Taking the closure of a relation

Let D be some fixed domain over which we define a relation $R \subseteq D \times D$. The P-closure of R is the smallest R' such that $R \subseteq R' \subseteq D \times D$ and R has property P. For example, the reflexive, symmetric, transitive closure of R is the smallest superset R' that is reflexive, symmetric, and transitive.

Example 1 Consider the relation $R := \{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,4 \rangle\}$. We want to compute its transitive closure. We first add $\langle 1,3 \rangle$ and $\langle 2,4 \rangle$, which we can construct from the existing pairs via transitivity. But the result is still not transitive The relation now contains both $\langle 1,3 \rangle$ and $\langle 3,4 \rangle$, so we also have to add $\langle 1,4 \rangle$. At this point, the relation is transitive, so we do not add any more edges.

Example 2 Consider the relation $R := \{\langle 1,2 \rangle, \langle 3,2 \rangle\}$. Its transitive closure is just R.

Example 3 Consider once more the relation $R := \{\langle 1,2 \rangle, \langle 3,2 \rangle\}$. Now we want to compute its symmetric, transitive closure. As R is not symmetric, we have to add additional pairs, in this case $\langle 2,1 \rangle$ and $\langle 2,3 \rangle$. But $R' := \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,3 \rangle, \langle 3,2 \rangle\}$ is not transitive. In order to make R' transitive, we have to add $\langle 1,3 \rangle$ and $\langle 3,1 \rangle$, but also $\langle 1,1 \rangle$, $\langle 2,2 \rangle$, and $\langle 3,3 \rangle$. The resulting $R'' := \{1,2,3\} \times \{1,2,3\}$ is transitive and symmetric, and even reflexive.

Exercise 1 Calculate all of the following, assuming that the relation's domain is $D := \{a,b,c,d\}$:

- the reflexive closure of $\{\langle a,b \rangle, \langle b,a \rangle\}$
- the transitive closure of $\{\langle a,b \rangle, \langle b,a \rangle\}$
- the transitive closure of $\{\langle a,b \rangle, \langle b,c \rangle, \langle c,d \rangle, \langle d,a \rangle\}$
- the reflexive, symmetric, transitive closure of $\{\langle a,b \rangle, \langle a,c \rangle, \langle d,c \rangle\}$
- the reflexive, symmetric, transitive closure of $\emptyset$