets and n a natural number. Then

- the multiset sum $A_M \uplus B_M$ maps every a to $A_M(a) + B_M(a)$,
- the scalar multiplication $n \otimes A_M$ maps every a to $n \times A_M(a)$ (where $\times$ denotes multiplication over natural numbers).

Example 2 As in the previous examples, we will look at the multisets $A_M := \{a:3, b:2, c:1\}$ and $B_M := \{a:1, b:1, c:2, d:1\}$. For those two sets, $A_M \uplus B_M := \{a:4, b:3, c:3, d:1\}$, which is identical to $B_M \uplus A_M$. Furthermore, $3 \otimes A_M = \{a:9, b:6, c:3\}$ whereas $3 \otimes B_M = \{a:3, b:3, c:6, d:3\}$.

Exercise 2 Calculate the final result of the equations below. If the result cannot be uniquely determined without additional assumptions, explain why.

- $3 \otimes \big((\{a:5, b:3\}_M \uplus \{a:1, c:5, d:6\}_M) - (\{a:5, b:3\}_M \cap \{a:1, c:5, d:6\}_M)\big)$
- $\{\text{John}, \text{John}\} \uplus 3 \otimes \{\text{Mary}, \text{Mary}, \text{Mary}, \text{John}\}_M$
- $\{a:5, c:1\}_M \cup \{a:3, b:3, c:3, d:3\}_M \uplus \{a:1\}_M$