

### Prerequisite

- functions (basics)

## Relations

Relations are similar to functions in that they establish connections between objects. But whereas a function associates only one output with every input, a relation is more flexible and allows connections to arbitrarily many elements.

**Example 1** The question *Is  $x$  a biological child of  $y$*  is a function because it maps any two  $x$  and  $y$  to either true or false, but never both. But if we slightly change the question to just *Name a biological child of  $y$* , we are no longer dealing with a function because multiple answers are possible if  $y$  has more than one child. Instead, we can talk about the *biological child* relation  $R$  such that  $x$  is related to  $y$  via  $R$  iff  $x$  is a biological child of  $y$ .

### Basic notation

Given some relation  $R$ , we write  $x R y$  to indicate that  $R$  relates  $x$  to  $y$ . Note that  $x R y$  does not imply that  $y R x$  also holds, nor that it doesn't hold.

**Example 2** One relation you know very well is the “less than” relation  $<$  over numbers. When we write  $5 < 7$ , we are saying that the relation  $<$  relates 5 to 7. However, it is not the case that 7 is related to 5 via  $<$ , since  $7 < 5$  does not hold.

**Example 3** Suppose *John* has exactly two siblings, *Mary* and *Sue*. Then the sibling relation establishes two connections for John: *John-Mary*, and *John-Sue*. Using  $S$  for the sibling relation, we have  $\text{John } S \text{ Mary}$  and  $\text{John } S \text{ Sue}$ . In contrast to  $<$ , the sibling relation is **symmetric**. That is to say, if Mary is a sibling of John, then John is also a sibling of Mary. Therefore we also have  $\text{Mary } S \text{ John}$  and  $\text{Sue } S \text{ John}$ . The relation is also **transitive**. If John is a sibling of Mary, and Sue is a sibling of John, then Sue is a sibling of Mary. So it also holds that  $\text{Mary } S \text{ Sue}$ , and via symmetry we also get  $\text{Sue } S \text{ Mary}$ . Overall, we have  $x S y$  where  $x, y \in \{\text{John, Mary, Sue}\}$  and  $x \neq y$ .

**Example 4** The **substring relation**  $\sqsubseteq$  holds between two strings  $u$  and  $v$  iff  $u$  is a substring of  $v$ . That is to say,  $u \sqsubseteq v$  iff there are  $w, w' \in \Sigma^*$  such that  $w \cdot u \cdot w' = v$ . Note that for any given string  $u$ , there are infinitely many  $v$  that  $u$  is a substring of. Even if the alphabet contains only  $a$ , the string  $aa$  is a substring of  $aaa$ ,  $aaaa$ ,  $aaaaa$ , and so on, ad infinitum. It is also a substring of itself (in this case,  $w = w' = \varepsilon$ ).

**Example 5** Relations can be defined over more complex objects like sets. An example of this is the subset relation  $\subseteq$ . The set  $\{1\}$  is a subset of infinitely many other sets:  $\{0, 1\}$ ,  $\{1, 2\}$ ,  $\{0, 1, 2\}$ ,  $\{1, 3\}$ , and so on.

**Example 6** Just like a function can take multiple arguments to return a single output, a relation can connect multiple elements. In the real world, the “jointly conceived” relation  $J$  would connect two individuals to their offspring. So the expression

$$\text{John, Mary } R \text{ Sue}$$

encodes that John and Mary are the biological parents of Sue (let’s just hope that those are not the same people as in the first example).

**Example 7** Here is an example of a very abstract relation. Consider the space of all possible functions from real numbers to real numbers — that’s a lot of functions. Now let’s define a boundedness relation  $B$  which relates function  $f$  to function  $g$  iff  $f(x) \leq g(x)$  for every natural number  $x$ . Suppose, for instance, that  $f(x) = x$  and  $g(x) = x^2$ . Then  $f B g$ , but not  $g B f$ .

Many functions aren’t related via  $B$  at all. One example of this is  $f(x) = -x + 1$  and  $g(x) = x^2$ . It is the case that  $f(x) \leq g(x)$  for all  $x \geq 1$ , but  $f(0) = 1 > 0 = g(0)$ .

We will mostly be dealing with the special case of **binary relations** where exactly one element is related to some other element.

**Example 8** Almost all relations above are binary relations. The only exception is the “jointly conceived” relation  $J$ , which is a ternary relation as it relates three elements.

Given a binary relation  $R$ ,  $a R$  is the set of objects that  $a$  is related to. Similarly,  $R b$  is the set of objects that are related to  $b$ .

$$a R := \{b \mid a R b\}$$

$$R b := \{a \mid a R b\}$$

**Example 9** Suppose that the parent-of relation  $P$  establishes the following relations between elements: John  $P$  Sue and Mary  $P$  Sue. Then John  $P = \{\text{Sue}\}$  and  $P \text{ Sue} = \{\text{John, Mary}\}$ .

**Exercise 1** Let  $R$  be the relation that connects words to their parts of speech (N for nouns, V for verbs, A for adjectives, P for prepositions, D for determiners, and so on). List the following for English:

- export  $R$
- apple  $R$
- $R P$

## Relations versus functions

Every function can be regarded as a relation.

**Example 10** Consider the function  $f : \mathbb{N} \rightarrow \mathbb{N}$  with  $x \mapsto 2x$ . It is identical to the relation  $R$  with  $x R y$  iff  $y = 2x$ .

The crucial difference between functions and relations is that functions are **right-unique relations**. That's a fancy way of saying that a function cannot provide more than one output, whereas a relation can unless it is right-unique. When we view a function  $f$  as a relation  $R$ , then it must hold for every  $a$  that  $a R$  is either empty or contains exactly one element. Hence the term right-unique: if we look at the expression  $a R b$ ,  $a$  is the left side and  $b$  the right side. If there cannot be more than one choice for  $b$ , then the right side of  $a R b$  is uniquely determined.

The bottom line: every function is a relation, but not every relation is a function. If  $a$  is related to two elements or more (i.e.  $|a R| \geq 2$ ), then  $R$  cannot be a function.

**Exercise 2** For each one of the following, say whether it is a function or just a relation.

- the parent-of relation (e.g.  $j P m$  for “John is a parent of Mary”)
- the parent-of relation in a world where the one-child policy is enforced globally
- the relation between a car's license plate and its owners
- the prefix relation, where  $u$  is a prefix of  $v$  iff there is some  $w \in \Sigma^*$  such that  $v = u \cdot w$

**Exercise 3** Is the following statement true or false? Justify your answer.  
Every relation  $R$  can be regarded as a function that maps  $x$  to  $x R$ .

## Recap

- Relations are a generalization of functions in that a single input can be related to many distinct outputs.
- Given some relation  $R$ , we write  $x R y$  to indicate that  $R$  relates  $x$  to  $y$ .
- We mostly focus on binary relations.
- For binary relations, we have the following notation:

$$a R := \{b \mid a R b\}$$

$$R b := \{a \mid a R b\}$$