14

Time and Astronomy

Ask my friend l'Abbé Sallier to recommend to you some meagre philomath, to teach you a little geometry and astronomy; not enough to absorb your attention, and puzzle your intellects, but only enough, not to be grossly ignorant of either. I have of late been a sort of an astronome malgré moy, *by bringing last Monday, into the house of Lords, a bill for reforming our present Calendar, and taking the New Style. Upon which occasion I was obliged to talk some astronomical jargon, of which I did not understand one word, but got it by heart, and spoke it by rote from a master. I wished that I had known a little more of it myself; and so much I would have you know.*

Letter from Philip Dormer Stanhope (Fourth Earl of Chesterfield)
to his son, February 28, 1751 C.E. (Julian)

The calendars in the second part of this book are based on accurate astronomical calculations. This chapter defines the essential astronomical terms and describes the necessary astronomical functions. A fuller treatment can be found in the references—an especially readable discussion is given in [14].

We begin with an explanation of how the positions of locations on Earth and of heavenly bodies are specified, followed by an examination of the notion of time itself. After discussing the 24-hour day, we summarize the different types of years and months used by various calendars along with algorithms that closely approximate the times of astronomical events—notably equinoxes, solstices, and new moons. These astronomical functions are adapted from those in [18] and [4] and require 64-bit arithmetic.

Most of the algorithms are centered around the present date, for which they are accurate to within about 2 minutes. Their accuracy decreases for the far-distant past or future. More accurate algorithms exist [3] but are extremely complex and not needed for our purposes.

Chapter 18 applies the methods of this chapter to several "speculative" astronomical calendars.

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14.1 Position

The cause of the error is very simple ... In journeying eastward he had gone towards the sun, and the days therefore diminished for him as many times four minutes as he crossed degrees in this direction. There are three hundred and sixty degrees in the circumference of the Earth; and these three hundred and sixty degrees, multiplied by four minutes, gives precisely twenty-four hours—that is, the day unconsciously gained.

Jules Verne: *Around the World in Eighty Days* (1873)

Locations on Earth are specified by giving their latitude and longitude. The (*terrestrial*) *latitude* of a geographic location is the angular distance on the Earth, measured in degrees from the equator, along the meridian of the location. Similarly, the (*terrestrial*) *longitude* of a geographic location is the angular distance on the Earth measured in degrees from the Greenwich meridian (which is defined as 0°), on the outskirts of London. Thus, for example, the location of Jerusalem is described as being 31.8° north, 35.2° east. In the algorithms, we take northern latitudes as positive and southern latitudes as negative. For longitudes, we take east from Greenwich as positive and west as negative;[1] thus a positive longitude means a time later than at Greenwich, and a negative longitude means a time earlier than at Greenwich.

As we will see in the next section, locations on Earth are also associated with a *time zone*, which is needed for determining the local clock time. For some calculations (local sunrise and sunset, in particular), the elevation above sea level is also a factor. Thus, the complete specification of a location that we use is

latitude	*longitude*	*elevation*	*zone*

We specify the time zone as the difference from Universal Time (U.T.; see Section 14.2) as a fraction of a day, and we measure the elevation above sea level in meters. For example, the specification of Urbana, Illinois, is

$$\textbf{urbana} \stackrel{\text{def}}{=} \boxed{40.1^\circ \mid -88.2^\circ \mid 225\text{ m} \mid -6^{\text{h}}} \tag{14.1}$$

because Urbana is at latitude 40.1° north, longitude 88.2° west, 225 meters above sea level, and 6 hours before U.T.; Greenwich is specified by

$$\textbf{greenwich} \stackrel{\text{def}}{=} \boxed{51.4777815^\circ \mid 0^\circ \mid 46.9\text{ m} \mid 0^{\text{h}}} \tag{14.2}$$

Muslims turn towards Mecca for prayer, Jews face Jerusalem, and the Bahá'í face Acre. Their locations are, respectively,

$$\textbf{mecca} \stackrel{\text{def}}{=} \boxed{21^\circ 25' 24'' \mid 39^\circ 49' 24'' \mid 298\text{ m} \mid 3^{\text{h}}} \tag{14.3}$$

$$\textbf{jerusalem} \stackrel{\text{def}}{=} \boxed{31.78^\circ \mid 35.24^\circ \mid 740\text{ m} \mid 2^{\text{h}}} \tag{14.4}$$

$$\textbf{acre} \stackrel{\text{def}}{=} \boxed{32.94^\circ \mid 35.09^\circ \mid 22\text{ m} \mid 2^{\text{h}}} \tag{14.5}$$

[1] This is in agreement with the standard of the International Astronomical Union but inconsistent with common sense and a century of common practice. See [18, p. 93].

If a spherical Earth is assumed, the direction (measured in degrees east of due north) of a location at latitude φ' and longitude ψ', along a great circle, when one stands at another location with latitude φ and longitude ψ, can be determined by spherical trigonometry[2] (see [16] for details):

$$\textbf{direction}\left(\boxed{\varphi \mid \psi \mid — \mid —}, \boxed{\varphi' \mid \psi' \mid — \mid —}\right) \stackrel{\text{def}}{=} \tag{14.6}$$

$$\begin{cases} 0° & \textbf{if } x = y = 0 \text{ or } \varphi' = 90° \\ 180° & \textbf{if } \varphi' = -90° \\ \textbf{arctan}\,(y, x) & \textbf{otherwise} \end{cases}$$

where

$$y = \sin(\psi' - \psi)$$
$$x = \cos\varphi \times \tan\varphi' - \sin\varphi \times \cos(\psi - \psi')$$

This formula uses the two-argument arctangent function

$$\textbf{arctan}\,(y, x) \stackrel{\text{def}}{=} \tag{14.7}$$

$$\begin{cases} \textbf{bogus} & \textbf{if } x = y = 0 \\ \left\{\begin{array}{ll} \text{sign}\,(y) \times 90° & \textbf{if } x = 0 \\ \alpha & \textbf{if } x \geqslant 0 \\ \alpha + 180° & \textbf{otherwise} \end{array}\right\} \bmod 360 & \textbf{otherwise} \end{cases}$$

where

$$\alpha = \arctan\left(\frac{y}{x}\right)$$

to find the arctangent of y/x in the appropriate quadrant; this angle changes when the two locations are on opposite sides of the globe. For example, in Urbana, Illinois, the *qibla* (direction of Mecca) is about 49° east of due north whereas Jerusalem is at 45° east.[3]

The positions of heavenly bodies can be measured in a manner corresponding to terrestrial longitude and latitude by reference to meridians (great circles passing through the two poles) of the celestial sphere. In this *equatorial* coordinate system, *right ascension* corresponds to longitude and *declination* to latitude. For marking the positions of the sun and moon, however, astronomers normally use an alternative coordinate system in which (*celestial* or *ecliptical*) *longitude* is measured along the ecliptic (the sun's apparent path among the stars) and (*celestial*) *latitude* is measured from the ecliptic. Zero longitude is at a position called the *First Point of Aries* (see page 219).

[2] Imperfect arithmetic accuracy can result in meaningless values of **direction** when *location* and *focus* are nearly coincident or antipodal.

[3] Despite the antiquity of such great-circle calculations in Muslim and Jewish sources, many mosques and synagogues are designed according to other conventions. See [1].

14.2 Time

What, then, is time? I know well enough what it is, provided that nobody asks me; but if I am asked what it is and try to explain, I am baffled.

Saint Augustine: *Confessions* (circa 400)

Three distinct methods of measuring time are in use today:[4]

- *Solar time* is based on the solar day, which measures the time between successive transits of the sun across the meridian (the north-south line, through the zenith, the point overhead in the sky). As we will see, this period varies because of the nonuniform motion of the Earth.
- *Sidereal time* varies less than solar time and indicates the orientation of the rotating Earth with respect to the stars. It is measured as the right ascension at a given moment of those points in the sky just crossing the meridian. Thus, *local* sidereal time depends on terrestrial longitude and differs from observatory to observatory.
- *Dynamical Time* is a uniform measure taking the frequency of oscillation of certain atoms as the basic building block. Various forms of Dynamical Time use different frames of reference, which makes a difference in a universe governed by relativity.

The ordinary method of measuring time is called *Universal Time* (U.T.). It is the local mean solar time, reckoned from midnight, at the observatory in Greenwich, England, the location of the 0° meridian.[5] The equivalent designation "Greenwich Mean Time," abbreviated G.M.T., has fallen into disfavor with astronomers because of confusion as to whether days begin at midnight or noon (before 1925, 00:00 G.M.T. meant noon; from 1925 onward it has meant midnight).

There are several closely related types of Universal Time. Civil time keeping uses Coordinated Universal Time (U.T.C.), which since 1972 has been atomic time adjusted periodically by leap seconds to keep it close to the prime meridian's mean solar time; see [17] and [25]. We use U.T.C. for calendrical purposes (except that we insert all leap seconds at the end of the year, whereas in actual practice they are often added during the year[6]) expressed as a fraction of a solar day.

From the start of the spread of clocks and pocket watches in Europe until the early 1800s, each locale would set its clocks to local mean time. Each longitudinal degree of separation gives rise to a 4 minute difference in local time. For example,

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[4] Ephemeris time, which takes the orbital motions in the solar system as the basic building block, is an outdated time scale as of 1984.

[5] The formal recognition of Greenwich as the "prime meridian" dates from the International Meridian Conference of 1884, but it had been informal practice from 1767. The French, however, continued to treat Paris as the prime meridian until 1911, when they switched to Greenwich, referring to it as "Paris Mean Time, minus nine minutes twenty-one seconds." France did not formally switch to Universal Time until 1978; see [30] and [11].

[6] The International Telecommunications Union states in ITU-R TF.460-6, sec. 2.1, that "A positive or negative leap-second should be the last second of a UTC month, but first preference should be given to the end of December and June, and second preference to the end of March and September." See [5] for a concise history of leap seconds.

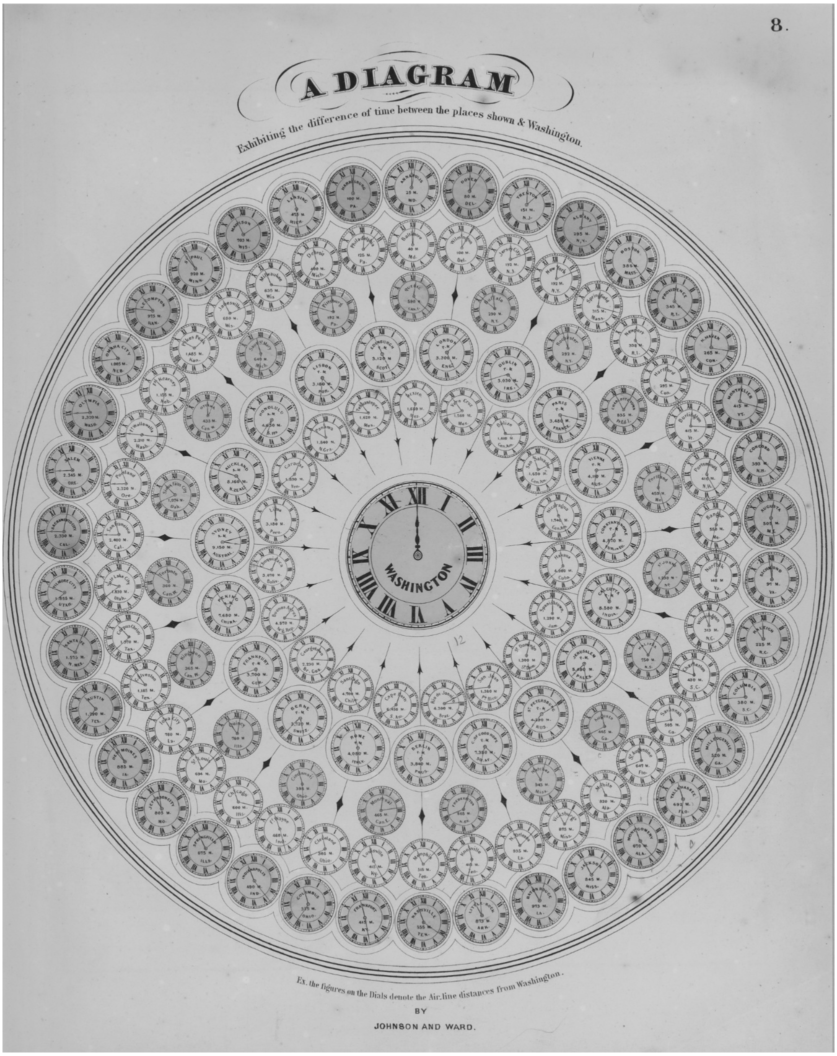

Figure 14.1 Differences in local time between Washington, D.C., and cities around the world: *Johnson's New Illustrated Family Atlas*, 1862. (Collection of E.M.R.)

because the meridian of Paris is $2°20'15''$ east, its local mean time is 9 minutes, 21 seconds, ahead of U.T. As another example, Beijing is $116°25'$ east; the time difference from U.T. is $7^{h}45^{m}40^{s}$. Figure 14.1 shows a helpful 1862 atlas page showing differences in local time before the advent of time zones.

"Standard time" was first used by British railway companies in 1840; time zones were first adopted by North American railway companies in the late 1800s [11]. The history of time measurement and time zones in the United States is discussed at length in [24]. Most of Western Europe is today in one zone; the 48 contiguous states of the United States are divided into four zones. A very extensive

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list of locations and the times that they use today and that they used historically appears in [29] for countries outside the United States and in [28] for the United States. We ignore the issue of daylight-saving (summer) time because it is irrelevant to the calendars we discuss.

The local *mean* time zone changes every 15°. We express time zones as a fraction of a day, and so we simply divide the longitude φ by a full circle:

$$\textbf{zone-from-longitude}\,(\varphi) \stackrel{\text{def}}{=} \frac{\varphi}{360^\circ} \tag{14.8}$$

Standard time zones are drawn more arbitrarily—see Figure 14.2.

Converting between Universal Time and local mean time is now an easy matter:

$$\textbf{universal-from-local}\,(t_\ell, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.9}$$
$$t_\ell - \textbf{zone-from-longitude}\left(\mathit{location}_{\textbf{longitude}}\right)$$

In the other direction,

$$\textbf{local-from-universal}\,(t_{\mathrm{u}}, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.10}$$
$$t_{\mathrm{u}} + \textbf{zone-from-longitude}\left(\mathit{location}_{\textbf{longitude}}\right)$$

where t_ℓ and t_{u} are local time and Universal Time, respectively. To convert between Universal Time and standard zone time, we need to use the time zone of the location:

$$\textbf{standard-from-universal}\,(t_{\mathrm{u}}, \mathit{location}) \stackrel{\text{def}}{=} t_{\mathrm{u}} + \mathit{location}_{\textbf{zone}} \tag{14.11}$$

In the other direction,

$$\textbf{universal-from-standard}\,(t_{\mathrm{s}}, \mathit{location}) \stackrel{\text{def}}{=} t_{\mathrm{s}} - \mathit{location}_{\textbf{zone}} \tag{14.12}$$

where time differences or zones are expressed as a fraction of a day after Greenwich time. To convert from local mean time to standard zone time, or vice versa, we combine the differences between Universal Time and standard time and between local mean time and Universal Time:

$$\textbf{standard-from-local}\,(t_\ell, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.13}$$
$$\textbf{standard-from-universal}\,\left(\textbf{universal-from-local}\,(t_\ell, \mathit{location}), \mathit{location}\right)$$

and in the other direction

$$\textbf{local-from-standard}\,(t_{\mathrm{s}}, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.14}$$
$$\textbf{local-from-universal}\,\left(\textbf{universal-from-standard}\,(t_{\mathrm{s}}, \mathit{location}), \mathit{location}\right)$$

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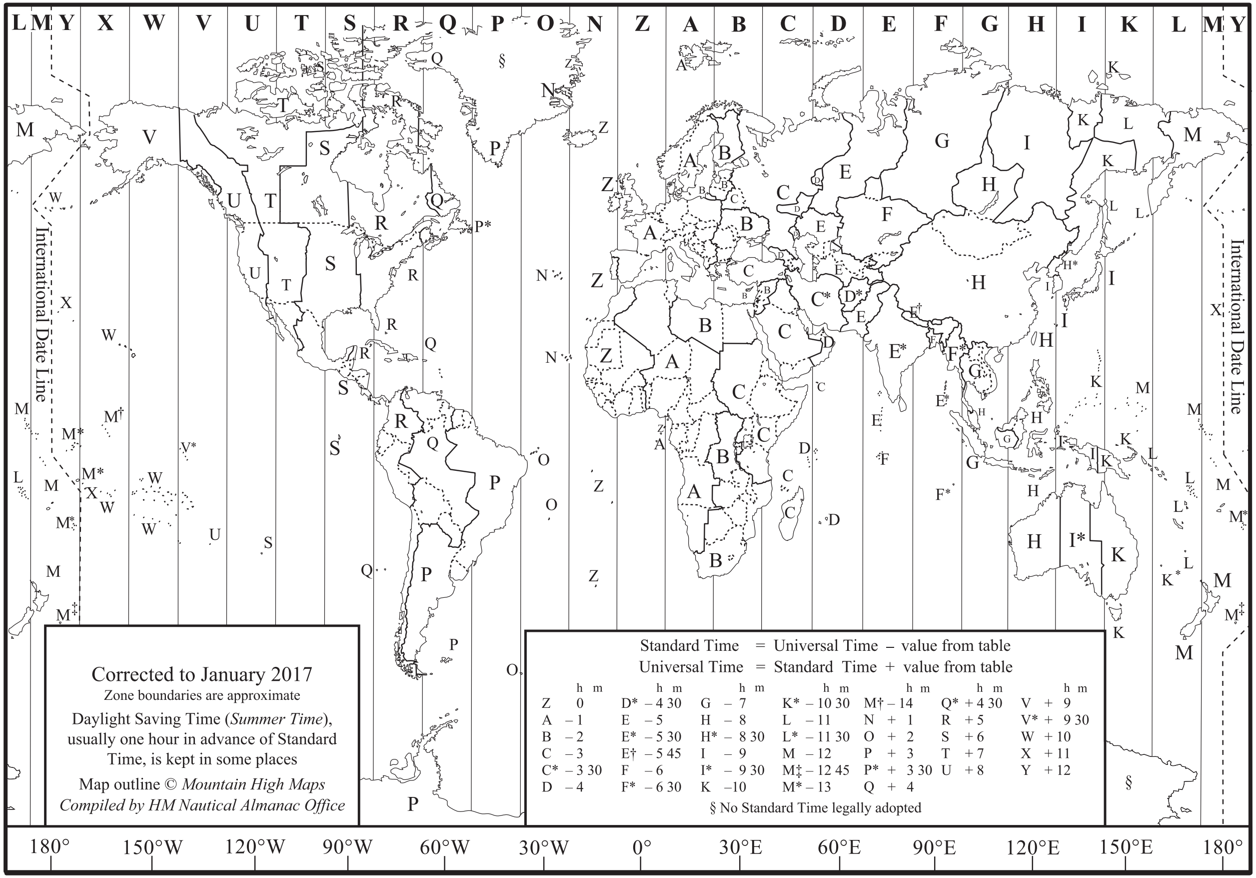

Figure 14.2 Standard time zones of the world as of January, 2017. © Crown Copyright and/or database rights. (Reproduced by permission of the Controller of Her Majesty's Stationery Office and the UK Hydrographic Office, `gov.uk/ukho`.)

For example, Jerusalem is 35.2° east of Greenwich; its time zone is U.T.+2^{h}. Therefore, to obtain standard time in Jerusalem from the local mean time, a net offset of $20^{\mathrm{m}}48^{\mathrm{s}}$ is added.

Astronomical calculations are typically done using Dynamical Time, with its unchanging time units. (There are various forms of Dynamical Time, but the differences are too small to be of concern to us.) Solar time units, however, are not constant through time, mainly because of the retarding effects of tides and the atmosphere, which cause a relatively steady lengthening of the day; they contribute what is called a "secular" (that is, steadily changing) term to its length. This slowdown causes the mean solar day to increase in length by about 1.7 milliseconds per century. Because Universal Time is based on the Earth's speed of rotation, which is slowly decreasing, the discrepancy between Universal and Dynamical Time is growing. It now stands at about 67 seconds and is currently increasing at about an average of 1 second per year. To account for the vagaries in the length of a U.T. day, every now and then a *leap second* is inserted (usually between December 31 and January 1), thereby keeping our clocks—which show Universal Time—in tune with the gradually slowing rotation of Earth. Because the accumulated discrepancy is not entirely predictable and is not accurately known for the years prior to 1600, we use the following ad hoc function for this *ephemeris correction*:

$$\textbf{ephemeris-correction}\,(t) \stackrel{\text{def}}{=} \begin{cases} c_{2051} & \textbf{if } 2051 \leqslant \mathit{year} \leqslant 2150 \\ c_{2006} & \textbf{if } 2006 \leqslant \mathit{year} \leqslant 2050 \\ c_{1987} & \textbf{if } 1987 \leqslant \mathit{year} \leqslant 2005 \\ c_{1900} & \textbf{if } 1900 \leqslant \mathit{year} \leqslant 1986 \\ c_{1800} & \textbf{if } 1800 \leqslant \mathit{year} \leqslant 1899 \\ c_{1700} & \textbf{if } 1700 \leqslant \mathit{year} \leqslant 1799 \\ c_{1600} & \textbf{if } 1600 \leqslant \mathit{year} \leqslant 1699 \\ c_{500} & \textbf{if } 500 \leqslant \mathit{year} \leqslant 1599 \\ c_{0} & \textbf{if } -500 < \mathit{year} < 500 \\ \mathit{other} & \textbf{otherwise} \end{cases} \tag{14.15}$$

where

$$\mathit{year} = \textbf{gregorian-year-from-fixed}\,(\lfloor t \rfloor)$$

$$c = \frac{1}{36525} \times \textbf{gregorian-date-difference}\left(\boxed{1900 \mid \textbf{january} \mid 1}, \boxed{\mathit{year} \mid \textbf{july} \mid 1} \right)$$

$$c_{2051} = \frac{1}{86400} \times \left(-20 + 32 \times \left(\frac{\mathit{year} - 1820}{100}\right)^{2} + 0.5628 \times (2150 - \mathit{year}) \right)$$

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$$
\begin{aligned}
y_{2000} &= year - 2000 \\
c_{2006} &= \frac{1}{86400} \times \left(62.92 + 0.32217 \times y_{2000} + 0.005589 \times y_{2000}^2 \right) \\
c_{1987} &= \frac{1}{86400} \times \Big(63.86 + 0.3345 \times y_{2000} - 0.060374 \times y_{2000}^2 \\
&\qquad + 0.0017275 \times y_{2000}^3 + 0.000651814 \times y_{2000}^4 \\
&\qquad + 0.00002373599 \times y_{2000}^5 \Big) \\
c_{1900} &= -0.00002 + 0.000297 \times c + 0.025184 \times c^2 \\
&\quad - 0.181133 \times c^3 + 0.553040 \times c^4 - 0.861938 \times c^5 \\
&\quad + 0.677066 \times c^6 - 0.212591 \times c^7 \\
c_{1800} &= -0.000009 + 0.003844 \times c + 0.083563 \times c^2 \\
&\quad + 0.865736 \times c^3 + 4.867575 \times c^4 + 15.845535 \times c^5 \\
&\quad + 31.332267 \times c^6 + 38.291999 \times c^7 + 28.316289 \times c^8 \\
&\quad + 11.636204 \times c^9 + 2.043794 \times c^{10} \\
y_{1700} &= year - 1700 \\
c_{1700} &= \frac{1}{86400} \times \Big(8.118780842 - 0.005092142 \times y_{1700} \\
&\qquad + 0.003336121 \times y_{1700}^2 - 0.0000266484 \times y_{1700}^3 \Big) \\
y_{1600} &= year - 1600 \\
c_{1600} &= \frac{1}{86400} \times \Big(120 - 0.9808 \times y_{1600} - 0.01532 \times y_{1600}^2 \\
&\qquad + 0.000140272128 \times y_{1600}^3 \Big) \\
y_{1000} &= \frac{year - 1000}{100} \\
c_{500} &= \frac{1}{86400} \times \Big(1574.2 - 556.01 \times y_{1000} + 71.23472 \times y_{1000}^2 \\
&\qquad + 0.319781 \times y_{1000}^3 - 0.8503463 \times y_{1000}^4 \\
&\qquad - 0.005050998 \times y_{1000}^5 + 0.0083572073 \times y_{1000}^6 \Big) \\
y_0 &= \frac{year}{100} \\
c_0 &= \frac{1}{86400} \\
&\quad \times \Big(10583.6 - 1014.41 \times y_0 + 33.78311 \times y_0^2 \\
&\qquad - 5.952053 \times y_0^3 - 0.1798452 \times y_0^4 + 0.022174192 \times y_0^5 \\
&\qquad + 0.0090316521 \times y_0^6 \Big)
\end{aligned}
$$

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$$y_{1820} = \frac{year - 1820}{100}$$
$$other = \frac{1}{86400} \times \left(-20 + 32 \times y_{1820}^2 \right)$$

We are using **gregorian-date-difference** (page 62) to calculate the number of centuries c before or after the beginning of 1900. The factor $1/86400$ converts seconds into a fraction of a day.

To convert from Universal Time to Dynamical Time, we add the correction

$$\textbf{dynamical-from-universal}\,(t_u) \stackrel{\text{def}}{=} t_u + \textbf{ephemeris-correction}\,(t_u) \tag{14.16}$$

where t is an R.D. moment measured in U.T. We approximate the inverse of (14.16) by

$$\textbf{universal-from-dynamical}\,(t) \stackrel{\text{def}}{=} t - \textbf{ephemeris-correction}\,(t) \tag{14.17}$$

The function **gregorian-date-difference** is given on page 62 and **gregorian-year-from-fixed** on page 61.

Figures 14.3 and 14.4 plot the difference between Universal Time and Dynamical Time for ancient and modern eras, respectively.

To keep the numbers within reasonable bounds, our astronomical algorithms usually convert dates and times (given in Universal Time) into "Julian centuries," that is, into the number (and fraction) of uniform-length centuries (36525 days, measured in Dynamical Time) before or after noon on January 1, 2000 (Gregorian):

$$\textbf{julian-centuries}\,(t) \stackrel{\text{def}}{=} \tag{14.18}$$
$$\frac{1}{36525} \times (\textbf{dynamical-from-universal}\,(t) - \textbf{j2000})$$

$$\textbf{j2000} \stackrel{\text{def}}{=} 12^{\text{h}} + \textbf{gregorian-new-year}\,(2000) \tag{14.19}$$

Sidereal time is discussed in the following section.

14.3 The Day

How could David have known exactly when [*true*] *midnight occurs? Even Moses didn't know!*

Babylonian Talmud (Beraḥot, 3a)

The Earth rotates around its axis, causing the sun, moon, and stars to move across the sky from east to west in the course of a day. The most obvious way of measuring days is from sunrise to sunrise or from sunset to sunset because sunrise and sunset are unmistakable. The Islamic, Hebrew, and Bahá'í calendars begin their days at sunset, whereas the Hindu day starts and ends with sunrise. The disadvantage of these methods of reckoning days is the wide variation over the year in the beginning and ending times. For example, in London sunrise occurs anywhere from 3:42 a.m.

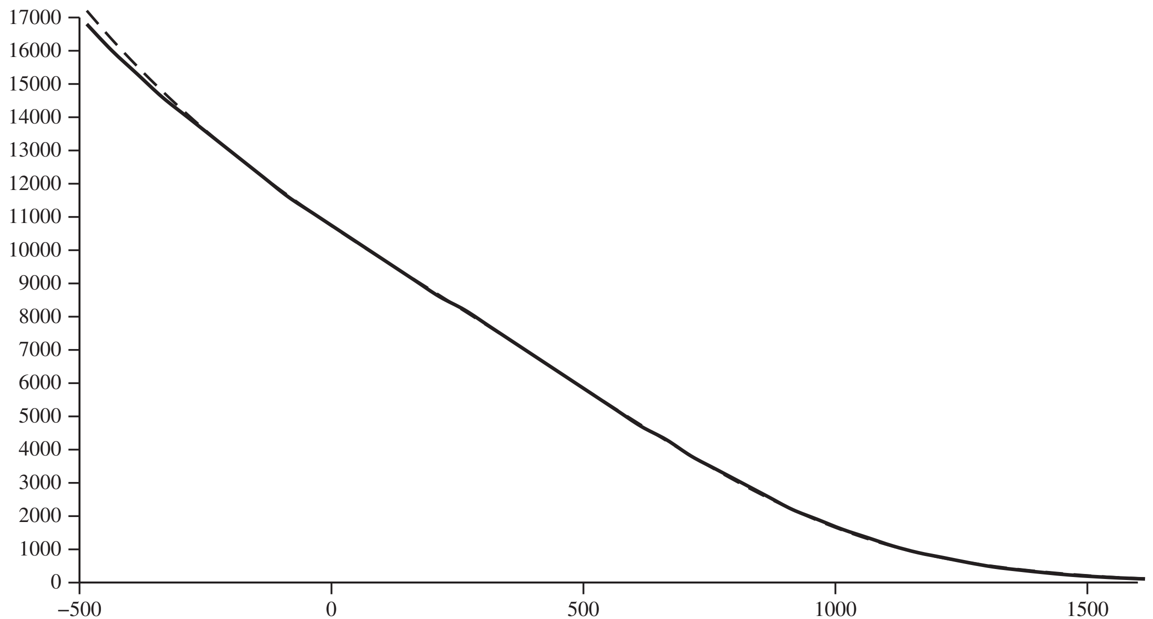

Figure 14.3 Difference between Dynamical (terrestrial) Time and Universal Time in atomic seconds plotted by Gregorian year. The dashed line shows the values of **ephemeris-correction**. Suggested by R. H. van Gent and based on [31, chap. 14].

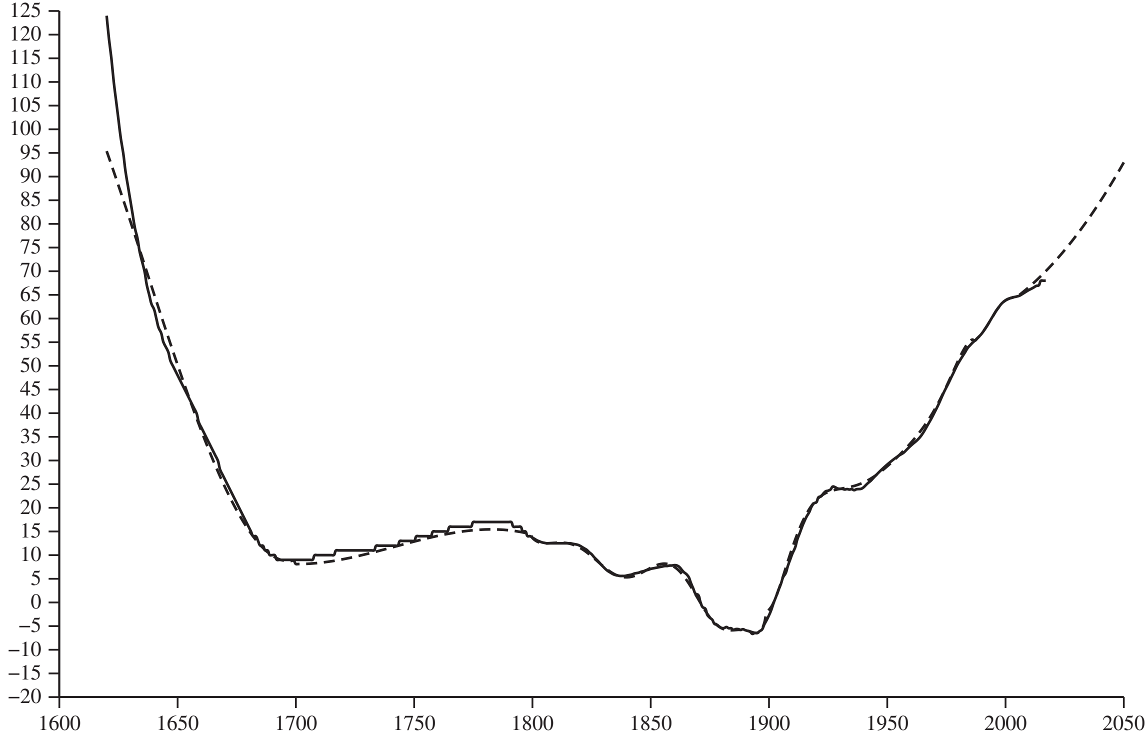

Figure 14.4 Difference between Dynamical (terrestrial) Time and Universal Time in atomic seconds plotted by Gregorian year (values for 2012−2017 are extrapolated). The dashed line shows the corresponding values computed by **ephemeris-correction**. Data supplied by R. H. van Gent based on *Astronomical Almanac for the Year 2014*, Nautical Almanac Office, United States Naval Observatory, Washington, D.C., pp. K8−K9.

to 8:06 a.m. and sunset varies from 3:51 p.m. to 8:21 p.m. By contrast, noon (the middle point of the day) and midnight (the middle point of the night) vary only by about half an hour in London or elsewhere (see below). Thus, in many parts of the world, sunset or sunrise definitions of the day have been superseded by a midnight-to-midnight day. For instance, the Chinese in the twelfth century B.C.E. began their day with the crowing of the rooster at 2 a.m., but more recently they have been using midnight. A noon-to-noon day is also plausible and indeed is used in the julian day system described in Section 1.5, but it has the disadvantage that the date changes in the middle of the working day.

Even with solar days measured from midnight to midnight there are seasonal variations. With the advent of mechanical clocks, introduced in the 1600s, the use of *mean* solar time, in which a day is 24 equal-length hours,[7] was preferred over the *apparent* (that is, true) time as measured by a sundial[8] (during the daytime, at least). The elliptical orbit of the Earth and the obliquity (inclination) of the Earth's equator with respect to its orbit cause a difference between the time the sun crosses the upper celestial meridian and 12 noon on a clock—the difference can be more than 16 minutes. This discrepancy is called the *equation of time*, where the term *equation* has its medieval meaning of "additive corrective factor."

The equation of time gives the difference between apparent midnight (when the sun crosses the lower meridian that passes through the nadir; this is virtually the same as the midpoint between sunset and sunrise) and mean midnight (0 hours on the 24-hour clock). Similarly, at other times of day the equation of time gives the difference between mean solar time and apparent solar time. In the past, when apparent time was the more readily available, the equation of time conventionally had the opposite sign.

The periodic pattern of the equation of time, shown in Figures 14.5 and 14.6, is sometimes inscribed as part of the analemma on sundials (usually in mirror image); the frontispiece for Chapter 18 (page 217) shows a three-dimensional image of the equation of time. During the twentieth century, the equation of time had zeroes around April 15, June 14, September 1, and December 25; it is at its maximum at the beginning of November and at its minimum in mid-February. The equation of time is needed for the French Revolutionary and Persian astronomical calendars, and a rough approximation is used in the modern Hindu calendars. We use the following function for the equation of time:

$$\textbf{equation-of-time}\,(t) \stackrel{\text{def}}{=} \qquad (14.20)$$
$$\text{sign}\,(\mathit{equation}) \times \min\left\{|\mathit{equation}|, 12^{\text{h}}\right\}$$

where

$$c = \textbf{julian-centuries}\,(t)$$
$$\lambda = 280.46645^\circ + 36000.76983^\circ \times c + 0.0003032^\circ \times c^2$$

[7] The 24-hour day is sometimes called a *nychthemeron* to distinguish it from the shorter period of daylight.

[8] The hands on early mechanical clocks were imitating the movement of the shadow of the gnomon (in the northern hemisphere where clocks were developed) as the sun crosses the sky. This is the origin of our notion of "clockwise." See "The Last Word," *New Scientist*, March 27, 1999.

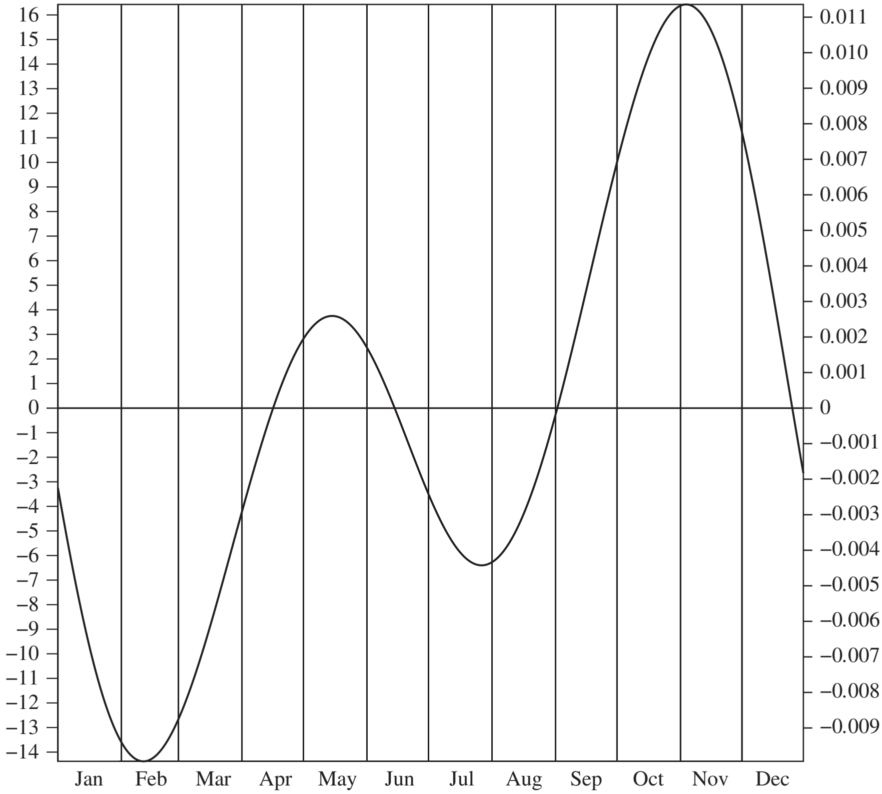

Figure 14.5 The current equation of time, as computed by **equation-of-time**. The left-hand vertical axis is marked in minutes and the right-hand vertical axis is marked in fractions of a day.

$$\begin{aligned}
\textit{anomaly} &= 357.52910^\circ + 35999.05030^\circ \times c \\
&\quad - 0.0001559^\circ \times c^2 - 0.00000048^\circ \times c^3 \\
\textit{eccentricity} &= 0.016708617 - 0.000042037 \times c - 0.0000001236 \times c^2 \\
\varepsilon &= \textbf{obliquity}\,(t) \\
y &= \tan^2\left(\frac{\varepsilon}{2}\right) \\
\textit{equation} &= \frac{1}{2 \times \pi} \\
&\quad \times \Big(y \times \sin(2 \times \lambda) - 2 \times \textit{eccentricity} \times \sin \textit{anomaly} \\
&\qquad + 4 \times \textit{eccentricity} \times y \times \sin \textit{anomaly} \times \cos(2 \times \lambda) \\
&\qquad - 0.5 \times y^2 \times \sin(4 \times \lambda) \\
&\qquad - 1.25 \times \textit{eccentricity}^2 \times \sin(2 \times \textit{anomaly}) \Big)
\end{aligned}$$

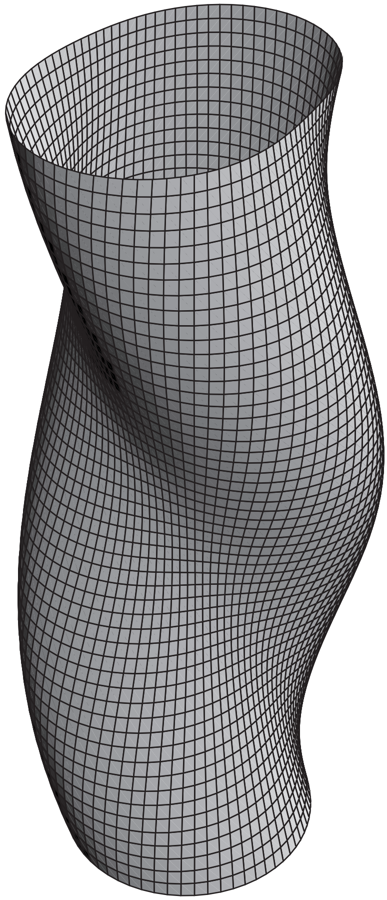

Figure 14.6 The equation of time wrapped onto a cylinder. The rotational range is 1 year; the axial range is Gregorian years 1500−12500. This rendering was converted into a three-dimensional cam by Stewart P. Dickson to be used as a mechanical cam in the "Clock of the Long Now" by W. Daniel Hillis and the Long Now Foundation. The clock is designed to keep local, absolute, and astronomical time over a span of 10000 years. The cam resynchronizes the clock at local solar noon via a thermal trigger. See *The Clock of the Long Now: Time and Responsibility* by Stewart Brand, Basic Books, New York, 1999 for more information. (Reproduced by permission.)

The parameter t is a moment (R.D. day and fraction); it is converted to "Julian centuries," c. The function for obliquity is given below (page 220). The preceding approximation of the equation of time is not valid for dates that are many millennia in the past or future; hence, for robustness, we limit the accuracy of the calculated value to half a day.

The equation of time permits us to convert easily to and from apparent time:

$$\textbf{apparent-from-local}\,(t_\ell, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.21}$$

$$t_\ell + \textbf{equation-of-time}\,(\textbf{universal-from-local}\,(t_\ell, \mathit{location}))$$

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In the other direction,

$$\textbf{local-from-apparent}\,(t, \mathit{location}) \stackrel{\text{def}}{=} \qquad (14.22)$$

$$t - \textbf{equation-of-time}\,(\textbf{universal-from-local}\,(t, \mathit{location}))$$

The latter function is slightly inaccurate because the function **equation-of-time** takes the local mean time, not the apparent time, as its argument; the difference in the value of the equation of time in those few minutes is negligible, however.

These functions may be composed with conversion to and from local time:

$$\textbf{apparent-from-universal}\,(t_{\text{u}}, \mathit{location}) \stackrel{\text{def}}{=} \qquad (14.23)$$

$$\textbf{apparent-from-local}\,(\textbf{local-from-universal}\,(t_{\text{u}}, \mathit{location}), \mathit{location})$$

and

$$\textbf{universal-from-apparent}\,(t, \mathit{location}) \stackrel{\text{def}}{=} \qquad (14.24)$$

$$\textbf{universal-from-local}\,(\textbf{local-from-apparent}\,(t, \mathit{location}), \mathit{location})$$

Using these time conversion functions, we can find the true middle of the night (true, or apparent, midnight) or the true middle of the day (apparent noon) in Universal Time:

$$\textbf{midnight}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \qquad (14.25)$$

$$\textbf{universal-from-apparent}\,(\mathit{date}, \mathit{location})$$

and

$$\textbf{midday}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \qquad (14.26)$$

$$\textbf{universal-from-apparent}\left(\mathit{date} + 12^{\text{h}}, \mathit{location}\right)$$

The Earth's *rotation period* with respect to the fixed celestial sphere is approximately equal to $23^{\text{h}}56^{\text{m}}4.09890^{\text{s}}$. This is marginally more than the length of a (*mean*) *sidereal* (or *tropical*) *day*, namely, $23^{\text{h}}56^{\text{m}}4.09054^{\text{s}}$, which is the time of rotation relative to the First Point of Aries. In the course of one rotation on its axis, the Earth has also revolved somewhat in its orbit around the sun, and thus the sun is not quite in the same position as it was one rotation prior. This accounts for the difference of almost 4 minutes with respect to the solar day. The sidereal day is employed in the Hindu calendar.

Like the solar day, the sidereal day is not constant; it is steadily growing longer. In practice, *sidereal time* is measured by the *hour angle* between the meridian (directly overhead) and the position of the First Point of Aries (see page 219). This definition of sidereal time is affected by the precession of the equinoxes—see

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page 219. Converting between mean solar and mean sidereal time amounts to evaluating a polynomial:

$$\textbf{sidereal-from-moment}\,(t) \stackrel{\text{def}}{=} \tag{14.27}$$

$$\left(280.46061837^\circ + 36525 \times 360.98564736629^\circ \times c \right.$$
$$\left. + 0.000387933^\circ \times c^2 - \frac{1^\circ}{38710000} \times c^3 \right) \bmod 360$$

where

$$c = \frac{t - \textbf{j2000}}{36525}$$

The modern Hindu lunar calendar uses an approximation to this conversion.

14.4 The Year

And the sun rises and the sun sets—then to its place it rushes; there it rises again. It goes toward the south and veers toward the north.[10]

Ecclesiastes 1, 5–6

The *vernal equinox* occurs at the moment when the sun's position crosses the *true celestial equator* (the line in the sky above the Earth's equator) from south to north, on approximately March 20 each year. At that time day and night are each 12 hours all over the world, and the Earth's axis of rotation is perpendicular to the line connecting the centers of the Earth and sun.[11] The point of intersection of the ecliptic (the sun's apparent path through the constellations of the zodiac), inclined from south to north, and the celestial equator is called the *true vernal equinox* or the "First Point of Aries," but it is currently in the constellation Pisces, not Aries, on account of a phenomenon called the *precession of the equinoxes*. In its gyroscopic motion, the Earth's rotational axis migrates in a slow circle mainly as a consequence of the moon's pull on a nonspherical Earth. This nearly uniform motion causes the position of the equinoxes to move backwards along the ecliptic in a period of about 25725 years. This precession has caused the vernal equinox to cease to coincide with the day when the sun enters Aries, as it did some 2300 years ago (however, since the absolute length of a day is getting longer—see page 210—the sun will be in the same position in calendar year 24500 as it was in 2000). Celestial longitude is measured from the First Point of Aries. As a consequence, the longitudes of the stars are constantly changing (in addition to the measurable motions of many of the "fixed" stars). This precession of the equinoxes also causes the celestial pole to rotate slowly in a circular pattern. This is why the identity of the "pole star" has changed over the course of history. In 13000 B.C.E., Vega was

[10] This translation follows the interpretation of Solomon ben Isaac.

[11] Perhaps this perpendicularity explains the odd belief that fresh eggs balance more easily on the day of the vernal equinox. This turned into a minor craze in the United States; see Martin Gardner's "Notes of a Fringe Watcher," *The Skeptical Inquirer*, May/June 1996.

near the pole; currently it is near the star Polaris. In contrast, the Hindu calendar is based on calculations in terms of the *sidereal* longitude, which ignores precession and remains fixed against the backdrop of the stars.

The equator is currently inclined at approximately

$$23.4443291^\circ = 23^\circ 26' 21.448''$$

with respect to the plane of revolution of the Earth (the ecliptic) around the sun.[12] As a result, the sun, in the course of a year, traces a path through the stars that moves towards the celestial North Pole, back towards the celestial equator, then towards the celestial South Pole and back again. The value of this inclination, called the *obliquity*, varies in a 100000-year cycle, ranging from 24.2° 10000 years ago to 22.6° in another 10000 years. The following function gives an approximate value:

$$\mathbf{obliquity}\,(t) \stackrel{\text{def}}{=} \tag{14.28}$$

$$23^\circ 26' 21.448'' + \left(-46.8150'' \times c - 0.00059'' \times c^2 + 0.001813'' \times c^3 \right)$$

where

$$c = \mathbf{julian\text{-}centuries}\,(t)$$

Given the obliquity, one can compute the declination and right ascension corresponding to celestial latitude β and longitude λ:

$$\mathbf{declination}\,(t, \beta, \lambda) \stackrel{\text{def}}{=} \tag{14.29}$$

$$\arcsin\left(\sin\beta \times \cos\varepsilon + \cos\beta \times \sin\varepsilon \times \sin\lambda\right)$$

where

$$\varepsilon = \mathbf{obliquity}\,(t)$$

and

$$\mathbf{right\text{-}ascension}\,(t, \beta, \lambda) \stackrel{\text{def}}{=} \tag{14.30}$$

$$\mathbf{arctan}\left(\sin\lambda \times \cos\varepsilon - \tan\beta \times \sin\varepsilon, \cos\lambda\right)$$

where

$$\varepsilon = \mathbf{obliquity}\,(t)$$

which is measured in degrees, not hours.

In addition to the precession, the axis of rotation of the Earth wobbles like a top in an 18.6-year period about its mean position. This effect is called *nutation*

[12] There has recently been much worry in the fringe science community about the change in the tilt of the Earth's axis causing a global calamity *New Scientist* (August 9, 2008, p. 56) refers to this as "fruitloopery."

and is caused by the gravitational pull of the moon and sun on the unevenly shaped Earth. Nutation causes slight changes in the celestial latitudes and longitudes of stars and planets. It also causes a periodic variation in the lengths of the sidereal and solar days of up to about 0.01 second. The *mean sidereal time* smoothes out (subtracts) this nutation, which can accumulate to a difference of about 1 second from the actual sidereal time. The moon also causes small oscillations in the length of the day, with periods ranging from 12 hours to 1 (sidereal) month, but these can safely be ignored.

The true position of the sun differs from the "mean sun" in both longitude and latitude. The (angular) speed of the longitude of the true sun oscillates markedly within a year, so that the lengths of the four annual seasons differ by as much as 5 days. The sun's mean latitude is 0°, but the true sun does not always stay on the ecliptic.

The *tropical year* is the time it takes for the "mean sun" to travel from one mean vernal equinox to the next. As the speed of the mean sun on the ecliptic is slowly increasing over the centuries, the tropical year as determined from the instantaneous speed of the mean sun for an arbitrary instant is slowly decreasing. The length of a tropical year is defined today with respect to a "dynamical" equinox [21];[13] its current value is 365.242177 mean solar days, and it is decreasing by about 1.3×10^{-5} solar days per century. We use the following older value in our calculations for estimation purposes:

$$\textbf{mean-tropical-year} \stackrel{\text{def}}{=} 365.242189 \qquad (14.31)$$

The time intervals between successive vernal equinoxes differ from those between successive autumnal equinoxes and also from the tropical year. Figure 14.7 shows for comparison the fluctuating equinox-to-equinox and solstice-to-solstice year lengths, measured in mean solar days at the same point in time, together with the mean year length used in various arithmetical calendars.

A *sidereal year* is the time it takes for the Earth to revolve once around the sun, that is, for the mean sun to return to the same position relative to the background of the fixed stars. The sidereal year is about 20 minutes more than the tropical:

$$\textbf{mean-sidereal-year} \stackrel{\text{def}}{=} 365.25636 \qquad (14.32)$$

The modern Hindu calendar (Chapter 20) uses approximations of the sidereal and tropical year.

To determine the times of equinoxes or solstices, as required for the French Revolutionary (Chapter 17), Chinese (Chapter 19), Persian astronomical (Chapter 15), and proposed Bahá'í (Section 16.3) calendars, we must calculate the longitude of the sun at any given time. The following function takes an astronomical time, given as an R.D. moment t, converts it to Julian centuries, sums a long sequence of periodic terms, and adds terms to compensate for aberration (the effect of the sun's

[13] This dynamical equinox is the intersection of the mean celestial equator with the ecliptic, where the movement of the ecliptic is derived from a dynamical model of the movement of the Earth-moon barycenter (center of gravity) within the solar system.

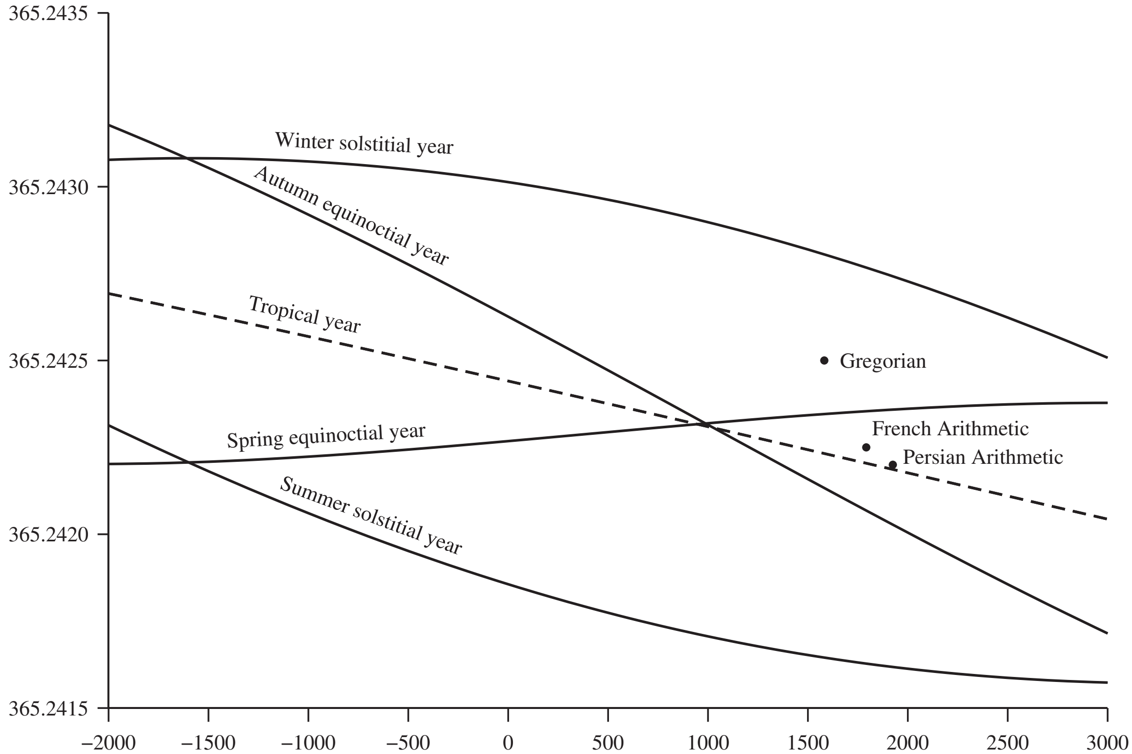

Figure 14.7 Length of the year, in contemporaneous mean solar days, plotted by Gregorian year. (The value for the Julian/Coptic/Ethiopic calendars, 365.25, is omitted because it is far above the values plotted.) Suggested by R. H. van Gent and based on formulas from [18, chap. 27] and the parabolic approximation from Figure 14.3.

apparent motion while its light is traveling towards Earth) and nutation (caused by the wobble of the Earth):

$$\textbf{solar-longitude}(t) \stackrel{\text{def}}{=} (\lambda + \textbf{aberration}(t) + \textbf{nutation}(t)) \bmod 360 \tag{14.33}$$

where

$$\begin{aligned}
c &= \textbf{julian-centuries}(t)\\
\lambda &= 282.7771834^\circ + 36000.76953744^\circ \times c\\
&\quad + 0.000005729577951308232^\circ \times \sum (\widetilde{x} \times \sin(\widetilde{y} + \widetilde{z} \times c))\\
\widetilde{x} &= \text{(see Table 14.1)}\\
\widetilde{y} &= \text{(see Table 14.1)}\\
\widetilde{z} &= \text{(see Table 14.1)}
\end{aligned}$$

To avoid cluttering the page with subscripts, we will use vector notation, with the intention that the operations within a sum are performed on like-indexed elements of $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$, displayed in the rows of Table 14.1. This function is accurate to within 2 minutes of arc for current times.

The effect of nutation on the longitude is approximately

$$\textbf{nutation}(t) \stackrel{\text{def}}{=} -0.004778^\circ \times \sin A - 0.0003667^\circ \times \sin B \tag{14.34}$$

where

$$\begin{aligned}
c &= \textbf{julian-centuries}(t)\\
A &= 124.90^\circ - 1934.134^\circ \times c + 0.002063^\circ \times c^2\\
B &= 201.11^\circ + 72001.5377^\circ \times c + 0.00057^\circ \times c^2
\end{aligned}$$

The *aberration*—the effect of the sun's moving about 20.47 seconds of arc during the 8 minutes during which its light is *en route* to Earth—is calculated as follows:

$$\textbf{aberration}(t) \stackrel{\text{def}}{=} \tag{14.35}$$

$$0.0000974^\circ \times \cos(177.63^\circ + 35999.01848^\circ \times c) - 0.005575^\circ$$

where

$$c = \textbf{julian-centuries}(t)$$

We determine the time of an equinox or solstice by giving a generic function that takes a moment t (in U.T.) and a number of degrees λ, indicating the season, and searches for the moment when the longitude of the sun is next equal to λ degrees. In effect, we search for the inverse of **solar-longitude**, using equation (1.36) on page 25, within an interval beginning 5 days before the estimate τ (or at the given moment, whichever comes later) and ending 5 days after:

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Table 14.1 Values of the arguments $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ in **solar-longitude** (page 223).

$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$
403406	270.54861	0.9287892	46	8	107997.405
195207	340.19128	35999.1376958	38	197.1	−4444.176
119433	63.91854	35999.4089666	37	250.4	151.771
112392	331.26220	35998.7287385	32	65.3	67555.316
3891	317.843	71998.20261	29	162.7	31556.080
2819	86.631	71998.4403	28	341.5	−4561.540
1721	240.052	36000.35726	27	291.6	107996.706
660	310.26	71997.4812	27	98.5	1221.655
350	247.23	32964.4678	25	146.7	62894.167
334	260.87	−19.4410	24	110	31437.369
314	297.82	445267.1117	21	5.2	14578.298
268	343.14	45036.8840	21	342.6	−31931.757
242	166.79	3.1008	20	230.9	34777.243
234	81.53	22518.4434	18	256.1	1221.999
158	3.50	−19.9739	17	45.3	62894.511
132	132.75	65928.9345	14	242.9	−4442.039
129	182.95	9038.0293	13	115.2	107997.909
114	162.03	3034.7684	13	151.8	119.066
99	29.8	33718.148	13	285.3	16859.071
93	266.4	3034.448	12	53.3	−4.578
86	249.2	−2280.773	10	126.6	26895.292
78	157.6	29929.992	10	205.7	−39.127
72	257.8	31556.493	10	85.9	12297.536
68	185.1	149.588	10	146.1	90073.778
64	69.9	9037.750			

$$\textbf{solar-longitude-after}\,(\lambda, t) \stackrel{\text{def}}{=} \textbf{solar-longitude}^{-1}(\lambda, [a \,..\, b]) \tag{14.36}$$

where

$$\begin{aligned} rate &= \frac{\textbf{mean-tropical-year}}{360^\circ} \\ \tau &= t + rate \times ((\lambda - \textbf{solar-longitude}\,(t)) \bmod 360) \\ a &= \max\{t, \tau - 5\} \\ b &= \tau + 5 \end{aligned}$$

Equinoxes and solstices occur when the sun's longitude is a multiple of 90°. Specifically, Table 14.2 gives the names, solar longitudes, and approximate Gregorian dates. The constants for the four seasons were defined in Chapter 3.

It will be convenient to be able to determine the times of the seasons within a given Gregorian year. So, we define

$$\textbf{season-in-gregorian}\,(season, g\text{-}year) \stackrel{\text{def}}{=} \tag{14.37}$$

$$\textbf{solar-longitude-after}\,(season, jan_1)$$

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Table 14.2 The solar longitudes and approximate current dates of equinoxes and solstices, along with the approximate length of the following season.

Name	Solar longitude	Approximate date	Season length
Vernal (spring) equinox	0°	March 20	92.76 days
Summer solstice	90°	June 21	93.65 days
Autumnal (fall) equinox	180°	September 22–23	89.84 days
Winter solstice	270°	December 21–22	88.99 days

where

$$jan_1 = \textbf{gregorian-new-year}\,(g\text{-}year)$$

To use this function to determine, say, the standard time of the winter solstice in Urbana, Illinois, we write

$$\textbf{urbana-winter}\,(g\text{-}year) \stackrel{\text{def}}{=} \tag{14.38}$$

$$\textbf{standard-from-universal}\,(\textbf{season-in-gregorian}\,(\textbf{winter}, g\text{-}year)\,, \textbf{urbana})$$

For year 2000 this gives us the answer R.D. 730475.31751, which is 7:37:13 a.m. on December 21.

To calculate sidereal longitude, we use the following computation for precession:

$$\textbf{precession}\,(t) \stackrel{\text{def}}{=} (p + P - arg) \bmod 360 \tag{14.39}$$

where

$$\begin{aligned}
c &= \textbf{julian-centuries}\,(t)\\
\eta &= \left(47.0029'' \times c - 0.03302'' \times c^2 + 0.000060'' \times c^3\right) \bmod 360\\
P &= \left(174.876384^\circ - 869.8089'' \times c \quad + 0.03536'' \times c^2\right) \bmod 360\\
p &= \left(5029.0966'' \times c + 1.11113'' \times c^2 + 0.000006'' \times c^3\right) \bmod 360\\
A &= \cos\eta \times \sin P\\
B &= \cos P\\
arg &= \textbf{arctan}\,(A, B)
\end{aligned}$$

To use **precession**, one needs to choose some moment, called **sidereal-start** [such as formula (20.41)], at which one considers that sidereal and ecliptic longitude coincide. Then we have

$$\textbf{sidereal-solar-longitude}\,(t) \stackrel{\text{def}}{=} \tag{14.40}$$

$$\left(\textbf{solar-longitude}\,(t) - \textbf{precession}\,(t) + \textbf{sidereal-start}\right) \bmod 360$$

Astronomical Hindu calendars (see Section 20.5) require the determination of this solar attribute.

Finally, the altitude of the sun above the horizon at any given time depends on the ecliptical position of the sun at that time and on the latitude φ and longitude ψ of the viewing location:

$$\textbf{solar-altitude}\left(t, \boxed{\varphi \mid \psi \mid — \mid —}\right) \stackrel{\text{def}}{=} \quad (14.41)$$

$$\textit{altitude} \bmod [-180\,..\,180)$$

where

$$\begin{aligned}
\lambda &= \textbf{solar-longitude}\,(t)\\
\alpha &= \textbf{right-ascension}\,(t, 0, \lambda)\\
\delta &= \textbf{declination}\,(t, 0, \lambda)\\
\theta_0 &= \textbf{sidereal-from-moment}\,(t)\\
H &= (\theta_0 + \psi - \alpha) \bmod 360\\
\textit{altitude} &= \arcsin(\sin\varphi \times \sin\delta + \cos\varphi \times \cos\delta \times \cos H)
\end{aligned}$$

Here α is the sun's right ascension, δ is its declination, and H is the local sidereal hour angle. The result is not corrected for parallax (the shift in observed position due to the change in position of the observer) or refraction, and ranges from $-90°$ to $+90°$.

14.5 Astronomical Solar Calendars

Astronomy [*lit. seasons*] *and geometry are accoutrements of wisdom.*
Pirkei Avoth III, 23

Astronomical solar calendars are based on the precise solar longitude at a specified time. For example, the astronomical Persian calendar begins its New Year on the day when the vernal equinox occurs before true noon (the middle point of the day, sundial time, not clock time) in Tehran; the start of the New Year is postponed to the next day if the equinox is after noon (see Chapter 15). Other calendars of this type include the astronomical form of the Bahá'í calendar (Chapter 16) and the original French Revolutionary calendar (Chapter 17).

The key to implementing an astronomical solar calendar is to determine the day of the New Year on or before a given fixed date. In general, the New Year begins on the day when the solar longitude reaches a certain value φ at some critical moment, such as noon or midnight. For this purpose, we first estimate the time using the current solar longitude:

$$\textbf{estimate-prior-solar-longitude}\,(\lambda, t) \stackrel{\text{def}}{=} \min\{t, \tau - \textit{rate} \times \Delta\} \quad (14.42)$$

where

$$\begin{aligned}
\textit{rate} &= \frac{\textbf{mean-tropical-year}}{360°}\\
\tau &= t - \textit{rate} \times ((\textbf{solar-longitude}\,(t) - \lambda) \bmod 360)\\
\Delta &= (\textbf{solar-longitude}\,(\tau) - \lambda) \bmod [-180\,..\,180)
\end{aligned}$$

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This is done in a two-step process. First we go back to the time when the sun, traveling at mean speed, was last at longitude φ; then the error Δ in the longitude is used to refine the estimate to within a day of the correct time. The only complication is handling the discontinuity from 360° to 0°; this is done using interval modulus.

Since this estimate is within a day of the actual occurrence, to determine when the year actually starts we need only carry out a short search of the form

$$\underset{day \geqslant \lfloor approx \rfloor - 1}{\mathbf{MIN}} \{\varphi \leqslant \mathbf{solar\text{-}longitude}(f(day)) \leqslant \varphi + 2°\} \tag{14.43}$$

where f is a function that returns the critical time of *day* for measuring longitude for the specific calendar. The upper bound $\varphi + 2°$ is only needed when one is looking for the spring equinox ($\varphi = 0°$), so that values close to 360°, which precede the equinox, do not stop the search prematurely. We use this method for the astronomical Persian, the astronomical Bahá'í, the original French Revolutionary, and Chinese calendars.

14.6 The Month

Should someone rather less skilled in calculation nonetheless be curious about the course of the moon, we have also for his sake devised a formula adapted to the capacity of his intelligence, so that he might find what he seeks.

The Venerable Bede: *De Temporum Ratione*

The *new moon* occurs when the sun and moon have the same celestial longitude; it is not necessarily the time of their closest encounter, as viewed from Earth, because the orbits of the Earth and moon are not coplanar. The time from new moon (the *conjunction* of the sun and the moon) to new moon, a *lunation*, is called the *synodic month*. Its value today ranges from approximately 29.27 to 29.84 days [19], with a mean of currently about 29.530588 mean solar days:

$$\mathbf{mean\text{-}synodic\text{-}month} \stackrel{\text{def}}{=} 29.530588861 \tag{14.44}$$

in days of 86400 atomic seconds. Approximations to this value are used in many lunar and lunisolar calendars. The Chinese calendar, however, uses actual astronomical values in its determinations. The mean and true times of the new moon can differ by up to about 14 hours. Figure 14.8 shows for comparison the changing length of the month, measured in mean solar days at the same point in time, with the values used in several arithmetic calendars.

The synodic month is not constant but is decreasing in mean length by about 3.6×10^{-7} solar days per century (though it is *increasing* in length by about 0.021 atomic seconds per century). The net effect of the decreases in synodic month and tropical year is to increase the number of months from its current value of about 12.3682670 per year by 0.3×10^{-6} months per century.

The full moon is the most visible feature of the night sky and has thus long fascinated human observers. Some cultures give names to the full moons—the "Harvest Moon" is the full moon closest to the autumnal equinox, for example. For obscure reasons, when four full moons occur within one solar season (from equinox to solstice or solstice to equinox), the third is termed a *blue moon*; see [23]. This event

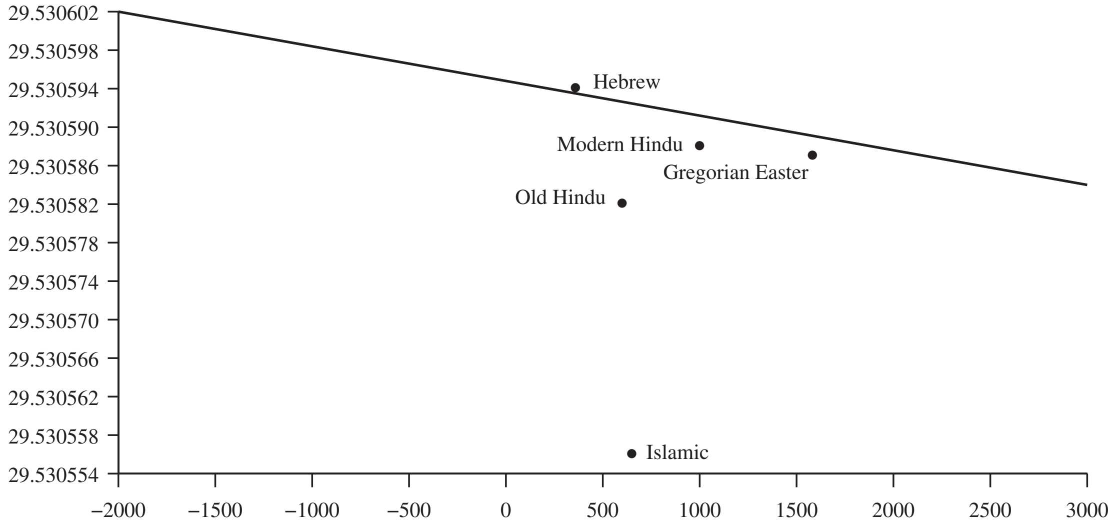

Figure 14.8 Length of the synodic month, in contemporaneous mean solar days, plotted by Gregorian year. (The value for Orthodox Easter, 29.530851, is omitted because it is far above the values plotted.) Suggested by R. H. van Gent and based on data from [18, Chap. 49] and [31, Chap. 14].

is similar to the conditions for a leap month on the Chinese calendar and Hindu lunisolar calendars, which mandate a leap month whenever two new moons occur within the same solar month (see Chapters 19 and 20).

The *sidereal month* is the time it takes the moon to make one revolution around the Earth. Its mean value is 27.32166 days. In the interim, the Earth has moved in its orbit around the sun, and thus the difference in longitude between the sun and moon has increased, which is why the synodic month is longer. The mean values of these types of month should satisfy the equation

$$\frac{1}{\text{sidereal month}} - \frac{1}{\text{synodic month}} = \frac{1}{\text{sidereal year}}$$

The *anomalistic month* is the time between consecutive perigees (points at which the moon is closest to Earth). The anomalistic month averages 27.55455 days. Approximations to these values are used in calculating the position of the moon for the modern Hindu lunisolar calendar.

We also use the notion of a *solar month*, the time for the sun's position in the sky to traverse one sign of the zodiac (30° of longitude). Its mean value is one-twelfth of a solar year and ranges from 29.44 days in Northern Hemisphere winter (to traverse Capricorn) to 31.43 days in Northern Hemisphere summer. Solar months play an important rôle in the Chinese calendar (which uses tropical longitude) and in the Hindu calendar (which uses sidereal longitude).

The time of new moon can be determined directly using sums of periodic terms. We use the following function for the moment (in U.T.) of the *n*th new moon after (before, if *n* is negative) the new moon of January 11, 1 (Gregorian), the first new moon after R.D. 0:

$$\textbf{nth-new-moon}\,(n) \stackrel{\text{def}}{=} \qquad (14.45)$$

$$\textbf{universal-from-dynamical}\,(\mathit{approx} + \mathit{correction} + \mathit{extra} + \mathit{additional})$$

where

$$\begin{aligned}
n_0 &= 24724\\
k &= n - n_0\\
c &= \frac{k}{1236.85}\\
\mathit{approx} &= \textbf{j2000} + \Big(5.09766\\
&\qquad + \textbf{mean-synodic-month} \times 1236.85 \times c\\
&\qquad + 0.00015437 \times c^2 - 0.000000150 \times c^3\\
&\qquad + 0.00000000073 \times c^4\Big)\\
E &= 1 - 0.002516 \times c - 0.0000074 \times c^2\\
\mathit{solar\text{-}anomaly} &= 2.5534^\circ + 1236.85 \times 29.10535670^\circ \times c\\
&\quad - 0.0000014^\circ \times c^2 - 0.00000011^\circ \times c^3
\end{aligned}$$

$$\begin{aligned}
\textit{lunar-anomaly} &= 201.5643^\circ + 385.81693528 \times 1236.85^\circ \times c \\
&\quad + 0.0107582^\circ \times c^2 + 0.00001238^\circ \times c^3 \\
&\quad - 0.000000058^\circ \times c^4 \\
\textit{moon-argument} &= 160.7108^\circ + 390.67050284 \times 1236.85^\circ \times c \\
&\quad - 0.0016118^\circ \times c^2 - 0.00000227^\circ \times c^3 \\
&\quad + 0.000000011^\circ \times c^4 \\
\Omega &= 124.7746^\circ + \left(-1.56375588 \times 1236.85\right)^\circ \times c \\
&\quad + 0.0020672^\circ \times c^2 + 0.00000215^\circ \times c^3 \\
\textit{correction} &= -0.00017 \times \sin\Omega \\
&\quad + \sum \Big(\widetilde{v} \times E^{\widetilde{w}} \\
&\qquad \times \sin\big(\widetilde{x} \times \textit{solar-anomaly} + \widetilde{y} \times \textit{lunar-anomaly} \\
&\qquad\quad + \widetilde{z} \times \textit{moon-argument} \big) \Big) \\
\textit{extra} &= 0.000325 \times \sin\big(299.77^\circ + 132.8475848^\circ \times c \\
&\qquad - 0.009173^\circ \times c^2 \big) \\
\textit{additional} &= \sum \left(\widetilde{l} \times \sin\left(\widetilde{\imath} + \widetilde{\jmath} \times k\right)\right) \\
\widetilde{v} &= \text{(see Table 14.3)} \\
\widetilde{w} &= \text{(see Table 14.3)} \\
\widetilde{x} &= \text{(see Table 14.3)} \\
\widetilde{y} &= \text{(see Table 14.3)} \\
\widetilde{z} &= \text{(see Table 14.3)} \\
\widetilde{\imath} &= \text{(see Table 14.4)} \\
\widetilde{\jmath} &= \text{(see Table 14.4)} \\
\widetilde{l} &= \text{(see Table 14.4)}
\end{aligned}$$

There were $n_0 = 24724$ months between January, 1 and January, 2000, upon which time this function is centered. The first new moon after **j2000** occurred 5.25952 days later. The value of E depends on the eccentricity of Earth's elliptical orbit; Ω is the longitude of the moon's "ascending node."

To find the time of the new moon preceding a given date or moment, we can use

$$\textbf{new-moon-before}\,(t) \stackrel{\text{def}}{=} \tag{14.46}$$

$$\textbf{nth-new-moon}\left(\underset{k \geqslant n-1}{\textbf{MAX}} \left\{ \textbf{nth-new-moon}\,(k) < t \right\} \right)$$

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Table 14.3 Values of the arguments $\widetilde{v}$, $\widetilde{w}$, $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ in **nth-new-moon** (page 229).

$\widetilde{v}$	$\widetilde{w}$	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$	$\widetilde{v}$	$\widetilde{w}$	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$
−0.40720	0	0	1	0	0.00038	1	1	0	−2
0.17241	1	1	0	0	−0.00024	1	−1	2	0
0.01608	0	0	2	0	−0.00007	0	2	1	0
0.01039	0	0	0	2	0.00004	0	0	2	−2
0.00739	1	−1	1	0	0.00004	0	3	0	0
−0.00514	1	1	1	0	0.00003	0	1	1	−2
0.00208	2	2	0	0	0.00003	0	0	2	2
−0.00111	0	0	1	−2	−0.00003	0	1	1	2
−0.00057	0	0	1	2	0.00003	0	−1	1	2
0.00056	1	1	2	0	−0.00002	0	−1	1	−2
−0.00042	0	0	3	0	−0.00002	0	1	3	0
0.00042	1	1	0	2	0.00002	0	0	4	0

Table 14.4 Values of the arguments $\widetilde{\imath}$, $\widetilde{\jmath}$, and $\widetilde{l}$ in **nth-new-moon** (page 229).

$\widetilde{\imath}$	$\widetilde{\jmath}$	$\widetilde{l}$	$\widetilde{\imath}$	$\widetilde{\jmath}$	$\widetilde{l}$
251.88	0.016321	0.000165	34.52	27.261239	0.000047
251.83	26.651886	0.000164	207.19	0.121824	0.000042
349.42	36.412478	0.000126	291.34	1.844379	0.000040
84.66	18.206239	0.000110	161.72	24.198154	0.000037
141.74	53.303771	0.000062	239.56	25.513099	0.000035
207.14	2.453732	0.000060	331.55	3.592518	0.000023
154.84	7.306860	0.000056			

where

$$t_0 = \textbf{nth-new-moon}\,(0)$$
$$\varphi = \textbf{lunar-phase}\,(t)$$
$$n = \text{round}\left(\frac{t - t_0}{\textbf{mean-synodic-month}} - \frac{\varphi}{360^\circ}\right)$$

For the following new moon, we have

$$\textbf{new-moon-at-or-after}\,(t) \stackrel{\text{def}}{=} \tag{14.47}$$
$$\textbf{nth-new-moon}\left(\underset{k\geqslant n}{\textbf{MIN}}\left\{\textbf{nth-new-moon}\,(k) \geqslant t\right\}\right)$$

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where

$$t_0 = \textbf{nth-new-moon}\,(0)$$
$$\varphi = \textbf{lunar-phase}\,(t)$$
$$n = \text{round}\left(\frac{t - t_0}{\textbf{mean-synodic-month}} - \frac{\varphi}{360^\circ}\right)$$

Alternatively, one can determine the time of new moon indirectly from the longitude of the moon. The moon's longitude is significantly more difficult to compute than that of the sun, because it is affected in a nonnegligible way by the pull of the sun, Venus, and Jupiter. The function for the longitude of the moon is given by

$$\textbf{lunar-longitude}\,(t) \stackrel{\text{def}}{=} \tag{14.48}$$
$$\left(L' + \textit{correction} + \textit{venus} + \textit{jupiter} + \textit{flat-earth} + \textbf{nutation}\,(t)\right) \bmod 360$$

where

$$c = \textbf{julian-centuries}\,(t)$$
$$L' = \textbf{mean-lunar-longitude}\,(c)$$
$$D = \textbf{lunar-elongation}\,(c)$$
$$M = \textbf{solar-anomaly}\,(c)$$
$$M' = \textbf{lunar-anomaly}\,(c)$$
$$F = \textbf{moon-node}\,(c)$$
$$E = 1 - 0.002516 \times c - 0.0000074 \times c^2$$
$$\textit{correction} = \frac{1^\circ}{1000000} \times \sum \left(\widetilde{v} \times E^{|\widetilde{x}|} \times \sin(\widetilde{w} \times D + \widetilde{x} \times M + \widetilde{y} \times M' + \widetilde{z} \times F)\right)$$
$$\textit{venus} = \frac{3958^\circ}{1000000} \times \sin(119.75^\circ + c \times 131.849^\circ)$$
$$\textit{jupiter} = \frac{318^\circ}{1000000} \times \sin(53.09^\circ + c \times 479264.29^\circ)$$
$$\textit{flat-earth} = \frac{1962^\circ}{1000000} \times \sin(L' - F)$$
$$\widetilde{v} = (\text{see Table 14.5})$$
$$\widetilde{w} = (\text{see Table 14.5})$$
$$\widetilde{x} = (\text{see Table 14.5})$$
$$\widetilde{y} = (\text{see Table 14.5})$$
$$\widetilde{z} = (\text{see Table 14.5})$$

This function and other lunar functions use the following auxiliary functions giving mean values of the moon's longitude, its elongation (angular distance from

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Table 14.5 Values of the arguments $\widetilde{v}$, $\widetilde{w}$, $\widetilde{x}$, $\widetilde{y}$, and $\vec{z}$ in **lunar-longitude** (page 232).

$\widetilde{v}$	$\widetilde{w}$	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$	$\widetilde{v}$	$\widetilde{w}$	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$
6288774	0	0	1	0	−2348	1	0	1	0
1274027	2	0	−1	0	2236	2	−2	0	0
658314	2	0	0	0	−2120	0	1	2	0
213618	0	0	2	0	−2069	0	2	0	0
−185116	0	1	0	0	2048	2	−2	−1	0
−114332	0	0	0	2	−1773	2	0	1	−2
58793	2	0	−2	0	−1595	2	0	0	2
57066	2	−1	−1	0	1215	4	−1	−1	0
53322	2	0	1	0	−1110	0	0	2	2
45758	2	−1	0	0	−892	3	0	−1	0
−40923	0	1	−1	0	−810	2	1	1	0
−34720	1	0	0	0	759	4	−1	−2	0
−30383	0	1	1	0	−713	0	2	−1	0
15327	2	0	0	−2	−700	2	2	−1	0
−12528	0	0	1	2	691	2	1	−2	0
10980	0	0	1	−2	596	2	−1	0	−2
10675	4	0	−1	0	549	4	0	1	0
10034	0	0	3	0	537	0	0	4	0
8548	4	0	−2	0	520	4	−1	0	0
−7888	2	1	−1	0	−487	1	0	−2	0
−6766	2	1	0	0	−399	2	1	0	−2
−5163	1	0	−1	0	−381	0	0	2	−2
4987	1	1	0	0	351	1	1	1	0
4036	2	−1	1	0	−340	3	0	−2	0
3994	2	0	2	0	330	4	0	−3	0
3861	4	0	0	0	327	2	−1	2	0
3665	2	0	−3	0	−323	0	2	1	0
−2689	0	1	−2	0	299	1	1	−1	0
−2602	2	0	−1	2	294	2	0	3	0
2390	2	−1	−2	0					

the sun), the solar anomaly (angular distance from perihelion), the lunar anomaly (angular distance from perigee), and the moon's "argument of latitude" (the distance from the moon's node, that point at which the moon's path crosses the ecliptic from the south to the north):

$$\textbf{mean-lunar-longitude}\,(c) \stackrel{\text{def}}{=} \tag{14.49}$$

$$\left(218.3164477^\circ + 481267.88123421^\circ \times c - 0.0015786^\circ \times c^2 + \frac{1^\circ}{538841} \times c^3 - \frac{1^\circ}{65194000} \times c^4\right) \bmod 360$$

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$$\textbf{lunar-elongation}\,(c) \stackrel{\text{def}}{=} \tag{14.50}$$

$$\left(297.8501921^\circ + 445267.1114034^\circ \times c - 0.0018819^\circ \times c^2 + \frac{1^\circ}{545868} \times c^3 - \frac{1^\circ}{113065000} \times c^4\right) \bmod 360$$

$$\textbf{solar-anomaly}\,(c) \stackrel{\text{def}}{=} \tag{14.51}$$

$$\left(357.5291092^\circ + 35999.0502909^\circ \times c - 0.0001536^\circ \times c^2 + \frac{1^\circ}{24490000} \times c^3\right) \bmod 360$$

$$\textbf{lunar-anomaly}\,(c) \stackrel{\text{def}}{=} \tag{14.52}$$

$$\left(134.9633964^\circ + 477198.8675055^\circ \times c + 0.0087414^\circ \times c^2 + \frac{1^\circ}{69699} \times c^3 - \frac{1^\circ}{14712000} \times c^4\right) \bmod 360$$

$$\textbf{moon-node}\,(c) \stackrel{\text{def}}{=} \tag{14.53}$$

$$\left(93.2720950^\circ + 483202.0175233^\circ \times c - 0.0036539^\circ \times c^2 - \frac{1^\circ}{3526000} \times c^3 + \frac{1^\circ}{863310000} \times c^4\right) \bmod 360$$

These all take, as their argument, a moment expressed in Julian centuries. (Several of these values also appear in **nth-new-moon**, but there they are centered around the year 2000.)

The following function shifts the distance from the equinoctial point into the range $[-90\,..\,90)$:

$$\textbf{lunar-node}\,(\mathit{date}) \stackrel{\text{def}}{=} \tag{14.54}$$

$$(\textbf{moon-node}\,(\textbf{julian-centuries}\,(\mathit{date}))) \bmod [-90\,..\,90)$$

If one wants the sidereal, rather than the equinoctial, lunar longitude, the following correction for precession may be used:

$$\textbf{sidereal-lunar-longitude}\,(t) \stackrel{\text{def}}{=} \tag{14.55}$$

$$(\,\textbf{lunar-longitude}\,(t) - \textbf{precession}\,(t) + \textbf{sidereal-start}\,) \bmod 360$$

Using **solar-longitude** and **lunar-longitude**, one can determine the phase of the moon—defined as the difference in longitudes of the sun and moon—at any moment t:

$$\textbf{lunar-phase}\,(t) \stackrel{\text{def}}{=} \begin{cases} \varphi' & \textbf{if } |\varphi - \varphi'| > 180° \\ \varphi & \textbf{otherwise} \end{cases} \tag{14.56}$$

where

$$\begin{aligned}
\varphi &= (\textbf{lunar-longitude}\,(t) - \textbf{solar-longitude}\,(t)) \bmod 360 \\
t_0 &= \textbf{nth-new-moon}\,(0) \\
n &= \text{round}\left(\frac{t - t_0}{\textbf{mean-synodic-month}}\right) \\
\varphi' &= 360° \times \left(\frac{t - \textbf{nth-new-moon}\,(n)}{\textbf{mean-synodic-month}} \bmod 1\right)
\end{aligned}$$

To ensure the robustness of our code, this function checks whether the phase obtained in this way conflicts with the time of new moon as calculated by the more precise **nth-new-moon** function. If it does, that is, if one method puts the time just before a new moon and the other just after, then an approximation based on the **nth-new-moon** moment is preferred.

To determine the time of the new moon, or other phases of the moon, we search using (1.36) for a time before moment t when the solar and lunar longitudes differ by the desired amount, φ:

$$\textbf{lunar-phase-at-or-before}\,(\varphi, t) \stackrel{\text{def}}{=} \textbf{lunar-phase}^{-1}(\varphi, [a\,..\,b]) \tag{14.57}$$

where

$$\begin{aligned}
\tau &= t - \textbf{mean-synodic-month} \times \frac{1}{360°} \times ((\textbf{lunar-phase}\,(t) - \varphi) \bmod 360) \\
a &= \tau - 2 \\
b &= \min\{t, \tau + 2\}
\end{aligned}$$

The search is centered around the last time the *mean* moon had that phase. That moment τ is calculated by a variant of equation (1.63) based on the average rate at which the phase changes by 1°.

The search for the next time the moon has a given phase is analogous:

$$\textbf{lunar-phase-at-or-after}\,(\varphi, t) \stackrel{\text{def}}{=} \textbf{lunar-phase}^{-1}(\varphi, [a\,..\,b]) \tag{14.58}$$

where

$$\begin{aligned}
\tau &= t + \textbf{mean-synodic-month} \times \frac{1}{360°} \times ((\varphi - \textbf{lunar-phase}\,(t)) \bmod 360) \\
a &= \max\{t, \tau - 2\} \\
b &= \tau + 2
\end{aligned}$$

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For the computation of specific phases of the moon, that is, new moon, first quarter, full moon, and last quarter, we can use **lunar-phase-at-or-before** and **lunar-phase-at-or-after**, along with the following set of constants:

$$\textbf{new} \stackrel{\text{def}}{=} 0^\circ \tag{14.59}$$

$$\textbf{first-quarter} \stackrel{\text{def}}{=} 90^\circ \tag{14.60}$$

$$\textbf{full} \stackrel{\text{def}}{=} 180^\circ \tag{14.61}$$

$$\textbf{last-quarter} \stackrel{\text{def}}{=} 270^\circ \tag{14.62}$$

Lunar latitude is computed in nearly the same way as longitude:

$$\textbf{lunar-latitude}\,(t) \stackrel{\text{def}}{=} \beta + \mathit{venus} + \mathit{flat\text{-}earth} + \mathit{extra} \tag{14.63}$$

where

$$\begin{aligned}
c &= \textbf{julian-centuries}\,(t)\\
L' &= \textbf{mean-lunar-longitude}\,(c)\\
D &= \textbf{lunar-elongation}\,(c)\\
M &= \textbf{solar-anomaly}\,(c)\\
M' &= \textbf{lunar-anomaly}\,(c)\\
F &= \textbf{moon-node}\,(c)\\
E &= 1 - 0.002516 \times c - 0.0000074 \times c^2\\
\beta &= \frac{1^\circ}{1000000}\\
&\quad \times \sum \left(\widetilde{v} \times E^{|\widetilde{x}|} \times \sin\left(\widetilde{w} \times D + \widetilde{x} \times M + \widetilde{y} \times M' + \widetilde{z} \times F\right) \right)\\
\mathit{venus} &= \frac{175^\circ}{1000000} \times \Big(\sin\left(119.75^\circ + c \times 131.849^\circ + F\right)\\
&\qquad + \sin\left(119.75^\circ + c \times 131.849^\circ - F\right) \Big)\\
\mathit{flat\text{-}earth} &= -\frac{2235^\circ}{1000000} \times \sin L' + \frac{127^\circ}{1000000} \times \sin\left(L' - M'\right)\\
&\quad - \frac{115^\circ}{1000000} \times \sin\left(L' + M'\right)\\
\mathit{extra} &= \frac{382^\circ}{1000000} \times \sin\left(313.45^\circ + c \times 481266.484^\circ\right)\\
\widetilde{v} &= \text{(see Table 14.6)}\\
\widetilde{w} &= \text{(see Table 14.6)}\\
\widetilde{x} &= \text{(see Table 14.6)}\\
\widetilde{y} &= \text{(see Table 14.6)}\\
\widetilde{z} &= \text{(see Table 14.6)}
\end{aligned}$$

Lunar latitude ranges from -6° to $+6^\circ$.

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Table 14.6 Values of the arguments $\widetilde{v}$, $\widetilde{w}$, $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ in **lunar-latitude** (page 236).

$\widetilde{v}$	$\widetilde{w}$	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$	$\widetilde{v}$	$\widetilde{w}$	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$
5128122	0	0	0	1	777	0	0	1	−3
280602	0	0	1	1	671	4	0	−2	1
277693	0	0	1	−1	607	2	0	0	−3
173237	2	0	0	−1	596	2	0	2	−1
55413	2	0	−1	1	491	2	−1	1	−1
46271	2	0	−1	−1	−451	2	0	−2	1
32573	2	0	0	1	439	0	0	3	−1
17198	0	0	2	1	422	2	0	2	1
9266	2	0	1	−1	421	2	0	−3	−1
8822	0	0	2	−1	−366	2	1	−1	1
8216	2	−1	0	−1	−351	2	1	0	1
4324	2	0	−2	−1	331	4	0	0	1
4200	2	0	1	1	315	2	−1	1	1
−3359	2	1	0	−1	302	2	−2	0	−1
2463	2	−1	−1	1	−283	0	0	1	3
2211	2	−1	0	1	−229	2	1	1	−1
2065	2	−1	−1	−1	223	1	1	0	−1
−1870	0	1	−1	−1	223	1	1	0	1
1828	4	0	−1	−1	−220	0	1	−2	−1
−1794	0	1	0	1	−220	2	1	−1	−1
−1749	0	0	0	3	−185	1	0	1	1
−1565	0	1	−1	1	181	2	−1	−2	−1
−1491	1	0	0	1	−177	0	1	2	1
−1475	0	1	1	1	176	4	0	−2	−1
−1410	0	1	1	−1	166	4	−1	−1	−1
−1344	0	1	0	−1	−164	1	0	1	−1
−1335	1	0	0	−1	132	4	0	1	−1
1107	0	0	3	1	−119	1	0	−1	−1
1021	4	0	0	−1	115	4	−1	0	−1
833	4	0	−1	1	107	2	−2	0	1

Finally, the altitude of the moon is determined in a similar fashion to that of the sun (page 226).

$$\textbf{lunar-altitude}\left(t,\ \boxed{\varphi \mid \psi \mid \text{—} \mid \text{—}}\right) \stackrel{\text{def}}{=} \tag{14.64}$$

$$altitude \bmod [-180\,..\,180)$$

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where

$$\begin{aligned}
\lambda &= \textbf{lunar-longitude}\,(t)\\
\beta &= \textbf{lunar-latitude}\,(t)\\
\alpha &= \textbf{right-ascension}\,(t, \beta, \lambda)\\
\delta &= \textbf{declination}\,(t, \beta, \lambda)\\
\theta_0 &= \textbf{sidereal-from-moment}\,(t)\\
H &= (\theta_0 + \psi - \alpha) \bmod 360\\
\mathit{altitude} &= \arcsin\,(\sin\varphi \times \sin\delta + \cos\varphi \times \cos\delta \times \cos H)
\end{aligned}$$

The result has not been corrected for parallax or refraction.

To convert geocentric altitude (viewed from the center of the Earth), as computed by **lunar-altitude**, into topocentric altitude (viewed from the surface), we first need to compute the parallax, for which we need to know the distance in meters between the centers of the Earth and the moon:

$$\textbf{lunar-distance}\,(t) \overset{\text{def}}{=} 385000560\text{ m} + \mathit{correction} \tag{14.65}$$

where

$$\begin{aligned}
c &= \textbf{julian-centuries}\,(t)\\
D &= \textbf{lunar-elongation}\,(c)\\
M &= \textbf{solar-anomaly}\,(c)\\
M' &= \textbf{lunar-anomaly}\,(c)\\
F &= \textbf{moon-node}\,(c)\\
E &= 1 - 0.002516 \times c - 0.0000074 \times c^2\\
\mathit{correction} &= \sum \left(\widetilde{v} \times E^{|\widetilde{x}|} \times \cos\,(\widetilde{w} \times D + \widetilde{x} \times M + \widetilde{y} \times M' + \widetilde{z} \times F) \right)\\
\widetilde{v} &= \text{(see Table 14.7)}\\
\widetilde{w} &= \text{(see Table 14.7)}\\
\widetilde{x} &= \text{(see Table 14.7)}\\
\widetilde{y} &= \text{(see Table 14.7)}\\
\widetilde{z} &= \text{(see Table 14.7)}
\end{aligned}$$

Now, we have:

$$\textbf{lunar-parallax}\,(t, \mathit{location}) \overset{\text{def}}{=} \arcsin \mathit{arg} \tag{14.66}$$

where

$$\begin{aligned}
\mathit{geo} &= \textbf{lunar-altitude}\,(t, \mathit{location})\\
\Delta &= \textbf{lunar-distance}\,(t)
\end{aligned}$$

Table 14.7 Values of the arguments $\widetilde{v}$, $\widetilde{w}$, $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ in **lunar-distance** (page 238).

$\widetilde{v}$	$\widetilde{w}$	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$	$\widetilde{v}$	$\widetilde{w}$	$\widetilde{x}$	$\widetilde{y}$	$\widetilde{z}$
−20905355	0	0	1	0	6322	1	0	1	0
−3699111	2	0	−1	0	−9884	2	−2	0	0
−2955968	2	0	0	0	5751	0	1	2	0
−569925	0	0	2	0	0	0	2	0	0
48888	0	1	0	0	−4950	2	−2	−1	0
−3149	0	0	0	2	4130	2	0	1	−2
246158	2	0	−2	0	0	2	0	0	2
−152138	2	−1	−1	0	−3958	4	−1	−1	0
−170733	2	0	1	0	0	0	0	2	2
−204586	2	−1	0	0	3258	3	0	−1	0
−129620	0	1	−1	0	2616	2	1	1	0
108743	1	0	0	0	−1897	4	−1	−2	0
104755	0	1	1	0	−2117	0	2	−1	0
10321	2	0	0	−2	2354	2	2	−1	0
0	0	0	1	2	0	2	1	−2	0
79661	0	0	1	−2	0	2	−1	0	−2
−34782	4	0	−1	0	−1423	4	0	1	0
−23210	0	0	3	0	−1117	0	0	4	0
−21636	4	0	−2	0	−1571	4	−1	0	0
24208	2	1	−1	0	−1739	1	0	−2	0
30824	2	1	0	0	0	2	1	0	−2
−8379	1	0	−1	0	−4421	0	0	2	−2
−16675	1	1	0	0	0	1	1	1	0
−12831	2	−1	1	0	0	3	0	−2	0
−10445	2	0	2	0	0	4	0	−3	0
−11650	4	0	0	0	0	2	−1	2	0
14403	2	0	−3	0	1165	0	2	1	0
−7003	0	1	−2	0	0	1	1	−1	0
0	2	0	−1	2	0	2	0	3	0
10056	2	−1	−2	0	8752	2	0	−1	−2

$$alt = \frac{6378140\text{ m}}{\Delta}$$
$$arg = alt \times \cos geo$$

and

$$\textbf{topocentric-lunar-altitude}\,(t, location) \stackrel{\text{def}}{=} \tag{14.67}$$

$$\textbf{lunar-altitude}\,(t, location) - \textbf{lunar-parallax}\,(t, location)$$

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14.7 Rising and Setting of the Sun and Moon

Some, like the Chaldees and the ancient Jews, define such a day as the time between two sunrises; others, like the Athenians, as that between two sunsets; or, like the Romans, from midnight to midnight; or like the Egyptians, from noon to noon . . . It was necessary . . . to choose some mean and equal day, by which it would be possible to measure regularity of movement without trouble.

Nicolaus Copernicus: *De revolutionibus orbium coelestium* (1543)

We occasionally need the time of sunrise or sunset for a location. Astronomical sunrise is nowadays defined as the time of first appearance of the upper limb of the sun; sunset is the moment of disappearance, again of the upper limb. This is also the definition used for calendars that begin their day at sunset (for example, the Islamic and Hebrew calendars) or sunrise (the Hindu calendar, according to some authorities). Because of the asymmetry involved, on the day of the equinox the intervals from sunrise to sunset and from sunset to sunrise differ by a few minutes. This discrepancy is further compounded by atmospheric refraction (the bending of the sun's light by the Earth's atmosphere), which makes the sun visible 2 to 3 minutes before a straight line to it is actually above the horizon and keeps it visible for a few minutes after it is geometrically below the horizon at sunset time.

We first write a general function to calculate the moment, in local mean time, when the "depression angle" of the geometric center of the sun is α degrees below (above, if the angle is negative) the *geometric horizon* at sea level at a given location around a fixed moment *t*. The same depression angle occurs both in the east (at around sunrise) and the west (at around sunset), so we use the variable *early?* to specify which we want: *early?* is **true** for the eastern horizon and **false** for the western horizon. First, an approximation is determined:

$$\textbf{approx-moment-of-depression}\,(t, \mathit{location}, \alpha, \mathit{early?}) \stackrel{\text{def}}{=} \tag{14.68}$$

$$\begin{cases} \textbf{local-from-apparent} \\ \quad \left(\mathit{date} + \begin{cases} 6^{\text{h}} - \mathit{offset} & \textbf{if } \mathit{early?} \\ 18^{\text{h}} + \mathit{offset} & \textbf{otherwise} \end{cases}, \mathit{location} \right) & \textbf{if } |\mathit{value}| \leqslant 1 \\ \textbf{bogus} & \textbf{otherwise} \end{cases}$$

where

$$\mathit{try} = \textbf{sine-offset}\,(t, \mathit{location}, \alpha)$$

$$\mathit{date} = \textbf{fixed-from-moment}\,(t)$$

$$\mathit{alt} = \begin{cases} \mathit{date} & \textbf{if } \alpha \geqslant 0 \text{ and } \mathit{early?} \\ \mathit{date} + 1 & \textbf{if } \alpha \geqslant 0 \\ \mathit{date} + 12^{\text{h}} & \textbf{otherwise} \end{cases}$$

$$\mathit{value} = \begin{cases} \textbf{sine-offset}\,(\mathit{alt}, \mathit{location}, \alpha) & \textbf{if } |\mathit{try}| > 1 \\ \mathit{try} & \textbf{otherwise} \end{cases}$$

$$\mathit{offset} = \frac{\arcsin \mathit{value}}{360^\circ} \bmod \left[-12^{\text{h}} \,..\, 12^{\text{h}}\right)$$

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Here, we have

$$\textbf{sine-offset}\,(t, \mathit{location}, \alpha) \stackrel{\text{def}}{=} \tan\varphi \times \tan\delta + \frac{\sin\alpha}{\cos\delta \times \cos\varphi} \tag{14.69}$$

where

$$\begin{aligned}\varphi &= \mathit{location}_{\textbf{latitude}}\\ t' &= \textbf{universal-from-local}\,(t, \mathit{location})\\ \delta &= \textbf{declination}\,(t', 0^\circ, \textbf{solar-longitude}\,(t'))\end{aligned}$$

The function **sine-offset** gives the sine of the angle α between where the sun is at *t* and where it is at its position of interest. An impossible value (that is, outside the range of [−1 .. 1]) is returned if the angle α is not reachable. That approximation is then repeatedly refined:

$$\textbf{moment-of-depression}\,(\mathit{approx}, \mathit{location}, \alpha, \mathit{early?}) \stackrel{\text{def}}{=} \tag{14.70}$$

$$\begin{cases}\textbf{bogus} & \textbf{if } t = \textbf{bogus}\\ t & \textbf{if } |\mathit{approx} - t| < 30^{\text{s}}\\ \textbf{moment-of-depression}\,(t, \mathit{location}, \alpha, \mathit{early?}) & \textbf{otherwise}\end{cases}$$

where

$$t = \textbf{approx-moment-of-depression}\,(\mathit{approx}, \mathit{location}, \alpha, \mathit{early?})$$

In polar regions, when the sun does not reach the stated depression angle this function returns the constant **bogus**.

The function **moment-of-depression** may then be used in the determination of the local time in the morning or evening when the sun reaches a specified angle below the true horizon. The result (for nonpolar regions) is then converted to standard time, using **standard-from-local** (page 208):

$$\textbf{morning} \stackrel{\text{def}}{=} \text{true} \tag{14.71}$$

$$\textbf{dawn}\,(\mathit{date}, \mathit{location}, \alpha) \stackrel{\text{def}}{=} \tag{14.72}$$

$$\begin{cases}\textbf{bogus} & \textbf{if } \mathit{result} = \textbf{bogus}\\ \textbf{standard-from-local}\,(\mathit{result}, \mathit{location}) & \textbf{otherwise}\end{cases}$$

where

$$\mathit{result} = \textbf{moment-of-depression}\left(\mathit{date} + 6^{\text{h}}, \mathit{location}, \alpha, \textbf{morning}\right)$$

Similarly for the evening we have:

$$\textbf{evening} \stackrel{\text{def}}{=} \text{false} \tag{14.73}$$

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$$\textbf{dusk}\,(\mathit{date}, \mathit{location}, \alpha) \stackrel{\text{def}}{=} \tag{14.74}$$

$$\begin{cases} \textbf{bogus} & \textbf{if } \mathit{result} = \textbf{bogus} \\ \textbf{standard-from-local}\,(\mathit{result}, \mathit{location}) & \textbf{otherwise} \end{cases}$$

where

$$\mathit{result} = \textbf{moment-of-depression}\left(\mathit{date} + 18^{\text{h}}, \mathit{location}, \alpha, \textbf{evening}\right)$$

The *visible horizon* depends on the elevation of the observer. The half-diameter of the sun is $16'$, while the average effect of refraction is $34'$, for a total depression angle of $50'$. If the observer is above sea level then the sun is even lower when its upper limb touches the observer's horizon.

A standard value of the refraction, taking elevation into account, is computed as follows:

$$\textbf{refraction}\,(t, \mathit{location}) \stackrel{\text{def}}{=} 34' + \mathit{dip} + 19'' \times \sqrt{h} \tag{14.75}$$

where

$$\begin{aligned} h &= \max\{0\text{ m}, \mathit{location}_{\textbf{elevation}}\} \\ R &= 6.372 \times 10^{6}\text{ m} \\ \mathit{dip} &= \arccos\left(\frac{R}{R+h}\right) \end{aligned}$$

The value for R is the radius of the Earth; $\mathit{dip} + 19''\sqrt{h}$ is the approximate contribution (in degrees) to the depression angle caused by an elevation of h meters [34]. This function ignores "elevations" that are below sea level or obstructions of the line of sight to the horizon. Also, it cannot be perfectly accurate because the observed position of the sun depends on atmospheric conditions, such as atmospheric temperature, humidity, and pressure (see [26] and [32]). The time parameter t is not used here, but could be used in a more refined calculation that takes average atmospheric conditions into account.

Hence, for sunrise we write

$$\textbf{sunrise}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \textbf{dawn}\,(\mathit{date}, \mathit{location}, \alpha) \tag{14.76}$$

where

$$\alpha = \textbf{refraction}\left(\mathit{date} + 6^{\text{h}}, \mathit{location}\right) + 16'$$

The extra $16'$ is needed because we want the time when the upper limb of the sun first becomes visible. Similarly, for sunset we have

$$\textbf{sunset}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \textbf{dusk}\,(\mathit{date}, \mathit{location}, \alpha) \tag{14.77}$$

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where

$$\alpha = \textbf{refraction}\left(date + 18^{\text{h}}, location\right) + 16'$$

For example, to calculate the standard time of sunset in Urbana, Illinois on a given Gregorian date we could write

$$\textbf{urbana-sunset}\,(g\text{-}date) \stackrel{\text{def}}{=} \tag{14.78}$$
$$\textbf{time-from-moment}\,(\textbf{sunset}\,(d, \textbf{urbana}))$$

where

$$d = \textbf{fixed-from-gregorian}\,(g\text{-}date)$$

On November 12, 1945, this gives sunset at 4:42 p.m. At the Canadian Forces Station Alert in Nunavut, the northernmost settled point in the western hemisphere, for which

$$\textbf{cfs-alert} \stackrel{\text{def}}{=} \boxed{82°30' \;|\; -62°19' \;|\; 0\text{ m} \;|\; -5^{\text{h}}} \tag{14.79}$$

we get **bogus** for an answer on the same date.

The contribution of 34′ to the depression angle, used above, is based on the average effect of refraction, but—as already mentioned—the refraction varies greatly, depending on atmospheric conditions. Thus, the times of sunrise and sunset can be calculated only to the nearest minute; for polar regions the uncertainty will be several minutes.[14] Furthermore, at high latitudes, because of the discrepancies between apparent, local, and standard time, dawn—or even sunrise—on *date* can actually occur on $date - 1$ before midnight, and dusk or sunset can occur on $date + 1$. There may even be two occurrences on the same civil day.

The times of occurrence of certain depression angles have religious significance for Jews and Muslims. Some Jews, for example, end Sabbath on Saturday night when the sun reaches a depression angle of 7°5′,

$$\textbf{jewish-sabbath-ends}\,(date, location) \stackrel{\text{def}}{=} \textbf{dusk}\,(date, location, 7°5') \tag{14.80}$$

but for other purposes they consider dusk to end earlier:

$$\textbf{jewish-dusk}\,(date, location) \stackrel{\text{def}}{=} \textbf{dusk}\,(date, location, 4°40') \tag{14.81}$$

Table 14.8 gives some depression angles and their significance.

The rising and setting times of the moon, for nonpolar regions, can be determined in a similar fashion to that of the sun. Refraction is used to adjust the topocentric altitude:

$$\textbf{observed-lunar-altitude}\,(t, location) \stackrel{\text{def}}{=} \tag{14.82}$$
$$\textbf{topocentric-lunar-altitude}\,(t, location) + \textbf{refraction}\,(t, location) + 16'$$

16′ being the approximate average half-diameter of the moon.

[14] A 12-minute discrepancy between the calculated and observed times of sunrise is documented in [26].

Table 14.8 Significance of various solar depression angles. The Islamic values are derived from [13]; the Jewish values are from [15], primarily, and from [8].

	Angle	Significance
Morning	20°	Alternative Jewish dawn (Rabbenu Tam)
	18°	Astronomical and Islamic dawn
	16°	Jewish dawn (Maimonides)
	15°	Alternative Islamic dawn
	12°	Nautical twilight begins
	6°	Civil twilight begins
	0°50′	Sunrise
Evening	0°50′	Sunset
	4°40′	Jewish dusk (Vilna Gaon)
	6°	Civil twilight ends
	7° 5′	Jewish sabbath ends (Cohn)
	8°30′	Alternative Jewish sabbath ends (Tykocinski)
	12°	Nautical twilight ends
	15°	Alternative Islamic dusk
	18°	Astronomical and Islamic dusk
	20°	Alternative Jewish dusk (Rabbenu Tam)

Moonrise and moonset are found by binary search, after estimating the time of the event, based on altitude at midnight and on whether the moon is waxing or waning:

$$\textbf{moonrise}\,(date, location) \stackrel{\text{def}}{=} \tag{14.83}$$

$$\begin{cases} \max\{\textbf{standard-from-universal}\,(rise, location), date\} & \textbf{if } rise < t+1 \\ \textbf{bogus} & \textbf{otherwise} \end{cases}$$

where

$$\begin{aligned}
t &= \textbf{universal-from-standard}\,(date, location) \\
waning &= \textbf{lunar-phase}\,(t) > 180^\circ \\
alt &= \textbf{observed-lunar-altitude}\,(t, location) \\
lat &= location_{\textbf{latitude}} \\
offset &= \frac{alt}{4 \times (90^\circ - |lat|)} \\
approx &= \begin{cases} t + 1 - offset & \textbf{if } waning \text{ and } offset > 0 \\ t - offset & \textbf{if } waning \\ t + \frac{1}{2} + offset & \textbf{otherwise} \end{cases} \\
rise &= \underset{x \in [approx - 6^{\text{h}} \,..\, approx + 6^{\text{h}})}{\overset{u-l<1^{\text{m}}}{\textbf{MIN}}} \left\{ \textbf{observed-lunar-altitude}\,(x, location) > 0^\circ \right\}
\end{aligned}$$

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$$\textbf{moonset}\,(date, location) \stackrel{\text{def}}{=} \tag{14.84}$$

$$\begin{cases} \max\left\{\textbf{standard-from-universal}\,(set, location), date\right\} & \textbf{if } set < t+1 \\ \textbf{bogus} & \textbf{otherwise} \end{cases}$$

where

$$\begin{aligned}
t &= \textbf{universal-from-standard}\,(date, location) \\
waxing &= \textbf{lunar-phase}\,(t) < 180^\circ \\
alt &= \textbf{observed-lunar-altitude}\,(t, location) \\
lat &= location_{\textbf{latitude}} \\
offset &= \frac{alt}{4 \times (90^\circ - |lat|)} \\
approx &= \begin{cases} t + offset & \textbf{if } waxing \text{ and } offset > 0 \\ t + 1 + offset & \textbf{if } waxing \\ t - offset + \frac{1}{2} & \textbf{otherwise} \end{cases} \\
set &= \underset{x \in [approx - 6^{\text{h}} .. approx + 6^{\text{h}})}{\overset{u-l<1^{\text{m}}}{\textbf{MIN}}} \left\{ \begin{array}{r} \textbf{observed-lunar-altitude} \\ (x, location) < 0^\circ \end{array} \right\}
\end{aligned}$$

A **bogus** value is returned if, on the day in question, the event does not occur, as happens about once a month: since the search for the moment when the moon is at the horizon can return a moment just before midnight of the day in question, we need to take the maximum of the result and the start of the day. This function is not robust in the sense that it returns the time at which the moon gets closest to the horizon in those cases where it does not appear to cross the horizon at all, as happens in polar latitudes.

We will need **moonset** for the Babylonian calendar (Section 18.1).

14.8 Times of Day

May the gods destroy that man who first discovered hours and who first set up a sundial here; who cut up my day piecemeal, wretched me.

Plautus: *The Boeotian Woman*

Now Peter and John went up together into the temple at the hour of prayer, being the ninth hour.

The Acts of the Apostles 3:1

This singular and inconvenient method[16] *had its defenders, and that even among the French; who have found that with pencil, and a little astronomical calculation, one may fix the hour of dinner with very little embarrassment.*

Jacques Ozanam: *Recreations in Science and Natural Philosophy* (1851)

[16] Counting hours from zero at sunset.

Our civil day is divided into 24 hours, counting from zero at midnight (so-called "French time"); each hour is divided into 60 minutes, and each minute is divided into 60 seconds (if we assume no leap second is added to that day). Accordingly, we represent the time of day as a triple

$$hour \ : \ minute \ : \ second$$

where *hour* is an integer in the range 0 to 23, *minute* is an integer in the range 0 to 59, and *second* is a nonnegative real number less than 60. (Sometimes we omit the third component and give only the hour and minute.) Other cultures subdivided the day differently. For instance, the ancient Egyptians—as well as the Greeks and Romans in classical times—divided the day and night *separately* into 12 equal "hours" each. Because, except at the equator, the lengths of daylight and nighttime vary with the seasons, the lengths of such daytime and nighttime hours also vary with the season. These seasonally varying *temporal* (or *seasonal*) hours (*horæ temporales*) are still used for ritual purposes among Jews. In London, for example, the length of such an hour varies from about 39 minutes in December to about 83 minutes in June.

Ancient Chinese civilization divided a day into 10 *shí* and 100 *kè* based on marks on dripping pot. In the first century B.C.E., Chinese astronomers started to divide a day into 12 *shí*, beginning at midnight. Although 100 *kè* cannot be divided equally into 12 *shí*, the *kè* was not changed until 1670, during the early Qīng dynasty, when it was redefined as an eighth of a *shí*, making 96 *kè* per day.

The Hindus divide the civil day into 60 *ghaṭikás* of 24-minute duration, each of which is divided into 60 *palas*, each of which is 24 seconds. They also divide the sidereal day into 60 *nádís*, each *nádí* into 60 *vinadis*, and each of the latter into 6 *asus*. The Hebrew calendar divides hours into 1080 *ḥalaqim* (parts) of $3\frac{1}{3}$ seconds each; each part is divided into 76 *regaim* (moments). The French Revolutionary calendar divided each day into 10 "hours," each "hour" into 100 "minutes," and each "minute" into 100 "seconds."

There have been various conventions for the start of the hour count of a day. In many places in the past, town clocks were reset to 0 h at sunset or at dusk. This is usually referred to as "Italian time" [7], but was the convention in many other places in Europe and the Middle East. An alternate convention, often seen on sundials, was to begin counting hours at sunrise; these were called "Babylonian hours." We can convert between Italian time and local time. For example the clock in Padua,

$$\mathbf{padua} \stackrel{\text{def}}{=} \boxed{45°24'28''} \boxed{11°53'9''} \boxed{18\text{ m}} \boxed{1^{\text{h}}} \qquad (14.85)$$

was reset every day at the moment of local dusk, taken to be 30 minutes after sunset (which was sometimes computed to occur at a solar depression angle of 16′). Thus we define

$$\mathbf{local\text{-}zero\text{-}hour}\,(t) \stackrel{\text{def}}{=} \qquad (14.86)$$

$$\mathbf{local\text{-}from\text{-}standard}\,(\mathbf{dusk}\,(date, \mathbf{padua}, 16') + 30^{\text{m}}, \mathbf{padua})$$

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where

$$date = \textbf{fixed-from-moment}\,(t)$$

To convert local time, measured from midnight, to and from Italian time, measured from dusk, we use:

$$\textbf{local-from-italian}\,(t) \stackrel{\text{def}}{=} t - date + z \tag{14.87}$$

where

$$\begin{aligned} date &= \textbf{fixed-from-moment}\,(t) \\ z &= \textbf{local-zero-hour}\,(t-1) \end{aligned}$$

In the opposite direction,

$$\textbf{italian-from-local}\,(t_\ell) \stackrel{\text{def}}{=} \begin{cases} t_\ell + date + 1 - z & \textbf{if } t_\ell > z \\ t_\ell + date - z_0 & \textbf{otherwise} \end{cases} \tag{14.88}$$

where

$$\begin{aligned} date &= \textbf{fixed-from-moment}\,(t_\ell) \\ z_0 &= \textbf{local-zero-hour}\,(t_\ell - 1) \\ z &= \textbf{local-zero-hour}\,(t_\ell) \end{aligned}$$

Thus when the clock struck 2:00 according to Italian apparent time in Padua on November 12, 1732 (Gregorian), it was 19:16 according to French apparent time, which was 7:01 p.m. by local mean time; this would be 7:13 p.m. on today's standard time clocks.

In Ethiopia and some neighboring regions, this style of time reckoning is still in use. Twelve daytime hours are counted from 6 a.m. until 6 p.m., and twelve nightime hours are counted from 6 p.m. until the next morning.

With the functions for local sunrise and sunset times of the previous section, we can also compute the time based on temporal (seasonal) hours, still used by Jews and Hindus. At a specified *location* on a particular fixed *date*, the lengths of daytime and nighttime temporal hours are given by

$$\textbf{daytime-temporal-hour}\,(date, location) \stackrel{\text{def}}{=} \tag{14.89}$$

$$\begin{cases} \textbf{bogus} & \textbf{if sunrise}\,(date, location) = \textbf{bogus} \text{ or} \\ & \quad \textbf{sunset}\,(date, location) = \textbf{bogus} \\ \frac{1}{12} \times (\textbf{sunset}\,(date, location) - \textbf{sunrise}\,(date, location)) & \\ & \textbf{otherwise} \end{cases}$$

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and

$$\textbf{nighttime-temporal-hour}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.90}$$

$$\begin{cases} \textbf{bogus} & \textbf{if } \textbf{sunrise}\,(\mathit{date}+1, \mathit{location}) = \textbf{bogus} \text{ or} \\ & \quad \textbf{sunset}\,(\mathit{date}, \mathit{location}) = \textbf{bogus} \\ \frac{1}{12} \times (\textbf{sunrise}\,(\mathit{date}+1, \mathit{location}) - \textbf{sunset}\,(\mathit{date}, \mathit{location})) \\ \qquad \textbf{otherwise} \end{cases}$$

This allows us to convert "sundial time" to standard time with

$$\textbf{standard-from-sundial}\,(t, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.91}$$

$$\begin{cases} \textbf{bogus} & \textbf{if } h = \textbf{bogus} \\ \textbf{sunrise}\,(\mathit{date}, \mathit{location}) + (\mathit{hour} - 6) \times h & \textbf{if } 6 \leqslant \mathit{hour} \leqslant 18 \\ \textbf{sunset}\,(\mathit{date} - 1, \mathit{location}) + (\mathit{hour} + 6) \times h & \textbf{if } \mathit{hour} < 6 \\ \textbf{sunset}\,(\mathit{date}, \mathit{location}) + (\mathit{hour} - 18) \times h & \textbf{otherwise} \end{cases}$$

where

$$\begin{aligned} \mathit{date} &= \textbf{fixed-from-moment}\,(t) \\ \mathit{hour} &= 24 \times \textbf{time-from-moment}\,(t) \\ h &= \begin{cases} \textbf{daytime-temporal-hour}\,(\mathit{date}, \mathit{location}) & \textbf{if } 6 \leqslant \mathit{hour} \leqslant 18 \\ \textbf{nighttime-temporal-hour}\,(\mathit{date} - 1, \mathit{location}) & \textbf{if } \mathit{hour} < 6 \\ \textbf{nighttime-temporal-hour}\,(\mathit{date}, \mathit{location}) & \textbf{otherwise} \end{cases} \end{aligned}$$

which in turn allows us to determine, say, the end of morning according to Jewish ritual:

$$\textbf{jewish-morning-end}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.92}$$

$$\textbf{standard-from-sundial}\left(\mathit{date} + 10^{\text{h}}, \mathit{location}\right)$$

Temporal hours were also used for the canonical hours of the Church breviary: Matins (midnight), Lauds (dawn), Prime (sunrise), Terce (9 a.m.), Sext (noon), None (3 p.m.), Vespers (sunset), and Compline (dusk).

The times of apparent noon and midnight could be calculated using temporal hours, but the times can differ by a few seconds from **midday** (14.26) and **midnight** (14.25) because the times of sunrise and sunset sometimes change relatively quickly.

An important time of day for Muslim prayer is *asr*, which is defined for Hanafi Muslims as the moment in the afternoon when the shadow of a gnomon

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has increased by double its own length over the shadow length at noon. By trigonometry, we get the following determination:

$$\textbf{asr}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \begin{cases} \textbf{bogus} & \textbf{if } \mathit{altitude} \leqslant 0^\circ \\ \textbf{dusk}\,(\mathit{date}, \mathit{location}, \; -h) & \textbf{otherwise} \end{cases} \tag{14.93}$$

where

$$\begin{aligned} \mathit{noon} &= \textbf{midday}\,(\mathit{date}, \mathit{location}) \\ \varphi &= \mathit{location}_{\textbf{latitude}} \\ \delta &= \textbf{declination}\,(\mathit{noon}, 0^\circ, \textbf{solar-longitude}\,(\mathit{noon})) \\ \mathit{altitude} &= \arcsin\,(\cos\delta \times \cos\varphi + \sin\delta \times \sin\varphi) \\ h &= \textbf{arctan}\,(\tan \mathit{altitude}, 2 \times \tan \mathit{altitude} + 1) \bmod [-90\,..\,90) \end{aligned}$$

and where δ is the solar declination at noon. Shafi'i Muslims use the moment when the length of the shadow doubles:

$$\textbf{alt-asr}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.94}$$

$$\begin{cases} \textbf{bogus} & \textbf{if } \mathit{altitude} \leqslant 0^\circ \\ \textbf{dusk}\,(\mathit{date}, \mathit{location}, \; -h) & \textbf{otherwise} \end{cases}$$

where

$$\begin{aligned} \mathit{noon} &= \textbf{midday}\,(\mathit{date}, \mathit{location}) \\ \varphi &= \mathit{location}_{\textbf{latitude}} \\ \delta &= \textbf{declination}\,(\mathit{noon}, 0^\circ, \textbf{solar-longitude}\,(\mathit{noon})) \\ \mathit{altitude} &= \arcsin\,(\cos\delta \times \cos\varphi + \sin\delta \times \sin\varphi) \\ h &= \textbf{arctan}\,(\tan \mathit{altitude}, \tan \mathit{altitude} + 1) \\ &\quad \bmod [-90\,..\,90) \end{aligned}$$

On certain dates in polar regions, there is no shadow.

14.9 Lunar Crescent Visibility

So patent are the evils of a purely lunar year whose length varies, owing to primitive methods of observation and determination of the new moon, that efforts to correct them have never ceased from the beginning to the present day.

K. Vollers: *Encyclopædia of Religion and Ethics*, vol. III, p. 127 (1911)[17]

Astronomical methods, as well as rules of thumb, for predicting the time of first visibility of the crescent moon (the *phasis*) have been developed over the millennia

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[17] المؤلفون ليس بالّضرورة موافقون للآراء الموجودة في الأقتباس.

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by the ancient Babylonians, medieval Muslim and Hindu scientists, and by modern astronomers. We will require such a method to simulate the observation-based calendars of Chapter 18.

One simple criterion for likely visibility of the crescent moon, proposed by S. K. Shaukat [2], requires a minimum difference in altitudes between the setting sun and moon (ignoring parallax and refraction, for simplicity), and a minimum-size crescent, which depends on the elongation (angular separation), *arc-of-light*, between the two bodies. The elongation is computed as follows:

$$\textbf{arc-of-light}\,(t) \stackrel{\text{def}}{=} \tag{14.95}$$

$$\arccos\left(\cos\left(\textbf{lunar-latitude}\,(t)\right) \times \cos\left(\textbf{lunar-phase}\,(t)\right)\right)$$

A good time for viewing the young moon is when the sun is 4.5° below the horizon:

$$\textbf{simple-best-view}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.96}$$

$$\textbf{universal-from-standard}\,(\mathit{best}, \mathit{location})$$

where

$$\mathit{dark} = \textbf{dusk}\,(\mathit{date}, \mathit{location}, 4.5^\circ)$$

$$\mathit{best} = \begin{cases} \mathit{date} + 1 & \textbf{if } \mathit{dark} = \textbf{bogus} \\ \mathit{dark} & \textbf{otherwise} \end{cases}$$

The following boolean function checks whether the moon was visible on the *eve* of *date* at *location*, according to Shaukat's method:

$$\textbf{shaukat-criterion}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.97}$$

$$\textbf{new} < \mathit{phase} < \textbf{first-quarter} \text{ and } 10.6^\circ \leqslant \mathit{ARCL} \leqslant 90^\circ \text{ and } h > 4.1^\circ$$

where

$$\begin{aligned} t &= \textbf{simple-best-view}\,(\mathit{date} - 1, \mathit{location}) \\ \mathit{phase} &= \textbf{lunar-phase}\,(t) \\ h &= \textbf{lunar-altitude}\,(t, \mathit{location}) \\ \mathit{ARCL} &= \textbf{arc-of-light}\,(t) \end{aligned}$$

This definition is not designed for high altitudes and polar regions (where dusk may not occur or where the moon may only become visible late in the month).

Scientists have continued working on improved criteria for predicting visibility. For example, one may prefer to base visibility on the topocentric altitude (page 239), rather than the geocentric altitude (page 237). Some proposed criteria use what is called the *arc of vision*, the angular difference in altitudes of the sun and moon at a given time and place:

$$\textbf{arc-of-vision}\,(t, location) \stackrel{\text{def}}{=} \tag{14.98}$$

$$\textbf{lunar-altitude}\,(t, location) - \textbf{solar-altitude}\,(t, location)$$

In particular, B. D. Yallop [33] suggested using the following ideal time for visibility, based on [6]:

$$\textbf{bruin-best-view}\,(date, location) \stackrel{\text{def}}{=} \tag{14.99}$$

$$\textbf{universal-from-standard}\,(best, location)$$

where

$$\begin{aligned}
sun &= \textbf{sunset}\,(date, location)\\
moon &= \textbf{moonset}\,(date, location)\\
best &= \begin{cases} date + 1 & \textbf{if } sun = \textbf{bogus} \text{ or } moon = \textbf{bogus}\\ \frac{5}{9} \times sun + \frac{4}{9} \times moon & \textbf{otherwise} \end{cases}
\end{aligned}$$

Yallop's criterion is as follows:

$$\textbf{yallop-criterion}\,(date, location) \stackrel{\text{def}}{=} \tag{14.100}$$

$$\textbf{new} < phase < \textbf{first-quarter} \text{ and } ARCV > q_1 + e$$

where

$$\begin{aligned}
t &= \textbf{bruin-best-view}\,(date - 1, location)\\
phase &= \textbf{lunar-phase}\,(t)\\
D &= \textbf{lunar-semi-diameter}\,(t, location)\\
ARCL &= \textbf{arc-of-light}\,(t)\\
W &= D \times (1 - \cos ARCL)\\
ARCV &= \textbf{arc-of-vision}\,(t, location)\\
e &= -0.14\\
q_1 &= 11.8371 - 6.3226 \times W + 0.7319 \times W^2 - 0.1018 \times W^3
\end{aligned}$$

To determine the angular width of the crescent, Yallop's criterion takes into account the moon's topocentric semi-diameter (in degrees):

$$\textbf{lunar-semi-diameter}\,(t, location) \stackrel{\text{def}}{=} \tag{14.101}$$

$$0.27245 \times p \times (\sin h \times \sin p + 1)$$

where

$$h = \textbf{lunar-altitude}\,(t, \mathit{location})$$
$$p = \textbf{lunar-parallax}\,(t, \mathit{location})$$

An approximation for the geocentric apparent lunar diameter is used, for example, by [22]:

$$\textbf{lunar-diameter}\,(t) \stackrel{\text{def}}{=} \frac{1792367000^\circ}{9 \times \textbf{lunar-distance}\,(t)} \tag{14.102}$$

For a recent synthesis of modern methods of determining visibility, see [10].

Adopting Shaukat's relatively simple criterion for the determination of first visibility, we define

$$\textbf{visible-crescent}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.103}$$
$$\textbf{shaukat-criterion}\,(\mathit{date}, \mathit{location})$$

Other criteria may, of course, be used instead.

With the function **visible-crescent**, we can calculate the day on which the new moon is first observable before—or after—any given *date* by checking for first visibility after the relevant new moon:

$$\textbf{phasis-on-or-before}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.104}$$
$$\underset{d \geqslant \tau}{\textbf{MIN}} \left\{ \textbf{visible-crescent}\,(d, \mathit{location}) \right\}$$

where

$$\mathit{moon} = \textbf{fixed-from-moment}\,(\textbf{lunar-phase-at-or-before}\,(\textbf{new}, \mathit{date}))$$
$$\mathit{age} = \mathit{date} - \mathit{moon}$$
$$\tau = \begin{cases} \mathit{moon} - 30 & \textbf{if } \mathit{age} \leqslant 3 \text{ and not } \textbf{visible-crescent}\,(\mathit{date}, \mathit{location}) \\ \mathit{moon} & \textbf{otherwise} \end{cases}$$

$$\textbf{phasis-on-or-after}\,(\mathit{date}, \mathit{location}) \stackrel{\text{def}}{=} \tag{14.105}$$
$$\underset{d \geqslant \tau}{\textbf{MIN}} \left\{ \textbf{visible-crescent}\,(d, \mathit{location}) \right\}$$

where

$$\mathit{moon} = \textbf{fixed-from-moment}\,(\textbf{lunar-phase-at-or-before}\,(\textbf{new}, \mathit{date}))$$

$$
\begin{aligned}
\mathit{age} &= \mathit{date} - \mathit{moon}\\
\tau &= \begin{cases} \mathit{moon} + 29 & \textbf{if } 4 \leqslant \mathit{age} \text{ or } \textbf{visible-crescent}\,(\mathit{date} - 1, \mathit{location})\\ \mathit{date} & \textbf{otherwise} \end{cases}
\end{aligned}
$$

This method will be used in Chapter 18 for the observation-based Islamic (Section 18.3) and Hebrew (Section 18.4) calendars.

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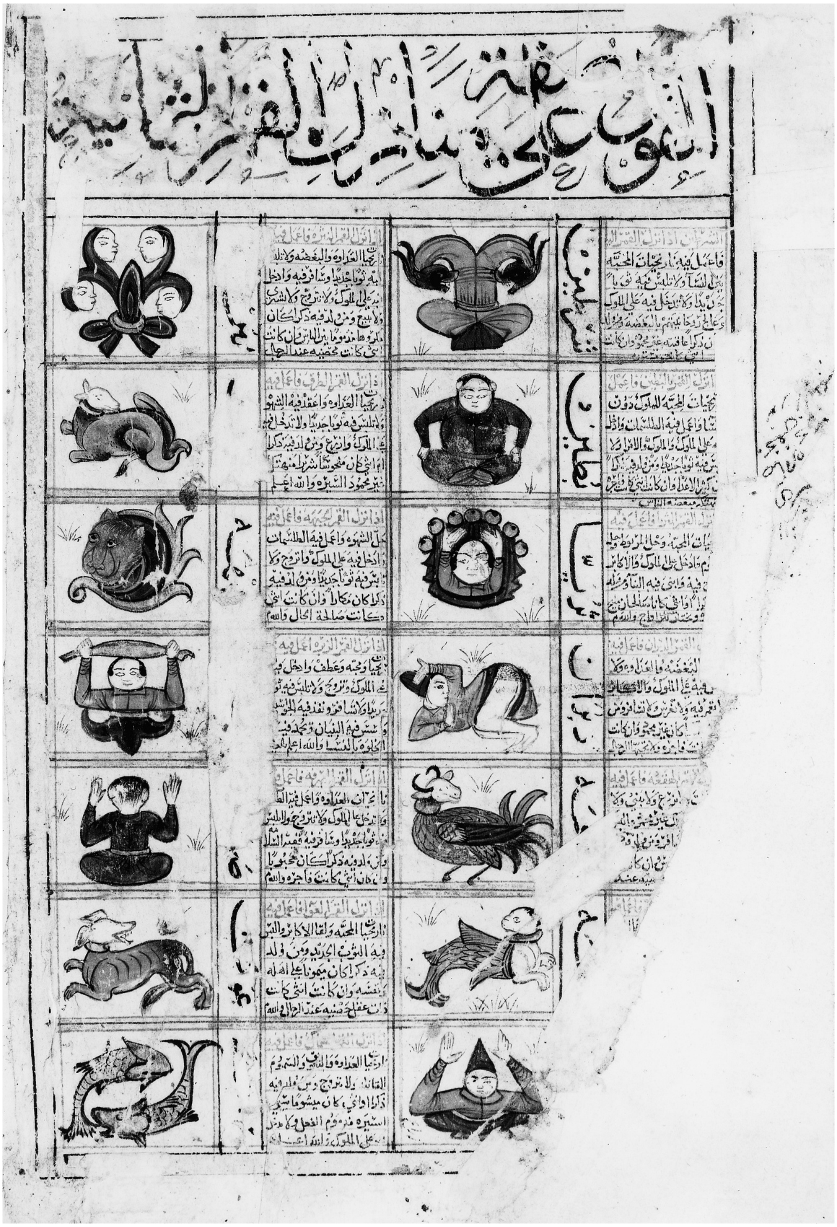

First 14 of 28 Arabian lunar stations from a late fourteenth-century manuscript of *Kitāb al-Bulhān* by the celebrated ninth-century Muslim astrologer Abu-Ma'shar al-Falaki (Albumazar) of Balkh, Khurasan, Persia. (Courtesy of the Bodleian Libraries, University of Oxford, Oxford; MS Bodley Or. 133, fol. 27b.)