STAT-511 **2018 Spring**

Lecture 2: Optimality of Hypothesis Tests

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2.1 Uniformly Most Powerful Tests

We test

$$H_0 : \theta \in \Theta_0 \qquad H_1 : \theta \in \Theta_1,$$

and our target is

$$\begin{aligned} &\max_{\phi(\mathbf{X})} E[\phi(\mathbf{X})], && \forall\theta \in \Theta_1 \\ &\text{s.t. } E[\phi(\mathbf{X})] \leq \alpha, && \forall\theta \in \Theta_0 \end{aligned} \tag{2.1}$$

Denote $\phi^*(\mathbf{X})$ as a solution of (2.1), then $\phi^*(\mathbf{X})$ is called a uniformly most powerful test. Genearlly, it is hard to find a UMP-test. But under a few limited situations, there is a recipe for us to construct the UMP-test.

2.1.1 Simple Versus Simple

We test

$$H_0 : \theta = \theta_0 \qquad H_1 : \theta = \theta_1.$$

Lemma 2.1 Suppose

1. $\phi^*(\mathbf{X})$ is a solution of $\max_{\phi(\mathbf{X})} E_{\theta_1}[\phi(\mathbf{X})] - cE_{\theta_0}[\phi(\mathbf{X})]$, where $c > 0$;

2. The size of $\phi^*(\mathbf{X})$ is exactly α, that is, $E_{\theta_0}\phi^*(\mathbf{X}) = \alpha$.

Then $\phi^*(\mathbf{X})$ is a UMP-test.

Based on Lemma 2.1, the construction of the $\phi^*(\mathbf{X})$ is straightforward.

$$\begin{aligned} &E_{\theta_1}[\phi(\mathbf{X})] - cE_{\theta_0}[\phi(\mathbf{X})] \\ &= \int \phi(\mathbf{X})f_{\theta_1}(x) - c\phi(\mathbf{X})f_{\theta_0}(x)dx \\ &= \int_{\{x|f_{\theta_1}(x)>cf_{\theta_0}(x)\}} \phi(\mathbf{X})[f_{\theta_1}(x) - cf_{\theta_0}(x)]dx + \int_{\{x|f_{\theta_1}(x)<cf_{\theta_0}(x)\}} \phi(\mathbf{X})[f_{\theta_1}(x) - cf_{\theta_0}(x)]dx \end{aligned} \tag{2.2}$$

To maxmize (2.2),

$$\phi^*(\mathbf{X}) = \begin{cases} 1 & , LR(\mathbf{X}) > c \\ \gamma(x) & , LR(\mathbf{X}) = c \\ 0 & , LR(\mathbf{X}) < c \end{cases},$$

where $LR(\mathbf{X})$ is the likelihood ratio as described in the Lecture 1, for here, it is just $\frac{f_{\theta_1}}{f_{\theta_0}}$ and c is chosen to make the size of $\phi^*(\mathbf{X})$ is exactly α.

$\phi^*(\mathbf{X})$ is called **likelihood ratio test (LRT)**, or equivalently, the test function of the Neyman-Pearson form.

Remark

1. $LR(\mathbf{X})$ is well-defined unless the numerator and denomintor are both 0. If it is the case, take $r(x) = \phi(x)$ with $c \in (0, +\infty]$.
2. If we take $\alpha = 0$, then we shall not reject the null as long as the denomintor of $LR(\mathbf{X})$ is positive. So the

$$\phi(\mathbf{X}) = \begin{cases} 1 & , LR(\mathbf{X}) = +\infty \\ 0 & , LR(\mathbf{X}) < +\infty \end{cases}.$$

3. From 2, we know the test function of the Neyman-Pearson form doesn't necessarily has to be ternary. Refer detailed examples in Section 21.3.1 (p375-p376) from John Marden's book, which imply

$$\phi(\mathbf{X}) = \begin{cases} 1 & , LR(\mathbf{X}) > c \\ r(x) & , LR(\mathbf{X}) = c \end{cases} \quad \text{and} \quad \phi(\mathbf{X}) = \begin{cases} r(x) & , LR(\mathbf{X}) = c \\ 1 & , LR(\mathbf{X}) < c \end{cases}.$$

4. Given the size α, the $r(x)$ may not be unique, hence, $\phi(\mathbf{X})$ is not unique.

Lemma 2.2 It is imposible that $\phi(\mathbf{X})$ is a UMP-test with size less than α.

Lemma 2.3 The UMP-test with size less than α has power at lease α.

2.1.2 Simple Versus Composite (One-Sided)

For tests of any of following forms,

1. $H_0 : \theta = \theta_0 \qquad H_1 : \theta > \theta_0$;
2. $H_0 : \theta \leq \theta_0 \qquad H_1 : \theta > \theta_0$;

The LRT is still a UMP-test. Beacuse for 1, the cutoff, c is determined under the the H_0, so any "one-side" change in H_1 won't affect the size of $\phi(\mathbf{X})$. For 2, any test that has level α under $H_0 : \theta \leq \theta_0$ is a level-α test under $H_0 : \theta = \theta_0$.

2.2 Uniformly Locally Most Powerful Tests

2.3 Uniformly Unbiased Most Powerful Tests