STAT-510 **2017 Fall**

Lecture 8: Convergence

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8.1 Convergence in probability

Definition 8.1 A sequence of random variables $\{X_n\}$ converges in probability towards the random variable X, denoted as $X_n \xrightarrow{p} X$, if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr\big(|X_n - X| > \varepsilon\big) = 0.$$

Remark:

1. X can also be a constant.
2. For random vectors $\{X_n\}$, the definition can be modified as $\lim_{n\to\infty} \Pr\big(||X_n - X|| > \varepsilon\big) = 0$. It turns out each coordinate $X_{i,n} \xrightarrow{p} X_i$.

Example 8.2 $X_1, .., X_n \overset{iid}{\sim} U[0,1]$, $Y_n = \min\{X_1, .., X_n\}$, then $Y_n \xrightarrow{p} 0$. $\forall 0 < \epsilon \leq 1$,

$$P(|Y_n - 0| \geq \epsilon) = P(Y_n \geq \epsilon) = P(\min_i X_i \geq \epsilon) = (1-\epsilon)^n \to 0.$$

Lemma 8.3 (Chebyshev's Inequality)

$$\forall \epsilon > 0, P(|X| \geq \epsilon) \leq \frac{EX^2}{\epsilon^2}.$$

Proof: $I(|X| \geq \epsilon) \leq \frac{X^2}{\epsilon^2}$. Take expectation w.r.t X concludes the proof. ∎

Lemma 8.4 (Markov's Inequality)

$$\forall \epsilon > 0, P(|X| \geq \epsilon) \leq \frac{E\phi(X)}{\phi(\epsilon)},$$

where $\phi(t)$ is a non-decreasing function.

Theorem 8.5 (Weak Law of Large Numbers) If $X_1, .., X_n \overset{iid}{\sim} X$, where $E(X) = \mu$,and $E(|X|), \mathrm{Var}(X)$ are finite. Then $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{p} \mu$.

Proof: By Chebyshev's inequality, we conclude the proof. ∎

Example 8.6 Suppose $\text{Var}(X) = \sigma^2, E(X) = \mu$ and $S_n^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{p} E(X^2) = \sigma^2 + \mu^2$. More generally, for a proper function f,

$$\frac{1}{n}\sum_{i=1}^n f(X_i) \xrightarrow{p} E(f(X)).$$

We can take one step further to derive the following lemma.

Lemma 8.7 If $X_n \xrightarrow{p} C$, where C is a constant, and $g(.)$ is a funcntion that is continuous at C, then

$$g(X_n) \xrightarrow{p} g(C).$$

Remark: Lemma 8.7 can be extended to the random vector sequence.

Theorem 8.8 (Strong Law of Large Numbers) If $X_1, .., X_n \overset{iid}{\sim} X$, where $E(|X|) = \mu$ is finite. Then $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{a.s.} \mu$, i.e.,

$$P[\lim_{n\to+\infty} \bar{X}_n = \mu] = 1.$$

8.2 Convergence in distribution

There'a another type of convergence.

Definition 8.9 (Convergence in distribution) Suppose X_n is a sequence of random variables, and X is a random variable. Let F_n be the distribution function of X_n, and F be the distribution function of X. Then X_n converges in distribution to X if

$$F_n(x) \to F(x)$$

for every $x \in R$ at which F is continuous. This convergence is written as

$$X_n \xrightarrow{D} X.$$

Example 8.10 (law of rare events) Suppose $X_n \overset{iid}{\sim} \text{Binomial}(n, \frac{\lambda}{n})$. Then the c.d.f of X_n is

$$F_n(x) = \begin{cases} 0, & x < 0 \\ \sum_{i=1}^{[x]} f(i; n, \frac{\lambda}{n}) & 0 \le x \le n \\ 1 & x > n \end{cases},$$

where $f(i; n, \frac{\lambda}{n}) = C_n^i (\frac{\lambda}{n})^i (1 - \frac{\lambda}{n})^{n-i}$.

Note that

$$\lim_{n\to+\infty} f(i; n, \frac{\lambda}{n}) = \lim_{n\to+\infty} \frac{\lambda^i}{i!} \frac{n!}{n^i (n-i)!} (1 - \frac{\lambda}{n})^{\frac{-n}{\lambda}(-\lambda)} (1 - \frac{\lambda}{n})^{-i} = \frac{\lambda^i}{i!} e^{-\lambda},$$

and no matter how big x is it will always less than n, when n goes to infinity. Therefore, $F_n \to F$, where

$$F(x) = \begin{cases} 0, & x < 0 \\ \sum_{i=1}^{[x]} \frac{\lambda^i}{i!} e^{-\lambda} & 0 \le x \end{cases}.$$

Sometimes, it might be diffcult to find the limit of $F_n(x)$, we can turn to the Moment Generating Functions (mgf), $M_X(t) = E[e^{tX}]$. Though the mgf may not necessarily exist, once we can find one, then the following lemma relive us from complicated computation.

Lemma 8.11

8.3 Connections between two types of convergence

Lemma 8.12 1. $X_n \overset{p}{\to} X \Rightarrow X_n \overset{D}{\to} X$

2. $X_n \overset{D}{\to} X \nRightarrow X_n \overset{p}{\to} X$

3. If C is a constatnt, then $X_n \overset{D}{\to} X \Rightarrow X_n \overset{p}{\to} X$

Remark A simple example to prove (2). Suppose $W_i \overset{iid}{\sim} N(0,1)$. Set $X_n = W_1, n = 1, 2, ..$ and $X = W_2$, then $X_n \overset{D}{\to} X$. However,

$$P(|X_n - X| \geq \epsilon) = P(|N(0,2)| \geq \epsilon) \nrightarrow 0 \ (n \to \infty).$$

8.4 Central Limit Theorem