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Convergence of the Pseudospectral Method for the Ginzburg–Landau Equation

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The convergence of the pseudospectral (Fourier) method for the Ginzburg–Landau equation in nonlinear wave theory is proved. The rate of convergence depends on the smoothness of the initial data. © 1990 Academic Press, Inc.

1. INTRODUCTION

The Ginzburg–Landau amplitude equation, also called the Stewartson–Stuart equation (cf. Kuramoto [7]), governs the evolution of the slowly varying amplitude of instability waves close to criticality in problems such as Rayleigh–Bénard convection [11] and Taylor–Couette flow [3] in fluid dynamics, drift dissipative waves in plasma physics [13], and turbulent motion in chemical reaction [8]. This equation, which has been the focus of various recent interesting studies on transition to chaos, can be written after suitable transforms in the form [12, 14]

$$iA_t + (1 - ic_0) A_{xx} = i\frac{c_0}{c_1} A - \left(1 + i\frac{c_0}{c_1}\right) |A|^2 A, \tag{1}$$

where t is the time variable, x is the one-dimensional spatial coordinate, $A(x, t)$ represents the complex amplitude under discussion, c_0 and c_1 are nonzero real parameters, and $i = \sqrt{-1}$. Spatially periodic solutions have been widely adopted in simulating the long-time behavior of the instability wave motion. The general initial value problem of Eq. (1) cannot be solved analytically, and therefore numerical integration has to be resorted to. One of the most important numerical integration methods for obtaining the spatially periodic solution to the initial value problem of (1) is the

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pseudospectral method introduced about 16 years ago by Kreiss and Oliger [5]. Using this method many authors (for example, Moon *et al.* [9, 10]; Keefe [2]) have investigated the global behavior of the spatially periodic solutions of Eq. (1) in the parameter region $c_0>0$, $c_1>0$ and the *route to chaos* (the appearance of limit circle, quasiperiodic, and chaotic motion as one varies a certain control parameter) has been observed. On the other hand, however, so far the convergence of the above integration method has not been studied. The purpose of the present note is to establish the convergence for the parameter regions $c_0>0$, $c_1>0$ and $c_0>0$, $c_1<0$. The rate of convergence depends on the smoothness of the initial data. In the next section we state our main result. In Section 3 some preliminary a priori estimates for higher order derivatives are obtained. The convergence of the pseudospectral approximation scheme is proved in Section 4.

2. The Pseudospectral Approximation

The initial value problem of Eq. (1) can be written

$$\begin{aligned} &A_t=(i+c_0)\,A_{xx}+\frac{c_0}{c_1}A+\left(i-\frac{c_0}{c_1}\right)|A|^2\,A, \qquad -\infty<x<\infty,\ t>0,\\ &A(x,0)=A_0(x), \qquad -\infty<x<\infty, \end{aligned} \tag{2}$$

where $A_0(x)$ is of period $L>0$.

We introduce for convenience the complex-valued function spaces for integers $k\geqslant 0$:

$$C_p^k[0,L]=\{U\in C^k[0,L]\mid U^{(j)}(0)=U^{(j)}(L),\ j=0,\ldots,k\}\subset C^k[0,L]$$

and

$$H_p^k(0,L)=\text{the completion of } C_p^k[0,L] \text{ in } W^{k,2}(0,L).$$

There is an obvious compact embedding

$$H_p^k(0,L)\to C_p^{k-1}[0,L]. \tag{3}$$

We shall use $|\ |_k$ and $\|\ \|_k$ to denote the norms of the spaces $C_p^k[0,L]$ and $H_p^k(0,L)$, respectively, and $D^k=\partial^k/\partial x^k$.

In [15] we have shown that if $A_0\in H_p^4(0,L)$ and

(i) $c_0>0$, $c_1>0$, or

(ii) $c_0 > 0$, $c_1 < 0$, and

$$\|A_0\|_0^2 < \frac{\sqrt{c_1^2 - 4L^2 c_1} + c_1}{2L},$$

then (2) has a unique classical solution $A(x, t)$ on $(-\infty, \infty) \times [0, \infty)$ with period L and satisfying

$$\sup_t \|A(t)\|_1 \leqslant M, \tag{4}$$

where $M > 0$ is a constant. The convergence of the pseudospectral method for Eq. (2) will be established under the condition (i) or (ii). (In the region $c_0 > 0$, $c_1 < 0$, if $\|A_0\|_0$ is sufficiently large, solutions of (2) with spatial period L may not exist globally for $t \geqslant 0$; in the region $c_0 < 0$, $c_1 > 0$, blowup motion can evolve from arbitrarily small initial states. For details, see [15].)

Let us now describe briefly the pseudospectral method for the numerical integration of Eq. (2) over the spatial period $[0, L]$.

First, the spatial interval $[0, L]$ is discretized by $2N + 1$ equidistant gridpoints $x_\nu = \nu L/(2N + 1)$, $\nu = 0, 1, ..., 2N$, with spacing $h = L/(2N + 1)$.

For an L-periodic function $v(x)$ whose values $v(x_\nu)$ are known at the gridpoints $x_\nu = \nu h$, $\nu = 0, 1, ..., 2N$, introduce the trigonometric interpolant

$$w(x) = \sum_{n=-N}^{N} a_n e^{inqx}, \qquad q = \frac{2\pi}{L}$$

of degree N of $v(x)$ such that $v(x_\nu) = w(x_\nu)$, $\forall \nu$. It can be shown that this interpolation problem is uniquely solvable (cf. Kreiss [4]). With the notation

$$\mathbf{v} = (v(x_0), ..., v(x_{2N}))^\perp,$$
$$\mathbf{v}^{(k)} = (w^{(k)}(x_0), ..., w^{(k)}(x_{2N}))^\perp, \qquad k = 1, 2, ...,$$

we have a complex $(2N + 1) \times (2N + 1)$ matrix S such that

$$\mathbf{v}^{(k)} = S^k \mathbf{v}, \qquad k = 1, 2, ...$$

Knowing $\mathbf{v}$, we can numerically calculate $\mathbf{v}^{(k)}$ by the fast Fourier transforms (Cooley and Tukey [1]).

Second, taking $S^2 \boldsymbol{\alpha}(t)$ as an approximation of

$$(D^2 A(x_0, t), ..., D^2 A(x_{2N}, t))^\perp,$$

where $\boldsymbol{\alpha}(t) = (\alpha_0(t), ..., \alpha_{2N}(t))^\perp$ is an approximation of $(A(x_0, t), ..., A(x_{2N}, t))^\perp$, we may discretize (2) into a system of ordinary differential equations in the form

$$\frac{d}{dt}\boldsymbol{\alpha} = (i + c_0)\, S^2\boldsymbol{\alpha} + \frac{c_0}{c_1}\boldsymbol{\alpha} + \left(i - \frac{c_0}{c_1}\right) Q_\alpha \boldsymbol{\alpha}, \qquad t > 0, \qquad (5)$$
$$\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0,$$

where $Q_\alpha \equiv \mathrm{diag}\{|\alpha_0(t)|^2, ..., |\alpha_{2N}(t)|^2\}$ is a diagonal matrix, and $\boldsymbol{\alpha}_0 = (\alpha_{0,0}, ..., \alpha_{0,2N})^\perp$ is given by

$$\alpha_{0,v} = A_0(x_v), \qquad v = 0, 1, ..., 2N.$$

This system can then be integrated numerically by a central difference or leapfrog scheme for instance.

Our main result is stated as follows.

2.1. THEOREM. *Suppose $A_0 \in H_p^k$, $k \geqslant 4$, and $A(x, t)$ the unique classical solution of Eq.* (2). *Then, under the condition* (i) ((ii)), *for any integer $N \geqslant 0$ (there is an integer $N_0 \geqslant 0$ such that for $N \geqslant N_0$), Eq.* (5) *has a unique solution $\boldsymbol{\alpha}(t)$ on $t \geqslant 0$; moreover, there are two constants δ, $\varepsilon > 0$ depending only on c_0, c_1, L, k, and $\|A_0\|_k$ such that for any $N > 0$ ($N > N_0$) and $T > 0$ we have the error estimate*

$$\sup_{0 \leqslant v \leqslant 2N,\, 0 \leqslant t \leqslant T} |A(x_v, t) - \alpha_v(t)| \leqslant \frac{\delta}{N^{k-5/2}} e^{\varepsilon T}. \qquad (6)$$

Note. The inequality (6) establishes the convergence of the pseudospectral method and measures the rate of convergence in terms of the regularity of the initial data $A_0(x)$.

We shall prove this theorem in Section 4. In the remaining part of this section we show only that the initial value problem (5) is globally and uniquely solvable under some suitable formulation.

In $\mathbf{C}^{2N+1}$, define the integral inner product and the associated norm by

$$(\mathbf{u}, \mathbf{v})_{(h)} = h \sum_{v=0}^{2N} u_v v_v^*, \qquad \|\mathbf{u}\|_{(h)}^2 = (\mathbf{u}, \mathbf{u})_{(h)}, \qquad \mathbf{u}, \mathbf{v} \in \mathbf{C}^{2N+1},$$

where $h = L/(2N+1)$. If $w(x)$ is the trigonometric interpolant of degree N of the function $u(x)$ defined at $x_v = vh$ by $u(x_v) = u_v$, $v = 0, 1, ..., 2N$, then

$$\|u\|_{(h)}^2 \equiv \|\mathbf{u}\|_{(h)}^2 = \|w\|_0^2. \qquad (7)$$

2.2. LEMMA (Cf. Kreiss [4], Kreiss and Oliger [6]). *S is skew-Hermitian. The eigenvalues of S are $\lambda_\nu = i\nu q$ and the corresponding eigenvectors are*

$$\mathbf{e}_\nu = (1, e^{iqx_\nu}, ..., e^{i2Nqx_\nu})^\perp, \qquad \nu = 0, \pm 1, ..., \pm N.$$

2.3. LEMMA. *Let $v(x) \in H_p^k(0, L)$ and $w(x)$ be the trigonometric interpolant of $v(x)$ at $x_\nu = \nu h$, $\nu = 0, 1, ..., 2N$, of degree N, then*

$$\|D^j v - D^j w\|_0 \leqslant \frac{1}{N^{k-j}} C(j, L, \|v\|_k), \qquad j = 0, ..., k-1,$$

where $C > 0$ is a constant independent of N.

For a proof of this lemma, see Kreiss and Oliger [6].

By virtue of 2.3 and $S^j\mathbf{v} = (w^{(j)}(x_0), ..., w^{(j)}(x_{2N}))^\perp$, we have

$$\|S^j\mathbf{v}\|_{(h)} = \|w^{(j)}\|_0 \leqslant \frac{1}{N^{k-j}} C(j, L, \|v\|_k) + \|v\|_j, \qquad j = 0, ..., k-1. \quad (8)$$

2.4. LEMMA. *Under the condition* (i), *there is a constant $C > 0$ independent of N such that the solution $\boldsymbol{\alpha}(t)$ of Eq.* (5) *satisfies*

$$\sup_t \|\boldsymbol{\alpha}(t)\|_{(h)} \leqslant C.$$

Proof. Multiplying $(5)_1$ by $h\boldsymbol{\alpha}^\dagger$, where $\boldsymbol{\alpha}^\dagger = (\alpha_0^*, ..., \alpha_{2N}^*)$, we have

$$\left(\frac{d}{dt}\boldsymbol{\alpha}, \boldsymbol{\alpha}\right)_{(h)} = (i + c_0)(S^2\boldsymbol{\alpha}, \boldsymbol{\alpha})_{(h)} + \frac{c_0}{c_1}\|\boldsymbol{\alpha}\|_{(h)}^2$$
$$+ \left(i - \frac{c_0}{c_1}\right)(Q_\alpha \boldsymbol{\alpha}, \boldsymbol{\alpha})_{(h)}, \qquad t \geqslant 0. \quad (9)$$

From the Schwarz inequality,

$$(Q_\alpha\boldsymbol{\alpha}, \boldsymbol{\alpha})_{(h)} = h \sum_{\nu=0}^{2N} |\alpha_\nu|^4 \geqslant \frac{1}{L}\left(h \sum_{\nu=0}^{2N} |\alpha_\nu|^2\right)^2 = \frac{1}{L}\|\boldsymbol{\alpha}\|_{(h)}^4. \quad (10)$$

Substituting (10) into (9) and using 2.2 we obtain the inequality

$$\frac{d}{dt}\|\boldsymbol{\alpha}\|_{(h)}^2 \leqslant -2c_0\|S\boldsymbol{\alpha}\|_{(h)}^2 + \frac{2c_0}{c_1}\|\boldsymbol{\alpha}\|_{(h)}^2 - \frac{2c_0}{Lc_1}\|\boldsymbol{\alpha}\|_{(h)}^4. \quad (11)$$

Consequently, it follows from (11) and an application of Perron's comparison principle for differential inequalities that

$$\|\boldsymbol{\alpha}(t)\|_{(h)}^2 \leqslant \min\{\|\boldsymbol{\alpha}(0)\|_{(h)}^2 e^{2c_0t/c_1}, \max\{L, \|\boldsymbol{\alpha}(0)\|_{(h)}\}\}.$$

Then 2.4 follows from (8) and the above inequality.

2.5. LEMMA. *Under the condition* (ii), *there are an* $N_0 \geqslant 0$ *and a constant* $C > 0$ *independent of* $N \geqslant N_0$ *such that the solution* $\boldsymbol{\alpha}(t)$ *of Eq.* (5) *satisfies*

$$\sup_t \|\boldsymbol{\alpha}(t)\|_{(h)} \leqslant C, \qquad \textit{for} \quad N \geqslant N_0.$$

Proof. Let $c_2 = -c_1 > 0$. For a fixed $t \geqslant 0$, choose v_0 such that $|\alpha_{v_0}(t)|^2 = \min\{|\alpha_v(t)|^2 \mid v = 0, 1, ..., 2N\}$. In particular we have $|\alpha_{v_0}(t)|^2 \leqslant \|\boldsymbol{\alpha}(t)\|^2_{(h)}/L$, and as a consequence, for any $v = 0, 1, ..., 2N$,

$$\begin{aligned}
|\alpha_v(t)|^2 = |w(x_v, t)|^2 &\leqslant \frac{1}{L}\|\boldsymbol{\alpha}(t)\|^2_{(h)} + 2\,\mathrm{Re}\left\{\int_{x_{v_0}}^{x_v} w^* \, Dw \, dx\right\} \\
&\leqslant \frac{1}{L}\|\boldsymbol{\alpha}(t)\|^2_{(h)} + 2\|Dw\|_0 \|w\|_0 \\
&= \frac{1}{L}\|\boldsymbol{\alpha}(t)\|^2_{(h)} + 2\|S\boldsymbol{\alpha}(t)\|_{(h)} \|\boldsymbol{\alpha}(t)\|_{(h)},
\end{aligned} \tag{12}$$

where we have used (7) and the notation that $w(x, t)$ is the trigonometric interpolant of degree N of $v(x, t)$ defined at x_v by $v(x_v, t) = \alpha_v$, $v = 0, 1, ..., 2N$.

From (12),

$$(Q_\alpha \boldsymbol{\alpha}, \boldsymbol{\alpha})_{(h)} \leqslant \frac{1}{L}\|\boldsymbol{\alpha}(t)\|^4_{(h)} + 2\|S\boldsymbol{\alpha}(t)\|_{(h)} \|\boldsymbol{\alpha}(t)\|^3_{(h)}. \tag{13}$$

Combining (9), (13) and using a simple interpolation inequality we obtain

$$\begin{aligned}
\frac{d}{dt}\|\boldsymbol{\alpha}(t)\|^2_{(h)} &\leqslant -2c_0\|S\boldsymbol{\alpha}\|^2_{(h)} - \frac{2c_0}{c_2}\|\boldsymbol{\alpha}\|^2_{(h)} \\
&\quad + \frac{2c_0}{c_2}\left[\frac{1}{L}\|\boldsymbol{\alpha}\|^4_{(h)} + 2\|S\boldsymbol{\alpha}\|_{(h)}\|\boldsymbol{\alpha}\|^3_{(h)}\right] \\
&\leqslant -\frac{2c_0}{c_2}\|\boldsymbol{\alpha}\|^2_{(h)}\left(1 - \frac{1}{L}\|\boldsymbol{\alpha}\|^2_{(h)} - \frac{1}{c_2}\|\boldsymbol{\alpha}\|^4_{(h)}\right).
\end{aligned} \tag{14}$$

Taking $N_0 \geqslant 0$ sufficiently large so that

$$\begin{aligned}
\|\boldsymbol{\alpha}(0)\|^2_{(h)} = \|\boldsymbol{\alpha}_0\|^2_{(h)} &= h\sum_{v=0}^{2N} |A_0(x_v)|^2 \\
&< \frac{\sqrt{c_2^2 + 4L^2c_2} - c_2}{2L} \equiv r, \qquad N \geqslant N_0.
\end{aligned}$$

From the structure of Eq. (14) we have $(d/dt)\,\|\boldsymbol{\alpha}(t)\|^2_{(h)} \leqslant 0$. In particular

$$\|\boldsymbol{\alpha}(t)\|^2_{(h)} \leqslant \|\boldsymbol{\alpha}(0)\|^2_{(h)} \leqslant r, \qquad t \geqslant 0.$$

This proves 2.5.

The global existence of the unique solution $\boldsymbol{\alpha}(t)$ on $t \geqslant 0$ of Eq. (5) for any $N \geqslant 0$ under the condition (i) (or for any $N \geqslant N_0$ under the condition (ii)) follows from 2.4 (or 2.5).

3. Estimates for Higher Order Derivatives

Let $\boldsymbol{\alpha}(t)$ be the unique solution of Eq. (5) defined on $t \geqslant 0$.

3.1. Lemma. *Under the assumption* (i) ((ii)) *there is* (*are an integer* $N_0 \geqslant 0$ *and*) *a constant* $C > 0$ *independent of* N ($N \geqslant N_0$) *such that*

$$\sup_t \|S\boldsymbol{\alpha}(t)\|_{(h)} \leqslant C.$$

Proof. We can get from Eq. (5) that

$$\begin{aligned}
\frac{d}{dt}\,\|S\boldsymbol{\alpha}(t)\|^2_{(h)} &= -2\,\mathrm{Re}\left(\frac{d}{dt}\,\boldsymbol{\alpha}(t),\, S^2\boldsymbol{\alpha}(t)\right)_{(h)} \\
&\leqslant -2c_0\|S^2\boldsymbol{\alpha}\|^2_{(h)} + \frac{2c_0}{|c_1|}\,\|S\boldsymbol{\alpha}\|^2_{(h)} \\
&\quad + 2\left(1 + \frac{c_0}{|c_1|}\right)|(Q_\alpha\boldsymbol{\alpha},\, S^2\boldsymbol{\alpha})_{(h)}| \\
&\leqslant \frac{2c_0}{|c_1|}\,\|S\boldsymbol{\alpha}\|^2_{(h)} + \frac{1}{2c_0}\left(1 + \frac{c_0}{|c_1|}\right)^2 \|Q_\alpha\boldsymbol{\alpha}\|^2_{(h)}. \qquad (15)
\end{aligned}$$

On the other hand, from (12), (13), and a simple interpolation inequality, we can derive

$$\begin{aligned}
\|Q_\alpha\boldsymbol{\alpha}\|^2_{(h)} &= h \sum_{v=0}^{2N} |\alpha_v(t)|^6 \\
&\leqslant (Q_\alpha\boldsymbol{\alpha},\, \boldsymbol{\alpha})_{(h)}\left(\frac{1}{L}\,\|\boldsymbol{\alpha}\|^2_{(h)} + 2\|S\boldsymbol{\alpha}\|_{(h)}\|\boldsymbol{\alpha}\|_{(h)}\right) \\
&\leqslant (1 + 2\|\boldsymbol{\alpha}\|^4_{(h)})\left(\frac{1}{L^2}\,\|\boldsymbol{\alpha}\|^6_{(h)} + 2\|S\boldsymbol{\alpha}\|^2_{(h)}\right). \qquad (16)
\end{aligned}$$

By virtue of (15), (16), and 2.4 (or 2.5), we can conclude that there are constants μ, $\zeta > 0$ independent of N (or $N \geqslant N_0$) such that

$$\frac{d}{dt}(\|S\boldsymbol{\alpha}(t)\|_{(h)}^2 + \mu\|\boldsymbol{\alpha}(t)\|_{(h)}^2)$$
$$\leqslant -(\|S\boldsymbol{\alpha}(t)\|_{(h)}^2 + \mu\|\boldsymbol{\alpha}(t)\|_{(h)}^2) + \zeta, \qquad t \geqslant 0.$$

Finally, 3.1 follows from solving the above inequality and using (8).

3.2. COROLLARY. *Under the condition* (i) ((ii)) *there is a constant* $C > 0$ *independent of* N ($N \geqslant N_0$) *such that*

$$\sup_{t \geqslant 0,\, 0 \leqslant v \leqslant 2N} |\alpha_v(t)| \leqslant C.$$

Proof. The uniform bound can be inferred directly from (12), 2.4 (or 2.5), and 3.1.

3.3. LEMMA. *Let* $A(x, t)$ *be the unique classical solution of* (2) *where* $A_0(x) \in H_p^k(0, L)$, $k \geqslant 4$. *Then there is a constant* $C > 0$ *such that*

$$\sup_t \|A(t)\|_k \leqslant C.$$

Proof. We will use induction on $\|A(t)\|_j$ for $j = 1, ..., k$.

The $j = 1$ case has been stated in (4). Suppose for $1 \leqslant j < k$ there is a constant $C_1 > 0$ such that

$$\sup_t \|A(t)\|_j \leqslant C_1.$$

Define $I_l(t) = \|D^l A(t)\|_0^2$. Then, for $t > 0$,

$$I'_{j+1}(t) = 2\,\mathrm{Re}(\partial_t D^{j+1}A, D^{j+1}A)_2 = 2(-1)^{j+1}\,\mathrm{Re}(\partial_t A, D^{2(j+1)}A)_2, \qquad (17)$$

where $(\ ,\ )_2$ is the usual inner product of $L^2(0, L)$.

From $(2)_1$ we have

$$(\partial_t A, D^{2(j+1)}A)_2 = (-1)^j (i + c_0)\, \|D^{j+2}A(t)\|_0^2$$
$$+ (-1)^{j+1} \cdot \frac{c_0}{c_1} \|D^{j+1}A(t)\|_0^2$$
$$+ \left(i - \frac{c_0}{c_1}\right) (|A|^2 A, D^{2(j+1)}A)_2. \qquad (18)$$

Substituting (18) into (17) we obtain

$$I'_{j+1}(t) = -2c_0 I_{j+2}(t) + \frac{2c_0}{c_1} I_{j+1}(t)$$
$$+ 2(-1)^{j+1} \operatorname{Re}\left\{\left(i - \frac{c_0}{c_1}\right)(|A|^2 A, D^{2(j+1)}A)_2\right\}, \qquad t > 0. \tag{19}$$

From the embedding (3), we can find a constant $C_2 > 0$ such that

$$|A(t)|_l \leqslant C_2 I_{l+1}^{1/2}(t), \qquad l = 0, ..., k-1. \tag{20}$$

By virtue of (20) and Leibniz' rule, there are constants $C_3, C_4 > 0$ such that

$$|(|A|^2 A, D^{2(j+1)}A)_2| = |(D^{j+1}(|A|^2 A), D^{j+1}A)_2|$$
$$\leqslant C_3 + C_4 I_{j+1}(t), \qquad t > 0. \tag{21}$$

Accordingly, from (19) and (21), we have

$$I'_{j+1}(t) \leqslant -2c_0 I_{j+2}(t) + C_5 I_{j+1}(t) + C_6, \qquad t > 0, \tag{22}$$

where $C_5, C_6 > 0$ are constants.

Similarly there are constants $C_7, C_8 > 0$ such that $I'_j(t) \leqslant -2c_0 I_{j+1}(t) + C_7 I_j(t) + C_8$. Using this inequality, (22), and the boundedness of $I_j(t)$ we can find two constants $\mu, \zeta > 0$ to achieve the inequality

$$I'_{j+1}(t) + \mu I'_j(t) \leqslant -(I_{j+1}(t) + \mu I_j(t)) + \zeta, \qquad t > 0,$$

which gives us the bound $I_{j+1}(t) \leqslant [I_{j+1}(0) + \mu I_j(0)]\, e^{-t} + \zeta(1 - e^{-t})$.

Lemma 3.3 is proved.

3.4. COROLLARY. *Under the condition of* 3.3, *there is a constant* $C > 0$ *such that*

$$\sup_t \|\partial_t A\|_{k-2} \leqslant C.$$

Proof. The above inequality follows from differentiating Eq. $(2)_1$ with respect to the spatial variable x and using 3.3.

4. Proof of Convergence

Let $A(x, t)$ be the unique classical solution of Eq. (2) of spatial period $L > 0$, where $A_0 \in H_p^k(0, L)$, $k \geqslant 4$. We have the Fourier mode expansion

$$A(x, t) = \sum_{n=-N}^{N} a_n(t)\, e^{inqx} + \sum_{|n|>N} a_n(t)\, e^{inqx}$$

$$\equiv A_N(x, t) + A_R(x, t).$$

$A_N(x, t)$ satisfies the equation

$$\partial_t A_N = (i + c_0)\, D^2 A + \frac{c_0}{c_1} A_N + \left(i - \frac{c_0}{c_1}\right) |A_N|^2 A_N + G_R,$$

where

$$G_R = -\partial_t A_R + (i + c_0)\, D^2 A_R + \frac{c_0}{c_1} A_R$$
$$+ \left(i - \frac{c_0}{c_1}\right) (|A|^2 A_R + (|A_R|^2 + 2 \operatorname{Re} A_R^* A_N)\, A_N).$$

Let us now set

$$\boldsymbol{\beta}(t) = (\beta_\nu(t)) = (A_N(x_0, t), ..., A_N(x_{2N}, t))^\perp,$$
$$\boldsymbol{\eta}(t) = (\eta_\nu(t)) = (G_R(x_0, t), ..., G_R(x_{2N}, t))^\perp.$$

Then, at the $2N + 1$ gridpoints $x_\nu = \nu h$, $\nu = 0, 1, ..., 2N$, we have the system

$$\begin{aligned} &\frac{d}{dt} \boldsymbol{\beta}(t) = (i + c_0)\, S^2 \boldsymbol{\beta} + \frac{c_0}{c_1} \boldsymbol{\beta} + \left(i - \frac{c_0}{c_1}\right) Q_\beta \boldsymbol{\beta} + \boldsymbol{\eta}, \qquad t > 0, \\ &\boldsymbol{\beta}(0) = \boldsymbol{\beta}_0, \end{aligned} \tag{23}$$

where $\boldsymbol{\beta}_0 = ((A_0)_N(x_0), ..., (A_0)_N(x_{2N}))^\perp$ and

$$Q_\beta = \operatorname{diag}\{|\beta_0(t)|^2, ..., |\beta_{2N}(t)|^2\}.$$

By virtue of (5) and (23), $\boldsymbol{\gamma}(t) \equiv (\boldsymbol{\alpha} - \boldsymbol{\beta})(t)$ satisfies the equation

$$\begin{aligned} &\frac{d}{dt} \boldsymbol{\gamma}(t) = (i + c_0)\, S^2 \boldsymbol{\gamma} + \frac{c_0}{c_1} \boldsymbol{\gamma} + \left(i - \frac{c_0}{c_1}\right) (Q_\alpha \boldsymbol{\gamma} + [Q_\alpha - Q_\beta]\, \boldsymbol{\beta}) - \boldsymbol{\eta}, \qquad t > 0, \\ &\boldsymbol{\gamma}(0) = \boldsymbol{\alpha}(0) - \boldsymbol{\beta}(0). \end{aligned} \tag{24}$$

Multiplying $(24)_1$ by $h\gamma^\dagger$ and using 2.2 and a simple interpolation inequality, we easily find

$$\frac{d}{dt}\|\gamma(t)\|^2_{(h)} \leqslant \left(1+\frac{2c_0}{|c_1|}+6k_1\left[1+\frac{c_0}{|c_1|}\right]\right)\|\gamma(t)\|^2_{(h)}+\frac{1}{4}k_2(N)$$

$$\equiv \sigma\|\gamma(t)\|^2_{(h)}+\frac{1}{4}k_2(N), \qquad t\geqslant 0, \tag{25}$$

where we have put

$$k_1=\max\{\sup_{t,v,N}|\alpha_v(t)|^2, \sup_{t,v,N}|\beta_v(t)|^2\}, \qquad k_2(N)=\sup_t \|\boldsymbol{\eta}\|^2_{(h)}.$$

From (25) we obtain

$$\|\gamma(t)\|^2_{(h)} \leqslant \|\gamma(0)\|^2_{(h)}e^{\sigma t}+\frac{1}{4\sigma}k_2(N)(e^{\sigma t}-1). \tag{26}$$

Using (24), in a similar manner we find

$$\frac{d}{dt}\|S\gamma(t)\|^2_{(h)} \leqslant -2c_0\|S^2\gamma\|^2_{(h)}+\frac{2c_0}{|c_1|}\|S\gamma\|^2_{(h)}$$

$$+6k_1\left(1+\frac{c_0}{|c_1|}\right)\|\gamma\|_{(h)}\|S^2\gamma\|_{(h)}+\|\boldsymbol{\eta}\|_{(h)}\|S^2\gamma\|_{(h)}$$

$$\leqslant \frac{2c_0}{|c_1|}\|S\gamma\|^2_{(h)}+k_3\|\gamma\|^2_{(h)}+k_4k_2(N), \qquad t>0,$$

where $k_3, k_4>0$ are constants depending on c_0, c_1, k_1. As a consequence, for $0\leqslant t\leqslant T$,

$$\|S\gamma(t)\|^2_{(h)} \leqslant \|S\gamma(0)\|^2_{(h)}e^{2c_0t/|c_1|}+\frac{|c_1|}{2c_0}(k_2(N)\,k_4+k_5(N)\,k_3)(e^{2c_0t/|c_1|}-1), \tag{27}$$

where

$$k_5(N)=\sup_{0\leqslant t\leqslant T}\|\gamma(t)\|^2_{(h)}.$$

Let us first estimate $\|\gamma(0)\|^2_{(h)}$.
From the definition,

$$\gamma_v(0)=\alpha_{0,v}-\beta_{0,v}=A_0(x_v)-(A_0)_N(x_v)=(A_0)_R(x_v).$$

Therefore, if $A_0(x)=\sum b_n e^{inqx}$, then

$$\begin{aligned}\|\gamma(0)\|_{(h)} &\leqslant L|(A_0)_R|_0 \leqslant L \sum_{|n|>N} |b_n| \\ &\leqslant L\Big(\sum_{|n|>N} |b_n|^2 (nq)^{2k}\Big)^{1/2} \Big(\sum_{|n|>N} \frac{1}{(nq)^{2k}}\Big)^{1/2} \\ &\leqslant \frac{M_k}{N^{k-1/2}} \|D^k A_0\|_0, \end{aligned} \tag{28}$$

where $M_k=(2k-1)^{-1/2}$.

Next let us estimate $\|S\gamma(0)\|_{(h)}$. As before, let $w(x)$ be the trigonometric interpolant of degree N of $A_0(x)$. Then $S\boldsymbol{\alpha}_0=(Dw(x_0), ..., Dw(x_{2N}))^{\perp}$. On the other hand, since $S\boldsymbol{\beta}_0=(D(A_0)_N(x_0), ..., D(A_0)_N(x_{2N}))^{\perp}$, so, using (7), 2.3, and (28), we have

$$\begin{aligned}\|S\gamma(0)\|_{(h)} &= \|D(w-(A_0)_N)\|_{(h)} = \|D(w-(A_0)_N)\|_0 \\ &\leqslant \|Dw-DA_0\|_0 + \|D(A_0)_R\|_0 \\ &\leqslant \frac{1}{N^{k-1}} C(L, \|A_0\|_k) + \frac{1}{N^{k-3/2}} \|D^k A_0\|_0 M_{k-1}. \end{aligned} \tag{29}$$

Finally, we need to estimate $k_2(N)$. In a manner similar to the derivation of (28), we easily see that

$$|D^2 A_R(t)|_0 \leqslant \frac{1}{LN^{k-5/2}} \|D^k A(t)\|_0 M_{k-2},$$

$$|\partial_t A_R(t)|_0 \leqslant \frac{1}{LN^{k-5/2}} \|\partial_t A(t)\|_{k-2} M_{k-2},$$

therefore, by virtue of 3.3 and 3.4,

$$\sup_t |G_R(t)|_0 \leqslant \frac{1}{N^{k-5/2}} C(k, L, c_0, c_1, \|A_0\|_k).$$

In particular we have

$$k_2(N) \leqslant L \sup_t |G_R(t)|_0 \leqslant \frac{C}{N^{k-5/2}}. \tag{30}$$

Now, after the above preparation, the error estimate (6) follows readily from using the inequality

$$|A(x_v, t)-\alpha_v(t)| \leqslant |A_R(t)|_0 + |\gamma_v(t)|,$$

applying (12) to $\gamma(t)$, and observing (26), (27), (28), (29), and (30).

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