Machine-assisted Formalisation of Parametrised Graph Algebra

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Describing hardware microcontrollers

Low level options:

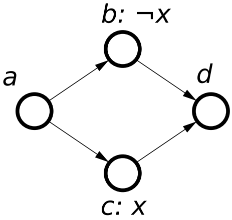
- Logic gate circuits
- State machines
- ...

High level options:

- Petri nets
- Process algebra
- High-level languages
- ...

Conditional Partial Order Graphs

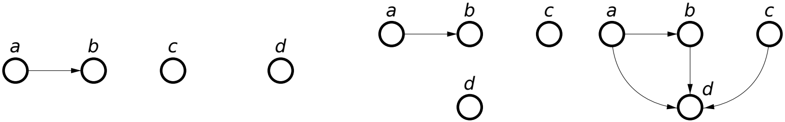
- Vertices represent events
- Edges represent causal dependencies
- Annotated with conditions



Parametrised Graph Algebra

PG Algebra is a generalisation of CPOGs

- Arbitrary set together with algebraic operations on it
- Equivalence relation satisfying certain laws



$G_1 = a \gg b + c$ $\quad$ $G_2 = d$ $\quad$ $G_1 + G_2$ $\quad$ $G_1 \gg G_2$

Desired PG software support

- Formula manipulations
- Conversions to/from different formalisms
- Hardware synthesis

Formal methods

- How do we know the theory is sound?
- How do we know the tools are correct?
- Need a way to statically ensure this.

Agda

Why Agda?

- A total functional programming language
- A proof environment based on Curry-Howard isomorphism
- Easy to learn when you know Haskell
- Newbie-friendly community

Graph Algebra

```
record GraphOps G : Set where
field
  ε : G
  _+_ : G → G → G
  _≫_ : G → G → G

record IsGraphAlgebra : Set where
field
  +assoc      : ∀ {p q r} → (p + q) + r ≈ p + (q + r)
  +comm       : ∀ {p q} → p + q ≈ q + p
  ≫ assoc     : ∀ {p q r} → (p ≫ q) ≫ r ≈ p ≫ (q ≫ r)
  ≫ identityˡ : ∀ {p}     → ε ≫ p ≈ p
  ≫ identityʳ : ∀ {p}     → p ≫ ε ≈ p
  distribˡ    : ∀ {p q r} → p ≫ (q + r) ≈ p ≫ q + p ≫ r
  distribʳ    : ∀ {p q r} → (p + q) ≫ r ≈ p ≫ r + q ≫ r
  decomposition : ∀ {p q r} → p ≫ q ≫ r ≈ p ≫ q + p ≫ r + q
```

Introducing conditions

$[\_]\_ \ : \ \mathsf{B} \to \mathsf{G} \to \mathsf{G}$

$$
\begin{array}{ll}
\text{boolean-algebra} & : \ \mathsf{BooleanAlgebra\ B} \\
\text{true-condition} & : \ \forall\, x \to [\top]\ x \approx x \\
\text{false-condition} & : \ \forall\, x \to [\bot]\ x \approx \varepsilon \\
\text{and-condition} & : \ \forall\, f\ g\ x \to [f \wedge g]\ x \approx [f]\ [g]\ x \\
\text{or-condition} & : \ \forall\, f\ g\ x \to [f \vee g]\ x \approx [f]\ x + [g]\ x \\
\text{conditional+} & : \ \forall\, f\ x\ y \to [f]\ (x + y) \approx [f]\ x + [f]\ y \\
\text{conditional}\gg & : \ \forall\, f\ x\ y \to [f]\ (x \gg y) \approx [f]\ x \gg [f]\ y
\end{array}
$$

PG Algebra theorems

The following theorems has been derived from the axioms:

$$+\text{identity} : \forall\ p \to p + \varepsilon \approx p$$

$$+\text{idempotence} : \forall\ p \to p + p \approx p$$

$$\text{absorption}^{\text{l}} : \forall\ p\ q \to p \gg q + p \approx p \gg q$$

$$\text{absorption}^{\text{r}} : \forall\ p\ q \to p \gg q + q \approx p \gg q$$

$$\text{choice-propagation}_1 : \forall\ b\ p\ q\ r \to$$
$$[b]\ (p \gg q) + [\neg\ b]\ (p \gg r) \approx p \gg ([b]\ q + [\neg\ b]\ r)$$

$$\text{choice-propagation}_2 : \forall\ b\ p\ q\ r \to$$
$$[b]\ (p \gg r) + [\neg\ b]\ (q \gg r) \approx ([b]\ p + [\neg\ b]\ q) \gg r$$

$$\text{condition-regularisation} : \forall\ f\ g\ p\ q \to$$
$$[f]\ p \gg [g]\ q \approx [f]\ p + [g]\ q + [f \wedge g]\ (p \gg q)$$

Formula data structure

Formula data structure mimics the algebra operations:

$$
\begin{array}{l}
\textbf{data}\ \mathsf{PGFormula}\ \mathsf{B}\ \mathsf{V}\ :\ \mathsf{Set}\ \textbf{where} \\
\quad \_+\_\ :\ (x\ y\ :\ \mathsf{PGFormula}) \to \mathsf{PGFormula} \\
\quad \_\gg\_\ :\ (x\ y\ :\ \mathsf{PGFormula}) \to \mathsf{PGFormula} \\
\quad \varepsilon\ :\ \mathsf{PGFormula} \\
\quad \mathsf{var}\ :\ (a\ :\ \mathsf{V}) \to \mathsf{PGFormula} \\
\quad [\_]\ \_\ :\ (c\ :\ \mathsf{B}) \to \mathsf{PGFormula} \to \mathsf{PGFormula}
\end{array}
$$

Making sense of the formulae

We need to give the formula semantics in terms of algebra.

$$
\begin{aligned}
&\mathsf{pg\text{-}eval} \;:\; \mathsf{PGFormula\ B\ V} \\
&\quad \to (\mathsf{V} \to \mathsf{G}) \\
&\quad \to \mathit{PGAlgebra}\ \mathsf{G\ V} \\
&\quad \to \mathsf{G}
\end{aligned}
$$

$$
\begin{aligned}
&\_{\approx_{\mathsf{f}}}\_ \;:\; \mathsf{PGFormula\ B\ V} \to \mathsf{PGFormula\ B\ V} \to \mathsf{Set} \\
&a \approx_{\mathsf{f}} b \;=\; \forall\ \mathit{assign}\ \mathit{alg} \\
&\quad \to \mathsf{pg\text{-}eval}\ a\ \mathit{assign}\ \mathit{alg} \approx \mathsf{pg\text{-}eval}\ b\ \mathit{assing}\ \mathit{alg}
\end{aligned}
$$

PG Formula normal form

$$
\begin{aligned}
\mathsf{BF} &= \mathsf{BoolFormula}\ \mathsf{B} \\
\mathsf{PG} &= \mathsf{PGFormula}\ \mathsf{BF}\ \mathsf{V} \\
\mathsf{Node} &= \mathsf{V} \uplus (\mathsf{V} \times \mathsf{V}) \\
\mathsf{Lit} &= \mathsf{Node} \times \mathsf{BF} \\
\mathsf{NF} &= \mathsf{List}\ \mathsf{Lit}
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{fromNode} : \mathsf{Node} \to \mathsf{PG} \\
&\mathsf{fromNode}\ (\mathsf{inj}_1\ x) = \mathsf{var}\ x \\
&\mathsf{fromNode}\ (\mathsf{inj}_2\ (x, y)) = \mathsf{var}\ x \gg \mathsf{var}\ y \\
&\mathsf{fromLit} : \mathsf{Lit} \to \mathsf{PG} \\
&\mathsf{fromLit}\ (node, cond) = [cond]\ \mathsf{fromNode}\ node \\
&\mathsf{fromNF} : \mathsf{NF} \to \mathsf{PG} \\
&\mathsf{fromNF} = \mathsf{foldr}\ \_{+}\_\ \varepsilon \circ \mathsf{map}\ \mathsf{fromLit}
\end{aligned}
$$

PG Formula normalisation

fromNF : NF → PG

normalise : PG → NF

... (*35 lines of implementation*)

normalise-correct : $\forall\, f \to f \approx$ fromNF (normalise f)

... (*100 lines of proof*)

Conclusions

- We have successfuly formalised the PG Algebra
- We have developed a simple verified program for converting formulae to normal forms
- We thank you for your attention