Interaction

To present the formula for the ANOVA table using repeated observations/cell, we shall consider a rectangular array/matrix with **r** rows, **c** columns, **n** observation/cell

i.e **rc** cells with **n** observations/cell

N = **rcn**

Rows	**Columns**					**Total**	**Mean**
	1	**2**	**3 …………..**	**j ………...**	**c**		
1	y_{111}	y_{121}	y_{131} ⋯ ⋯ . .	y_{1j1} ⋯ ⋯ .	y_{1c1}		
	y_{112}	y_{122}	y_{132} ⋯ ⋯ . .	y_{1j2} ⋯ ⋯ .	y_{1c2}	y_{1oo}	$\bar{y}_{1oo}$
	:	:	:	:	:		
	y_{11n}	y_{12n}	y_{13n} ⋯ ⋯ . .	y_{1jn} ⋯ ⋯ .	y_{1cn}		
2	y_{211}	y_{221}	y_{231} ⋯ ⋯ . .	y_{2j1} ⋯ ⋯ .	y_{2c1}		
	y_{212}	y_{222}	y_{232} ⋯ ⋯ . .	y_{2j2} ⋯ ⋯ .	y_{2c2}	y_{2oo}	$\bar{y}_{2oo}$
	:	:	:	:	:		
	y_{21n}	y_{22n}	y_{23n} ⋯ ⋯ . .	y_{2jn} ⋯ ⋯ .	y_{2cn}		
i	:	:	:	:	:		
	y_{i11}	y_{i21}	y_{i31} ⋯ . ⋯ . .	y_{ij1} ⋯ ⋯ .	y_{ic1}		
	y_{i12}	y_{i22}	y_{i32} ⋯ ⋯ ⋯ .	y_{ij2} ⋯ ⋯ .	y_{ic2}	y_{ioo}	$\bar{y}_{ioo}$
	:	:	:	:	:		
	y_{i1n}	y_{i2n}	y_{i3n} ⋯ ⋯ . . .	y_{ijn} ⋯ ⋯ .	y_{icn}		
R	y_{r11}	y_{r21}	y_{r31} ⋯ ⋯ . .	y_{rj1} ⋯ ⋯ .	y_{rc1}		
	y_{r12}	y_{r22}	y_{r32} ⋯ ⋯ . .	y_{rj2} ⋯ ⋯ .	y_{rc2}	y_{roo}	$\bar{y}_{roo}$
	:	:	:	:	:		

	y_{r1n} y_{r2n} $y_{r3n} \cdots \cdots . . y_{rjn} \cdots \cdots . y_{rcn}$		
Total	$\boldsymbol{y_{ojo}}$ $\boldsymbol{y_{o2o}}$ $\boldsymbol{y_{o3o}} \cdots \cdots . . \boldsymbol{y_{ojo}} \cdots \cdots . \boldsymbol{y_{oco}}$	$\boldsymbol{y_{ooo}}$	
Mean	$\boldsymbol{\bar{y}_{ojo}}$ $\boldsymbol{\bar{y}_{o2o}}$ $\boldsymbol{\bar{y}_{o3o}} \cdots \cdots . . \boldsymbol{\bar{y}_{ojo}} \cdots \cdots . \boldsymbol{\bar{y}_{oco}}$		$\boldsymbol{\bar{y}_{ooo}}$

y_{ijk} is the k^{th} observation in the i^{th} row and the j^{th} column
y_{ijo} is the sum of the observations in the ij^{th} cell
y_{ioo} is the sum of the observations in the i^{th} row
y_{ojo} is the sum of the observations in the j^{th} column
y_{ooo} is the grand total of all the N = rcn observations
$\bar{y}_{ooo}$ is the grand mean

Model is

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

Where y_{ijk} – k^{th} observation for the i^{th} row and the j^{th} column.
μ - general mean
α_i - i^{th} row effect
β_j - j^{th} column effect
$(\alpha\beta)_{ij}$ - interaction between the i^{th} row and the j^{th} column
ϵ_{ijk} – error term

Hypotheses

H_{o1}: $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_r = 0$ Vs H_{AI}: Some $\alpha_i \neq 0$

H_{o2}: $\beta_1 = \beta_2 = \ldots\ldots = \beta_c = 0$ Vs H_{A2}: Some $\beta_j \neq 0$

H_{o3}: $(\alpha\beta)_{11} = (\alpha\beta)_{12} = \ldots\ldots = (\alpha\beta)_{rc} = 0$ Vs H_{A3}: Some $(\alpha\beta)_{ij} \neq 0$

Computations

Note that total sums of squares (SST) is split into 4 components, that is,
sums of squares due to rows
sums of squares due to columns
sums of squares due to interaction, and
sums of squares due to error, that is,
SST = SSR + SSC + SS(RC) + SSE

The computational formula for the sums of squares are:

$$SST = \sum_i^r \sum_j^c \sum_k^n y_{ijk}^2 - \frac{y_{ooo}^2}{N} \quad \text{from} \quad \sum_i^r \sum_j^c \sum_k^n \left(y_{ijk} - \bar{y}_{ooo}\right)^2$$

$$SSR = \sum_i^r \frac{y_{ioo}^2}{cn} - \frac{y_{ooo}^2}{N} \quad \text{from} \quad cn\sum_{i=1}^r (\bar{y}_{ioo} - \bar{y}_{ooo})^2$$

$$SSC = \sum_{j=1}^c \frac{y_{ojo}^2}{rn} - \frac{y_{ooo}^2}{N} \quad \text{from} \quad rn\sum_{j=1}^c \left(\bar{y}_{ojo} - \bar{y}_{ooo}\right)^2$$

$$SS(RC) = \frac{\sum\sum y_{ijo}^2}{n} - \frac{\sum y_{ioo}^2}{cn} - \frac{\sum y_{ojo}^2}{rn} + \frac{y_{ooo}^2}{N}$$

$$SSE = SST - SSR - SSC - SS(RC)$$

ANOVA table

Source of variation	**df**	**ss**	**ms**	**F-ratio**
Rows	(r – 1)	SSR	$MSR = \frac{SSR}{(r-1)}$	$F_{1c} = \frac{MSR}{MSE}$
Columns	(c – 1)	SSC	$MSC = \frac{SSC}{(c-1)}$	$F_{2c} = \frac{MSC}{MSE}$
Interaction	(r – 1) (c – 1)	SS(RC)	$MS(RC) = \frac{SS(RC)}{(r-1)(c-1)}$	$F_{3c} = \frac{MS(RC)}{MSE}$
Error	rc(n – 1)	SSE	$MSE = \frac{SSE}{rc(n-1)}$	-
Total	**N – 1 or (rcn) – 1**	**SST**	-	-

Table Y: Yield of beans with 3 observations/cell

Fertilisers	**Varieties of beans**		
	V_1	**V_2**	**V_3**
t_1	64	72	74
	66	81	51
	70	64	65
t_2	55	57	47
	63	43	58

	68	52	67
t_3	59 68 65	66 71 59	58 39 42
t_4	58 41 46	57 61 53	53 59 38

r = 4, c = 3, n = 3, N = rcn = 4x3x3 = 36

Test the hypothese**s** that

a) The average yield of the beans is the same when different fertilisers are used.
b) There is no difference in the average yield for the different varieties of the beans.
c) There is no interaction between fertilisers and varieties.

Model

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

Where y_{ijk} – k^{th} observation for the i^{th} fertiliser and the j^{th} variety

μ - general mean

α_i - i^{th} fertiliser effect

β_j - j^{th} variety effect

$(\alpha\beta)_{ij}$ - interaction between the i^{th} fertiliser and j^{th} variety

ϵ_{ijk} – error term

Hypotheses

H_{o1}: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ Vs H_{AI}: Some $\alpha_i \neq 0$

H_{o2}: $\beta_1 = \beta_2 = \beta_3 = 0$ Vs H_{A2}: Some $\beta_j \neq 0$

H_{o3}: $(\alpha\beta)_{11} = (\alpha\beta)_{12} = (\alpha\beta)_{13} = \ldots\ldots\ldots\ldots.. = (\alpha\beta)_{43} = 0$

H_{A3}: Some $(\alpha\beta)_{ij} \neq 0$

Rejection criteria

Reject H_{o1} if $F_{Ic} \geq F_{\alpha, (r-1), rc(n-1)} = F_{0.05, 3, 24} = 3.01$

Reject H_{o2} if $F_{2c} \geq F_{\alpha, (c-1), rc(n-1)} = F_{0.05, 2, 24} = 3.40$

Reject H_{o3} if $F_{3c} \geq F_{\alpha, (r-1)(c-1), rc(n-1)} = F_{0.05, 6, 24} = 2.51$

Computations

Construct a table of cell totals

Fertilisers	**Varieties of beans**			y_{ioo}
	V1	**V2**	**V3**	
t1	200	217	190	607
t2	186	152	172	510
t3	192	196	139	527
t4	145	171	150	466
y_{ojo}	**723**	**736**	**651**	y_{ooo} **= 2110**

C.F = $\frac{y_{ooo}^2}{N} = \frac{2110^2}{36}$ = 123,669

SST = $[64^2 + 66^2 + \cdots + 38^2] -$123,669 = 127,508 – 123,669 = 3839

SSR = $\frac{[607^2+510^2+527^2+466^2]}{3\times3} -$ C.F = 124,826 – 123,669 = 1157

SSC = $\frac{[723^2+736^2+651^2]}{4\times3} -$ C.F = 124,019 – 123,669 = 350

SS(RC) = $\frac{[200^2+186^2+\cdots+150^2]}{3} -$ 124,826 – 124,019 + 123,669 = 771

SSE = SST – SSR – SSC – SS(RC) = 3839 – 1157 – 350 – 771 = 1561

ANOVA table

Source of variation	**df**	**ss**	**ms**	**F-ratio**
Fertilisers	3	1157	385.67	$F_{1c} = 5.92$
Varieties	2	350	175.00	$F_{2c} = 2.69$
Interaction	6	771	128.50	$F_{3c} = 1.98$
Error	24	1561	65.04	-
Total	**35**	**3839**	**-**	**-**

Conclusion

F_{1c} (5.92) > F_{1T} (3.01), Reject H_{o1}
There is a significant difference in the yield of beans due to the fertiliser effect.

F_{2c} (2.69) < F_{2T} (3.40), Accept H_{o2}
The variety effect is not significant.

F_{3c} (1.98) < F_{3T} (2.51), Accept H_{o3}
The interaction effect between fertilisers and varieties is not significant.