Prerequisite

- Tuples (basics)

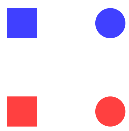
Crossproduct

One often wants to define an entire set of tuples. This can be done with set-builder notation.

Example 1 Let N be a set of names and P a set of phone numbers. Then we might define an address book as the set

$$A := \{\langle n, p\rangle \mid n \in N, p \in P, \text{ and } p \text{ is } n\text{'s phone number}\}$$

But when we want to allow all possible combinations, there is an easier option. Consider the colored objects depicted below:



We can represent each object as a pair $\langle s, c\rangle$ where s and c are drawn from a set $S := \{\text{square}, \text{circle}\}$ of shapes and a set $C := \{\text{blue}, \text{red}\}$ of colors, respectively. The figure above contains every possible combination of those shapes and colors. We can still use set-builder notation in this case: $\{\langle s, c\rangle \mid s \in S, c \in C\}$.

Exercise 1 Why shouldn't we use a set $\{s, c\}$ instead of the pair $\langle s, c\rangle$? What might go wrong in this case depending on our choice of S and C?

A more elegant alternative to set-builder notation, however, is the **crossproduct** or **Cartesian product**.

Definition 1. For any two sets S and T, their crossproduct $S \times C$ is defined as $\{\langle s, c\rangle \mid s \in S, c \in C\}$. In general, $A_1 \times A_2 \times \cdots \times A_n := \{\langle a_1, a_2, \ldots, a_n\rangle \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$.

Example 2 For $S := \{\text{square}, \text{circle}\}$ and $C := \{\text{blue}, \text{red}\}$, $S \times C$ contains the pairs

- $\langle\text{square}, \text{blue}\rangle$
- $\langle\text{circle}, \text{blue}\rangle$
- $\langle\text{square}, \text{red}\rangle$
- $\langle\text{circle}, \text{red}\rangle$

This is different from $C \times S$, which contains

- $\langle\text{blue}, \text{square}\rangle$
- $\langle\text{blue}, \text{circle}\rangle$
- $\langle\text{red}, \text{square}\rangle$
- $\langle\text{red}, \text{circle}\rangle$

Exercise 2 Suppose S consists of *John*, *Mary*, and *the old man*, whereas V contains only *slept* and *left*. Compute $S \times V$.

Example 3 Now suppose that we also have a set $A = \{\text{awesome}\}$. Then $S \times C \times A$ would be a set containing the following triples:

- $\langle \text{square}, \text{blue}, \text{awesome} \rangle$
- $\langle \text{circle}, \text{blue}, \text{awesome} \rangle$
- $\langle \text{square}, \text{red}, \text{awesome} \rangle$
- $\langle \text{circle}, \text{red}, \text{awesome} \rangle$

Exercise 3 List all 8 members of $A \times C \times S \times A \times C \times A$.

Exercise 4 In a certain sense, the crossproduct is the result of generalizing concatenation from tuples to sets of 1-tuples. Explain why.

Exercise 5 If A has m members and B has n members, then the number of tuples in $A \times B$ is m multiplied by n. Explain why.

Remark. The name Cartesian product makes more sense if you consider the special case of $\mathbb{N} \times \mathbb{N}$. Here $\mathbb{N} := \{0, 1, 2, 3, ...\}$ is the set of all natural numbers. So $\mathbb{N} \times \mathbb{N}$ is the set of all possible pairs of natural numbers. We can take these two components to represent (x, y)-coordinates in the upper right quadrant of a coordinate system. Such a coordinate system is also called a **Cartesian plane**, and that is why the crossproduct is sometimes called the Cartesian product.

Just like tuple concatenation, the crossproduct operation is not commutative.

Example 4 Let $A := \{a, b\}$ and $B := \{1\}$. Then $A \times B = \{\langle a, 1\rangle, \langle b, 1\rangle\}$ whereas $B \times A = \{\langle 1, a\rangle, \langle 1, b\rangle\}$.

But whereas tuple concatenation is associative, the crossproduct operation is not. Most of the time, $A \times B \times C$ and $A \times (B \times C)$ and $(A \times B) \times C$ yield different results.

Example 5 Let $A := \{a, b, c\}$, $B := \{T, F\}$, and $C := \{1\}$. Then $A \times (B \times C)$ contains 6 pairs:

- $\langle a, \langle T, 1\rangle\rangle$
- $\langle a, \langle F, 1\rangle\rangle$
- $\langle b, \langle T, 1\rangle\rangle$
- $\langle b, \langle F, 1\rangle\rangle$
- $\langle c, \langle T, 1\rangle\rangle$
- $\langle c, \langle F, 1\rangle\rangle$

While $(A \times B) \times C$ also contains 6 pairs, they are different pairs:

- $\langle\langle a, T\rangle, 1\rangle$
- $\langle\langle a, F\rangle, 1\rangle$
- $\langle\langle b, T\rangle, 1\rangle$
- $\langle\langle b, F\rangle, 1\rangle$
- $\langle\langle c, T\rangle, 1\rangle$
- $\langle\langle c, F\rangle, 1\rangle$

Exercise 6 Continuing the previous example, list all elements of $A \times B \times C$. Does this set also contain 6 tuples? Are they also pairs?

Recap

- The crossproduct (or Cartesian product) generalizes concatenation from tuples to sets:

$$A_1 \times A_2 \times \cdots \times A_n := \{\langle a_1, a_2, \dots, a_n \rangle \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}$$

- The crossproduct operation is not commutative. Never confuse $A \times B$ and $B \times A$.
- The crossproduct operation is not associative. Never confuse $A \times B \times C$, $A \times (B \times C)$, and $(A \times B) \times C$.