2

The Gregorian Calendar

For some ridiculous reason, to which, however, I've no desire to be disloyal,
Some person in authority, I don't know who, very likely the Astronomer Royal,
Has decided that, although for such a beastly month as February,
twenty-eight as a rule are plenty.
One year in every four his days shall be reckoned as nine-and-twenty.

Gilbert and Sullivan: *Pirates of Penzance*, Act II (1879)

2.1 Structure

The calendar in use today in most of the world is the Gregorian or *new-style* calendar designed by a commission assembled by Pope Gregory XIII[1] in the sixteenth century. The main author of the new system was the Naples astronomer Aloysius Lilius; see [4], [6], [16], and [18] for mathematical and historical details. This strictly solar calendar is based on a 365-day common year divided into 12 months of lengths 31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30, and 31 days with 366 days in leap years, the extra day being added to make the second month 29 days long:

(1) January	31 days	(7) July	31 days
(2) February	28 {29} days	(8) August	31 days
(3) March	31 days	(9) September	30 days
(4) April	30 days	(10) October	31 days
(5) May	31 days	(11) November	30 days
(6) June	30 days	(12) December	31 days

The leap-year structure is given in curly brackets. A year is a leap year if it is divisible by 4 and is not a century year (a multiple of 100) or if it is divisible by 400. For example, 1900 is not a leap year; 2000 is. The Gregorian calendar differs

[1] Gregory was also responsible for a bull *Vices eius nos* (September 1, 1577) organizing regular missionizing sermons by apostate Jews, which the Jewish community of Rome was forced to attend and subsidize. His bull *Sancta mater ecclesia* (September 1, 1584) specified more precise conditions: beadles armed with rods made sure the Jews paid attention and checked that they had not put wax in their ears. These sermons took place throughout the Papal States and much of the Roman Catholic world, as well in the church nearest the Jewish Quarter in Rome, San Gregorio della Divina Pietà. (The front of this church has an inscription in Hebrew and Latin, beside an image of the crucified Jesus, quoting from Isaiah 65:2–3, "I have spread out My hands all the day unto a rebellious people, that walk in a way that is not good, after their own thoughts; a people that provoke me to my face continually.")

https://doi.org/10.1017/9781107415058.005 Published online by Cambridge University Press

from its predecessor, the old-style or Julian calendar, only in that the Julian calendar did not include the century rule for leap years—all century years were leap years. It is the century rule that causes the leap year structure to fall outside the cycle-of-years paradigm of Section 1.14 (but Gregorian-like leap year rules have their own interesting mathematical properties; see [19]). Days on both calendars begin at midnight.

Although the month lengths seem arbitrarily arranged, they would precisely satisfy the cyclic formulas of Section 1.14 with $c = 12$, $l = 7$, $\Delta = 11$, and $L = 30$, if February always had 30 days. In other words, if we assume February has 30 days, formula (1.86) tells us that there are

$$\left\lfloor \frac{7m - 2}{12} \right\rfloor + 30(m - 1) = \left\lfloor \frac{367m - 362}{12} \right\rfloor \tag{2.1}$$

days in the months $1, \ldots, m - 1$, and formula (1.90) tells us that day n of the year falls in month number

$$\left\lfloor \frac{12n + 373}{367} \right\rfloor \tag{2.2}$$

where, as in the derivation of (1.90), the first day of the year is $n = 0$; that is, n is the number of prior days in the year rather than the day number in the usual sense. The values $c = 12$ and $L = 30$ leading to (2.1) and (2.2) are obvious: There are 12 months and the ordinary length is 30 days. The value $l = 7$ comes from the 7 long months of 31 days; the value $\Delta = 11$ forces January to be month number 1 (rather than 0), necessary for the applicability of formulas (1.86) and (1.90). It is a simple matter to use the formulas (2.1) and (2.2) and to correct for the mistaken assumption that February has 30 days; we do just that in the next section.

The Julian calendar dates from January 1, 709 A.U.C.[2] (45 B.C.E.) and is by Julius Cæsar, with the help of Alexandrian astronomer Sosigenes; it was a modification of the Roman Republican (see [15]) and ancient Egyptian calendars. Because every

[2] *Ab Urbe Condita*; from the founding of the city (of Rome). Varro's statements imply that the year of the founding of Rome was 753 B.C.E., which gives 709 A.U.C. = 45 B.C.E. as the year of institution of the Julian calendar; this year is commonly, but not universally, accepted. The counting of years according to the Christian era was instituted by Eusebius, a fourth-century bishop of Cæsarea, and then used by the sixth-century Roman monk and scholar Dionysius Exiguus; it only became commonplace a few centuries later—Eusebius erred by a few years in his determination of the year of Jesus's birth (see D. P. McCarthy, "The Emergence of *Anno Domini*," pp. 31–53 in *Time and Eternity*: *The Medieval Discourse*, G. Jaritz and G. Moreno-Riaño, eds., Brepols, Turnhout, Belgium, 2003). Much of the Christian world used "Anno Diocletiani" for many years (the Julian calendar with Diocletian's reign as the origin—the same origin as the Coptic calendar discussed in Chapter 4). Eusebius's innovation was to substitute his estimate of Jesus's birth year for the origin, starting his count at 1. The "1 B.C.E. is the year before 1 C.E." problem was a result of the system introduced and popularized by the Venerable Bede around 731. Bede did not know about 0, so he did not use it [Bede's work *De Temporum Ratione* was translated by Faith Wallis as *Bede: The Reckoning of Time*, Liverpool University Press, Liverpool, 1999 (also University of Pennsylvania Press, Philadelphia, 2000)], and the custom of omitting a year 0 in the Julian calendar's year count became well established. Astronomers do use a year 0 preceding year 1 on the Gregorian calendar—this is due to Cassini in 1740 [5]; see also Dick Teresi, "Zero," *The Atlantic Monthly*, vol. 280, no. 1, pp. 88–94, July 1997.

fourth year was a leap year, a cycle of 4 years contained $4 \times 365 + 1 = 1461$ days, giving an average length of year of 365.25 days. This is somewhat more than the mean length of the tropical year (the year measured between successive vernal equinoxes), and over the centuries the calendar slipped with respect to the seasons. By the sixteenth century, the true date of the vernal (spring) equinox had shifted from around March 21 in the fourth century when the date of Easter was fixed (see Chapter 9) to around March 11. If this error were not corrected, then Easter, whose date depends on the ecclesiastical approximation of March 21 for the vernal equinox, would gradually migrate through the seasons, eventually to become a summer holiday.

Pope Gregory XIII instituted only a minor change in the calendar—century years not divisible by 400 would no longer be leap years. (He also modified the rules for Easter; see Chapter 9.) Thus, 3 out of 4 century years are common years, giving a cycle of 400 years containing $400 \times 365 + 97 = 146097$ days and an average year length of $146097/400 = 365.2425$ days. He also corrected the accumulated 10-day error in the date of the equinox by proclaiming that Thursday, October 4, 1582 C.E. according to the calendar then in use (Julian) would be followed by Friday, October 15, 1582, the first day of the new-style (Gregorian) calendar. Catholic countries followed his rule: Spain, Portugal, and Italy adopted it immediately, as did the Catholic states in Germany. However, Protestant countries resisted. The Protestant parts of Germany waited until 1700 to adopt it. The various cantons of Switzerland changed at different times. Sweden began a gradual changeover in 1699, omitting February 29 in 1700. At that point the plan was abandoned, leaving the Swedish calendar one day off from the Julian. This was only rectified in 1712 by adding a February 30 to that year—see the frontispiece for this chapter! The Swedish calendar stayed in tune with the Julian until 1753, when the Gregorian was adopted.[3] Great Britain and her colonies (including the United States) waited until 1752 (see [17] for an interesting description of the effect); Russia held out until 1918, after the Bolshevik Revolution, which is also known as the October Revolution because it occurred on October 25–26, 1917 C.E. (Julian) = November 7–8, 1917 (Gregorian).[4] Different parts of what is now the United States changed over at different dates; Alaska, for example, changed only when it was purchased by the United States in 1867.[5] Turkey did not change to the Gregorian calendar

[3] See [9, p. 275]. We are indebted to Tapani Tarvainen and Donald Knuth for pointing out this anomaly.

[4] In 1923 the Congress of the Orthodox Oriental Churches adopted a slightly more accurate leap-year rule: Century years are leap years only if they leave a remainder of 2 or 6 when divided by 9; this "Revised Julian" rule agrees with the usual Gregorian rule for 1700–2700 (see M. Milankovitch, "Das Ende des julianischen Kalenders und der neue Kalender der orientalischen Kirche," *Astronomische Nachrichten*, vol. 220, pp. 379–384, 1924). The Soviet Union and some orthodox churches (the New Calendarists) adopted this rule at that time. Like the rest of the world, we ignore this "improvement."

[5] Alaska skipped only 11 days instead of 12 (as we might expect) but with a repeated weekday because it also jumped the International Date Line when it became United States territory in 1867: Friday, October 6, 1867 C.E. (Julian) was followed by Friday, October 18, 1867 (Gregorian)! Even without the change from the Julian to the Gregorian calendar, jumping the date line causes bizarre situations. In 1892 Samoa jumped the date line and also switched from "Asian Time" to "American Time," causing the Fourth of July to be celebrated for 2 consecutive days; the reverse happened

until 1927. An extensive list of dates of adoption of the Gregorian calendar can be found in [1].

The Gregorian calendar is not fully accurate in its alignment with the solar cycle because its approximation to the year, $365\frac{97}{400} = 365.2425$ is slightly too large (see the discussion in Section 14.4). This was known as early as 1700, so various modifications have been suggested, but none accepted. For example, the astronomer John Herschel (and others) proposed making years divisible by 4000 ordinary years, not leap years; such a modification is simple to incorporate into our functions in the following sections. Isaac Newton had much earlier proposed a radically different approach (see [3]) with a 5000-year cycle in which years divisible by 4 would be leap years (February would have 29 days), except that years divisible by 100 would not be leap years, except that years divisible by 500 would be leap years; furthermore, years divisible by 5000 would be "double leap years" with 30 days in February. Implementing Newton's calendar is a nice exercise for the reader.

By universal current custom, the new Gregorian year number begins on January 1. There have, however, been other beginnings—parts of Europe began the New Year variously on March 1, Easter, September 1, Christmas, and March 25 (see, for example, [11]). This is no small matter in interpreting dates between January 1 and the point at which the number of the year changed. For example, in England under the Julian calendar, the commencement of the ecclesiastical year on March 25 in the sixteenth and seventeenth centuries means that a date like February 1, 1660 leaves the meaning of the year in doubt. Such confusion led to the practice of writing a hyphenated year giving both the legal year first and the calendar year number second: February 1, 1660-1. The same ambiguity occurs even today when we speak of the "fiscal year," which can run from July to July or from October to October, but we would always give the calendar year number, not the fiscal year number in specifying dates.

Although the Gregorian calendar did not exist prior to the sixteenth century, we can extrapolate backwards using its rules to obtain what is sometimes called the "proleptic Gregorian calendar,"[6] which we implement in the next section. Unlike the Julian calendar, we implement this proleptic calendar with a year 0, as is common among astronomers—see the footnote on page 56. By our choice of the starting point of our fixed counting of days, we define

$$\textbf{gregorian-epoch} \stackrel{\text{def}}{=} \text{R.D. } 1 \tag{2.3}$$

when the Philippines jumped the date line in the other direction in 1844: Monday, December 30, 1844, was followed by Wednesday, January 1, 1845. On December 29, 2011 Samoa again changed its time zone to align itself with Australia and New Zealand, moving from the eastern side of the international date line to the western side. Samoans lost a day, going straight from December 29 to December 31.

[6] The name is really a misnomer because "proleptic" refers to the future, not the past.

2.2 Implementation

Les protestants de toutes les communions s'obstinérent à ne pas recevoir des mains du pape une vérité qu'il aurait fallu recevoir des Turcs, s'ils l'avaient proposée. [*The Protestants of all denominations insist on rejecting a truth from the hands of the Pope, which they would have accepted even from the Turks had they proposed it.*]

Voltaire: *Essai sur les Mœurs et l'esprit des nations* (1756)

For convenience, we define 12 numerical constants by which we will refer to the 12 months of the Gregorian and Julian calendars:

$$\textbf{january} \stackrel{\text{def}}{=} 1 \tag{2.4}$$

$$\textbf{february} \stackrel{\text{def}}{=} 2 \tag{2.5}$$

$$\textbf{march} \stackrel{\text{def}}{=} 3 \tag{2.6}$$

$$\textbf{april} \stackrel{\text{def}}{=} 4 \tag{2.7}$$

$$\textbf{may} \stackrel{\text{def}}{=} 5 \tag{2.8}$$

$$\textbf{june} \stackrel{\text{def}}{=} 6 \tag{2.9}$$

$$\textbf{july} \stackrel{\text{def}}{=} 7 \tag{2.10}$$

$$\textbf{august} \stackrel{\text{def}}{=} 8 \tag{2.11}$$

$$\textbf{september} \stackrel{\text{def}}{=} 9 \tag{2.12}$$

$$\textbf{october} \stackrel{\text{def}}{=} 10 \tag{2.13}$$

$$\textbf{november} \stackrel{\text{def}}{=} 11 \tag{2.14}$$

$$\textbf{december} \stackrel{\text{def}}{=} 12 \tag{2.15}$$

To convert from a Gregorian date to an R.D. date, we first need a function that tells us whether a year is a leap year. We write

$$\textbf{gregorian-leap-year?}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{2.16}$$
$$(g\text{-}year \bmod 4) = 0 \text{ and } (g\text{-}year \bmod 400) \notin \{100, 200, 300\}$$

The calculation of the R.D. date from the Gregorian date (which was described in [12] as "impractical") can now be done by counting the number of days in prior years (both common and leap years), the number of days in prior months of the current year, and the number of days in the current month:

$$\textbf{fixed-from-gregorian}\left(\begin{array}{|c|c|c|}\hline \textit{year} & \textit{month} & \textit{day} \\ \hline\end{array}\right) \stackrel{\text{def}}{=} \tag{2.17}$$

$$\textbf{gregorian-epoch} - 1 + 365 \times (\textit{year} - 1) + \left\lfloor \frac{\textit{year} - 1}{4} \right\rfloor - \left\lfloor \frac{\textit{year} - 1}{100} \right\rfloor$$
$$+ \left\lfloor \frac{\textit{year} - 1}{400} \right\rfloor + \left\lfloor \frac{1}{12} \times (367 \times \textit{month} - 362) \right\rfloor$$
$$+ \begin{Bmatrix} 0 & \textbf{if } \textit{month} \leqslant 2 \\ -1 & \textbf{if gregorian-leap-year?}\,(\textit{year}) \\ -2 & \textbf{otherwise} \end{Bmatrix} + \textit{day}$$

The explanation of this function is as follows. We start at the R.D. number of the last day before the epoch (**gregorian-epoch** $- 1 = 0$, but we do it explicitly so that the dependency on our arbitrary starting date is clear); to this, we add the number of nonleap days (positive for positive years, negative otherwise) between R.D. 0 and the last day of the year preceding the given year, the corresponding (positive or negative) number of leap days, the number of days in prior months of the given year, and the number of days in the given month up to and including the given day. The number of leap days between R.D. 0 and the last day of the year preceding the given year is determined by the mathematical principle of "inclusion and exclusion" [13, chapter 4]: add all Julian-leap-year-rule leap days (multiples of 4), subtract all the century years (multiples of 100), and then add back all multiples of 400. The number of days in prior months of the given year is determined by formula (2.1), corrected by 0, -1, or -2 for the assumption that February always has 30 days.

For example, to compute the R.D. date of November 12, 1945 (Gregorian), we compute $365 \times (1945 - 1) = 709560$ prior nonleap days, $\lfloor (1945 - 1)/4 \rfloor = 486$ prior Julian-rule leap days (multiples of 4), $-\lfloor (1945-1)/100 \rfloor = -19$ prior century years, $\lfloor (1945-1)/400 \rfloor = 4$ prior 400-multiple years, $\lfloor (367 \times 11 - 362)/12 \rfloor = 306$ prior days, corrected by -2 because November is beyond February and 1945 is not a Gregorian leap year. Adding these values and the day number 12 together gives $709560 + 486 - 19 + 4 + 306 - 2 + 12 = 710347$.

The function **fixed-from-gregorian** allows us to calculate the first and last days of the Gregorian year, and the range of dates between them:

$$\textbf{gregorian-new-year}\,(\textit{g-year}) \stackrel{\text{def}}{=} \tag{2.18}$$
$$\textbf{fixed-from-gregorian}\left(\begin{array}{|c|c|c|}\hline \textit{g-year} & \textbf{january} & 1 \\ \hline\end{array}\right)$$

$$\textbf{gregorian-year-end}\,(\textit{g-year}) \stackrel{\text{def}}{=} \tag{2.19}$$
$$\textbf{fixed-from-gregorian}\left(\begin{array}{|c|c|c|}\hline \textit{g-year} & \textbf{december} & 31 \\ \hline\end{array}\right)$$

$$\textbf{gregorian-year-range}\,(\textit{g-year}) \stackrel{\text{def}}{=} \tag{2.20}$$
$$[\textbf{gregorian-new-year}\,(\textit{g-year}) \,..\, \textbf{gregorian-new-year}\,(\textit{g-year} + 1))$$

https://doi.org/10.1017/9781107415058.005 Published online by Cambridge University Press

We will need these functions to determine holidays on other calendars that fall within a specific Gregorian year for example.

Calculating the Gregorian date from the R.D. *date* involves sequentially determining the year, month, and day of the month. Because the century rule for Gregorian leap years allows an occasional 7-year gap between leap years, we cannot use the methods of Section 1.14—in particular, formula (1.90)—to determine the Gregorian year. Rather, exact determination of the Gregorian year from the R.D. *date* involves the decomposition of the number of days into units of 1, 4, 100, and 400 years.

$$\textbf{gregorian-year-from-fixed}\,(date) \stackrel{\text{def}}{=} \tag{2.21}$$

$$\begin{cases} year & \textbf{if } n_{100} = 4 \text{ or } n_1 = 4 \\ year + 1 & \textbf{otherwise} \end{cases}$$

where

$$\begin{aligned} d_0 &= date - \textbf{gregorian-epoch} \\ n_{400} &= \left\lfloor \frac{d_0}{146097} \right\rfloor \\ d_1 &= d_0 \bmod 146097 \\ n_{100} &= \left\lfloor \frac{d_1}{36524} \right\rfloor \\ d_2 &= d_1 \bmod 36524 \\ n_4 &= \left\lfloor \frac{d_2}{1461} \right\rfloor \\ d_3 &= d_2 \bmod 1461 \\ n_1 &= \left\lfloor \frac{d_3}{365} \right\rfloor \\ year &= 400 \times n_{400} + 100 \times n_{100} + 4 \times n_4 + n_1 \end{aligned}$$

Alternatively, the year may be calculated by means of base conversion in a mixed-radix system (Section 1.10); see formula (2.30) in the next section.

This function can be extended to compute the ordinal day of *date* in its Gregorian year:

$$\text{Ordinal day of } date \text{ in its Gregorian year} \tag{2.22}$$

$$= \begin{cases} (d_3 \bmod 365) + 1 & \textbf{if } n_1 \neq 4 \text{ and } n_{100} \neq 4 \\ 366 & \textbf{otherwise} \end{cases}$$

That is, if $n_{100} = 4$ or $n_1 = 4$, then *date* is the last day of a leap year (day 146097 of the 400-year cycle or day 1461 of a 4-year cycle); in other words, *date* is December 31 of *year*. Otherwise, *date* is the ordinal day $(d_3 \bmod 365) + 1$ in $year + 1$.

This calculation of the Gregorian year of R.D. *date* is correct even for nonpositive years. In that case, n_{400} gives the number of 400-year cycles from *date* until the start of the Gregorian calendar—*including* the current cycle—as a *negative* number because the floor function always gives the largest integer smaller than its argument. Then the rest of the calculation yields the number of years from the *beginning* of that cycle, as a *positive* integer, because the modulus is always nonnegative for positive divisor—see equations (1.20) and (1.21).

Now that we can determine the year of an R.D. date, we can find the month by formula (2.2), corrected by 0, 1, or 2 for the assumption that February always has 30 days. Knowing the year and month, we determine the day of the month by subtraction. Putting these pieces together, we have

$$\textbf{gregorian-from-fixed}\,(\textit{date}) \stackrel{\text{def}}{=} \boxed{\textit{year} \mid \textit{month} \mid \textit{day}} \tag{2.23}$$

where

$$\begin{aligned}
\textit{year} &= \textbf{gregorian-year-from-fixed}\,(\textit{date})\\
\textit{prior-days} &= \textit{date} - \textbf{gregorian-new-year}\,(\textit{year})\\
\textit{correction} &= \begin{cases} 0 & \textbf{if } \textit{date} < \textbf{fixed-from-gregorian}\left(\boxed{\textit{year} \mid \textbf{march} \mid 1}\right)\\ 1 & \textbf{if gregorian-leap-year?}\,(\textit{year})\\ 2 & \textbf{otherwise} \end{cases}\\
\textit{month} &= \left\lfloor \frac{1}{367} \times (12 \times (\textit{prior-days} + \textit{correction}) + 373) \right\rfloor\\
\textit{day} &= \textit{date} - \textbf{fixed-from-gregorian}\left(\boxed{\textit{year} \mid \textit{month} \mid 1}\right) + 1
\end{aligned}$$

We can use our fixed numbering of days to facilitate the calculation of the number of days difference between two Gregorian dates:

$$\textbf{gregorian-date-difference}\,(\textit{g-date}_1, \textit{g-date}_2) \stackrel{\text{def}}{=} \tag{2.24}$$

$$\textbf{fixed-from-gregorian}\,(\textit{g-date}_2) - \textbf{fixed-from-gregorian}\,(\textit{g-date}_1)$$

This function can then be used to compute the ordinal day number of a date on the Gregorian calendar within its year:

$$\textbf{day-number}\,(\textit{g-date}) \stackrel{\text{def}}{=} \tag{2.25}$$

$$\textbf{gregorian-date-difference}\left(\boxed{\textit{g-date}_{\textbf{year}} - 1 \mid \textbf{december} \mid 31}, \textit{g-date}\right)$$

The ordinal day number could also be computed directly using equation (2.22) in a modified version of **gregorian-year-from-fixed**. It is easy to determine the number of days remaining after a given date in the Gregorian year:

$$\textbf{days-remaining}\,(\textit{g-date}) \stackrel{\text{def}}{=} \tag{2.26}$$

https://doi.org/10.1017/9781107415058.005 Published online by Cambridge University Press

$$\textbf{gregorian-date-difference}\left(g\text{-}date,\ \boxed{g\text{-}date_{\textbf{year}} \mid \textbf{december} \mid 31} \right)$$

Finally, we can compute the last day of a Gregorian month in a similar fashion:

$$\textbf{last-day-of-gregorian-month}\,(g\text{-}year, g\text{-}month) \stackrel{\text{def}}{=} \tag{2.27}$$

$$\textbf{gregorian-date-difference}\left(\boxed{g\text{-}year \mid g\text{-}month \mid 1},\ \boxed{\left\{\begin{array}{ll} g\text{-}year + 1 & \textbf{if } g\text{-}month = 12 \\ g\text{-}year & \textbf{otherwise} \end{array}\right\} \mid (g\text{-}month + 1) \bmod [1 \,..\, 12] \mid 1} \right)$$

2.3 Alternative Formulas

... premature, unnecessary, and likely to produce upheavals, and bewilderment of mind and conscience among the people.

Prince Carl Christoph von Lievenin in his denouncement to the Tsar of a plan to switch Russia to the Gregorian calendar (1829)

We noted in Section 2.1 that if we pretend that February always has 30 days and we count months starting from December, the month lengths satisfy the cycle-of-years formulas of Section 1.14 with $c = 12$, $l = 7$, $\Delta = 11$, and $L = 30$; we used the resulting formulas (2.1) and (2.2) to convert Gregorian dates to and from fixed dates. The fraction $7/12$ occurring on the left-hand side of (2.1) is not critical; we will see below that we can use the fraction $4/7$ instead. This leads us to see that the values $c = 7$, $l = 4$, $\Delta = 6$, and $L = 30$ also work, and thus we could substitute

$$\left\lfloor \frac{4m-1}{7} \right\rfloor + 30(m-1) = \left\lfloor \frac{214m - 211}{7} \right\rfloor$$

and

$$\left\lfloor \frac{7n + 217}{214} \right\rfloor$$

respectively, for (2.1) and (2.2) in **fixed-from-gregorian** and **gregorian-from-fixed**.

The justification of the change of $7/12$ to $4/7$ is worth examining in detail because it is typical of arguments used to derive and simplify calendrical formulas. Note that formulas (2.1) and (2.2) are applied only to month numbers 1 through 12. The sum on the left-hand side of equation (2.1) has a corrective term, the floor of

$$C(m) = \frac{7m-2}{12}$$

This has values

m	1	2	3	4	5	6	7	8	9	10	11	12
$\lfloor C(m) \rfloor$	0	1	1	2	2	3	3	4	5	5	6	6

https://doi.org/10.1017/9781107415058.005 Published online by Cambridge University Press

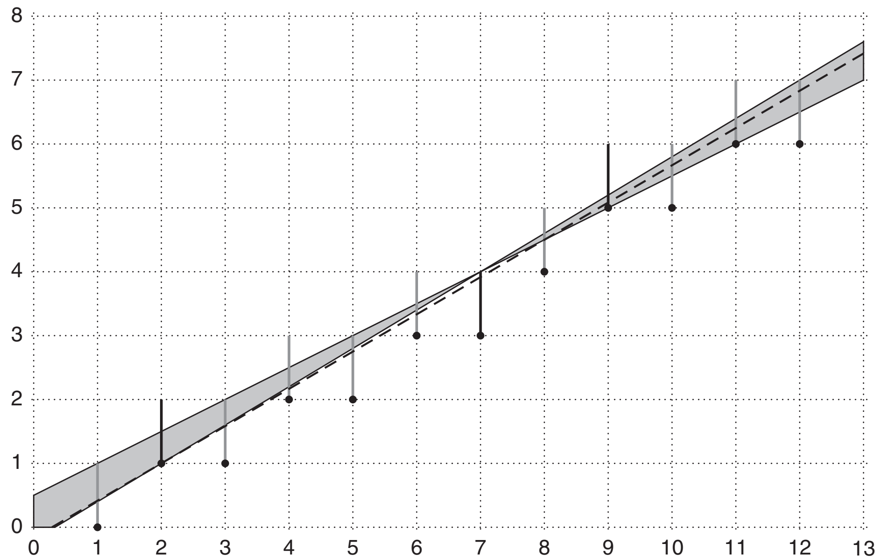

Figure 2.1 The 12 half-open line segments giving the ranges that the corrective line must transect, along with the correction of equation (2.1), that is, the dashed line $C(m) = (7m - 2)/12$. The limiting line segments, $[(2, 1) \,..\, (2, 2))$, $[(7, 3) \,..\, (7, 4))$, and $[(9, 5) \,..\, (9, 6))$, are shown darker than the others. The limiting region, bounded by slopes $1/2$ and $3/5$, is shaded light gray. (Suggested by M. H. Deckers.)

which we show as a set of points $(m, \lfloor C(m) \rfloor)$ in Figure 2.1. Each point can be moved upward by any amount less than 1 without changing the value of $\lfloor C(m) \rfloor$; each range is represented as a half-open vertical line segment in the figure. The problem is to determine lines $L(m) = am + b$ such that $\lfloor L(m) \rfloor = \lfloor C(m) \rfloor$ for the 12 integer values, $1 \leqslant m \leqslant 12$. In other words, we want to determine the lines that transect each of the 12 half-open line segments in the figure. The line we know about, $C(m) = (7m - 2)/12$, is shown dashed. The critical line segments, shown in bold, are $[(2, 1) \,..\, (2, 2))$, $[(7, 3) \,..\, (7, 4))$, and $[(9, 5) \,..\, (9, 6))$. To cut both the half-open line segments $[(7, 3) \,..\, (7, 4))$ and $[(9, 5) \,..\, (9, 6))$, a line $L(m) = am + b$ must have slope $a > 1/2$; to cut both the half-open line segments $[(2, 1) \,..\, (2, 2))$ and $[(7, 3) \,..\, (7, 4))$, a line $L(m) = am + b$ must have slope $a < 3/5$. The fraction $4/7$ has the smallest denominator in the acceptable range. It is clear from the figure that any slope $1/2 < a < 3/5$ (the shaded region) is possible—take the line of the desired slope that goes through the point $(7, 4)$. We make this precise by giving an explicit line for each slope in that range: $a = 1/2 + \varepsilon$, $b = 1/2 - 8\varepsilon$ works for $0 < \varepsilon \leqslant 1/12$, and $a = 3/5 - \varepsilon$, $b = -1/5 + 2\varepsilon$ works for $0 < \varepsilon \leqslant 1/35$. Because $1/2 + 1/12 > 3/5 - 1/35$, there exists a b for each value of a, $1/2 < a < 3/5$.

More significant use of the cycle-of-years formulas is also possible. Instead of pretending that February has 30 days and correcting for the pretense, we could instead consider the annual period from March 1 to the end of February of the following year (see, for example, [2] and [21]). For this shifted year, the cycle-of-years formulas with $c = 12$, $l = 7$, $\Delta = 1$, and $L = 30$ work perfectly because the formulas are never applied in cases for which the length of February matters. Again, as above, the fraction $7/12$ can be replaced by any fraction in the open range $(4/7, 5/8)$; the fraction of smallest denominator in the allowable range, $3/5$, leads to $c = 5$, $l = 3$, $\Delta = 4$, and $L = 30$. The well-known "Zeller's congruence," [22], [23], derived in the next section, is based on this idea, as are calendar formulas such as [20] (see [14, pp. 61–63]), and many others.

The shifted-year formulas are then applied as follows. The number of days in months starting in March prior to month m (where March is $m = 1$, April is $m = 2$, ..., February is $m = 12$) is

$$\left\lfloor \frac{3m-1}{5} \right\rfloor + 30(m-1) = \left\lfloor \frac{153m-151}{5} \right\rfloor$$

To consider March ($month = 3$) of $year$ to be month $m = 1$ of year $y = year + 1$, April ($month = 4$) of $year$ to be month $m = 2$ of year $y = year + 1$, ..., February ($month = 2$) of $year + 1$ to be month $m = 12$ of year $y = year + 1$, we shift the month numbers using

$$m = (month - 2) \bmod [1 \,..\, 12]$$

and adjust the year using

$$y = year + \left\lfloor \frac{month + 9}{12} \right\rfloor$$

We can simplify this further by expressing the calculations in terms of $m' = m - 1$ and $y' = y - 1$ calculated as

$$\begin{aligned} m' &= (month - 3) \bmod 12 \\ y' &= year - \left\lfloor \frac{m'}{10} \right\rfloor \end{aligned}$$

Because there are 306 days in the period March–December, we can write

$$\textbf{alt-fixed-from-gregorian}\left(\boxed{\mathit{year} \mid \mathit{month} \mid \mathit{day}} \right) \stackrel{\text{def}}{=} \tag{2.28}$$

$$\textbf{gregorian-epoch} - 1 - 306 + 365 \times y' + \sum \widetilde{y} \times \widetilde{a} + \left\lfloor \frac{3 \times m' + 2}{5} \right\rfloor + 30 \times m' + day$$

where

$$\begin{aligned} m' &= (month - 3) \bmod 12 \\ y' &= year - \left\lfloor \frac{m'}{10} \right\rfloor \end{aligned}$$

$$\begin{aligned}\widetilde{y} &= y' \xrightarrow{\text{rad}} \langle 4, 25, 4\rangle\\ \widetilde{a} &= \langle 97, 24, 1, 0\rangle\end{aligned}$$

The number of leap years under the Gregorian rule depends on the number of quadrennia, centuries, and 400-year periods. We compute the number of elapsed periods of 4, 100, and 400 years, using the mixed-radix notation of Section 1.10. Accordingly, the approximation for the year is expressed in base $\langle 4, 25, 4\rangle$, there being 4 years in a quadrennium, 25 quadrennia in a century, and 4 centuries in 400 years. Each quadrennium contributes 1 leap day, each century contributes 24, every 400 years contribute 97, while an ordinary year contributes none. So we count the total number of leap days by taking the sum of the products of the individual contributions $\widetilde{a} = \langle 97, 24, 1, 0\rangle$ with the corresponding components of $\widetilde{y} = \langle n_{400}, n_{100}, n_4, n_1\rangle$, which is the year number expressed in base $\langle 4, 25, 4\rangle$ using the same variable names as in equation (2.21) for the counts. To avoid subscripts the formula employs vector notation, $\sum \widetilde{y} \times \widetilde{a}$, with the intention that the operation in the sum is performed on like-indexed elements of $\widetilde{y}$ and $\widetilde{a}$ (as explained on page 31).

In the reverse direction, the same ideas lead to

$$\textbf{alt-gregorian-from-fixed}\,(date) \stackrel{\text{def}}{=} \boxed{year \mid month \mid day} \tag{2.29}$$

where

$$\begin{aligned}
y &= \textbf{gregorian-year-from-fixed}\\
&\qquad(\textbf{gregorian-epoch} - 1 + date + 306)\\
prior\text{-}days &= date - \textbf{fixed-from-gregorian}\left(\boxed{y-1 \mid \textbf{march} \mid 1}\right)\\
month &= \left(\left\lfloor \frac{1}{153} \times (5 \times prior\text{-}days + 2)\right\rfloor + 3\right) \bmod [1 \,..\, 12]\\
year &= y - \left\lfloor \frac{month + 9}{12} \right\rfloor\\
day &= date - \textbf{fixed-from-gregorian}\left(\boxed{year \mid month \mid 1}\right) + 1
\end{aligned}$$

All these alternative functions are simpler in appearance than our original functions converting Gregorian dates to and from fixed dates, but intuition has been lost with a negligible gain in efficiency. Versions of these alternative functions are the basis for the conversion algorithms in [7] (see [5, p. 604]) and many others because, by using formulas (1.17) and (1.29) to eliminate the modulus and adjusted-remainder operators, **alt-fixed-from-gregorian** and **alt-gregorian-from-fixed** can be written as single arithmetic expressions over the integer operations of addition, subtraction, multiplication, and division with no conditionals.

Finally, we can give an alternative version of **gregorian-year-from-fixed** by doing a simple but approximate calculation and correcting it when needed. The approximate year is found by dividing the number of days from the epoch until 2 days after the given fixed date by the average Gregorian year length. The fixed date

https://doi.org/10.1017/9781107415058.005 Published online by Cambridge University Press

of the start of the next year is then found; if the given date is before the start of that next year, then the approximation is correct; otherwise the correct year is the year after the approximation:

$$\textbf{alt-gregorian-year-from-fixed}\,(date) \stackrel{\text{def}}{=} \tag{2.30}$$

$$\begin{cases} approx & \textbf{if } date < start \\ approx + 1 & \textbf{otherwise} \end{cases}$$

where

$$\begin{aligned} approx &= \left\lfloor \frac{400}{146097} \times (date - \textbf{gregorian-epoch} + 2) \right\rfloor \\ start &= \textbf{gregorian-epoch} + 365 \times approx + \sum \widetilde{y} \times \widetilde{a} \\ \widetilde{y} &= approx \xrightarrow{\text{rad}} \langle 4, 25, 4 \rangle \\ \widetilde{a} &= \langle 97, 24, 1, 0 \rangle \end{aligned}$$

The adjustment by 2 days is needed because the number of days in years 1 through n of the 400-year cycle can fall short of $365.2425 \times n$ by as much as 1.4775 days (for $n = 303$). Thus 2 is the smallest integer we can add that guarantees that, for the first day of any year n, $approx \geqslant n - 1$.

2.4 The Zeller Congruence

These examples [of errors in published works] show that, even for the possessor of such reference books, formulæ are not completely superfluous, as they make it possible to double check the handbooks by means of calculations, without very much trouble.

Rektor Chr. Zeller: "Kalender-Formeln,"
Mathematisch-naturwissenschaftliche Mitteilungen des mathematisch-naturwissenschaftlichen Vereins in Württemberg (1885)

Zeller's congruence [22], [23] (see [21]), due to Christian Zeller, a Protestant minister and seminary director in Germany in the second half of the nineteenth century, is a relatively simple method—often used in feats of "mental agility"—for determining the day of the week, given any Gregorian calendar date. The main idea is to add up elapsed days from the epoch of the Gregorian until the day in question but, since we are interested only in the day of the week, always discarding multiples of 7, leaving numbers in the range 0–6 to represent Sunday through Saturday.

We can use the simplification of **fixed-from-gregorian** suggested on page 65 to derive Zeller's congruence. We substitute the number of elapsed days into the definition of **day-of-week-from-fixed** from page 33, ignoring the zero-valued

$$\textbf{gregorian-epoch} - 1$$

https://doi.org/10.1017/9781107415058.005 Published online by Cambridge University Press

to obtain the day of the week

$$
\begin{aligned}
&-306 + 365y' + \left\lfloor \frac{y'}{4} \right\rfloor - \left\lfloor \frac{y'}{100} \right\rfloor + \left\lfloor \frac{y'}{400} \right\rfloor \\
&+ \left\lfloor \frac{3m' + 2}{5} \right\rfloor + 30m' + day
\end{aligned} \tag{2.31}
$$

taken modulo 7, where

$$
\begin{aligned}
m' &= (month - 3) \bmod 12 \\
y' &= year - \left\lfloor \frac{m'}{10} \right\rfloor
\end{aligned}
$$

Zeller used month numbers 3–14 for March–February (that is, he renumbered January as 13 and February as 14), and dealt separately with centuries and the year within a century:

$$
\begin{aligned}
m &= m' + 3 = \begin{cases} month + 12 & \textbf{if } month < \textbf{march} \\ month & \textbf{otherwise} \end{cases} \\
y &= y' \bmod 100 \\
c &= \lfloor y'/100 \rfloor
\end{aligned}
$$

Making these substitutions in (2.31) we get

$$
\begin{aligned}
&-306 + 365(100c + y) + \left\lfloor \frac{100c + y}{4} \right\rfloor - c + \left\lfloor \frac{c}{4} \right\rfloor \\
&+ \left\lfloor \frac{3m - 7}{5} \right\rfloor + 30(m - 3) + day
\end{aligned}
$$

taken modulo 7. Discarding multiples of 7, dividing by 10 instead of 5, and regrouping, this becomes

$$
-5 + (2c + y) + 4c + \left\lfloor \frac{y}{4} \right\rfloor - c + \left\lfloor \frac{c}{4} \right\rfloor + \left\lfloor \frac{6m - 14}{10} \right\rfloor + 2m - 6 + day
$$

Finally, rearranging terms, we get

$$
(day - 1) + \left\lfloor \frac{(m + 1)26}{10} \right\rfloor + y + \left\lfloor \frac{y}{4} \right\rfloor + \left\lfloor \frac{c}{4} \right\rfloor - 2c
$$

taken modulo 7, which is Zeller's congruence as he wrote it in [23], except that he numbered the days Sunday–Saturday 1–7 so he had *day* not (*day* − 1).

Other versions of this formula often attributed to Zeller can be obtained by algebraic manipulation. Zeller [23] also gave a similar formula for Julian calendar dates.

2.5 Holidays

The information in this book has been gathered from many sources. Every effort has been made to insure its accuracy. Holidays sometimes are subject to change, however, and Morgan Guaranty cannot accept responsibility should any date or statement included prove to be incorrect.

Morgan Guaranty: *World Calendar* (1978)

Secular holidays on the Gregorian calendar are either on fixed days or on a particular day of the week relative to the beginning or end of a month. (An extensive list of secular holidays can be found in [10].) Fixed holidays are trivial to deal with; for example, to determine the R.D. date of United States Independence Day in a given Gregorian year we would use

$$\textbf{independence-day}\,(\textit{g-year}) \stackrel{\text{def}}{=} \tag{2.32}$$

$$\textbf{fixed-from-gregorian}\left(\boxed{\textit{g-year} \mid \textbf{july} \mid 4}\right)$$

Other holidays are on the nth occurrence of a given day of the week, counting from either the beginning or the end of the month. The U.S. Labor Day, for example, is the first Monday in September, and U.S. Memorial Day is the last Monday in May. To find the R.D. date of the nth k-day ($n \neq 0$, k is the day of the week) on, or after or before, a given Gregorian date (counting forward when $n > 0$, backward when $n < 0$), we write

$$\textbf{nth-kday}\,(n, k, \textit{g-date}) \stackrel{\text{def}}{=} \tag{2.33}$$

$$\begin{cases} 7 \times n + \textbf{kday-before}\,(k, \textbf{fixed-from-gregorian}\,(\textit{g-date})) & \textbf{if } n > 0 \\ 7 \times n + \textbf{kday-after}\,(k, \textbf{fixed-from-gregorian}\,(\textit{g-date})) & \textbf{if } n < 0 \\ \textbf{bogus} & \textbf{otherwise} \end{cases}$$

using the functions of Section 1.12 (page 34); when $n = 0$ the special constant **bogus** is returned, signifying a nonexistent value. It is convenient to define two special cases for use with this function:

$$\textbf{first-kday}\,(k, \textit{g-date}) \stackrel{\text{def}}{=} \textbf{nth-kday}\,(1, k, \textit{g-date}) \tag{2.34}$$

gives the fixed date of the first k-day on or after a Gregorian date;

$$\textbf{last-kday}\,(k, \textit{g-date}) \stackrel{\text{def}}{=} \textbf{nth-kday}\,(-1, k, \textit{g-date}) \tag{2.35}$$

gives the fixed date of the last k-day on or before a Gregorian date.

Now we can define holiday dates, such as U.S. Labor Day,

$$\textbf{labor-day}\,(\textit{g-year}) \stackrel{\text{def}}{=} \tag{2.36}$$

$$\textbf{first-kday}\left(\textbf{monday}, \boxed{\textit{g-year} \mid \textbf{september} \mid 1}\right)$$

U.S. Memorial Day,

$$\textbf{memorial-day}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{2.37}$$

$$\textbf{last-kday}\left(\textbf{monday},\ \boxed{g\text{-}year \mid \textbf{may} \mid 31}\right)$$

or U.S. Election Day (the Tuesday falling after the first Monday in November, which is the first Tuesday on or after November 2),

$$\textbf{election-day}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{2.38}$$

$$\textbf{first-kday}\left(\textbf{tuesday},\ \boxed{g\text{-}year \mid \textbf{november} \mid 2}\right)$$

Further, we can determine the starting and ending dates of U.S. daylight saving time (as of 2007, the second Sunday in March and the first Sunday in November, respectively):

$$\textbf{daylight-saving-start}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{2.39}$$

$$\textbf{nth-kday}\left(2, \textbf{sunday},\ \boxed{g\text{-}year \mid \textbf{march} \mid 1}\right)$$

$$\textbf{daylight-saving-end}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{2.40}$$

$$\textbf{first-kday}\left(\textbf{sunday},\ \boxed{g\text{-}year \mid \textbf{november} \mid 1}\right)$$

The main Christian holidays are Christmas, Easter, and various days connected with them (Advent Sunday, Ash Wednesday, Good Friday, and others; see [11, vol. V, pp. 844–853]). The date of Christmas on the Gregorian calendar is fixed and hence easily computed:

$$\textbf{christmas}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{2.41}$$

$$\textbf{fixed-from-gregorian}\left(\boxed{g\text{-}year \mid \textbf{december} \mid 25}\right)$$

The related dates of Advent Sunday (the Sunday closest to November 30) and Epiphany (the first Sunday after January 1)[7] are computed by

$$\textbf{advent}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{2.42}$$

$$\textbf{kday-nearest}\left(\textbf{sunday}, \textbf{fixed-from-gregorian}\left(\boxed{g\text{-}year \mid \textbf{november} \mid 30}\right)\right)$$

[7] Outside the United States, Epiphany is celebrated on January 6.

https://doi.org/10.1017/9781107415058.005 Published online by Cambridge University Press

$$\textbf{epiphany}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{2.43}$$

$$\textbf{first-kday}\left(\textbf{sunday},\ \boxed{g\text{-}year \mid \textbf{january} \mid 2}\right)$$

The date of the Assumption (August 15), celebrated in Catholic countries, is fixed and presents no problem. We defer the calculation of Easter and related "movable" Christian holidays, which depend on lunar events, until Chapter 9.

To find all instances of Friday the Thirteenth within a range of fixed dates *range*, we mimic (1.39) as follows:

$$\textbf{unlucky-fridays-in-range}\,([a \,..\, b)) \stackrel{\text{def}}{=} \tag{2.44}$$

$$\begin{cases} \left\{ \begin{array}{ll} \langle fri \rangle & \textbf{if } date_{\textbf{day}} = 13 \\ \langle\,\rangle & \textbf{otherwise} \end{array} \right\} \,\|\, \textbf{unlucky-fridays-in-range}\,([fri + 1 \,..\, b)) \\ \qquad \textbf{if } fri \in range \\ \langle\,\rangle \quad \textbf{otherwise} \end{cases}$$

where

$$\begin{aligned} range &= [a \,..\, b) \\ fri &= \textbf{kday-on-or-after}\,(\textbf{friday}, a) \\ date &= \textbf{gregorian-from-fixed}\,(fri) \end{aligned}$$

Then, to list the "unlucky" Fridays in a given Gregorian year, we use that year as the range:

$$\textbf{unlucky-fridays}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{2.45}$$

$$\textbf{unlucky-fridays-in-range}\,(\textbf{gregorian-year-range}\,(g\text{-}year))$$

References

[1] *Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac*, Her Majesty's Stationery Office, London, 1961.

[2] J. A. Ball, *Algorithms for RPN Calculators*, John Wiley & Sons, New York, 1978.

[3] A. Belenkiy and E. V. Echagüe, "History of One Defeat: Reform of the Julian calendar as Envisaged by Isaac Newton," *Notes & Records of the Royal Society*, vol. 59, pp. 223–254, 2005.

[4] G. V. Coyne, M. A. Hoskin, and O. Pedersen, *Gregorian Reform of the Calendar: Proceedings of the Vatican Conference to Commemorate Its 400th Anniversary, 1582–1982*, Pontifica Academica Scientiarum, Specola Vaticana, Vatican, 1983.

[5] L. E. Doggett, “Calendars,” *Explanatory Supplement to the Astronomical Almanac*, P. K. Seidelmann, ed., University Science Books, Mill Valley, CA, pp. 575–608, 1992.

[6] J. Dutka, “On the Gregorian Revision of the Julian Calendar,” *Mathematical Intelligencer*, vol. 10, pp. 56–64, 1988.

[7] H. F. Fliegel and T. C. van Flandern, “A Machine Algorithm for Processing Calendar Dates,” *Communications of the ACM*, vol. 11, p. 657, 1968.

[8] J. K. Fotheringham, “The Calendar,” *The Nautical Almanac and Astronomical Ephemeris*, His Majesty’s Stationery Office, London, 1931–1934; revised 1935–1938; abridged 1939–1941.

[9] F. K. Ginzel, *Handbuch der mathematischen und technischen Chronologie*, vol. 3, J. C. Hinrichs’sche Buchhandlung, Leipzig, 1914. Reprinted by F. Ullmann Verlag, Zwickau, 1958.

[10] R. W. Gregory, *Special Days*, Citadel, Secaucus, NJ, 1975. Previous editions appeared under the title *Anniversaries and Holidays*.

[11] J. Hastings, ed., *Encyclopædia of Religion and Ethics*, Charles Scribner’s Sons, New York, 1911.

[12] L. Lamport, “On the Proof of Correctness of a Calendar Program,” *Communications of the ACM*, vol. 22, pp. 554–556, 1979.

[13] C. L. Liu, *Introduction to Combinatorial Mathematics*, McGraw-Hill, New York, 1968.

[14] J. Meeus, *Astronomical Algorithms*, 2nd edn., Willmann-Bell, Richmond, VA, 1998.

[15] A. K. Michels, *The Calendar of the Roman Republic*, Princeton University Press, Princeton, NJ, 1967.

[16] G. Moyer, “The Gregorian Calendar,” *Scientific American*, vol. 246, no. 5, pp. 144–152, May 1982.

[17] R. Poole, “‘Give Us Our Eleven Days!’: Calendar Reform in Eighteenth-Century England,” *Past and Present*, no. 149, pp. 95–139, November 1995.

[18] V. F. Rickey, “Mathematics of the Gregorian Calendar,” *Mathematical Intelligencer*, vol. 7, pp. 53–56, 1985.

[19] J. Shallit, “Pierce Expansions and Rules for the Determination of Leap Years,” *Fibonacci Quarterly*, vol. 32, pp. 416–423, 1994.

[20] R. G. Tantzen, "Algorithm 199: Conversions Between Calendar Date and Julian Day Number," *Communications of the ACM*, vol. 6, p. 444, 1963.

[21] J. V. Uspensky and M. A. Heaslet, *Elementary Number Theory*, McGraw-Hill, New York, 1939.

[22] C. Zeller, "Problema duplex Calendarii fundamentale," *Bulletin Société Mathématique*, vol. 11, pp. 59–61, March 1883.

[23] C. Zeller, "Kalender-Formeln," *Acta Mathematica*, vol. 9, pp. 131–136, November 1886.

Illustration from Lichtenberg's 1757 *Göttinger Taschen Kalender*: a reverse copy of a man pouring gin over the head of another, and a flag reading "Give us our eleven days," in protest at the British abandonment of the Julian calendar in September 1752. From the first plate of William Hogarth's 1755 "An Election Entertainment." (Courtesy of the British Museum, London.)

https://doi.org/10.1017/9781107415058.005 Published online by Cambridge University Press