Domains and co-domains

Every function has a **domain** and a **co-domain**. The domain is the set of objects from which its arguments can be drawn, and the **co-domain** is the set of objects from which outputs can be drawn. A function is undefined on any arguments that do not belong to its domain. One commonly writes $f : D \to C$ to indicate that f is a function from domain D to co-domain C.

Example 1 Consider the function $f(x) = x + 1$. This actually represents multiple functions depending on how one picks the domain and co-domain. Suppose $f : \mathbb{N} \to \mathbb{N}$. Then f is a function from natural numbers (0, 1, 2, ...) to natural numbers. In this case we have, for instance, $f(0) = 1$ and $f(500) = 501$. However, $f(-1)$ or $f(2.5)$ would be undefined because -1 and 2.5 are not natural numbers.

Example 2 Now suppose that we have $f(x) = x + 1$ with $f : \mathbb{R} \to \mathbb{R}$, i.e. f is a function from real numbers to real numbers. (We haven't encountered real numbers yet, just assume that $\mathbb{R}$ includes pretty much number you have encountered in high school, e.g. 1, 1.38702, $-\frac{5}{17}$, $\sqrt{2}$, and so on.) Now $f(-1) = 0$\$ and $f(2.5) = 3.5$.

Exercise 1 Suppose that f is still defined by $f(x) = x + 1$, but we have $f : \mathbb{R} \to \mathbb{N}$. For each one of the following, say whether it is defined or undefined.

1. $f(0)$
2. $f(-1)$
3. $f(-2)$
4. $f(2.5)$

Example 3 When a car wash is viewed as a function, its domain is the set of all cars (both dirty and clean), whereas the co-domain only contains clean cars.

Exercise 2 What would be the domain and co-domain of a broken car wash that fails to remove even the tiniest speck of dirt?

Since it is so important to know the domain and co-domain of a function, those are usually specified before the precise mapping from inputs to outputs is given.

Example 4 Let E be the set of English first names. Then the function $f : E \to \{0, 1\}$ maps n to 1 iff n contains at least three syllables.

The mapping from arguments to outputs can be defined in various ways, e.g. in plain English, or as a formula like $f(x) = \frac{(x+x^2+5)^{x+1}}{1000^x}$. For very simple functions whose name was already mentioned, one often writes $x \mapsto y$ instead of $f(x) = y$.

Example 5 Instead of $f(x) = 5 \times x - 3$, we may simple write $x \mapsto 5 \times x - 3$.

Caution: Notice the difference between $\to$ and $\mapsto$. The first is used when specifying the domain and co-domain, whereas the latter indicates the concrete mapping from an argument to an output.