IE-411 **2018 Fall**

Lecture 1 Geometry of Linear Programming

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Disclaimer: *This homework has not been subjected to the usual scrutiny reserved for formal publications.*

1.1 Convex sets and polyhedral sets

Definition 1.1 (convex sets) *A set $X \in R^n$ is called a convex set if given any two points x_1 and x_2 in X, then $\lambda x_1 + (1-\lambda)x_2 \in X$ for each $\lambda \in [0,1]$.*

Example 1.2

1. $\{x; Ax = b\}$ 2. $\{x; Ax = b, x \geq 0\}$ 3. $\{x; Ax \leq b\}$

Definition 1.3 (convex cones) *A convex cone C is a convex set with the property that $\lambda x \in C$ for every $x \in X, \lambda \geq 0$.*

Remark:

1. Note that a convex cone always contains the origin.
2. Given a set of vectors, $\{x_i\}_{i=1}^n$, we can form the convex cone C generated by these vectors.

$$C = \{\sum_{i=1}^{n} \lambda_i x_i; \quad \lambda_i \geq 0\}.$$

Definition 1.4 (convex hulls) *A convex hull H is a convex combination of vectors in R^n. That is, given $\{x_i\}_{i=1}^n, \{\lambda_i\}_{i=1}^n$ then $H = \{\sum_i \lambda_i x_i; \lambda_i \geq 0, \sum_i \lambda_i = 1, x_i \in R^n\}$.*

Remark: Given the a set of $x_1, ..., x_n \in R^n$ and a set of scalar $\lambda_1, ..., \lambda_n$.

1. **Non-negative combination**: $\sum_i \lambda_i x_i$, where $\lambda_i \geq 0$;
2. **Affine combination**: $\sum_i \lambda_i x_i$, where $\sum_i \lambda_i = 1$;
3. **Convex combination**: $\sum_i \lambda_i x_i$, where $\lambda_i \geq 0, \sum_i \lambda_i = 1$.

Definition 1.5 (polyhedral set) *A polyhedral set or a polyhedron is the intersection of a finite number of half-spaces. A bounded polyhedral set is called a* polytope.

Remark:

1. A polyhedral set can be represented by $\{x; Ax \leq b\}$. Since an equation can be written as two inequalities, a polyhedral set can be represented by a finite number of linear inequalities and/or equations.

2. If a inequality in $Ax \leq b$ can be disregarded, but the polyhedral set is not affected. Such an inequality is called (geometrically) **redundant** or irrelevant to the polyhedral set.

Definition 1.6 (polyhedral cone) *A polyhedral cone is the intersection of a finite number of half-spaces, whose hyper-planes pass through the origin. In other words, C is a polyhedral cone if it can be represented as $\{x; Ax \leq 0\}$.*

Theorem 1.7

1. *The intersection of convex sets is convex.*
2. *Every polyhedral set is convex.*
2. *The convex hull of a finite number of vectors is a convex set.*

1.2 Extreme points and extreme directions

Definition 1.8 (extreme points) *A point x in a convex set X is called an extreme point of X if x cannot be represented as a* strict convex *combination of two distinct points in X. That is, if $x = \lambda x_1 + (1-\lambda)x_2$ with $\lambda \in (0,1), x_1, x_2 \in X$, then $x = x_1 = x_2$.*

Remark: By the definition, an extreme point of X is a point in X that cannot be made to lie in the interior of a line segment contained within X.

Definition 1.9 (rays and directions) *A ray is a collection of points of the form $\{x_0 + \lambda d; \lambda \geq 0\}$, where d is a nonzero vector. Here, x_0 is called the vertex of the ray, and d is the direction of the ray.*

Example 1.10 *The non-zero direction d of a polyhedral set defined by $X = \{x; Ax \leq b, x \geq 0\}$ must satisfy: for each $\lambda \geq 0, x \in X$,*

$$\begin{cases} A(x + \lambda d) \leq b \\ x + \lambda d \geq 0, \end{cases}$$

which implies $d \geq 0, d \neq 0, Ad \leq 0$. Here, we slightly abuse the usage of the notation. $d \geq 0$ is a element-wise comparison and $d \neq 0$ means there is a least one component of d is nonzero.

Definition 1.11 (directions of a convex set) *Given a convex set, a nonzero vector d is called (recession) direction of the set, if for each x_0 in the set, the ray $\{x_0 + \lambda d; \lambda \geq 0\}$, also belongs to the set.*

Remark: Starting at any point x_0 in the set, one can recede along d for any step length $\lambda \geq 0$ and remain within the set. Clearly, if the set is bounded, then it has no directions.

Definition 1.12 (extreme directions of a convex set) *An extreme direction of a convex set is a direction of the set that* cannot be *represented as a positive combination of two distinct directions of the set.*

Definition 1.13 (extreme ray) *Any ray that is contained in the convex set and whose direction is an extreme direction is called an extreme ray.*

Remark: Since a convex cone is formed by its rays, then a convex cone can be entirely characterized by its directions. In fact, not all directions are needed, since a non-extreme direction can be represented as a positive combination of extreme directions. In other words, a convex cone is fully characterized by its extreme directions.

1.2.1 Extreme points, and Extreme directions of polyhedra sets

Extreme points

Consider a polyhedral set defined as $X = \{x | A_{m\times n}x_{n\times 1} \leq b, x \geq 0\}$. Then X is the intersection of $m+n$ half-spaces. And we denote the corresponding hyper-planes of these half-space as defining hyper-planes of X.

Definition 1.14 *$x^* \in X$ is said to be an extreme point of X, if x^* lies on the intersection of n linear independent defining planes.*

- If more than n defining hyper-planes pass through an extreme point, then such an extreme point is called a **degenerate extreme point**. The excess number of planes over n is called its order of degeneracy.
- If there are less than n linear independent defining hyper-planes cross $x^* \in X$, then x^* can not be an extreme point. (See the proof on *p72 BJS*.)

Extreme directions

Learnt from Example 1.10, the set

$$D = \{d : Ad \leq 0, 1^T d = 1, d \geq 0\},$$

characterized all the directions of the polyhedral set X and the extreme points of D are exactly the extreme directions of X.

Remark: To eliminate duplication, these directions may be normalized. To maintain linearity, it is convenient to normalize the directions using the ℓ_1 norm.

Exsitence of extreme points

A polyhedron $P \subset R^n$ contains a line if there exists a vector $x \in P$ and a nonzero vector d such that $x + \lambda d \in P, \forall \lambda \in R$. Then the following theorem gives the existence of an extreme point.

Theorem 1.15 *Suppose that the polyhedron $P = \{x \in R^n | a_i^T x \geq b_i, i = 1, 2, ..., m\}$ is nonempty. Then the following are equivalent:*

1. The polyhedron P has at least one extreme point; 2. The polyhedron doesn't contain a line; 3. There exists n vectors out of the family $a_1, .., a_m$, which are linearly independent.

Corollary 1.16 *Every nonempty bounded polyhedron has at least one basic feasible solution.*

1.3 Extreme points, vertices and basic solutions

Definition 1.17 (vertex) *The $x \in X, x \in R^n$ is said to be the vertex of X if there exists some $c \in R^n$ such that $c^T x < c^T y$ for all $y \in X, y \neq x$.*

Remark: The definition of the vertex x_0 requires that the hyper-plane $\{y | c^T y = c^T x_0\}$ meets with X at and only at x_0. See Figure 1.1 for example, where A is a vertex while B is not.

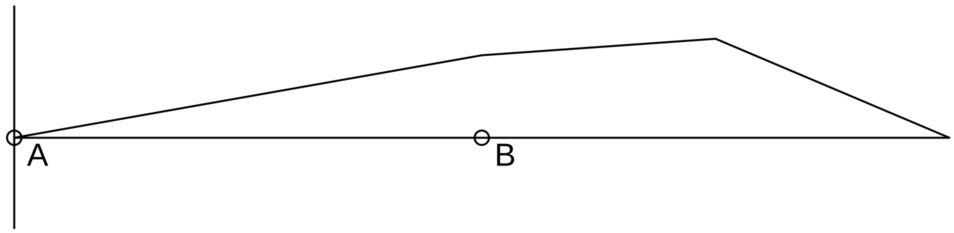


Figure 1.1: Demonstration of the vertex.

Definition 1.18 (active constraints) *Consider three types of constraints for $x_i \in R^n, i = 1, 2, .., n$.*

1. $a_i^T x = b_i, i \in I_1$; 2. $a_i^T x \geq b_i, i \in I_2$; 3. $a_i^T x \leq b_i, i \in I_3$,

where $a_i \in R^n, b_i \in R$. If there exists x^* such that $a_i^T x = b_i$ for some i in I_1, I_2 or I_3. Then we say corresponding constraint(a_i) is active.

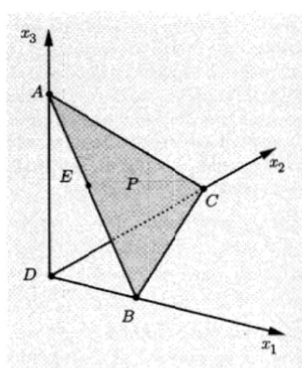
Definition 1.19 (basic solution) *Consider a polyhedron P defined by linear equality and linear inequality constraints, let $x^* \in R^n$.*

1. x^* is a basic solution *if: a). All equality constraints are active at x^*; b). Out of the constraints that are active at x^*, there are n of them are linear independent.*

2. If x^* is a basic solution that satisfies all constraints, then it is called a basic feasible solution.

Remark: If the number of constraints $m \leq n$, then there is no either basic solution nor basic feasible solution.

Example 1.20 *In Figure 1.2, A, B, C are basic feasible solutions while D is not. Point E is feasible but not basic.*



Let $P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\}$. There are three constraints that are active at each one of the points A, B, C and D. There are only two constraints that are active at point E, namely $x_1 + x_2 + x_3 = 1$ and $x_2 = 0$.

Figure 1.2: Demonstration of basic solutions.

Theorem 1.21 *Let P be a non-empty polyhedron and $x^* \in P$. Then x^* is a vertex $\Longleftrightarrow$ x^* is an extreme point $\Longleftrightarrow$ x^* is a basic feasible solution.*

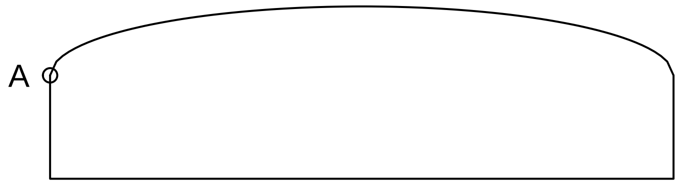


Figure 1.3: A counter example.

Remark: The Theorem 1.21 holds only under the polyhedron, not in the general convex set. For example, refer to Figure 1.3. A is an extreme point not a vertex. In general, we have

$$\text{extreme point} \not\Rightarrow \text{vertex}; \text{vertex} \Rightarrow \text{extreme point}.$$

1.4 Polyhedra in standard form

In this section, the polyhedral set is defined as $X = \{x | A_{m\times n} x_{n\times 1} = b, x \geq 0\}$.

Theorem 1.22 *WLOG, assume A is full row rank. $x^* \in X$ is said to be an extreme point of X iff*

1. $Ax^* = b$;

2. There exists indices, $B_1, ..., B_m$ such that

** The columns $\{A_{B_1}, ..., A_{B_m}\}$ are linear independent * $x_i = 0$ if $i \notin \{B_1, ..., B_m\}$.*

Please refer to *P53 of BT* for the proof.

The basic solution of X can be easily constructed, in light of the Theorem 1.22.

Remark According to the indices $B_1, ..., B_m$, we can partition and rearrange A and x as $A = [B_{m\times m}, N_{m\times(n-m)}]$ and $x^T = [x_B^T, x_N^T]$. The matrix B and $x_B \in R^m$ are called the *basic matrix* and *basic variables* respectively. Also, $x_B = B^{-1}b$.

basic matrix and basic solution:

- A *basic* matrix B can only be linked to one basic solution. This straightforward by the definition.
- A basic solution can correspond to different basic matrix B. For example, consider the polyhedral set $\{x_1 + x_2 = 0, x_1, x_2 \geq 0\}$. The bases can either be A_1 or A_2, but they all correspond to the basic solution $(0, 0)^T$.
- If there exists more than one basis representing an extreme point, then this extreme point is degenerate. However, the converse is not necessarily true. For example,

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ -x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{cases},$$

the point $(0, 1, 0)$ is a degenerated point but only has one basic matrix.

Baisc feasible solution:

The point $x^* = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix}$ is a basic feasible solution iff $B^{-1}b \geq 0$.

However, for the same setting we discussed above, that is, $X = \{x | A_{m\times n} x_{n\times 1} = b, x \geq 0\}$ and A is full row rank. If there is a $x' \in X$ such that has exactly m positive components, x' is not necessarily a basic feasible solution. Just consider the following polyhedron set

$$X = \{(x_1, x_2, x_3 | x_2 + x_3 = 0, x_1, x_2, x_3 \geq 0\},$$

and a point $x' = (1, 0, 0)$. Clearly $x' \in X$ and is not an extreme point.

Corollary 1.23 *The nonempty polyhedron in standard form has at least one basic feasible solution.*

This is a result of Theorem 1.15.

1.5 Representation of polyhedron

Let $X = \{x | Ax \leq b, x \geq 0\}$ be a nonempty polyhedral set. Then the set of extreme points is nonempty and has a finite number of elements, say $x_1, .., x_k$. Furthermore, the set of extreme directions is empty if and only if X is bounded. If X is not bounded, then the set of extreme directions is nonempty and has a finite number of elements, say $d_1, d_2, ..., d_l$. Moreover, $x \in X$ if and only if it can be represented as a convex combination of $x_1, .., x_k$ plus a non-negative linear combination of $d_1, d_2, ..., d_l$, that is,

$$x_0 = \sum_{j=1}^{k} \lambda_j x_j + \sum_{j=1}^{l} \mu_j d_j,$$

where $\sum_j \lambda_j = 1, \lambda_j \geq 0, \mu_j \geq 0$.