מטלה 3 - שאלות תיאורטיות

שאלה 1

נוכיח את השוויונות הבאים:

(1ב) $E(\hat{\mu}_{spike}) = E\left(\frac{1}{N}\sum_{1}^{N} n_i\right) = \frac{1}{N}\sum_{1}^{N} E(n_i) = \frac{1}{N}\sum_{1}^{N}(\mu)$

$= \frac{1}{N} \cdot N\mu = \mu.$

ז"ל: $B = \theta - E(\hat{\theta}_N) = \mu - \mu = 0$

B ← הטיה

$E(\hat{\theta}_N)$ ← ממוצע האומדן

μ ← הפרמטר האמיתי

(2) $E(\hat{\mu}_{axon}) = E\left(\frac{1}{N}\sum_{1}^{N}(n_i - \bar{n})^2\right)$

$= \frac{1}{N}\sum_{1}^{N} E\left((n_i - \mu) - (\bar{n} - \mu)\right)^2$

$= \frac{1}{N}\left\{\sum_{1}^{N} E\left\{(n_i-\mu)^2 - 2(n_i-\mu)(\bar{n}-\mu) + (\bar{n}-\mu)^2\right\}\right\}$

$= \frac{1}{N}\left\{\sum_{1}^{N} E(n_i-\mu)^2 - 2(\bar{n}-\mu)\sum_{1}^{N} E(n_i-\mu) + \sum_{1}^{N} E(\bar{n}-\mu)^2\right\}$

$= \frac{1}{N}\left\{\sum_{1}^{N} E(n_i-\mu)^2 - 2(\bar{n}-\mu)\left(\sum_{1}^{N} E(n_i) - \sum_{1}^{N} E(\mu_x)\right) + NE(\bar{n}-\mu)^2\right\}$

$= \frac{1}{N}\left\{\sum_{1}^{N} E(n_i-\mu)^2 - 2(\bar{n}-\mu)N E(\bar{n}-\mu) + N \cdot E(\bar{n}-\mu)^2\right\}$

$$= E\left\{\frac{1}{N}\sum_{1}^{N}(n_i-\mu)^2 - 2(\bar{n}-\mu)^2 + (\bar{n}-\mu)^2\right\}$$

$$= E\left\{\frac{1}{N}\sum_{1}^{N}(n_i-\mu)^2 - (\bar{n}-\mu)^2\right\}$$

$$= \frac{1}{N}\cdot\sum_{1}^{N}E(n_i-\mu)^2 - E(\bar{n}-\mu)^2$$

$$= \frac{1}{N}\cdot N\cdot Var(n_i) - \frac{Var(\bar{n})}{N} = \mu - \frac{\mu}{N} = \mu\left(1-\frac{1}{N}\right)$$

$$= \mu\left(\frac{N-1}{N}\right) = E(\hat{\mu}_{axon})$$

<u>הוכ"ה</u>: $B = \mu - \mu\left(\frac{N-1}{N}\right) = \mu\left(1-\frac{N-1}{N}\right)$

$$= \mu\left(\frac{N-N+1}{N}\right) = \frac{\mu}{N}$$

(ג) נשתמש בהתפלגות אמ"ס:

$$\hat{\mu}_{ax\text{-}fixed} = \frac{N}{N-1}\cdot\frac{1}{N}\sum(n_i-\bar{n})^2 = \frac{N}{N-1}\hat{\mu}_{axon}$$

התוחלת אמ"ס:

$$E(\hat{\mu}_{ax\text{-}fixed}) = E\left(\frac{N}{N-1}\hat{\mu}_{axon}\right) = \frac{N}{N-1}E(\hat{\mu}_{axon})$$

$$= \frac{N}{N-1}\cdot\frac{N-1}{N}\mu = \mu$$

ונשתמש ביה"ת מסכ הוכ"ה: $B' = \mu - \mu = 0$ //

(3)

$$\hat{\mu}^{axon}_{fixed} = \frac{N}{N-1}\hat{\mu}_{axon}$$

לפי תרגיל בכיתה

$$Var\left(\hat{\mu}^{axon}_{fixed}\right) = \left(\frac{N}{N-1}\right)^2 Var\left(\hat{\mu}_{axon}\right)$$

$$\Rightarrow Var\left(\hat{\mu}_{axon}\right) = \left(\frac{N-1}{N}\right)^2 Var\left(\hat{\mu}^{axon}_{fixed}\right)$$

$$\Rightarrow Var\left(\hat{\mu}_{axon}\right) = \left(\frac{N-1}{N}\right)^2\left(\frac{\sigma^2}{N} + \frac{2\mu^2}{N-1}\right)$$

(2) עבור $\hat{\mu}_{spike}$

$$\hat{\mu}_{spike} = \frac{1}{N}\sum_{1}^{N} n_i \qquad / Var(\,)$$

$$Var\left(\hat{\mu}_{spike}\right) = \frac{1}{N^2} Var\left(\sum_{1}^{N} n_i\right) = \frac{1}{N^2}\left\{\sum_{1}^{N} Var(n_i) + 2\sum_{1\le i<j\le N}\sum Cov(n_i, n_j)\right\}$$

$$= \frac{1}{N^2}\sum_{1}^{N} Var(n_i) = \frac{1}{N^2}\sum_{1}^{N}\mu = \frac{N\mu}{N^2} = \frac{\mu}{N} \;//$$

```
clc;clear;clf;
N = [2,5,10,50,100];
M = 1000;
mu = 10;

r1 = poissrnd(mu,N(1),M);
r2 = poissrnd(mu,N(2),M);
r3 = poissrnd(mu,N(3),M);
r4 = poissrnd(mu,N(4),M);
r5 = poissrnd(mu,N(5),M);

mu_spike = [mean(r1);mean(r2);mean(r3);mean(r4);mean(r5)];
mu_axon = [mean([r1-mu_spike(1,:)].^2);mean([r2-mu_spike(2,:)].^2);mean([r3-
mu_spike(3,:)].^2);mean([r4-mu_spike(4,:)].^2);mean([r5-mu_spike(5,:)].^2);];
mu_axon_fixed = [(N(1)/(N(1)-1)).*mu_axon(1,:);(N(2)/(N(2)-1))*mu_axon(2,:);
(N(3)/(N(3)-1))*mu_axon(3,:);(N(4)/(N(4)-1))*mu_axon(4,:);(N(5)/
(N(5)-1))*mu_axon(5,:)];

E_mu_spike = mean(mu_spike');
E_mu_axon = mean(mu_axon');
E_mu_axon_fixed = mean(mu_axon_fixed');

figure(1); plot([1:1000],mu_axon_fixed(1,:),[1:1000],mu_axon(1,:),
[1:1000],mu_spike(1,:)); title("N = 2"); grid on; % N = 2
ylabel("Estimated {\mu}"); xlabel("Amount of Itterations through random
 function"); yline(mu,'b--');
legend("{\mu} axon fixed","{\mu} axon","{\mu} spike");
figure(2); plot([1:1000],mu_axon_fixed(2,:),[1:1000],mu_axon(2,:),
[1:1000],mu_spike(2,:)); title("N = 5"); grid on; % N = 5
ylabel("Estimated {\mu}"); xlabel("Amount of Itterations through random
 function"); yline(mu,"b--")
legend("{\mu} axon fixed","{\mu} axon","{\mu} spike");
figure(3); plot([1:1000],mu_axon_fixed(3,:),[1:1000],mu_axon(3,:),
[1:1000],mu_spike(3,:)); title("N = 10"); grid on; % N = 10
ylabel("Estimated {\mu}"); xlabel("Amount of Itterations through random
 function"); yline(mu,"b--")
legend("{\mu} axon fixed","{\mu} axon","{\mu} spike");
figure(4); plot([1:1000],mu_axon_fixed(4,:),[1:1000],mu_axon(4,:),
[1:1000],mu_spike(4,:)); title("N = 50"); grid on; % N = 50
ylabel("Estimated {\mu}"); xlabel("Amount of Itterations through random
 function"); yline(mu,"b--")
legend("{\mu} axon fixed","{\mu} axon","{\mu} spike");
figure(5); plot([1:1000],mu_axon_fixed(5,:),[1:1000],mu_axon(5,:),
[1:1000],mu_spike(5,:)); title("N = 100"); grid on; % N = 100
ylabel("Estimated {\mu}"); xlabel("Amount of Itterations through random
 function"); yline(mu,"b--")
legend("{\mu} axon fixed","{\mu} axon","{\mu} spike");

figure(6);

subplot(311); stem(N,E_mu_spike); grid on;
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```
title("E({\mu}_s_p_i_k_e)"); xlabel("N"); ylabel("E({\mu})"); xticks([ 2 5 10
 50 100]); xticklabels({'N = 2','N = 5','N = 10', 'N = 50', 'N = 100'});
subplot(312); stem(N,E_mu_axon); grid on;
title("E({\mu}_a_x_o_n)"); xlabel("N"); ylabel("E({\mu})"); xticks([ 2 5 10 50
 100]); xticklabels({'N = 2','N = 5','N = 10', 'N = 50', 'N = 100'});
subplot(313); stem(N,E_mu_axon_fixed); grid on;
title("E({\mu}_a_x_o_n _f_i_x_e_d)"); xlabel("N"); ylabel("E({\mu})");
 xticks([ 2 5 10 50 100]); xticklabels({'N = 2','N = 5','N = 10', 'N = 50', 'N
 = 100'});

B_mu_spike = mu - E_mu_spike;
B_mu_axon = mu - E_mu_axon;
B_mu_axon_fixed = mu - E_mu_axon_fixed;

var_mu_spike = var(mu_spike');
var_mu_axon = var(mu_axon');
var_mu_axon_fixed = var(mu_axon_fixed');

mse_spike_vec = []; mse_axon_vec = []; mse_axon_fixed_vec = [];
for i = [1:5]
    mse_spike_vec = [mse_spike_vec, immse(E_mu_spike(1,i),mu)];
    mse_axon_vec = [mse_axon_vec, immse(E_mu_axon(1,i),mu)];
    mse_axon_fixed_vec = [mse_axon_fixed_vec, immse(E_mu_axon_fixed(1,i),mu)];
end

figure(7); clf(7);

hold on; stem(N,B_mu_spike); stem(N,B_mu_axon); stem(N,B_mu_axon_fixed);
 grid on;
xlabel("N"); ylabel("B({\mu})"); xticks([ 2 5 10 50 100]); xticklabels({'N =
 2','N = 5','N = 10', 'N = 50', 'N = 100'});
legend("{\mu} - E({\mu}_s_p_i_k_e)","{\mu} - E({\mu}_a_x_o_n)","{\mu} -
 E({\mu}_a_x_o_n _f_i_x_e_d)");
title("Bias for all Estimations"); hold off;

figure(8); clf(8);
hold on; stem(N,var_mu_spike); stem(N,var_mu_axon); stem(N,var_mu_axon_fixed);
 grid on;
xlabel("N"); ylabel("Var({\mu})"); xticks([ 2 5 10 50 100]); xticklabels({'N =
 2','N = 5','N = 10', 'N = 50', 'N = 100'});
legend("Var({\mu}_s_p_i_k_e)","Var({\mu}_a_x_o_n)","Var({\mu}_a_x_o_n
 _f_i_x_e_d)");
title("Variance for all Estimations"); hold off;

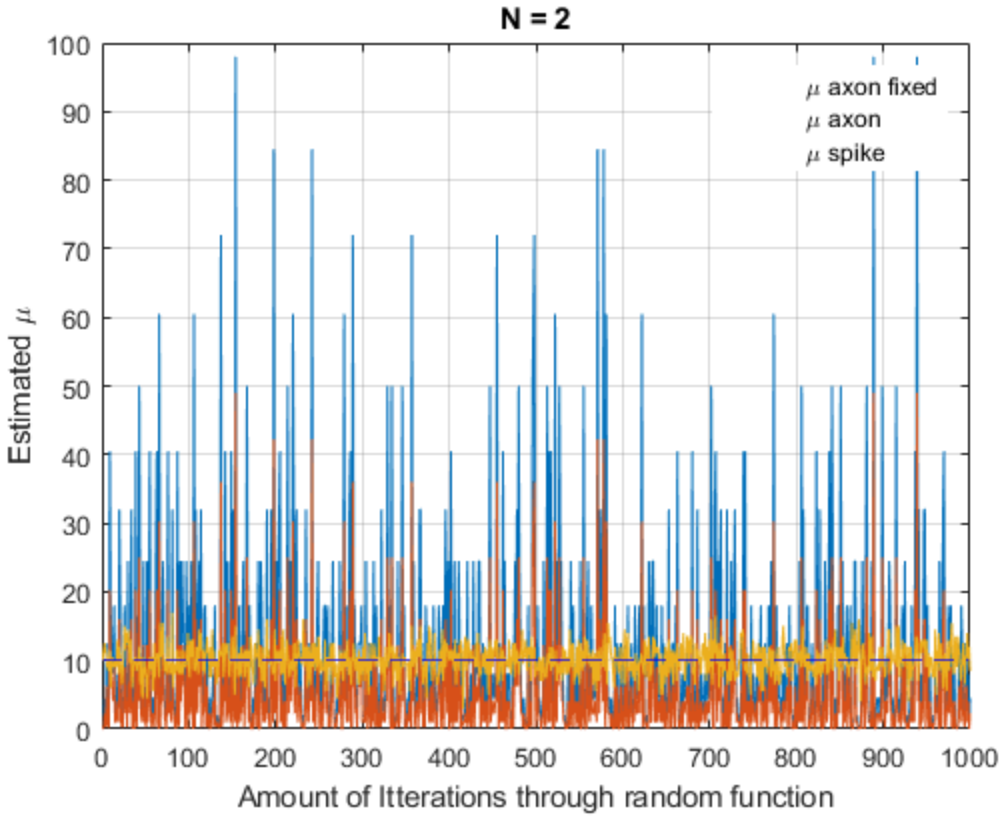
figure(9); clf(9)
hold on; stem(N,mse_spike_vec); stem(N,mse_axon_vec);
 stem(N,mse_axon_fixed_vec); grid on;
xlabel("N"); ylabel("MSE({\mu})"); xticks([ 2 5 10 50 100]); xticklabels({'N =
 2','N = 5','N = 10', 'N = 50', 'N = 100'});
legend("MSE({\mu}_s_p_i_k_e)","MSE({\mu}_a_x_o_n)","MSE({\mu}_a_x_o_n
 _f_i_x_e_d)");
title("MSE for all Estimations"); hold off;
mse_spike_vec
mse_axon_fixed_vec
```

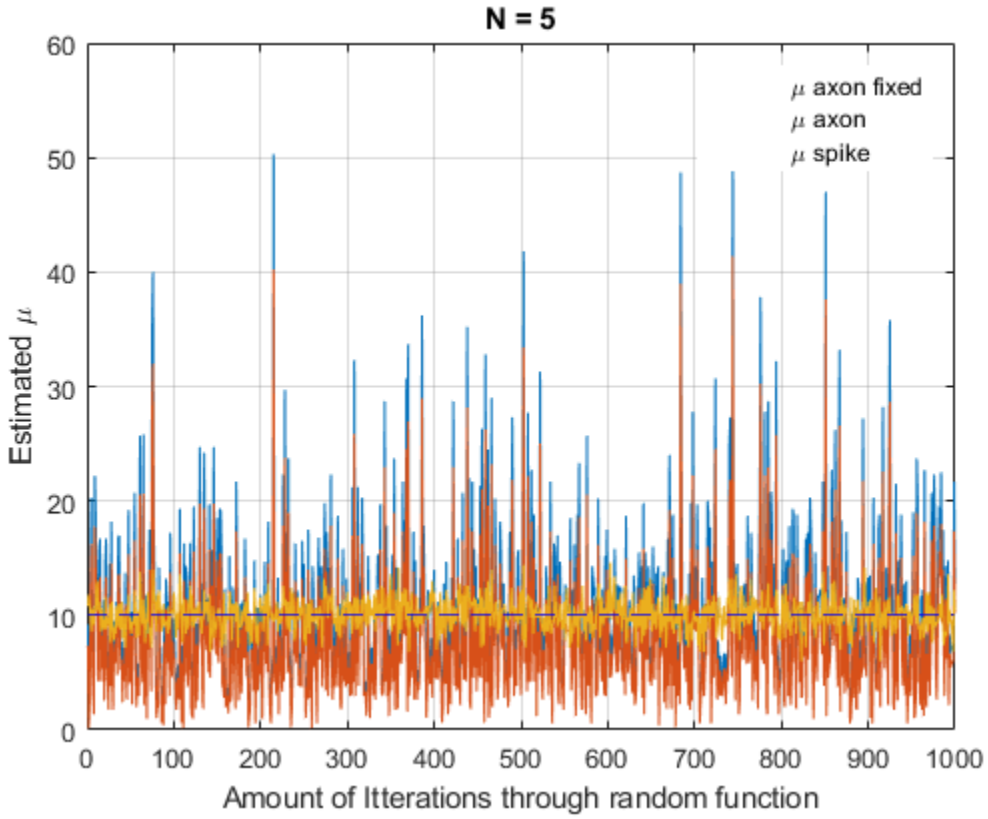
```
mse_spike_vec =

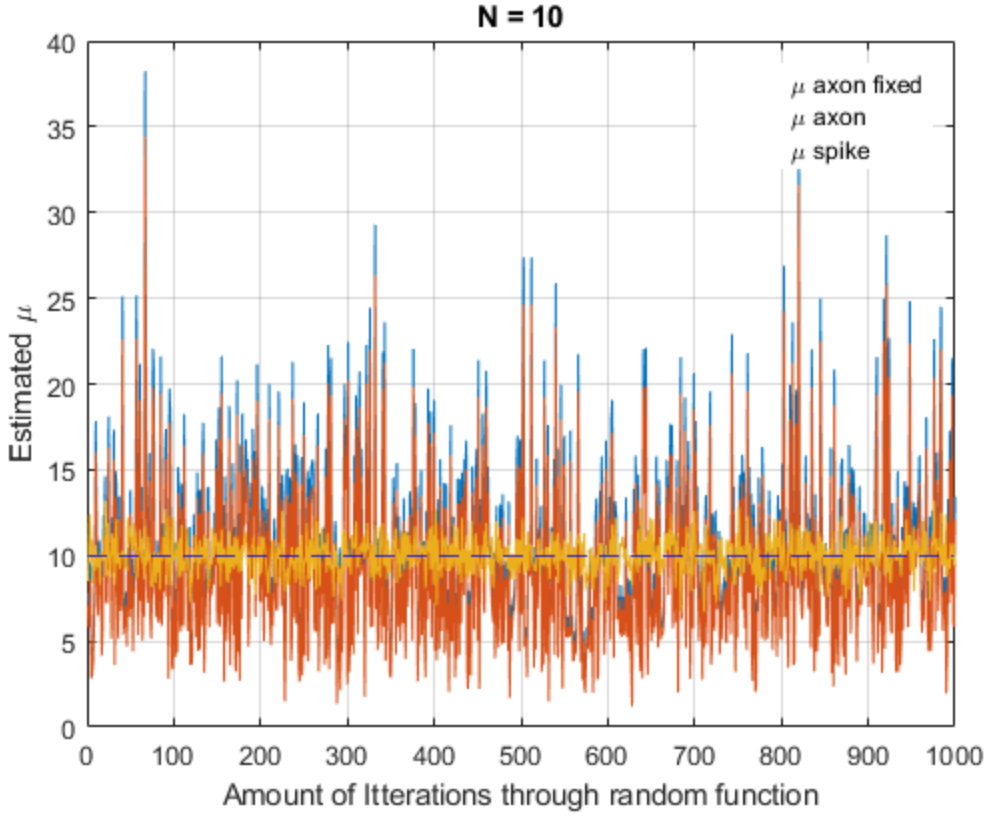
    0.0025    0.0064    0.0000    0.0003    0.0002

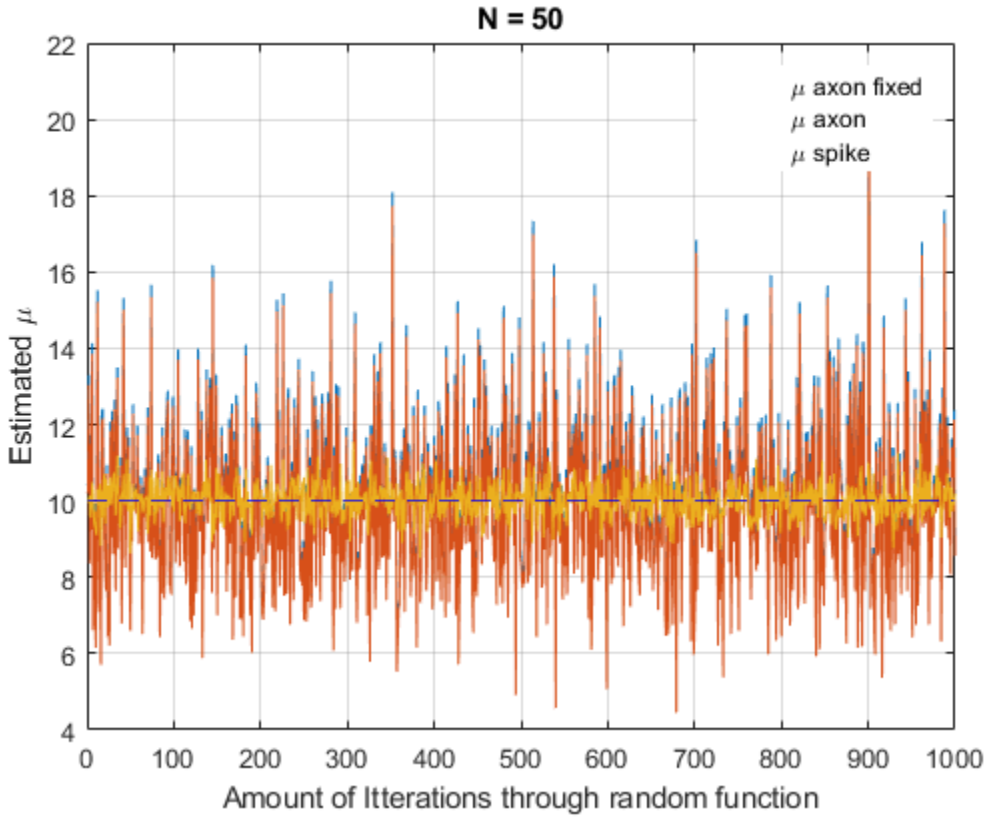
mse_axon_fixed_vec =

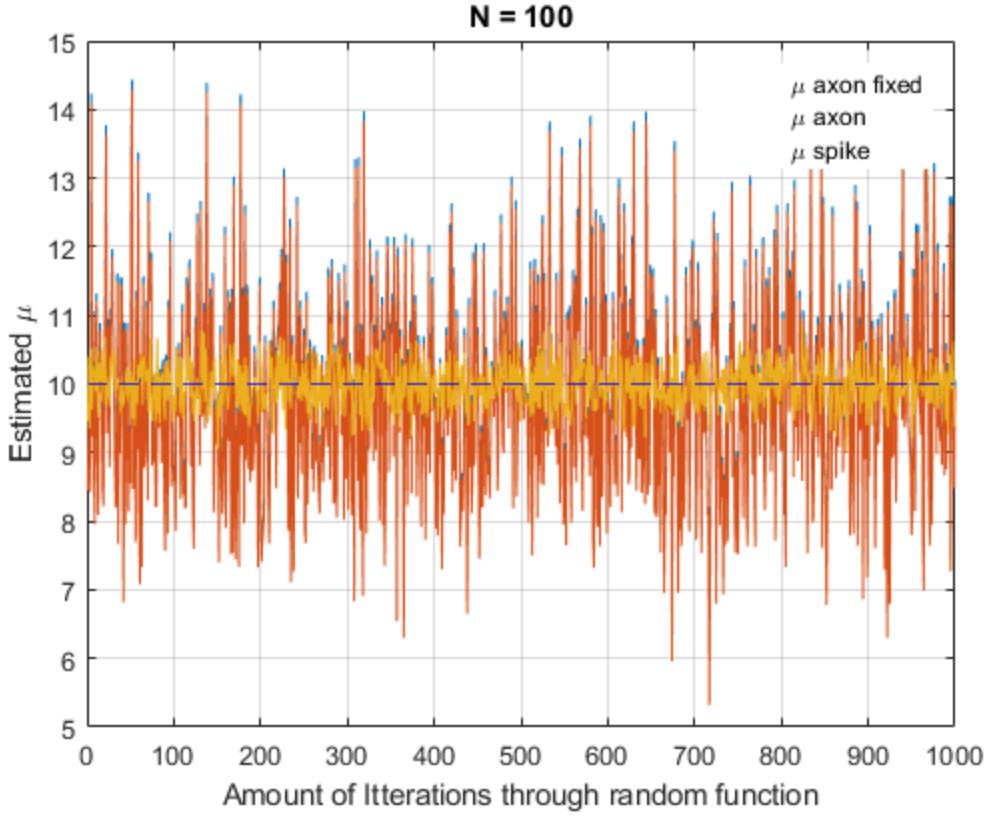
    0.5952    0.0004    0.0255    0.0055    0.0113
```

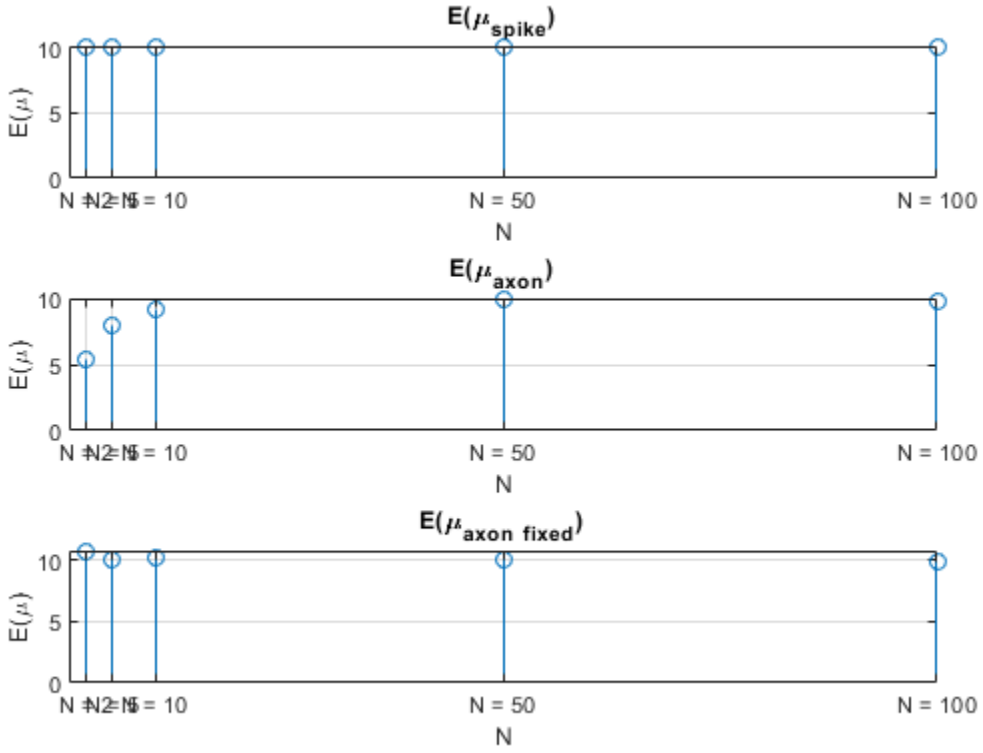

N = 2
μ axon fixed
μ axon
μ spike
Estimated μ
Amount of Itterations through random function

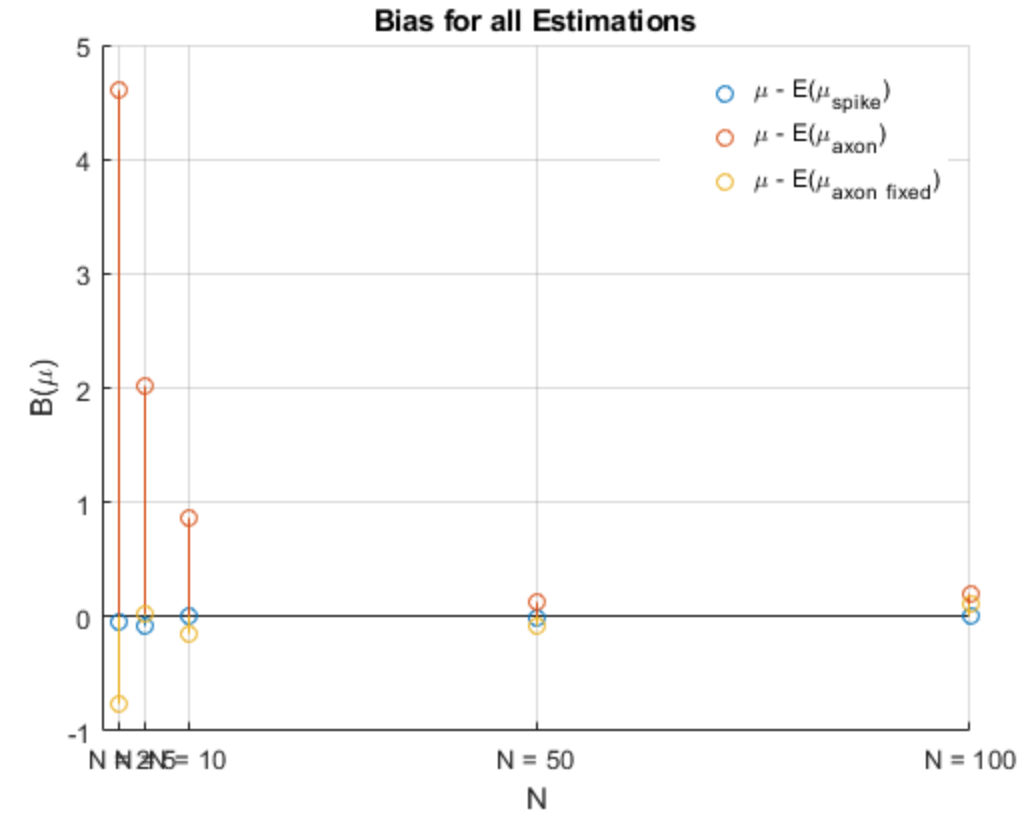

N = 5
μ axon fixed
μ axon
μ spike
Estimated μ
Amount of Itterations through random function

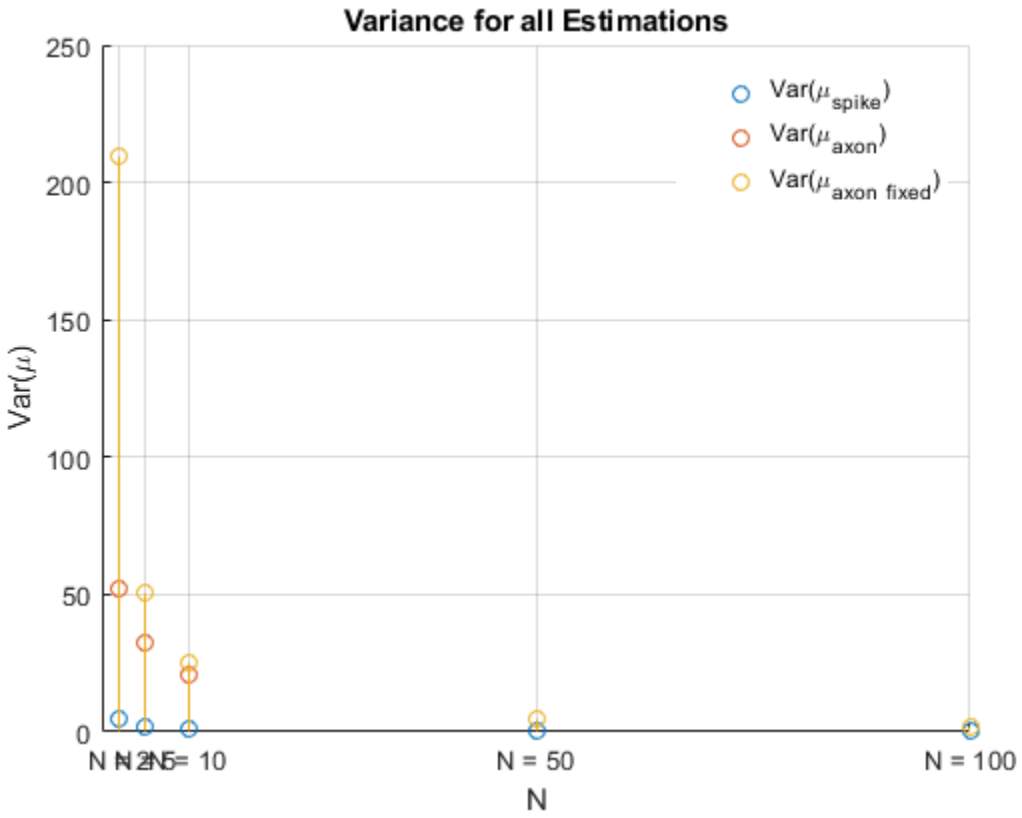

N = 10
μ axon fixed
μ axon
μ spike
Estimated μ
Amount of Itterations through random function

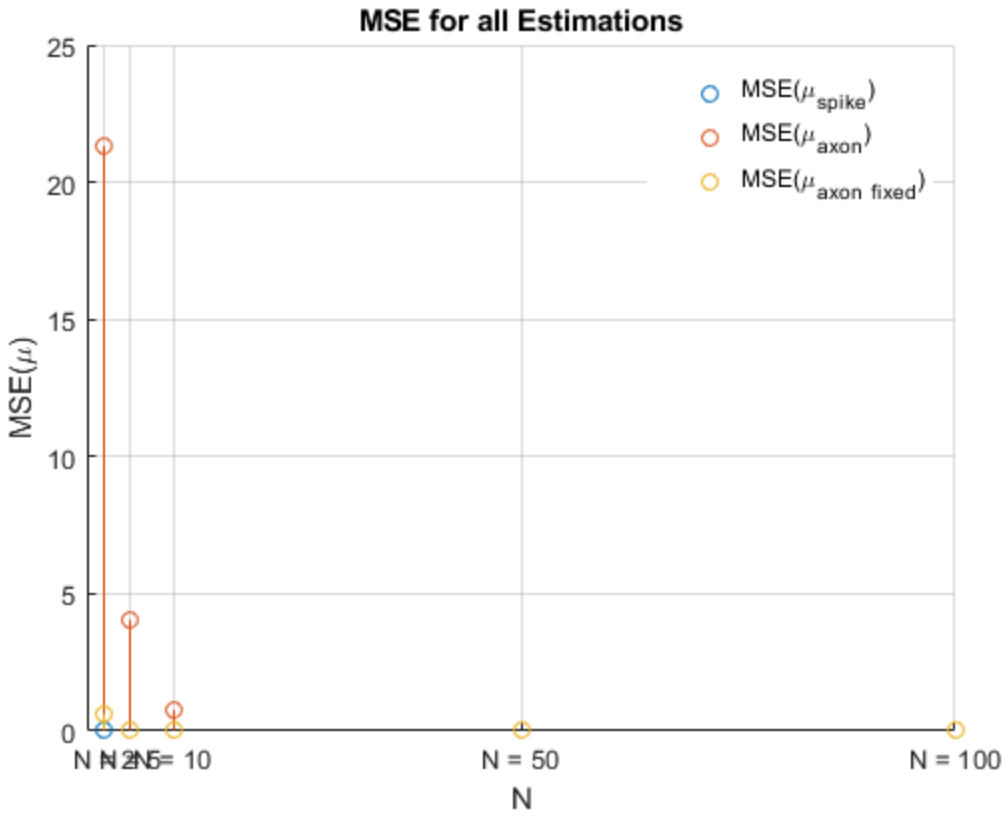

N = 50
μ axon fixed
μ axon
μ spike
Estimated μ
Amount of Itterations through random function


N = 100
μ axon fixed
μ axon
μ spike
Estimated μ
Amount of Itterations through random function


E(μspike)
E(μaxon)
E(μaxon fixed)
E(μ)
N = 10
N = 50
N = 100
N


Bias for all Estimations
μ - E(μ_{spike})
μ - E(μ_{axon})
μ - E($\mu_{axon\ fixed}$)
B(μ)
5
4
3
2
1
0
-1
N = 10
N = 50
N = 100
N


Variance for all Estimations
Var(μ_{spike})
Var(μ_{axon})
Var($\mu_{axon\ fixed}$)
Var(μ)
250
200
150
100
50
0
N = 10
N = 50
N = 100
N



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סעיף מטלבי

(ה) ערכים תיאורטיים לתוחלות

$E(\hat{\mu}_{spike}) = \mu$

$E(\hat{\mu}_{axon}) = \mu \cdot \left(\frac{N-1}{N}\right)$

$E(\hat{\mu}_{axon\ fixed}) = \mu$

לכן עבור $\mu = 10$ ו-$N = (2,5,10,50,100)$

$E(\hat{\mu}_{spike}) = 10$ קבוע לכל N

$$E(\hat{\mu}_{axon}) = \begin{cases} \frac{1}{2} \cdot 10 = 5 & N = 2 \\ 8 & N = 5 \\ 9 & N = 10 \\ 9.8 & N = 50 \\ 9.9 & N = 100 \end{cases}$$

$E(\hat{\mu}_{axon\ fixed}) = 10$ קבוע לכל N

הערכים המחושבים שהתקבלו ב-MATLAB (עבור הגרלה ספציפית)

$E(\hat{\mu}_{spike}) = [9.9405, 10.052, 10.0102, 9.9960, 10.000]$

$E(\hat{\mu}_{axon}) = [4.812, 8.212, 8.8984, 9.7895, 9.9306]$

$E(\hat{\mu}_{axon\ fixed}) = [9.625, 10.265, 9.8871, 9.9893, 10.0309]$

ניתן לראות כי ישנן שגיאות חישוב נומריות אך המגמה התקבלה כמצופה.

שני המשערכים $\hat{M}_{spline}$, $\hat{M}_{hotdeck\ fixed}$

תוחלת קבועה, בעוד $\hat{M}_{hotdeck}$ בעל

תוחלת שמתנהגת כמו $(\frac{N-1}{N})\cdot M$.

1+2:

מסקנות:

גרף Bias:

עפ"י התוצאות ניתן לראות כי המשערך

הכי מוטה הוא $\hat{M}_{hotdeck}$,

אחריו $\hat{M}_{hotdeck\ fixed}$ והמשערך הכי "פחות"

מוטה $\hat{M}_{spline}$.

גרף Variance:

עפ"י המודל האנליטי שפיתחנו

ניתן לראות התנהגות זהה למודל עבור

$Var(\hat{M}_{spline}) \propto \frac{1}{M}$

עבור השאר קצת יותר קשה לפתח אינטואיציה

מהמודל האנליטי אך ניתן לראות כי הם

דועכים עם גודל המדגם (לכל אחד קצב

דעיכה ותלות עם n שונה).

גרף MSE:

ניתן לראות כי השגיאה מתנהגת

בהתאם לשאר הפקטורים המרכיבה.

מורכב מ Bias ומרחק, ממוקד ממוצע

משונות (Var) - כלומר ← ככל שהשונות

וההטיה גדלים כך גם השגיאה.

ניתן לקבוע כי המשערך הכי טוב לפרמטר M

קולו ~~[illegible]~~ מעולה. (שלנו מסתדר טוב
כי הסולו עצמו הכי דומה לביצוע! מקורי
ומוכשר ...