Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză, Curs: Dynamical Systems, Primăvara 2024

Laboratory 5. Orbits of nonlinear planar systems

I. Hyperbolic equilibria.

1. We consider the planar nonlinear system $\dot{x} = 2x - x^2 - xy, \quad \dot{y} = -y + xy.$

a) Its equilibria are:

b) The matrix of the linearized system around the first (second and third, respectively) equilibrium point is (see section 8 in the tutorial):

and its eigenvalues are:

Notice that the equilibrium point is hyperbolic (this means that there is no eigenvalue with 0 real part).

The linearized system is the following:

and it has the type and stability:

Finally, using the linearization method, we deduce the stability of the first (second and third, respectively) equilibrium point:

c) Represent the direction field of the system in the box $[-1, 2.2] \times [-1, 1.2]$ of the phase plane. (In Maple use **dfieldplot**-see section 5 in the tutorial, while in Sage use **plot_vector_field**.) Notice that this box contains all the equilibria. Localize them. Now focus the image on smaller boxes around each equilibrium, like $[-0.2, 0.2] \times [-0.2, 0.2]$, $[1.2, 2.2] \times [-0.2, 0.2]$ and, respectively, $[0.2, 1.2] \times [0.2, 1.2]$. We will show in a lecture that the direction field is tangent to the orbits. Can you identify the shape of the orbits around each equilibrium point?

d) Using **DEplot** in Maple, and **desolve_odeint** in Sage, represent few orbits near each equilibrium point. For better results, first represent simultaneously only orbits near the same equilibrium point. For the variable time t take, for example, the interval $[0, 1]$ for various reasons: to identify better on the picture the initial and, respectively the future states (that is why we take $t \geq 0$) and, on the other hand, the interval is sufficiently small to not accumulate errors. Try also on larger intervals, like $[0, 20]$, sometimes it works. Choose the initial states to obtain a nice picture.

e) Gluing all the information obtained until now, sketch the phase portrait of this system in your notebook. Also, write the equilibria and the corresponding stability character.

2. This exercise has the same requirements as the previous one for the planar system $\dot{x} = x - 2xy, \quad \dot{y} = x^2/2 - y$. Of course, at c) choose suitable boxes for this system.

II. Non-hyperbolic equilibria.

3. We consider the conservative (or undamped) pendulum system

$$\dot{x} = y, \quad \dot{y} = -4\sin x.$$

a) Notice that $(0, 0)$ is an equilibrium point (we check in our mind).
Perform the necessary computations to justify that it is not hyperbolic:

b) Recall that the differential equation of the orbits of this system is

$$\frac{dy}{dx} = -\frac{4\sin x}{y}.$$

Find the general solution of this equation. Do not forget that here y is a function of x, and in Maple you have to write $y(x)$, while in Sage only y. Notice that the general solution can be written as

$$y^2 - 8\cos x = c, \quad c \in \mathbb{R}.$$

c) Consider $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = y^2 - 8\cos x$. Check that H is a first integral of the pendulum system (in fact, it is the total mechanical energy of the pendulum). Recall that we have to check the equality

$$\frac{\partial H}{\partial x} * y - 4\frac{\partial H}{\partial y} * \sin x = 0 \quad \text{in} \quad \mathbb{R}^2.$$

Then, the orbits of the pendulum system lie on the level curves of H. Represent the level curves of H in the box $[-5, 5] \times [-5, 5]$. Notice that the orbits which starts near the equilibrium point $(0, 0)$ are closed curves. If you want to convince yourself of this, represent the level curves of H in smaller boxes. From here deduce that the oscillations of the pendulum with small initial data are periodic.

4. We consider the Lotka-Volterra system (also called the predator-prey system; let's say that the foxes are the predators and the rabbits are the preys)

$$\dot{x} = x - xy, \quad \dot{y} = -0.3y + 0.3xy.$$

Moreover, we have that $100 * x(t)$ is the number of rabbits at time t, while $100 * y(t)$ is the number of foxes at time t in a given common habitat. So, the foxes eat rabbits and only rabbits, while the rabbits eat carrots. There are enough carrots, thus the rabbits can not die of starvation. We also assume that the borders of the habitat are closed, thus no one can enter or go out.

a) Notice that $(1,1)$ is an equilibrium point and show that it is non-hyperbolic.

b) Let $H : (0,\infty) \times (0,\infty) \to \mathbb{R}$, $H(x,y) = y - \ln y + 0.3(x - \ln x)$. Check that H is a first integral of the Lotka-Volterra system in the region $(0,\infty) \times (0,\infty)$, i.e. check that

$$\frac{\partial H}{\partial x} * (x - xy) + \frac{\partial H}{\partial y} * (-0.3y + 0.3xy) = 0.$$

c) Represent the level curves of H.

d) Deduce that the orbits around $(1,1)$ are periodic orbits. Describe the dynamics of the numbers of rabbits and foxes in the case that, at time $t = 0$ we know that there are 100 rabbits and 50 foxes. Describe the dynamics reading the phase portrait by detecting four crucial moments: - when one of the populations has a critical moment (e.g. before it increased, while after this critical moment, it will decrease). This behavior seems realistic to you? Try an explanation.