STAT-511 **2018 Spring**

Lecture 6: Bayesian Testing

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We are interested in the situation that with the data observed, what is the probability of the null hypothesis holds?

6.1 Global Prior Model

Under **Global Prior Model**, we impose prior distributions on two hypotheses, or equivalently the parameter space.

6.1.1 Simple Versus Simple

Setup:

We would like to test
$$H_0 : \theta = \theta_0 \qquad H_1 : \theta = \theta_1$$
with global prior $\mathcal{P}(H_0 \text{ is true}) = \pi_0, \mathcal{P}(H_1 \text{ is true}) = \pi_1$ and $\pi_0 + \pi_1 = 1$. This is also equivalent to $\mathcal{P}(\theta = \theta_0) = \pi_0, \mathcal{P}(\theta = \theta_1) = \pi_1$. The conditional distributions are given by $X|H_0 \sim f_{\theta_0}, X|H_1 \sim f_{\theta_1}$ respectively.

Solution:

To make decision, we need to calculate two quantities,

$$\mathcal{P}(H_0|X=x) = \frac{\mathcal{P}(X=x|H_0)\mathcal{P}(H_0)}{\mathcal{P}(X=x)} = \frac{\mathcal{P}(X=x|H_0)\mathcal{P}(H_0)}{\mathcal{P}(X=x|H_0)+\mathcal{P}(X=x|H_1)} = \frac{\pi_0 f_0(x)}{\pi_0 f_0(x)+\pi_1 f_1(x)} = \frac{\pi_0}{\pi_0+\pi_1 LR(x)} \tag{6.1}$$

$$P(H_1|X=x) = \frac{\mathcal{P}(X=x|H_1)\mathcal{P}(H_1)}{\mathcal{P}(X=x)} = \frac{P(X=x|H_1)p(H_1)}{P(X=x|H_0)+P(X=x|H_1)} = \frac{\pi_1 f_1(x)}{\pi_0 f_0(x)+\pi_1 f_1(x)} = \frac{\pi_1 LR(x)}{\pi_0+\pi_1 LR(x)} \tag{6.2}$$

We can also associate these two quantities by introducing the concepts of Odds.

Definition 6.1 (Odds and Posterior Odds)

$$\text{Odds}(Event) = \frac{P(Event)}{1-P(Event)}.$$

Therefore the postierior odds of $H_1|X=x$ is defined as

$$\text{Odds}[H_1|X=x] = \frac{P(H_1|X=x)}{P(H_0|X=x)} = \frac{\pi_1}{\pi_0}LR(x) = \text{Odds}(H_1)LR(x).$$

We can reject H_1 if the postierior odds exceeds some constant c.

6.1.2 Composite

For compostie, we start with imposing prior on the parameter space.

We would like to test

$$H_0 : \theta \in \Theta_0 \qquad H_1 : \theta \in \Theta_1.$$

with global prior $\theta \sim \boldsymbol{\pi}(\theta)$, and this results in $\mathcal{P}(H_0) = \mathcal{P}(\theta \in \Theta_0)$ and $\mathcal{P}(H_1) = \mathcal{P}(\theta \in \Theta_1)$.

Similarly, the posterior odds can be written as

$$\frac{\mathcal{P}(H_0|X=x)}{\mathcal{P}(H_1|X=x)} = \frac{\int_{\Theta_0} \boldsymbol{\pi}(\theta) f(x|\theta) d\theta}{\int_{\Theta_1} \boldsymbol{\pi}(\theta) f(x|\theta) d\theta} = \frac{\int_{\Theta_0} \boldsymbol{\pi}(\theta) d\theta}{\int_{\Theta_1} \boldsymbol{\pi}(\theta) d\theta} \frac{\int_{\Theta_0} \frac{\boldsymbol{\pi}(\theta)}{\int_{\Theta_0} \boldsymbol{\pi}(\theta) d\theta} f(x|\theta) d\theta}{\int_{\Theta_1} \frac{\boldsymbol{\pi}(\theta)}{\int_{\Theta_1} \boldsymbol{\pi}(\theta) d\theta} f(x|\theta) d\theta} \triangleq \frac{\mathcal{P}(H_0)}{\mathcal{P}(H_1)} \frac{\int_{\Theta_0} w_0(\theta) f(x|\theta) d\theta}{\int_{\Theta_1} w_1(\theta) f(x|\theta) d\theta},$$

where $w_0 = \frac{\boldsymbol{\pi}(\theta)}{\int_{\Theta_0} \boldsymbol{\pi}(\theta) d\theta}$ and $w_1 = \frac{\boldsymbol{\pi}(\theta)}{\int_{\Theta_1} \boldsymbol{\pi}(\theta) d\theta}$.

Remark This result is consistent with the simple versus simple case with additional averaing the likelihood over the parameter space.

Example 6.2 Suppose $x_i \overset{iid}{\sim} N(\theta, \sigma^2)$ and $\theta \sim N(\mu, \tau^2)$, where σ^2, μ, τ^2 are known. We want to test

$$H_0 : \theta \le 0 \qquad H_1 : \theta > 0.$$

It's easy to verify that

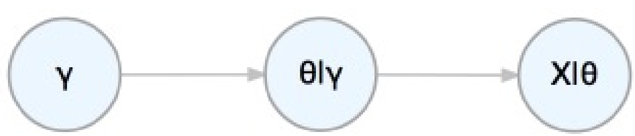
$$\theta | \underset{\sim}{\mathbf{X}} \sim N\left(\frac{\frac{n}{\sigma^2}\bar{x} + \frac{1}{\tau^2}\mu}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right) \triangleq N(m(\underset{\sim}{\mathbf{X}}), S^2(\underset{\sim}{\mathbf{X}})),$$

and

$$\frac{\mathcal{P}(H_0|X)}{\mathcal{P}(H_1|X)} = \frac{\Phi(0; m(\underset{\sim}{\mathbf{X}}), S^2(\underset{\sim}{\mathbf{X}}))}{1 - \Phi(0; m(\underset{\sim}{\mathbf{X}}), S^2(\underset{\sim}{\mathbf{X}}))}.$$

6.2 Hierarchical Prior Model

Sometimes for the null and alternative parameter space, we would like to impose different priors, that is, we can't impose a global prior on the entire parameter space. Then we build a hierarchical model to address this issue.



In the figure above,

- γ is a random variable defined as

$$\gamma = \begin{cases} 0, & H_0 \text{ is true with prob } p_0 \\ 1, & H_1 \text{ is true with prob } p_1 \end{cases}$$

- $\theta|\gamma = 0 \sim \boldsymbol{\pi}_0(\theta)$ and $\theta|\gamma = 1 \sim \boldsymbol{\pi}_1(\theta)$, where $\boldsymbol{\pi}_0(\theta), \boldsymbol{\pi}_1(\theta)$ are defined only on Θ_0 and Θ_1 respectively.
- $\underset{\sim}{\mathbf{X}}|\theta \sim f(x_1, ..., x_n|\theta)$

Some Importatnt Distributions

- The joint distribution defined by the graph is given by

$$f(\underset{\sim}{\mathbf{X}}, \theta, \gamma);$$

- $$\gamma, \theta|\underset{\sim}{\mathbf{X}} \propto \boldsymbol{\pi}(r)\boldsymbol{\pi}(\theta|\gamma)f(\underset{\sim}{\mathbf{X}}|\theta)$$

- $$\gamma|\underset{\sim}{\mathbf{X}} \propto \int \boldsymbol{\pi}(r)\boldsymbol{\pi}(\theta|\gamma)f(\underset{\sim}{\mathbf{X}}|\theta)d\theta$$

Example 6.3 We would like to test

$$H_0 : p = 0.5 \qquad H_1 : p \neq 0.5.$$

Suppose $x_1, ..., x_n \overset{iid}{\sim} \text{Bin}(m, p), \mathcal{P}(\gamma = 0) = q = 1 - \mathcal{P}(\gamma = 1), p|H_1 \sim \text{Beta}(\alpha, \beta)$ and $\mathcal{P}(p = 0.5|H_0) = 1$. Then we have

$$\gamma, p|\underset{\sim}{\mathbf{X}} \propto \boldsymbol{\pi}(r)\boldsymbol{\pi}(p|\gamma)f(\underset{\sim}{\mathbf{X}}|p) \propto q^{1-\gamma}(1-q)^{\gamma}[\delta(p)I(\gamma = 0) + \boldsymbol{\pi}(\theta|\gamma = 1)I(\gamma = 1)]p^{\sum_i x_i}(1-p)^{nm-\sum_i x_i}.$$

where $\delta(p) = 1$ when $p = 0$ and $\delta(p) = 0$ elsewhere. Then

$$\begin{aligned} \gamma|\underset{\sim}{\mathbf{X}} &\propto \int \boldsymbol{\pi}(r)\boldsymbol{\pi}(p|\gamma)f(\underset{\sim}{\mathbf{X}}|p) \propto q^{1-\gamma}(1-q)^{\gamma}[\delta(p)I(\gamma = 0) + \boldsymbol{\pi}(\theta|\gamma = 1)I(\gamma = 1)]p^{\sum_i x_i}(1-p)^{nm-\sum_i x_i}dp \\ &= q^{1-\gamma}(1-q)^{\gamma}[(\frac{1}{2})^{nm}I(\gamma = 0) + \text{Beta}(\alpha + \sum_i x_i, \beta + nm - \sum_i x_i)I(\gamma = 1)]p^{\sum_i x_i}(1-p)^{nm-\sum_i x_i} \end{aligned}$$

Then it's easy to calculate

$$\frac{\mathcal{P}(H_0|X = x)}{\mathcal{P}(H_1|X = x)} = \frac{q(\frac{1}{2})^{mn}}{(1-q)\text{Beta}(\alpha + \sum_i x_i, \beta + nm - \sum_i x_i)}$$

$$
\begin{aligned}
p|\underset{\sim}{\mathbf{X}} &\propto \int \boldsymbol{\pi}(r)\boldsymbol{\pi}(p|\gamma)f(\underset{\sim}{\mathbf{X}}|p) \propto q^{1-\gamma}(1-q)^{\gamma}[\delta(p)I(\gamma=0)+\boldsymbol{\pi}(\theta|\gamma=1)I(\gamma=1)]p^{\sum_i x_i}(1-p)^{nm-\sum_i x_i}d\gamma \\
&= [q^2\delta(p)+(1-q)^2\text{Beta}(p;\alpha,\beta)]p^{\sum_i x_i}(1-p)^{nm-\sum_i x_i}
\end{aligned}
$$

So

$$
p|\underset{\sim}{\mathbf{X}} \sim \begin{cases} \text{Beta}(\sum_i x_i+1, nm-\sum_i x_i+1), & p=0.5 \\ \text{Beta}(\sum_i x_i+\alpha, nm-\sum_i x_i+\beta), & p\neq 0.5 \end{cases}
$$