IE-411 **2017 Fall**

Lecture 3 Duality Theory

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Disclaimer: *This homework has not been subjected to the usual scrutiny reserved for formal publications.*

3.1 Duality

3.1.1 Motivation

For problem in the form of

$$\begin{aligned} &\min_x c^T x \\ &s.t \quad Ax \geq b \\ &\qquad\ x \geq 0, \end{aligned} \tag{3.1}$$

we can use simplex method to do the successive minimization, e.g., repeatedly finding a smaller upper-bound of the problem. We can also approch this problem by successivly maximizing the lower-bound of the problem. This leads to the following formulation.

Definition 3.1: Primal-Dual Problem

$$\begin{cases} \min \quad c^T x \\ s.t \quad Ax \geq b \qquad (P) \\ \qquad x \geq 0, \end{cases} \qquad \begin{cases} \max \quad b^T w \\ s.t \quad A^T w \leq c \qquad (D) \\ \qquad w \geq 0, \end{cases}$$

where $A \in R^{m\times n}, x \in R^n, w \in R^m$.

3.1.2 Primal-Dual Problem Conversion

Follow the convention, we always set the primal problems as minimization problems and dual probelms as maximizaiton problems.

We have following conversion rules:

$$
\begin{aligned}
&\min_x c^T x & \qquad & \max_w b^T w & \\
&s.t \quad A_{[i,]}x \geq b_i, \quad i \in M_1 & & s.t \quad w_i \geq 0, \quad i \in M_1 & \\
&\qquad A_{[i,]}x \leq b_i, \quad i \in M_2 & & w_i \leq 0, \quad i \in M_2 & \\
&\qquad A_{[i,]}x = b_i, \quad i \in M_3 & & w_i \text{ is free }, \quad i \in M_3 & \\
&\qquad x_j \geq 0, \quad j \in N_1 & & w^T A_{[,j]} \leq c_j, \quad j \in N_1 & \\
&\qquad x_j \leq 0, \quad j \in N_2 & & w^T A_{[,j]} \geq c_j, \quad j \in N_2 & \\
&\qquad x_j, \text{ is free} \quad j \in N_3 & & w^T A_{[,j]} = c_j, \quad j \in N_3 & \qquad (3.2)
\end{aligned}
$$

Remark 3.2

Note that for each constraints in the primal problem, we introduce a dual variable. And the first three sets of constraints in the dual problem are set to maintain the sign constraints. For each variable in the primal, we introduce a constraint. And the last three sets of constraints in the dual problem are set to maintain guarantee each feasible solution of the primal provides a lower-bound for the primal.

By this conversion rule, we know that **The dual of dual is primal**.

3.2 Fundamentals in duality

Theorem 3.3: Weak Duality

Any feasible solution to the *primal/dual* problem provides a *upper/lower* bound on the optimal objective function value of the *dual/primal* problem. This can also be written as

$$b^T w^* \leq c^T x^*,$$

where x^*, w^* are feasible solutions for primal and dual problems respectively.

Corollary 3.4

1. If the *primal/dual* problem has an unbounded objective, then the *dual/primal* has no feasible solution.
2. If x^*, w^* are feasible solutions for primal and dual problems respectively and $b^T w^* = c^T x^*$, then x^*, w^* are optimal solution for the *primal/dual* problem respectively.

Lemma 3.5: Gross Slackness Condition

If x^*, w^* are feasible solutions for primal and dual problems respectively and $b^T w^* = c^T x^*$, then

$$w^{*T}(Ax^* - b) = 0 \qquad (c^T - w^{*T}A)x^* = 0.$$

Theorem 3.6: Complementary Slackness Condition

If x^*, w^* are feasible solutions for primal and dual problems respectively. x^*, w^* are optimal, that is $b^T w^* = c^T x^*$, if and only if:

- For $i = 1, 2, ..., m$, $w_i^*(A_{[i,.]}x - b_i) = 0$.
- For $j = 1, 2, ..., n$, $x_j^*(c_j - (w^*)^T A_{[.,j]}) = 0$.

If we denote $x_{n+i} = A_{[i,.]}x - b_i, i = 1, 2..., m$ and $w_{m+j} = c_j - w^T A_{[.,j]}, j = 1, 2..., n$. Then the complementary slackness conditon can be rewritten as

- For $i = 1, 2, ..., m$, $w_i^* x_{n+i}^* = 0$.
- For $j = 1, 2, ..., n$, $x_j^* w_{m+j}^* = 0$.

We can (x_j, w_{m+j}) and (x_{n+i}, w_i) as complementary pairs of variables.

Theorem 3.7: Strong Duality

If the *primal/dual* problem has an optimal solution, then so does the *dual/primal* problem and their objective values are equal.

Proof: Consider the problem

$$\begin{aligned} &\min_x c^T x \\ s.t \quad &Ax \geq b \\ &x \geq 0, \end{aligned} \tag{3.3}$$

and we change it to the standard form

$$\begin{aligned} &\min_x c^T x \\ s.t \quad &Ax - s = b \\ &x, s \geq 0. \end{aligned} \tag{3.4}$$

We apply simplex method on this problem and end up with the optimal solution (x^*, s^*).

So the reduced cost for

1. x_j^*: $c_j - c_B^T B^{-1} A_{[,j]}] \geq 0$. Denote $w^* = c_B^T B^{-1}$, then we have $w^* A_{[,j]}] \leq c_j$.
2. s_i^*: $0 - w^*(-e_i) \geq 0$. This implies $w_i^* \geq 0, \forall i = 1, 2, ..., m$.

From 1 and 2, we know that $w^* = c_B^T B^{-1}$ is a feasible solution for the dual problem.

Note that $b^T w^* = c_B^T B^{-1} b = c_B^T x_B^* = c^T x^*$, and by the weak duality theorem, we know w^* is the optimal for the dual problem.

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Theorem 3.8: KKT condition

x^* is an optimal solution to the linear programming $\{\min c^T x \quad \text{s.t} \quad Ax \geq b, x \geq 0\}$ if there exists a vector w^* such that
1. [Primal Feasibility] $Ax^* \geq b, x^* \geq 0$
2. [Dual Feasibility] $A^T w^* \leq c, w^* \geq 0$
3. [Gross Slackness Condition] $w^{*T}(Ax^* - b) = 0 \qquad (c^T - w^{*T} A)x^* = 0$

Theorem 3.9: Strict Complementary Slackness

Consider the following primal and dual problem

$$\begin{cases} \min & c^T x \\ s.t & Ax \geq b \\ & x \geq 0, \end{cases} \quad (P) \qquad \begin{cases} \max & b^T w \\ s.t & A^T w \leq c \\ & w \geq 0, \end{cases} \quad (D)$$

where $A \in R^{m \times n}$. Assume that both questions have an optimal solution. Then there exist optimal solutions to the primal and to the dual, repectively, that satisfy,

- For every $j = 1, ..., n$, we have either $x_j > 0$ or $w^T A_{[,j]} < c_j$.
- For every $i = 1, ..., m.$, we have either $w_i \geq 0$ or $A_{[i,]} x > b_i$.

3.2.1 Duality Gap

- If the Primal probelm is optimal, then the Dual problem is also optimal, hence no gap.
- If the Primal probelm is unbounded, then the Dual problem is infeasible, hence no gap.
- If the Primal probelm is infeasible, then the Dual problem can be unbounded, hence no gap.
- If the Primal probelm is infeasible, then the Dual problem can be infeasible, hence infinite gap.

The case in which both primal and dual problems are infeasible is the sense in which there is a duality gap. If we take the view that the maximum of a function over the empty set is $-\infty$, and the minimum of a function over the empty set is ∞, then when both problems are infeasible we have a duality gap of ∞.