Frequency Filtering

Convolution Property of the Fourier Transform

Let functions $\mathbf{f}(r, c)$ and $\mathbf{g}(r, c)$ have Fourier Transforms $\mathrm{F}(u, v)$ and $\mathrm{G}(u, v)$.

Then,

$$\mathrm{F}\{\mathbf{f} \circ \mathbf{g}\} = \mathbf{F} .\!* \mathbf{G}.$$

Moreover,

$$\mathrm{F}\{\mathbf{f} .\!* \mathbf{g}\} = \mathbf{F} \circ \mathbf{G}.$$

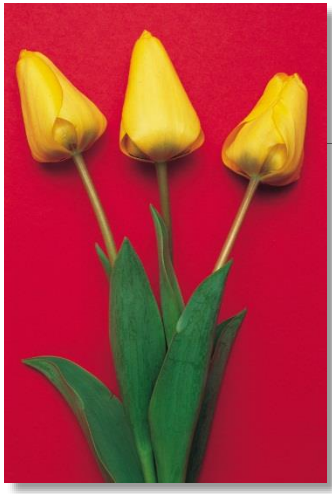
$\circ$ = convolution
.* = dot product

The Fourier Transform of a convolution equals the dot product of the Fourier Transforms.

Similarly, the Fourier Transform of a dot product is the convolution of the Fourier Transforms

Convolution via Fourier Transform

Spatial domain



Freq. domain

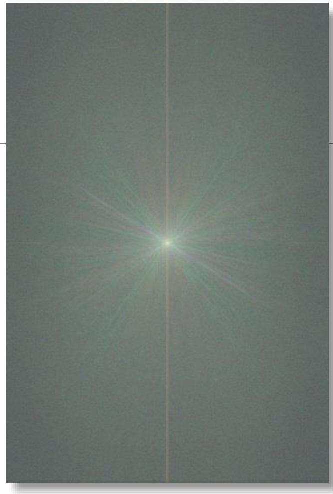
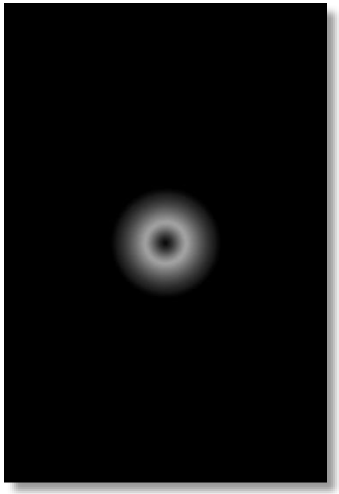
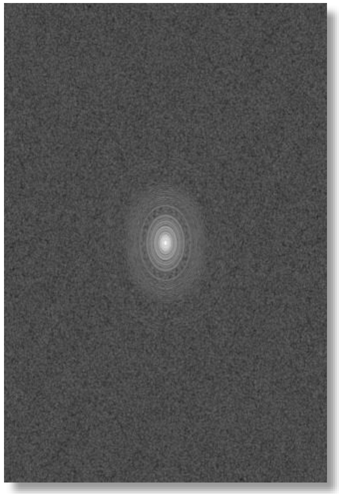
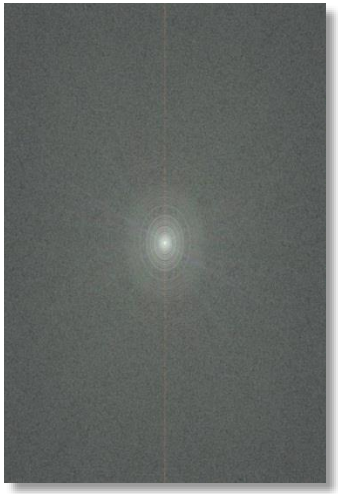


Image & Mask



Transforms





Pixelwise Product

Spatial domain

Inverse Transform

How to Convolve via FT in Matlab

1. Read the image from a file into a variable, say `I`.
2. Read in or create the convolution matrix, `h`.
3. Compute the sum of the matrix: `s = sum(h(:));`
4. If `s == 0,` set `s = 1;`
5. Replace `h` with `h = h/s;`
6. Create: `H = zeros(size(I));`
7. Copy `h` into the middle of `H`.
8. Shift `H` into position: `H = ifftshift(H);`
9. Take the 2D FT of `I` and `H`: `FI=fft2(double(I));` `FH=fft2(H);`
10. Pointwise multiply the FTs: `FJ=FI.*FH;`
11. Compute the inverse FT: `J = abs(ifft2(FJ));`

The matrix is usually 1-band

If h is a one-band matrix and I is multi-band, you must copy h into all the bands of H.

How to Convolve via FT in Matlab

1. Read the image from a file into a variable, say **`I`**.
2. Read in or create the convolution matrix, **`h`**.
3. Compute the sum of the matrix: **`s = sum(h(:));`**
4. If **`s == 0,`** set **`s = 1;`**
5. Replace **`h`** with **`h = h/s;`**
6. Create: **`H = zeros(size(I));`**
7. Copy **`h`** into the middle of **`H`**.
8. Shift **`H`** into position: **`H = ifftshift(H);`**
9. Take the 2D FT of **`I`** and **`H`**: **`FI=fft2(double(I));`** **`FH=fft2(H);`**
10. Pointwise multiply the FTs: **`FJ=FI.*FH;`**
11. Compute the inverse FT: **`J = abs(ifft2(FJ));`**

fftshift and ifftshift must be done separately for each band. fft2 transforms all the bands of a multiband image separately.

Blurring: Averaging / Lowpass Filtering

Blurring results from:

Pixel averaging in the spatial domain:

- Each pixel in the output is a weighted average of its neighbours.
- Is a convolution whose weight matrix sums to 1.

Lowpass filtering in the frequency domain:

- High frequencies are diminished or eliminated
- Individual frequency components are multiplied by a non-increasing function of ω such as $1/\omega = 1/\sqrt{(u^2+v^2)}$.

The values of the output image are all non-negative.

Sharpening: Differencing / Highpass Filtering

Sharpening results from adding to the image, a copy of itself that has been:

Pixel-differenced in the spatial domain:

- Each pixel in the output is a difference between itself and a weighted average of its neighbors.
- Is a convolution whose weight matrix sums to 0.

Highpass filtered in the frequency domain:

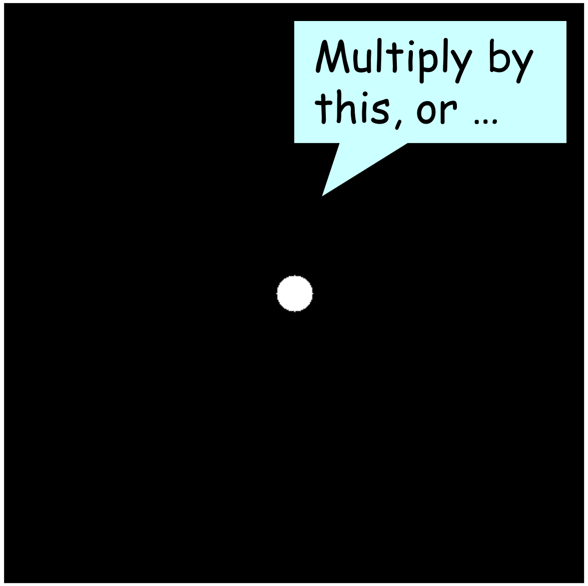
- High frequencies are enhanced or amplified.
- Individual frequency components are multiplied by an increasing function of ω such as $\alpha\omega = \alpha\sqrt{(u^2+v^2)}$, where α is a constant.

The values of the output image could be negative.

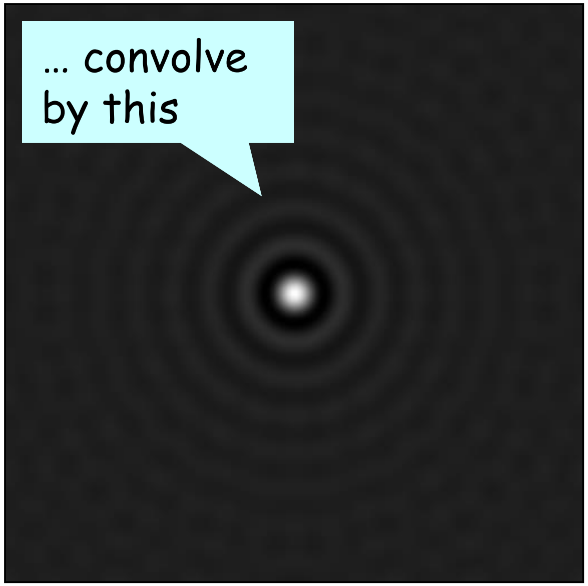
Ideal Lowpass Filter

Ideal Lowpass Filter

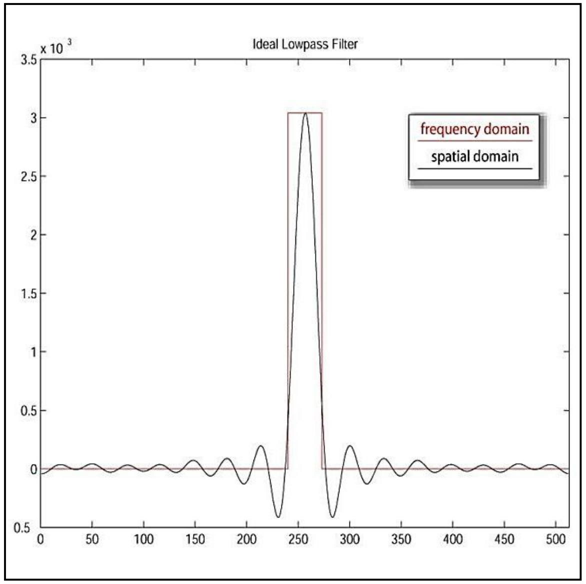
Image size: 512x512
FD filter radius: 16



Fourier Domain Rep.



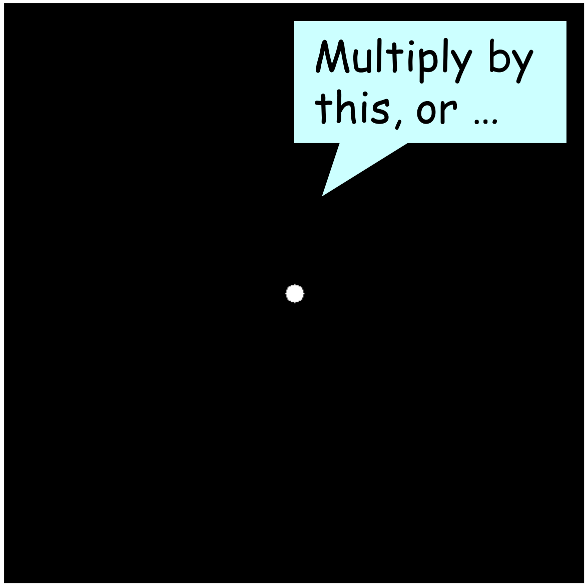
Spatial Domain Representation

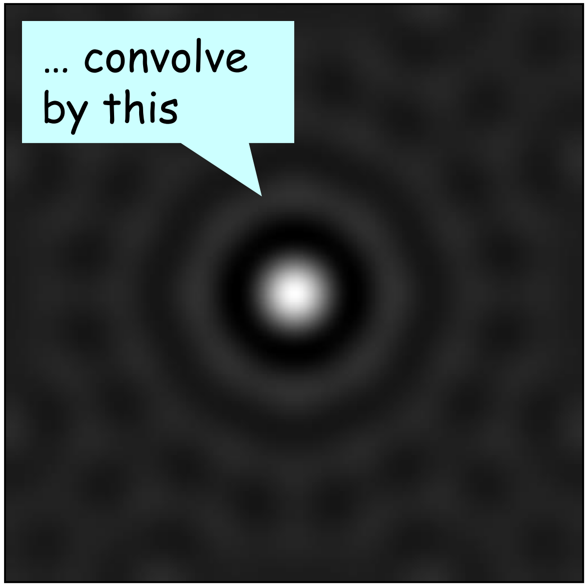


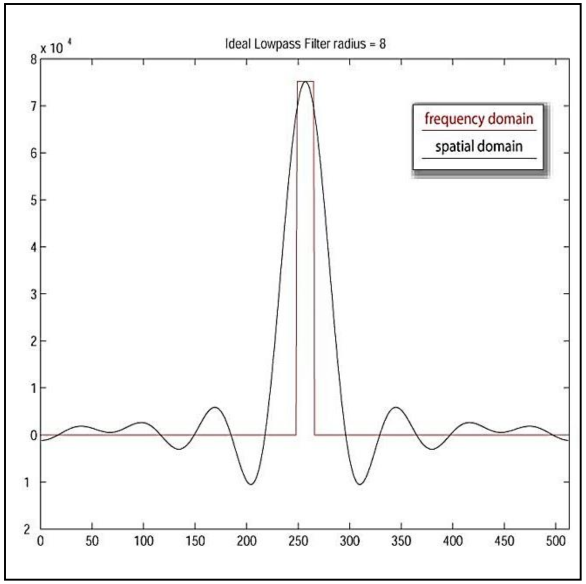
Central Profile

Ideal Lowpass Filter

Image size: 512x512
FD filter radius: 8





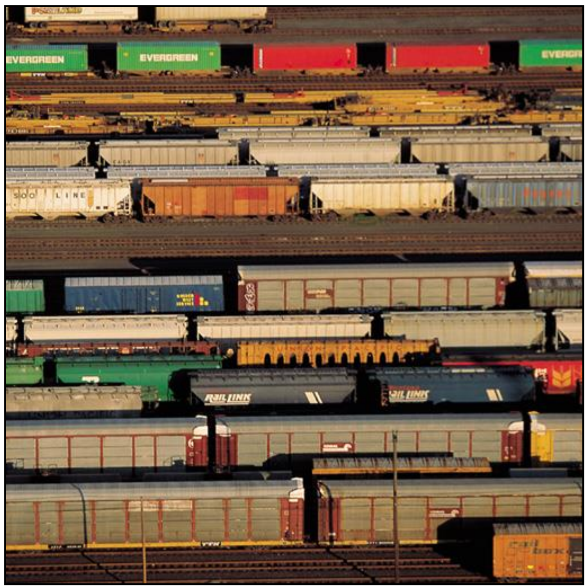


Fourier Domain Rep.

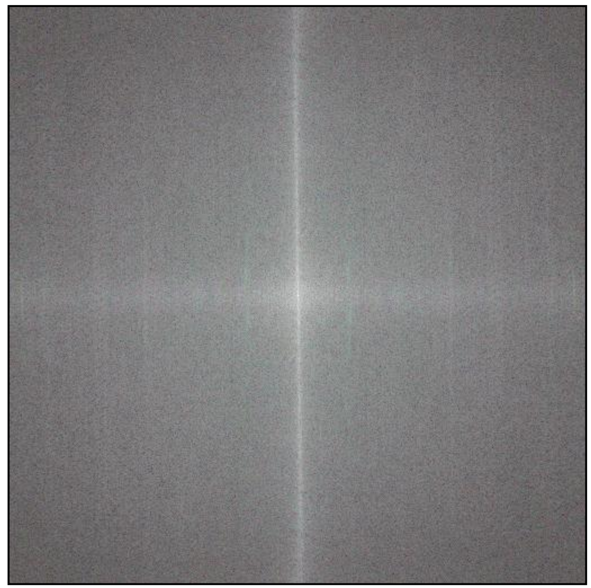
Spatial Domain Representation

Central Profile

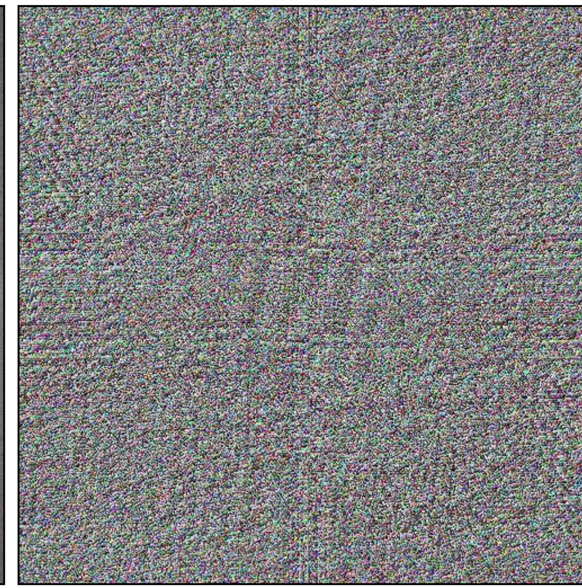
Power Spectrum and Phase of an Image



Original Image



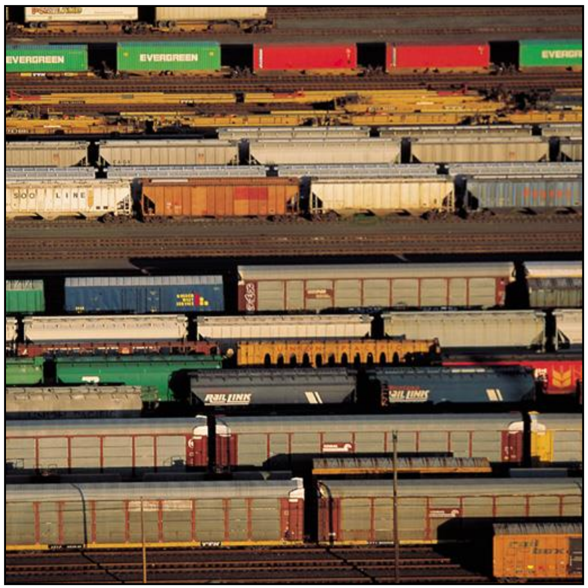
Power Spectrum



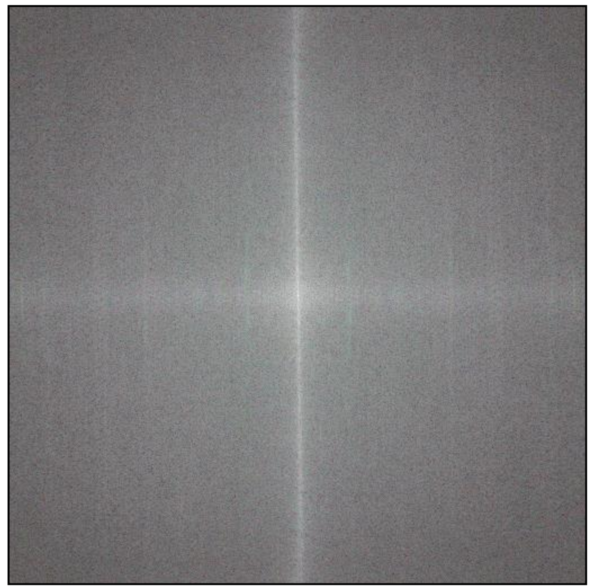
Phase

Ideal Lowpass Filter

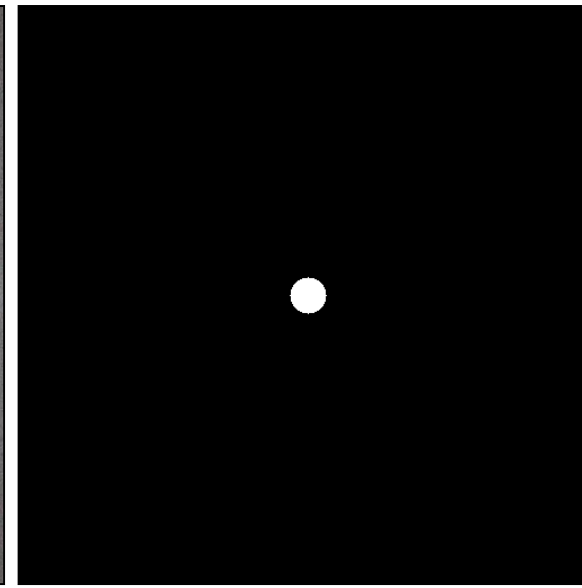
Image size: 512x512
FD filter radius: 16



Original Image



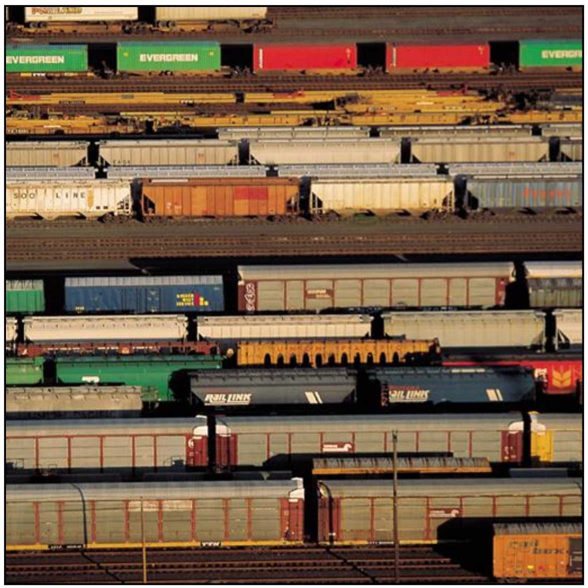
Power Spectrum

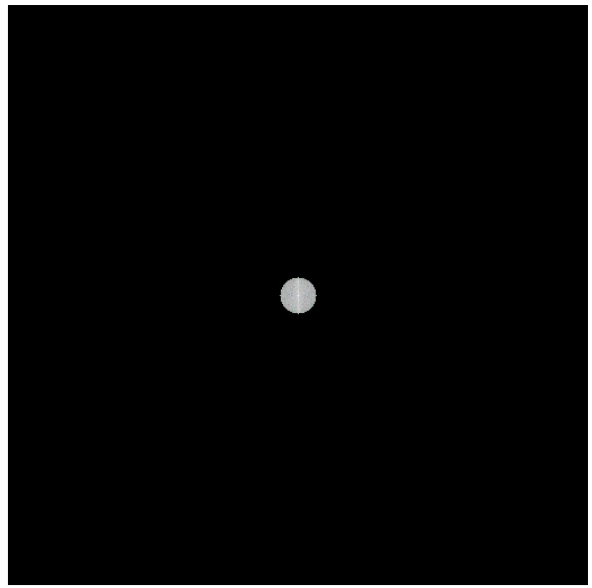


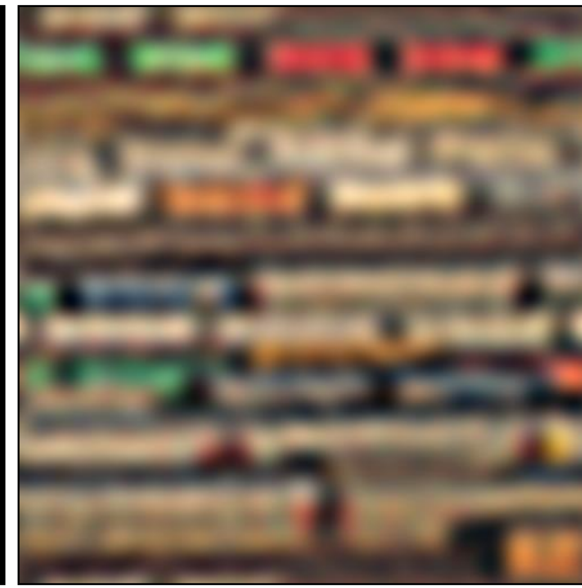
Ideal LPF in FD

Ideal Lowpass Filter

Image size: 512x512
FD filter radius: 16







Original Image

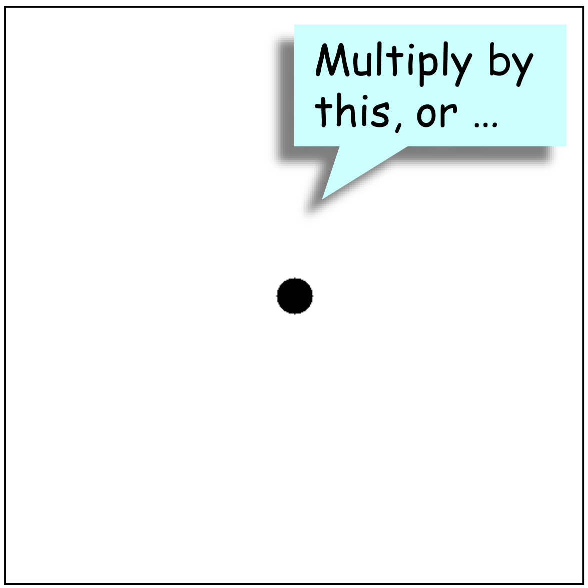
Filtered Power Spectrum

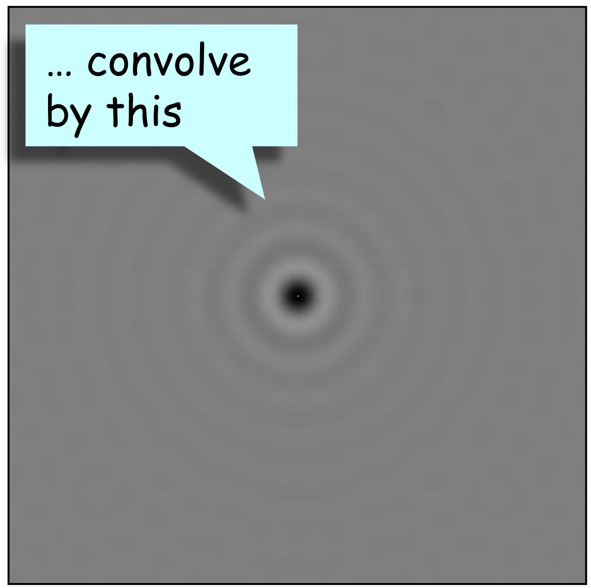
Filtered Image

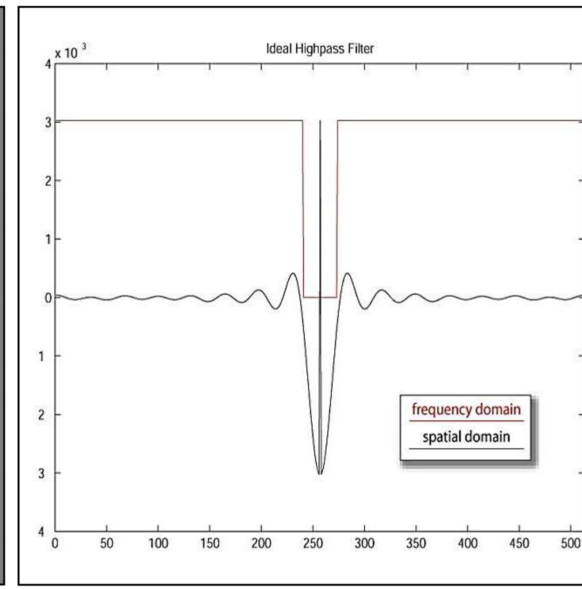
Ideal Highpass Filter

Ideal Highpass Filter

Image size: 512x512
FD notch radius: 16







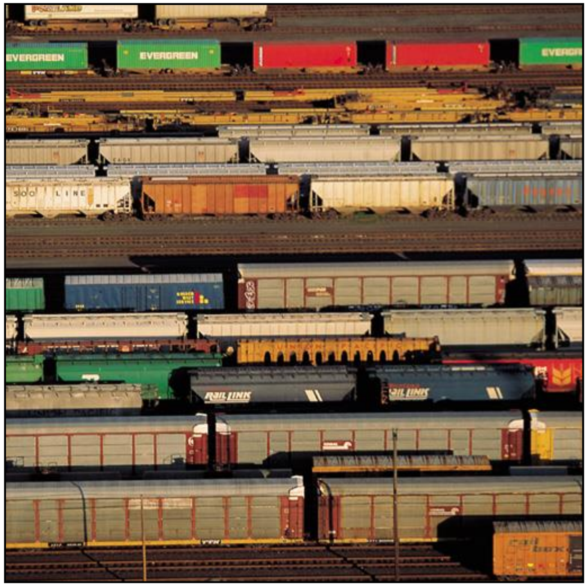
Fourier Domain Rep.

Spatial Representation

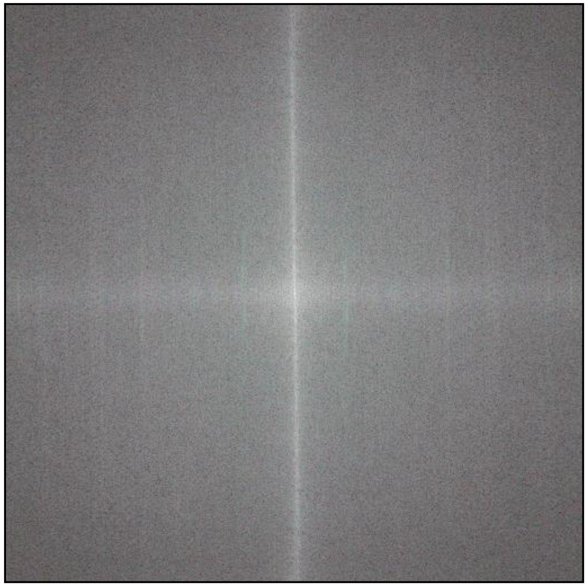
Central Profile

Ideal Highpass Filter

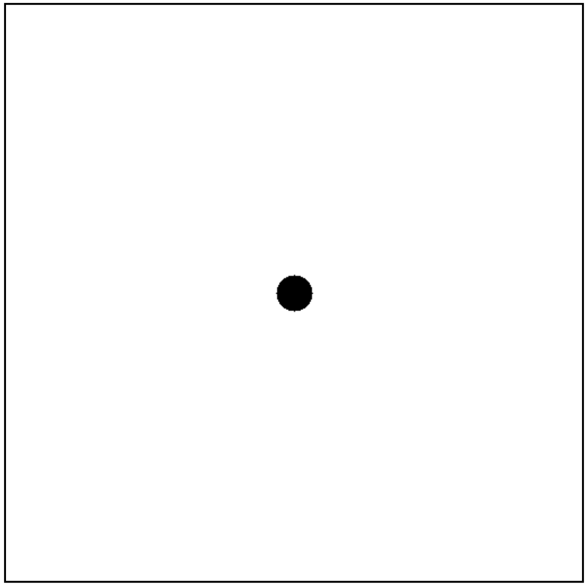
Image size: 512x512
FD notch radius: 16



Original Image



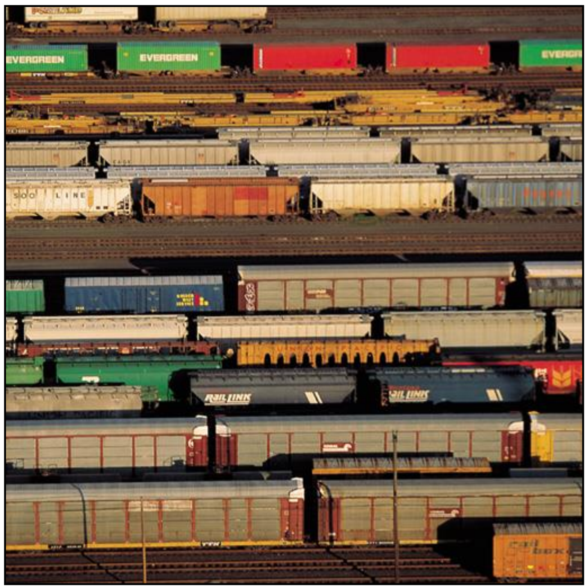
Power Spectrum

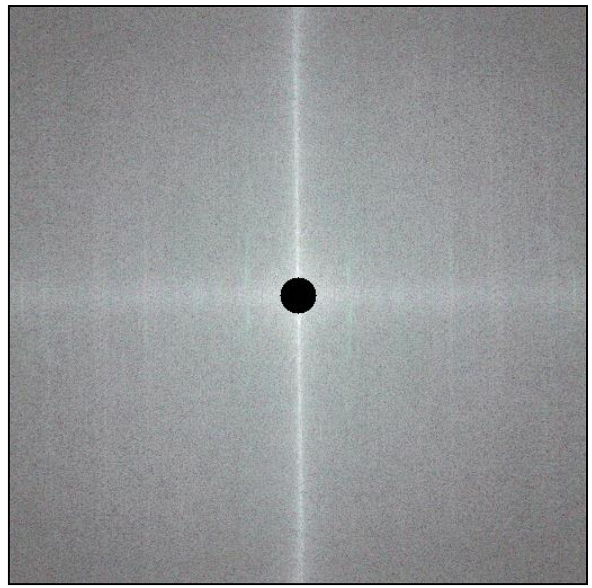


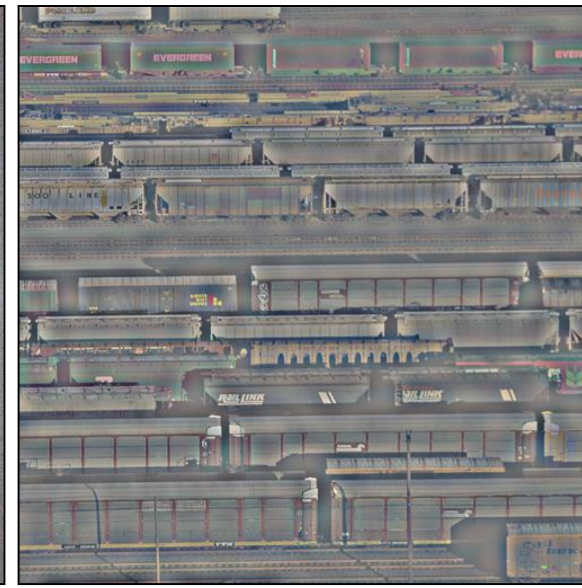
Ideal HPF in FD

Ideal Highpass Filter

Image size: 512x512
FD notch radius: 16







Original Image

Filtered Power Spectrum

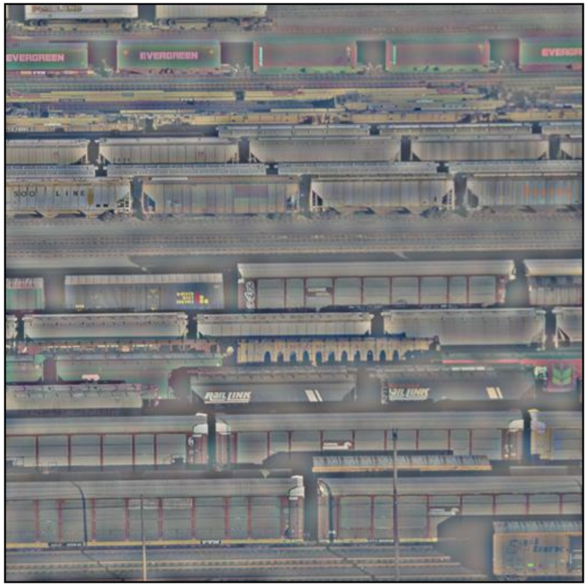
Filtered Image*

*signed image: 0 mapped to 128

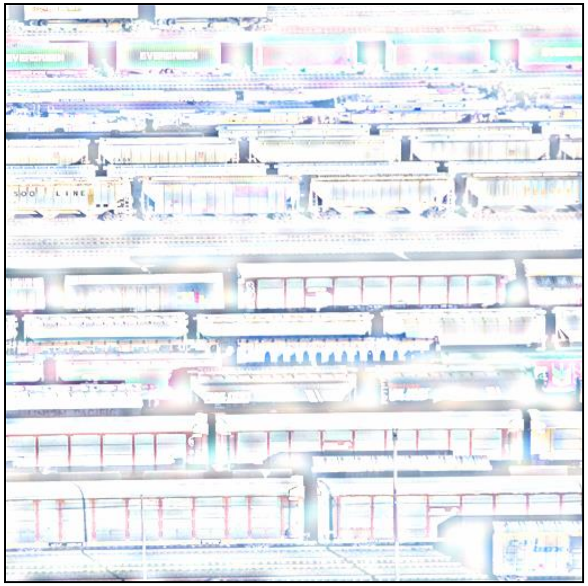
Ideal Highpass Filter

Image size: 512x512
FD notch radius: 16

Positive Pixels



Filtered Image*



Negative Pixels

*signed image: 0 mapped to 128

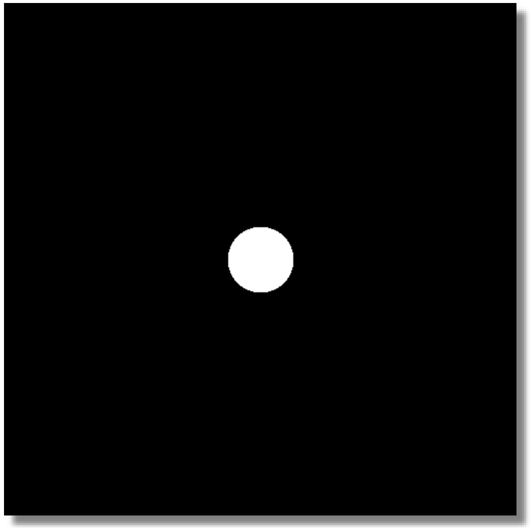
Ideal Bandpass Filter

Ideal Bandpass Filter

A bandpass filter is created by

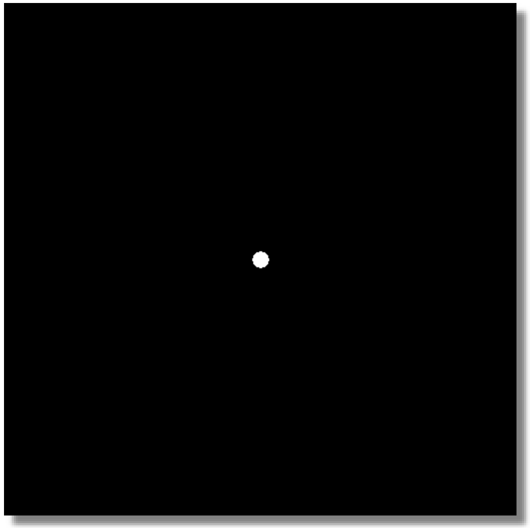
(1) subtracting a FD radius ρ_2 lowpass filtered image from a FD radius ρ_1 lowpass filtered image, where $\rho_2 < \rho_1$, or

(2) convolving the image with a matrix that is the difference of the two lowpass matrixs.



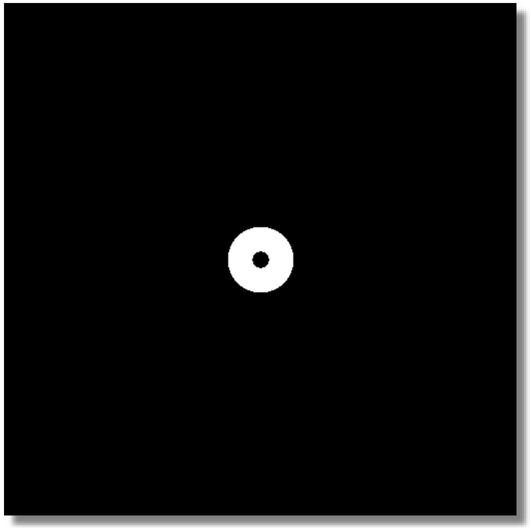
FD LP matrix with radius ρ_1

-



FD LP matrix with radius ρ_2

=



FD BP matrix ρ_1 - ρ_2

Ideal Bandpass Filter

image LPF radius ρ1

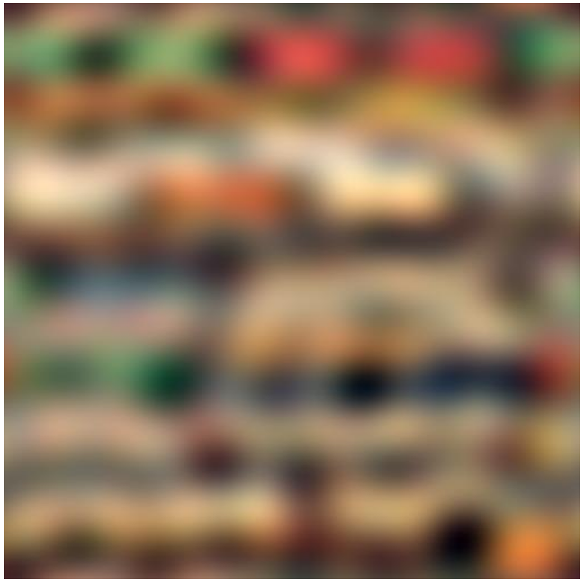


image LPF radius ρ2

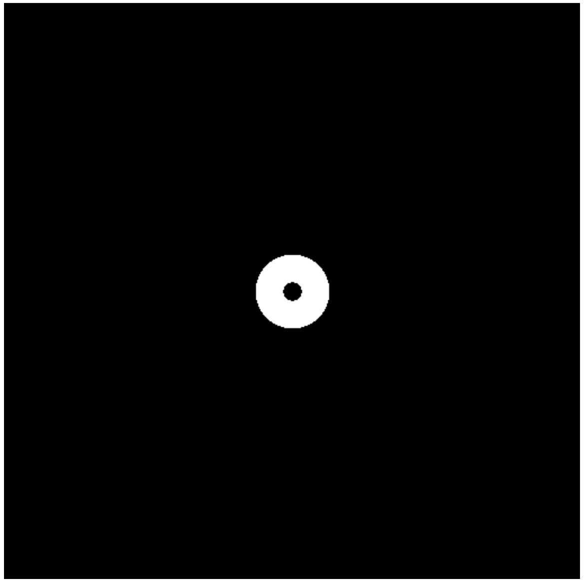
image BPF radii ρ1, ρ2*

*signed image: 0 mapped to 128

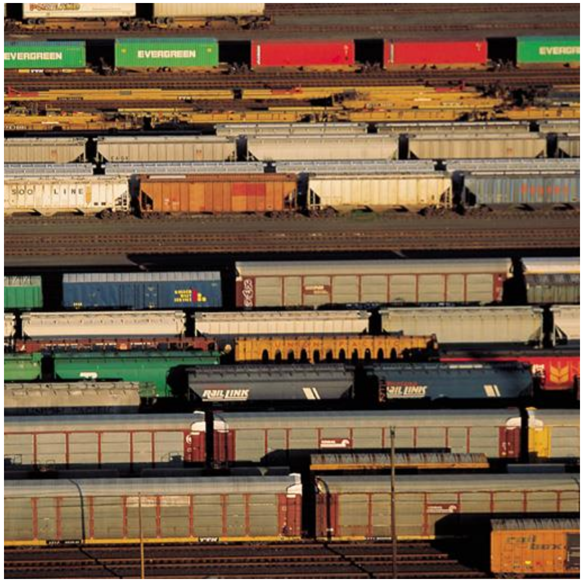
Ideal Bandpass Filter

filtered image*

*signed image: 0 mapped to 128

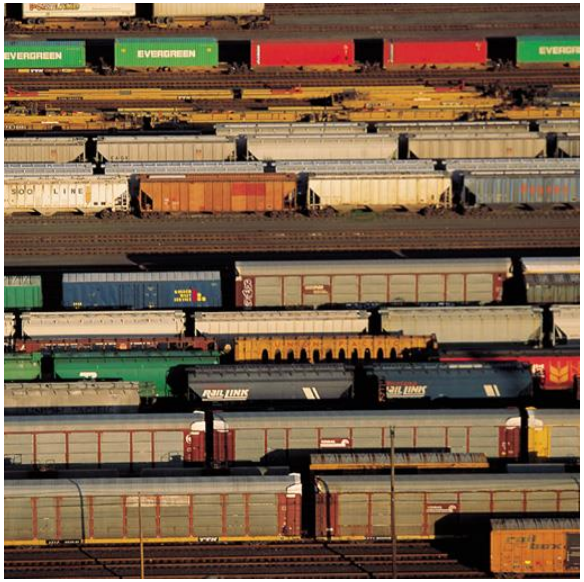


filter power spectrum

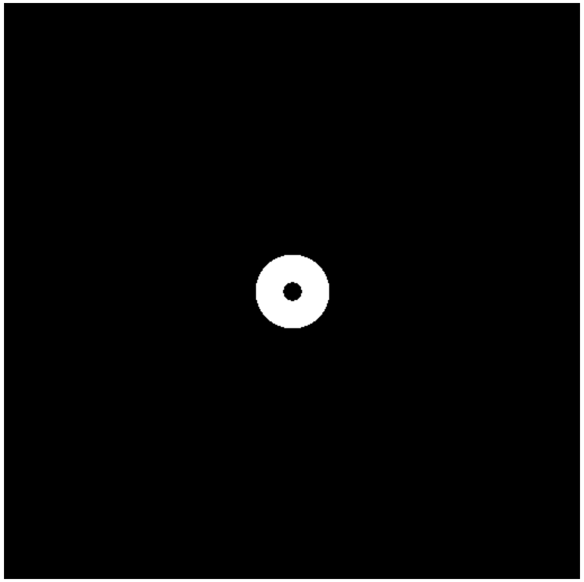


original image

Ideal Bandpass Filter



original image

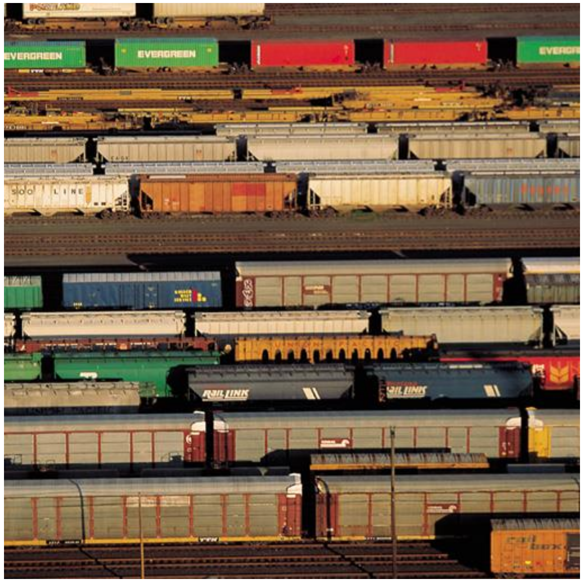


filter power spectrum

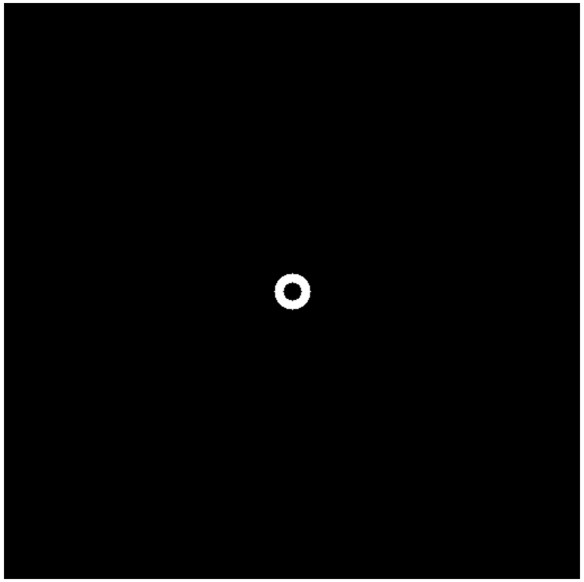
filtered image*

*signed image: 0 mapped to 128

A Different Ideal Bandpass Filter



original image



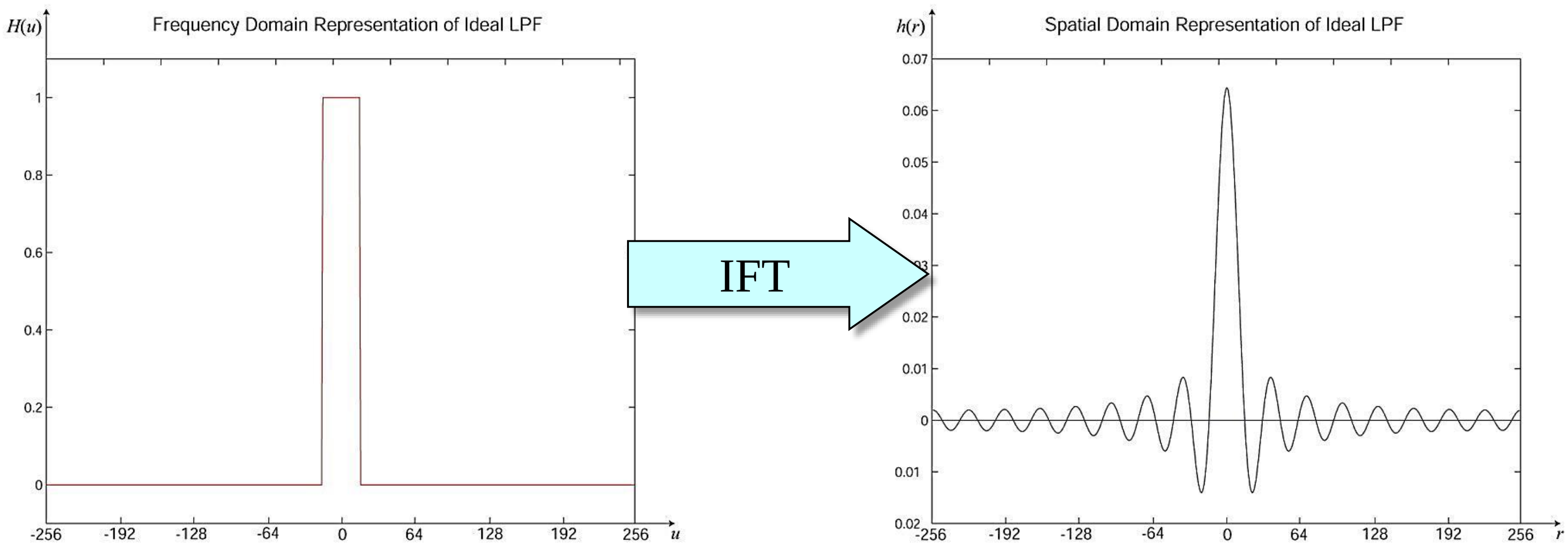
filter power spectrum

filtered image*

*signed image: 0 mapped to 128

Gaussian Lowpass Filter

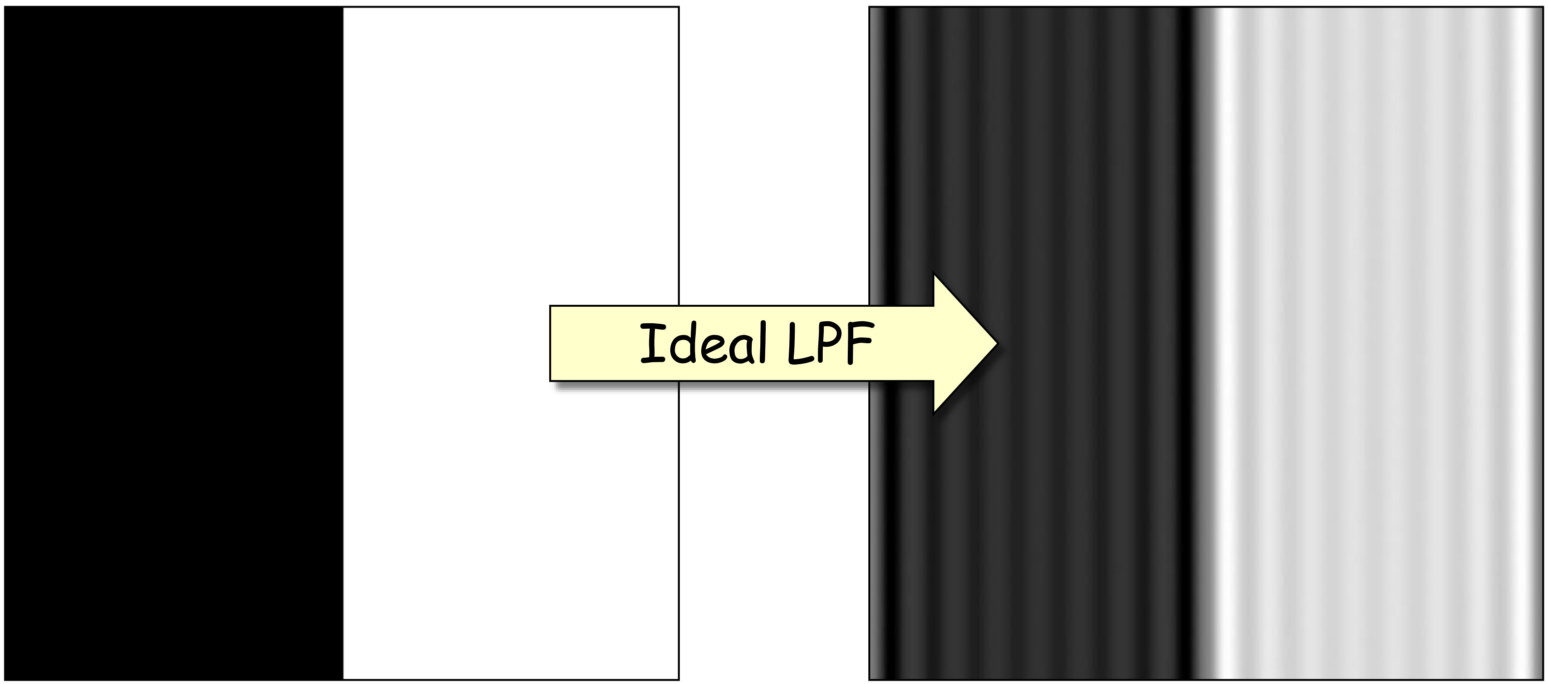
Ideal Filters Do Not Produce Ideal Results



A sharp cutoff in the frequency domain…

…causes ringing in the spatial domain.

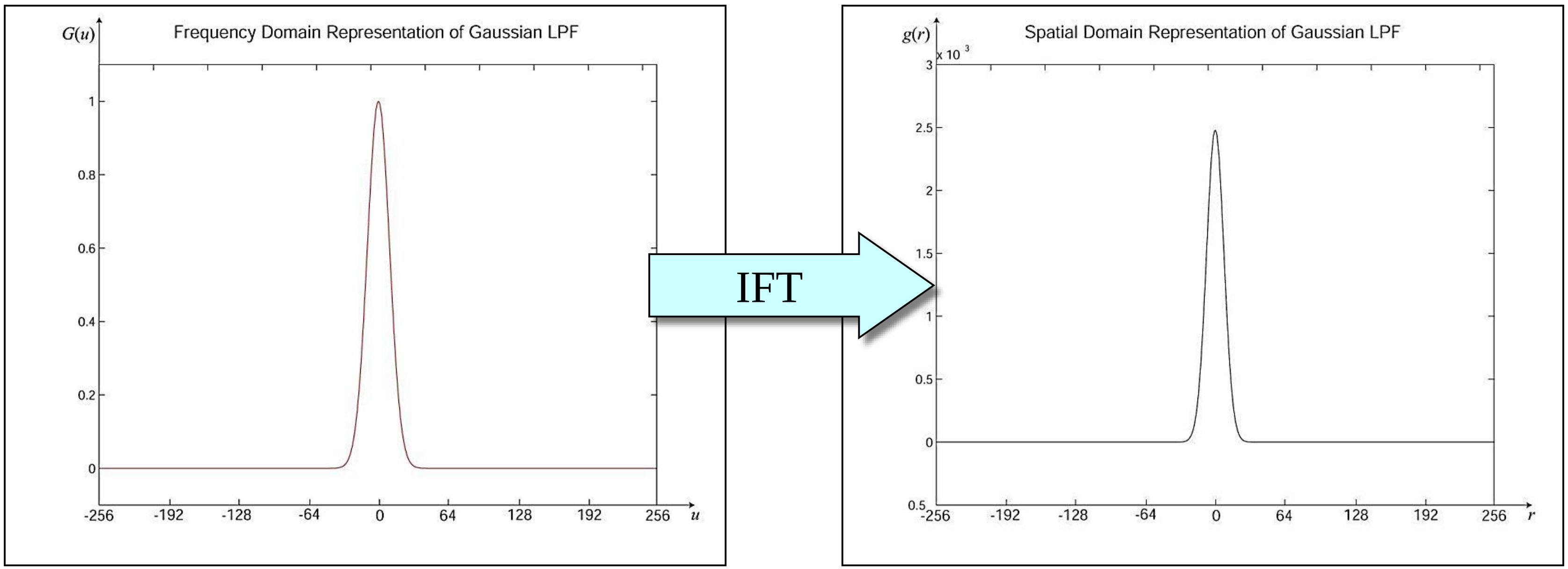
Ideal Filters Do Not Produce Ideal Results



Blurring the image above w/ an ideal lowpass filter…

…distorts the results with ringing or ghosting.

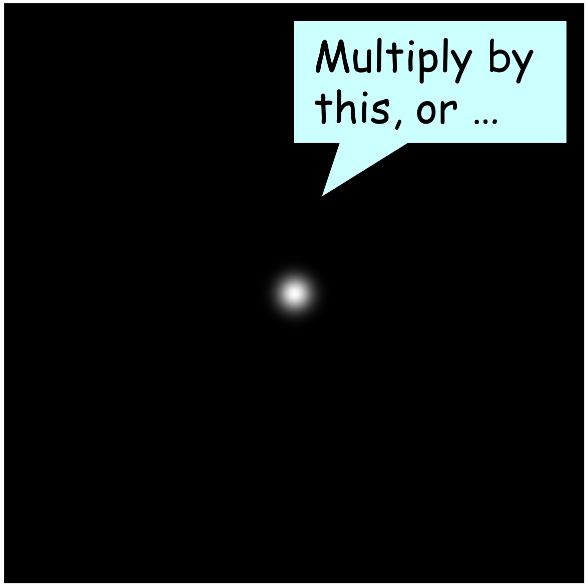
Optimal Filter: The Gaussian

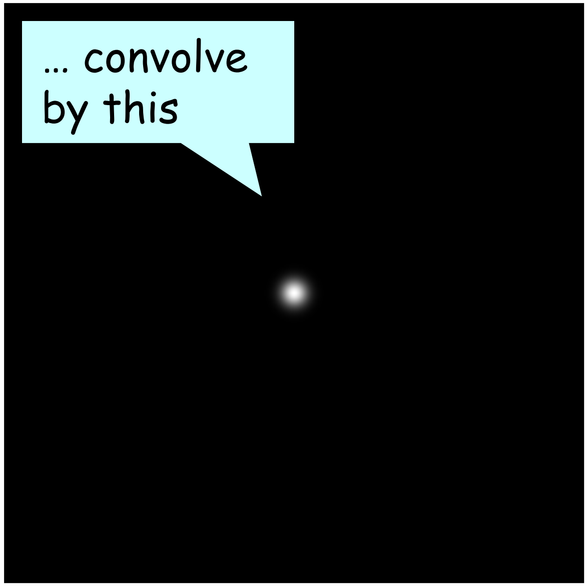


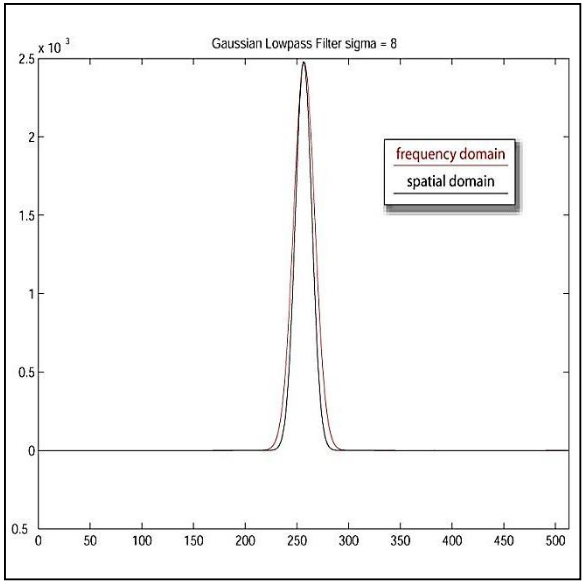
The Gaussian filter optimizes the uncertainty relation. It provides the sharpest cutoff possible without ringing.

Gaussian Lowpass Filter

Image size: 512x512
SD filter sigma = 8







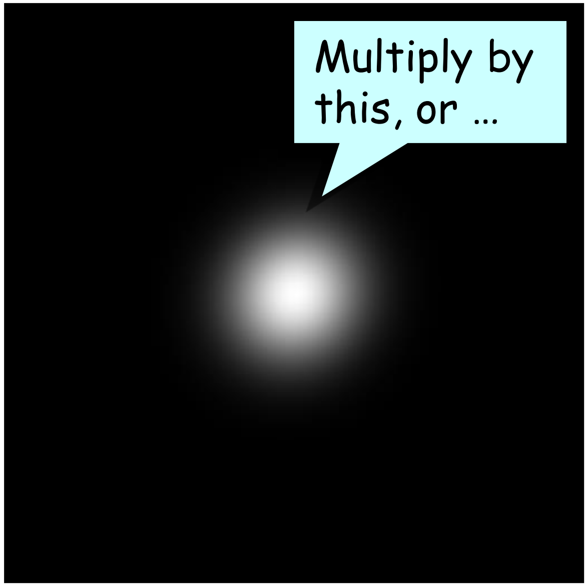
Fourier Domain Rep.

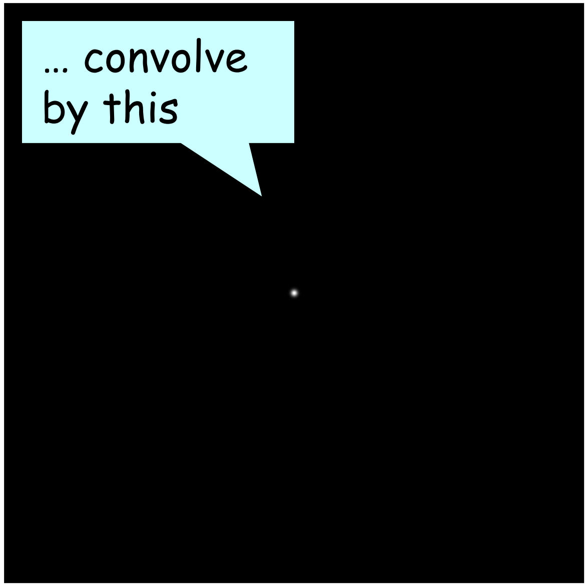
Spatial Representation

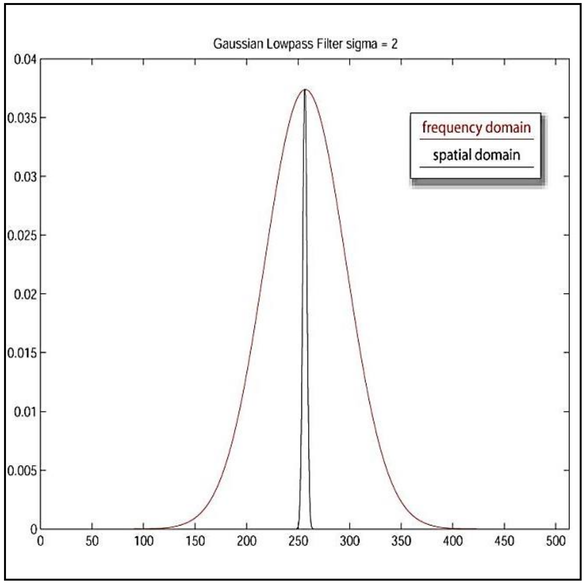
Central Profile

Gaussian Lowpass Filter

Image size: 512x512
SD filter sigma = 2







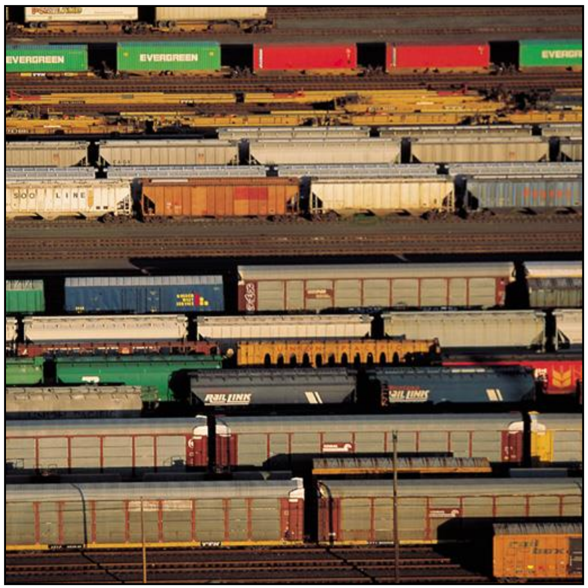
Fourier Domain Rep.

Spatial Representation

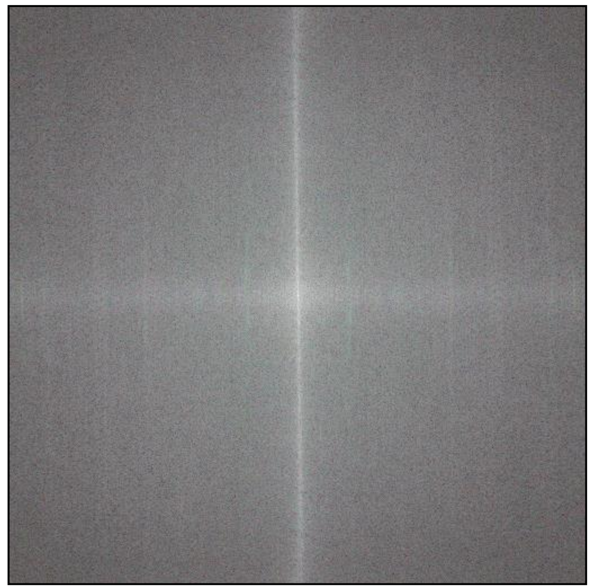
Central Profile

Gaussian Lowpass Filter

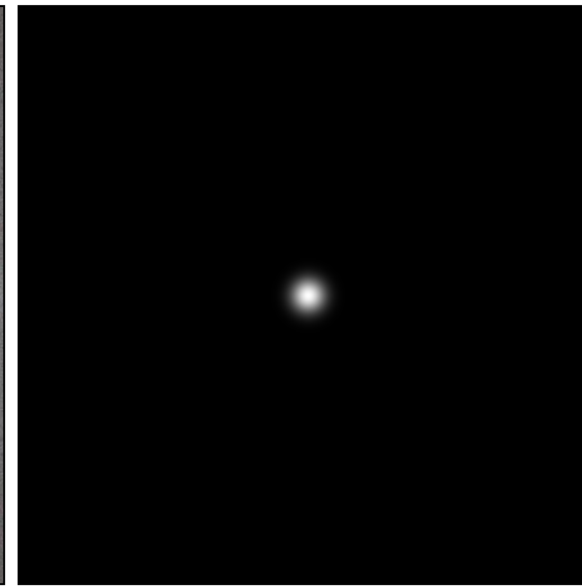
Image size: 512x512
SD filter sigma = 8



Original Image



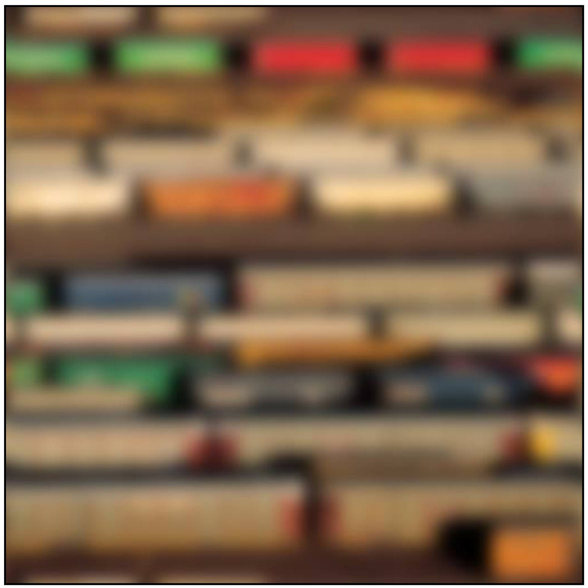
Power Spectrum

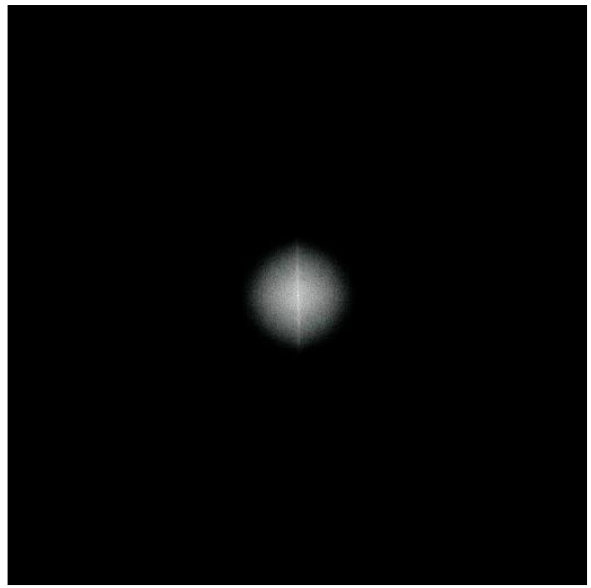


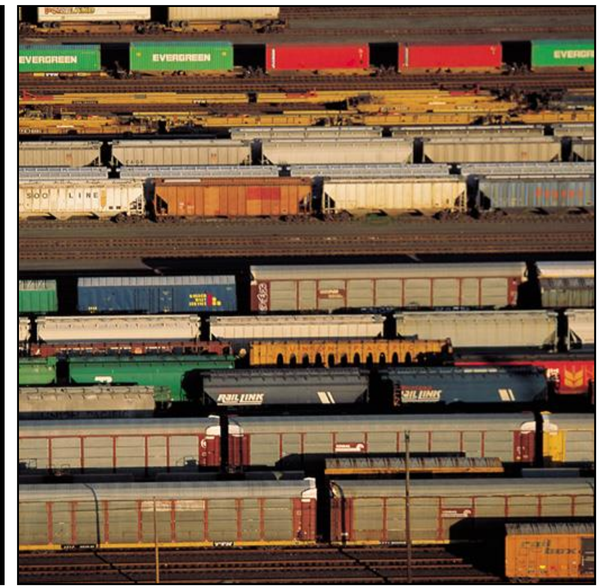
Gaussian LPF in FD

Gaussian Lowpass Filter

Image size: 512x512
SD filter sigma = 8







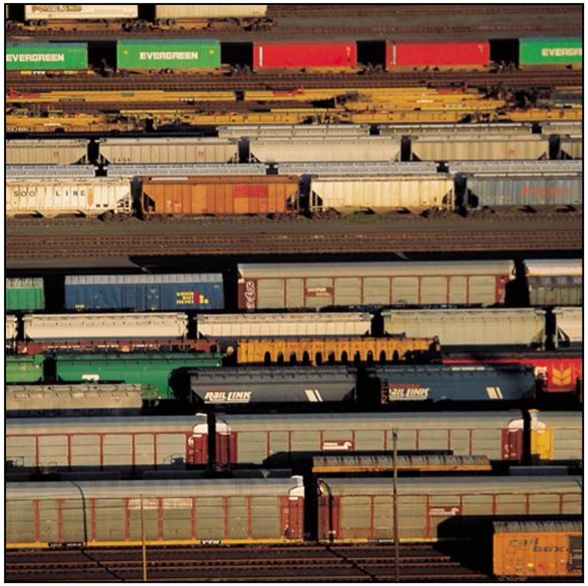
Filtered Image

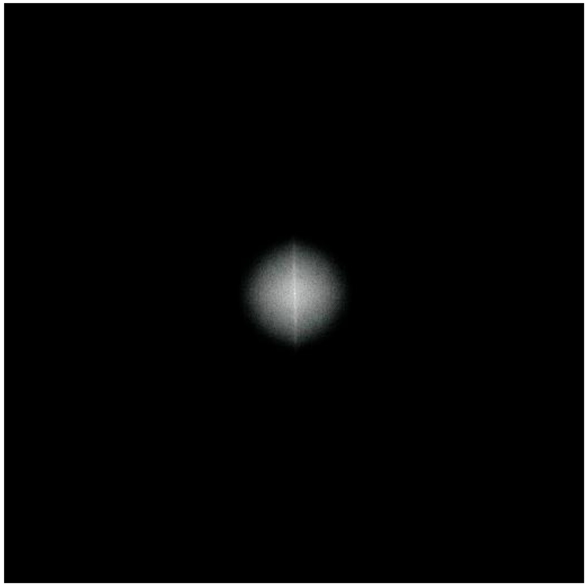
Filtered Power Spectrum

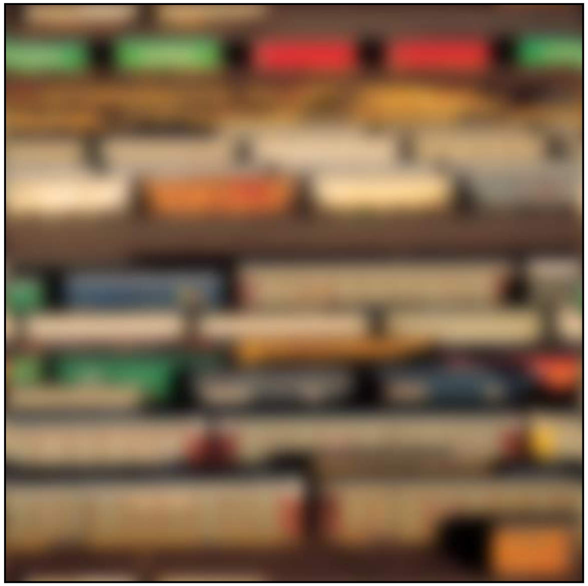
Original Image

Gaussian Lowpass Filter

Image size: 512x512
SD filter sigma = 8







Original Image

Filtered Power Spectrum

Filtered Image

Resolution Sequence

Original Image

$\sigma_0 = 0$

Resolution Sequence

Gaussian LPF

$\sigma_1 = 1$

Resolution Sequence

Gaussian LPF

$\sigma_2 = 2$

Resolution Sequence

Gaussian LPF

$\sigma_3 = 4$

Resolution Sequence

Gaussian LPF

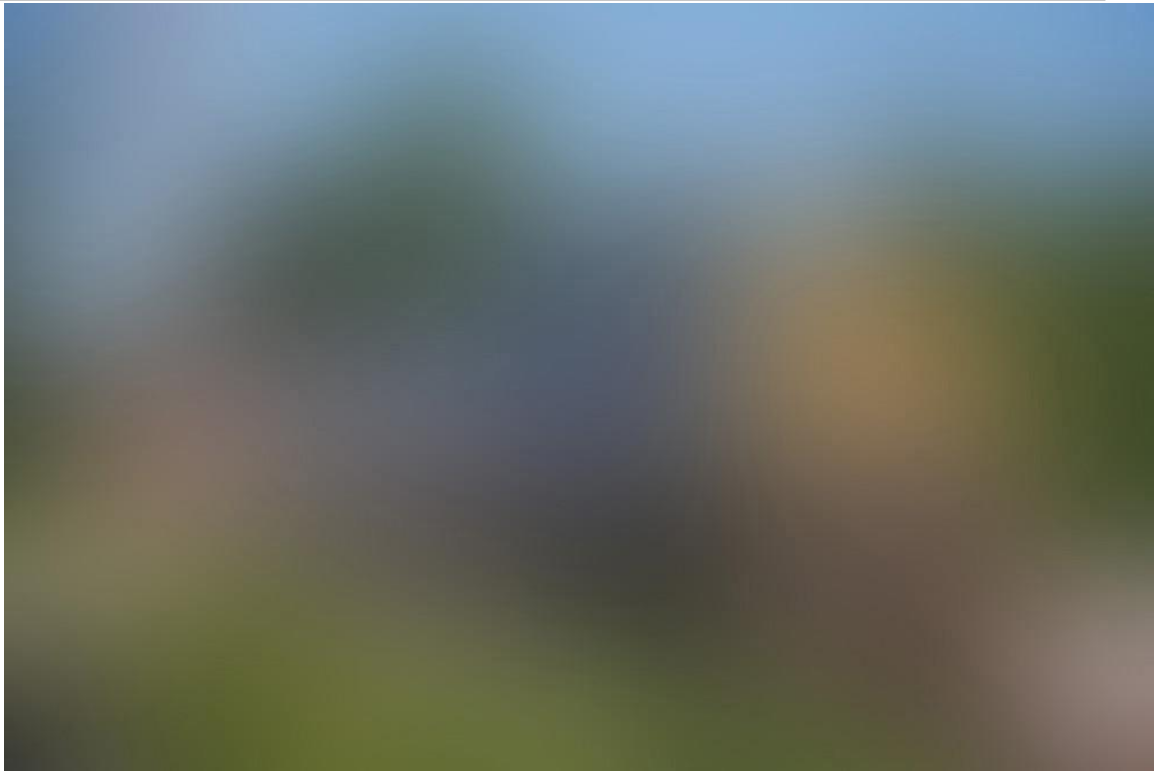
$\sigma_4 = 8$

Resolution Sequence

Gaussian LPF

$\sigma_5 = 16$

Resolution Sequence



Gaussian LPF

$\sigma_6 = 32$

Resolution Sequence

Gaussian LPF

$\sigma_7 = 64$

Resolution Sequence

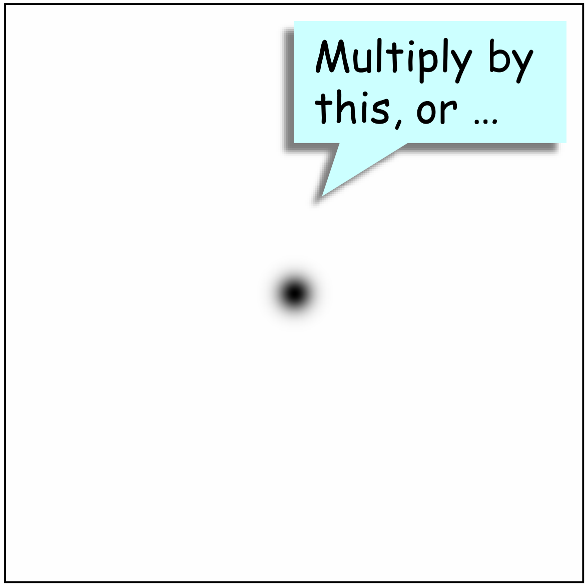
Gaussian LPF

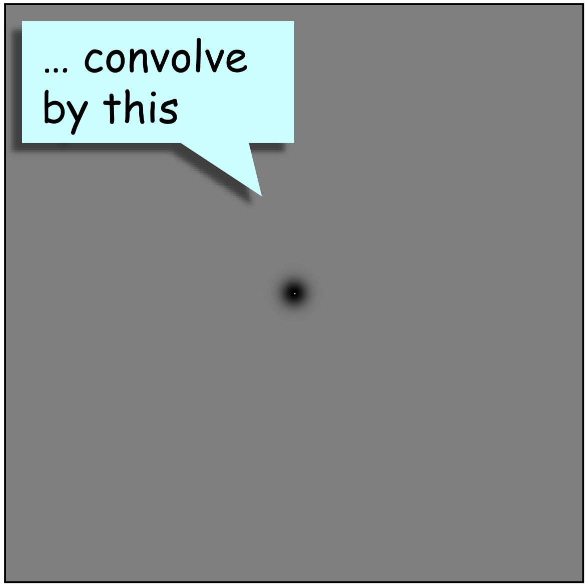
$\sigma_8 = 128$

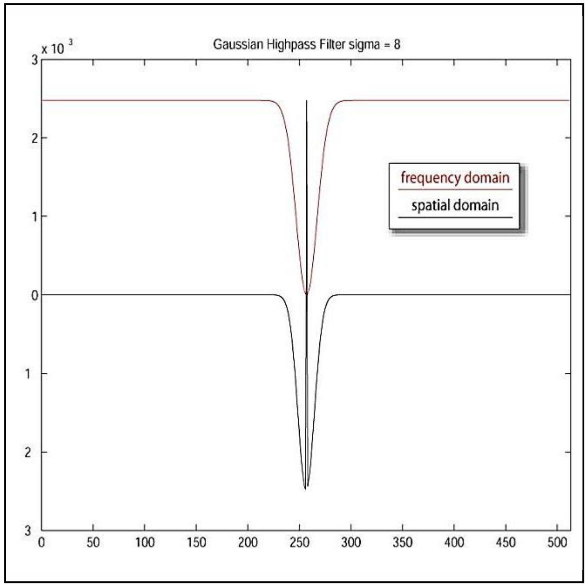
Gaussian Highpass Filter

Gaussian Highpass Filter

Image size: 512x512
FD notch sigma = 8







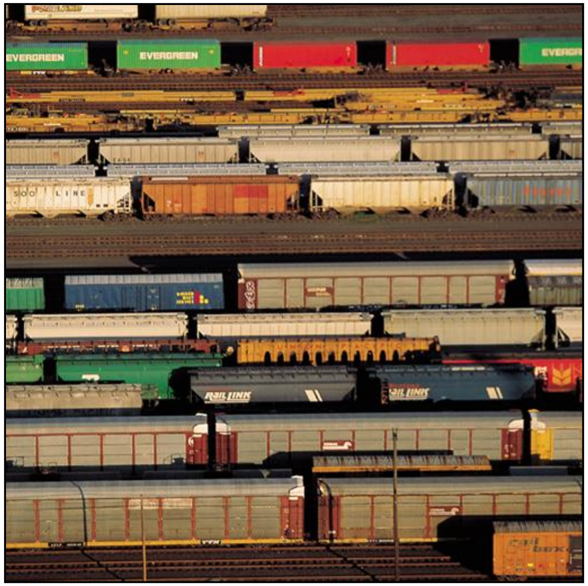
Fourier Domain Rep.

Spatial Representation

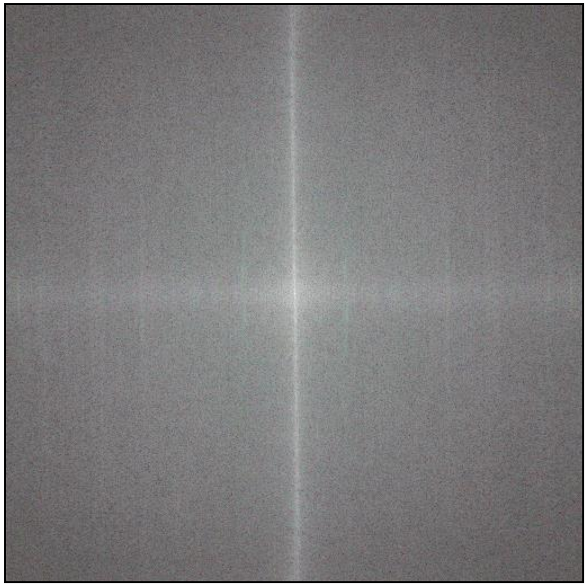
Central Profile

Gaussian Highpass Filter

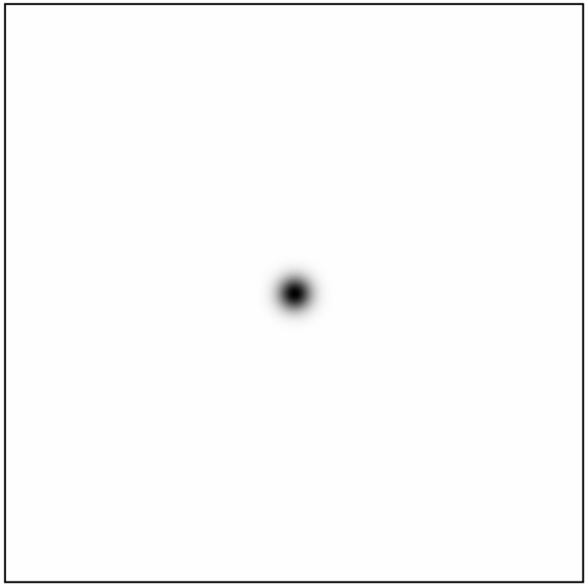
Image size: 512x512
FD notch sigma = 8



Original Image



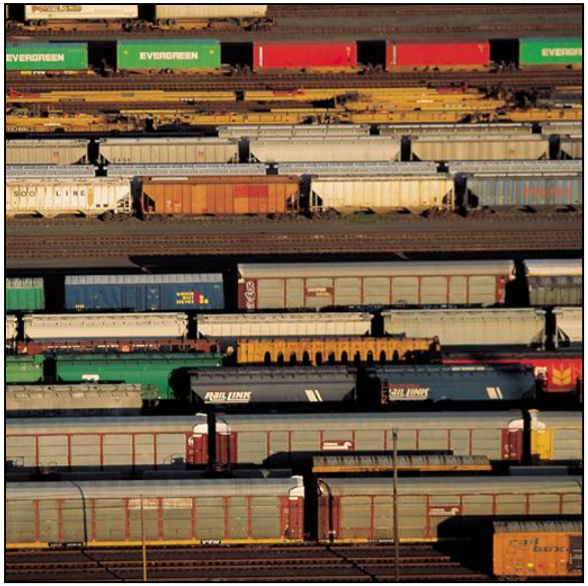
Power Spectrum

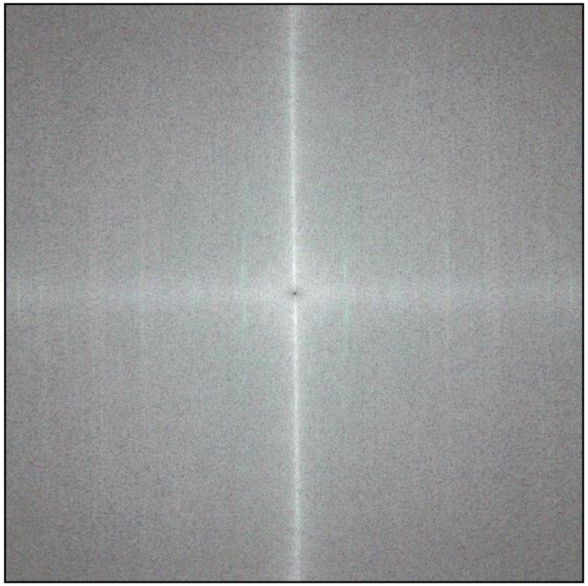


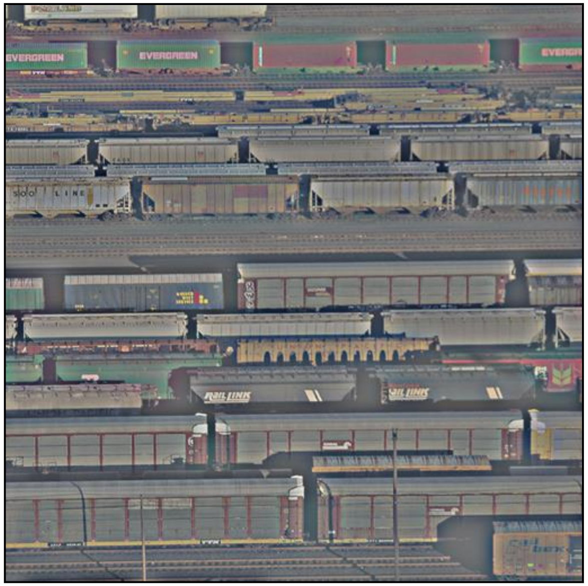
Gaussian HPF in FD

Gaussian Highpass Filter

Image size: 512x512
FD notch sigma = 8







Original Image

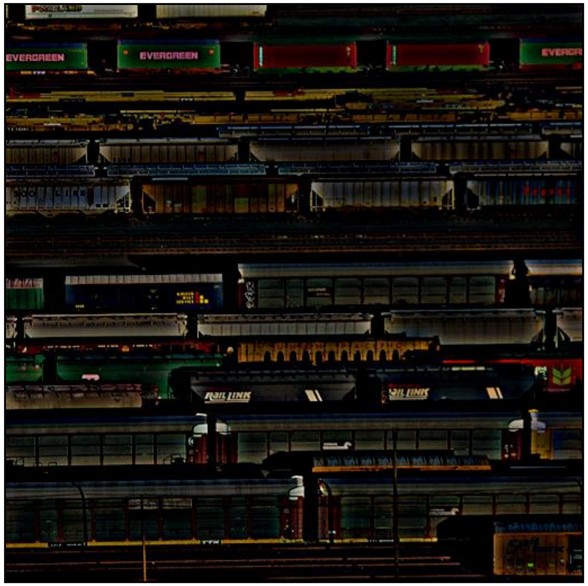
Filtered Power Spectrum

Filtered Image*

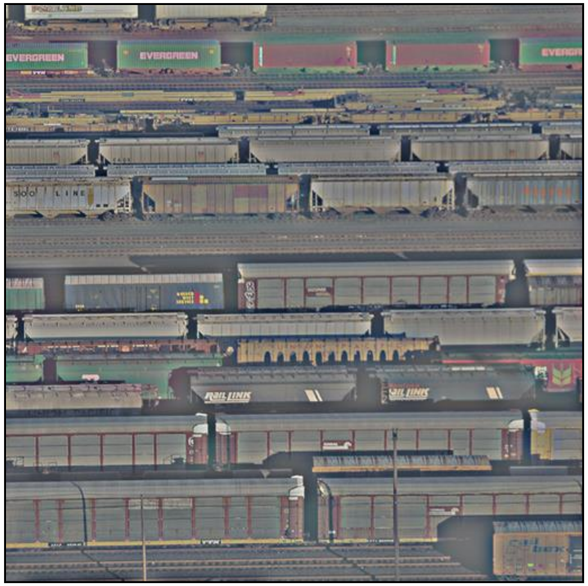
*signed image: 0 mapped to 128

Gaussian Highpass Filter

Image size: 512x512
FD notch sigma = 8

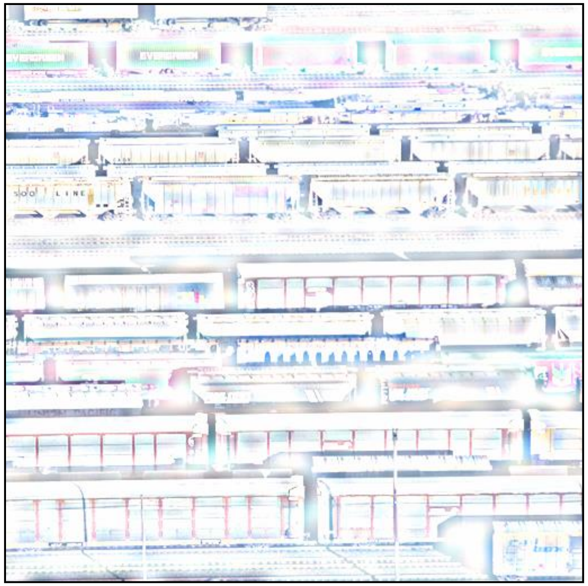


Positive Pixels



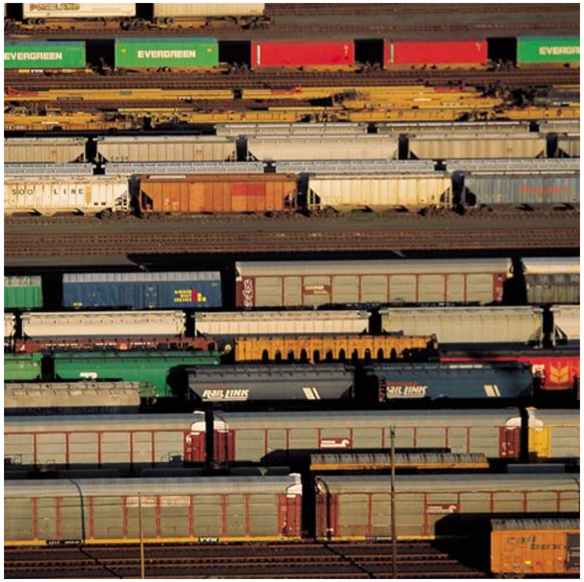
Filtered Image*

*signed image: 0 mapped to 128

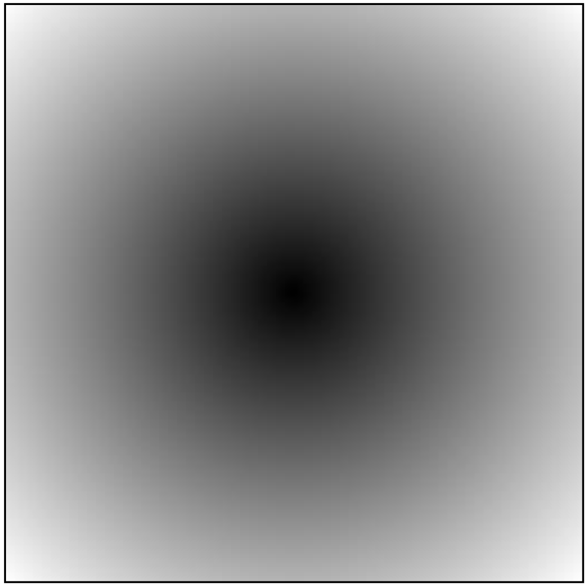


Negative Pixels

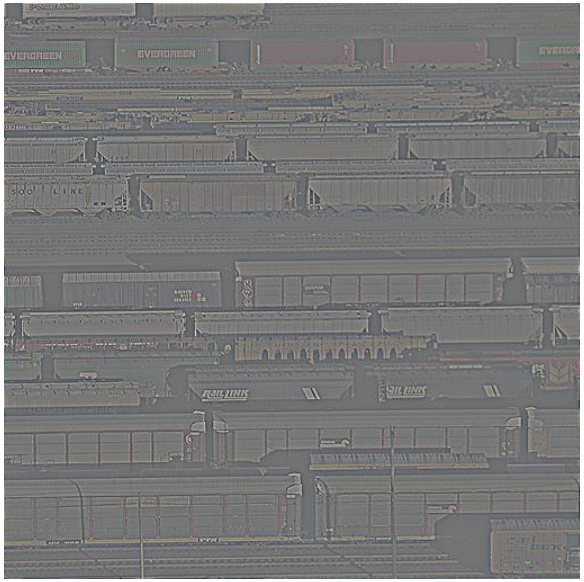
Another Gaussian Highpass Filter



Original Image



Filter Power Spectrum



Filtered Image*

*signed image: 0 mapped to 128

Highpass Sequence

Difference between original image and Gaussian LPF image at $\sigma_9 = 256$.

$$\mathbf{J} = \mathbf{I} - \left[\mathbf{I} * g\left(\sigma_9\right) \right].$$

Highpass Sequence

Difference between original image and Gaussian LPF image at $\sigma_8 = 128$.

$$\mathbf{J} = \mathbf{I} - \left[\mathbf{I} * g\left(\sigma_8\right)\right].$$

Highpass Sequence

Difference between original image and Gaussian LPF image at $\sigma_7 = 64$.

$$\mathbf{J} = \mathbf{I} - \left[\mathbf{I} * g\left(\sigma_7\right) \right].$$

Highpass Sequence

Difference between original image and Gaussian LPF image at $\sigma_6 = 32$.

$$\mathbf{J} = \mathbf{I} - \left[\mathbf{I} * g\left(\sigma_6\right)\right].$$

Highpass Sequence

Difference between original image and Gaussian LPF image at $\sigma_5 = 16$.

$$\mathbf{J} = \mathbf{I} - \left[\mathbf{I} * g\left(\sigma_5\right)\right].$$

Highpass Sequence

Difference between original image and Gaussian LPF image at $\sigma_4 = 8$.

$$\mathbf{J} = \mathbf{I} - \left[\mathbf{I} * g\left(\sigma_4\right)\right].$$

Highpass Sequence

Difference between original image and Gaussian LPF image at $\sigma_3 = 4$.

$$\mathbf{J} = \mathbf{I} - \left[\mathbf{I} * g\left(\sigma_3\right)\right].$$

Highpass Sequence

Difference between original image and Gaussian LPF image at $\sigma_2 = 2$.

$$\mathbf{J} = \mathbf{I} - \left[\mathbf{I} * g\left(\sigma_2\right)\right].$$

Highpass Sequence

Difference between original image and Gaussian LPF image at $\sigma_1 = 1$.

$$\mathbf{J} = \mathbf{I} - \left[\mathbf{I} * g\left(\sigma_1\right)\right].$$

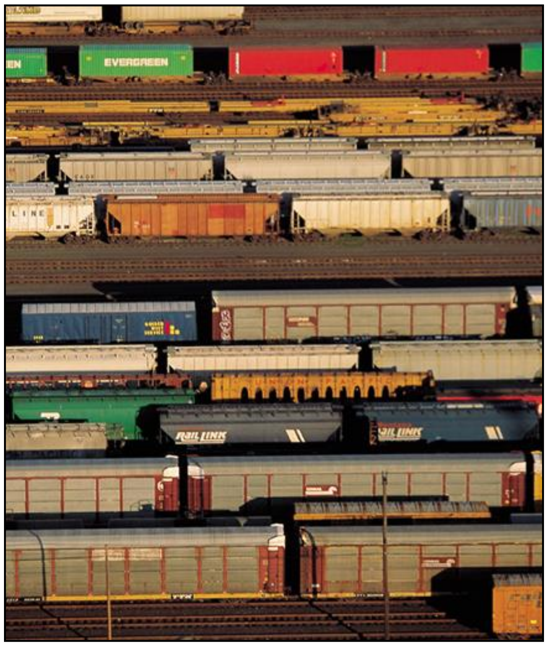
Highpass Sequence

Original Image

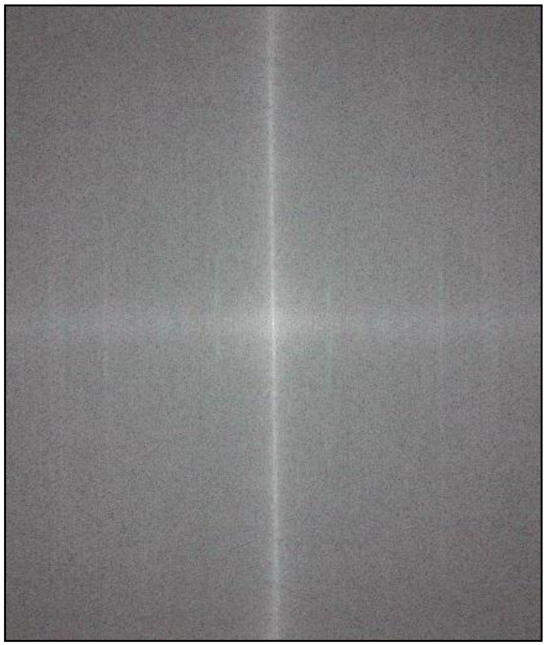
$\sigma_0 = 0$

Effects on Power Spectrum

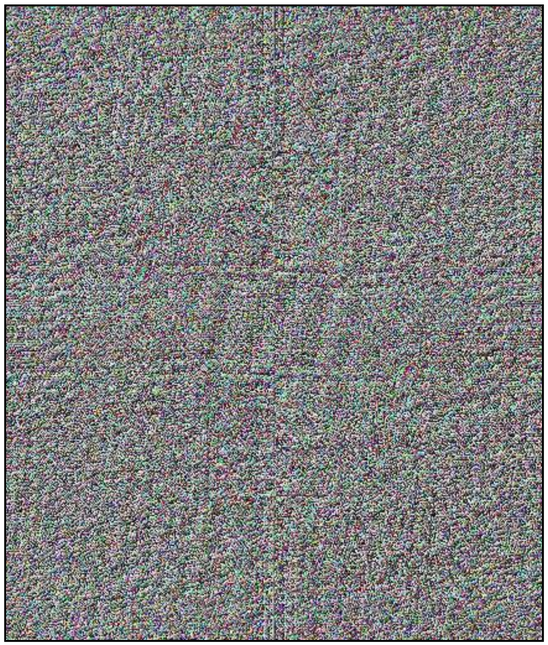
Power Spectrum and Phase of an Image



original image

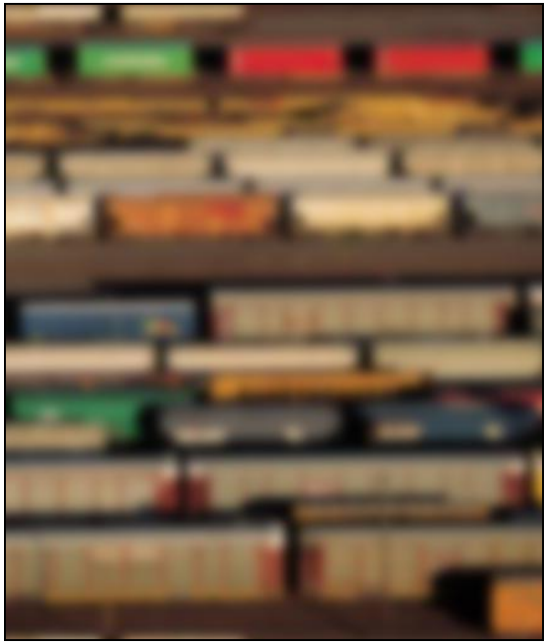


power spectrum

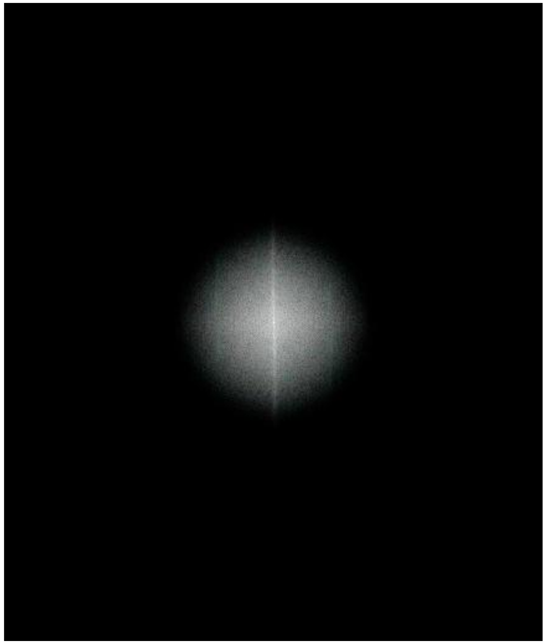


phase

Power Spectrum and Phase of a Blurred Image



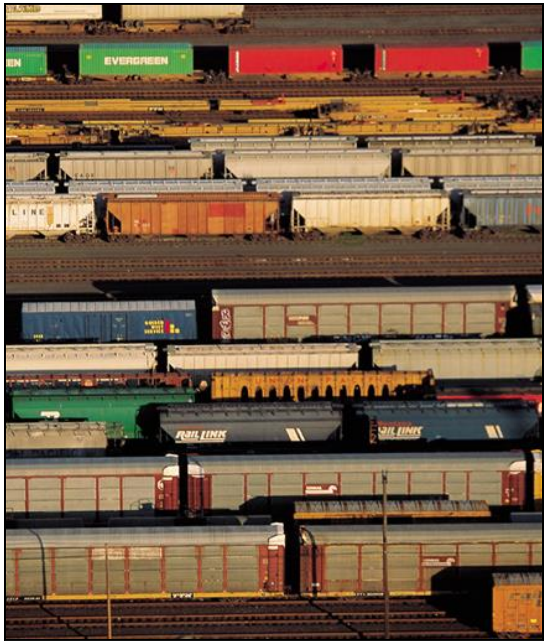
blurred image



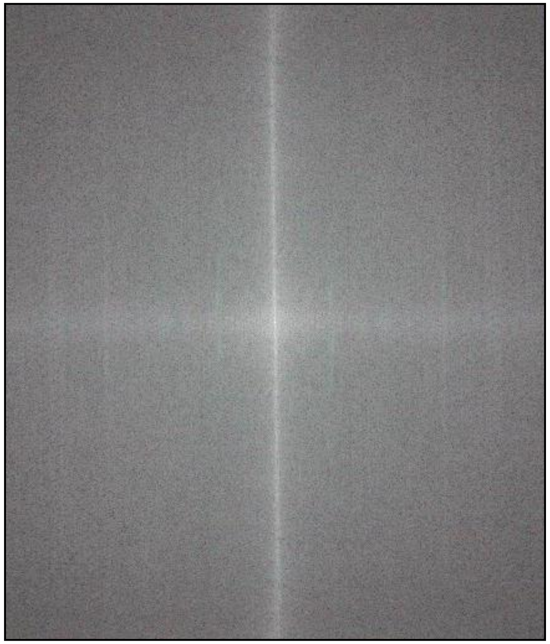
power spectrum

phase

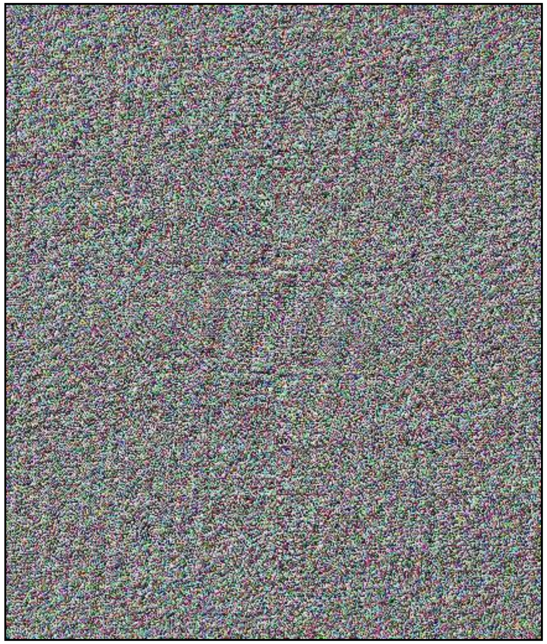
Power Spectrum and Phase of an Image



original image

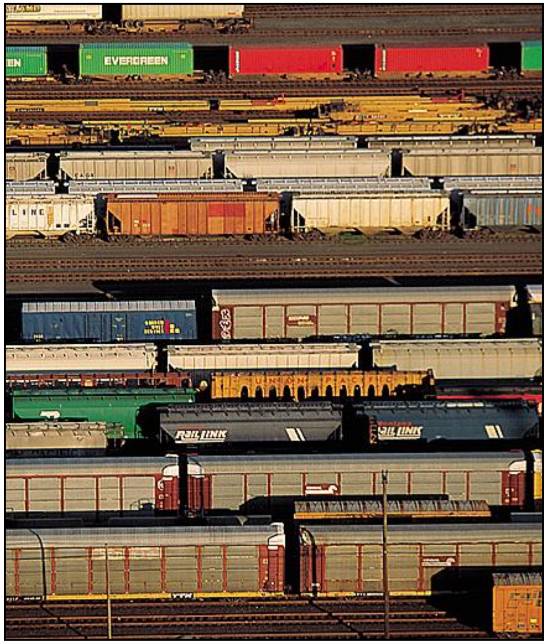


power spectrum

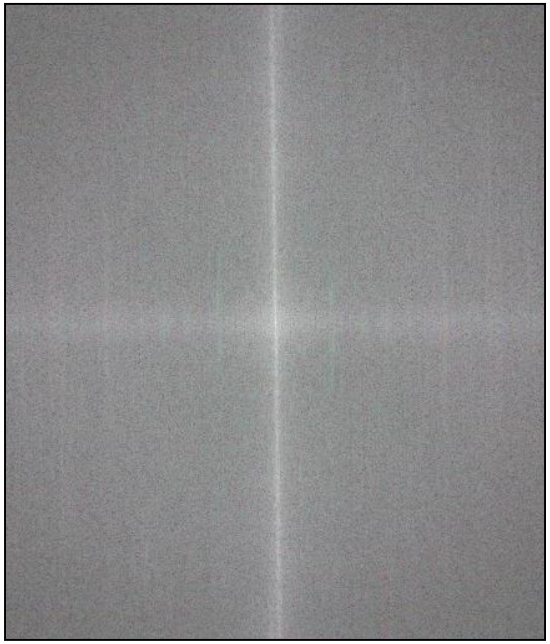


phase

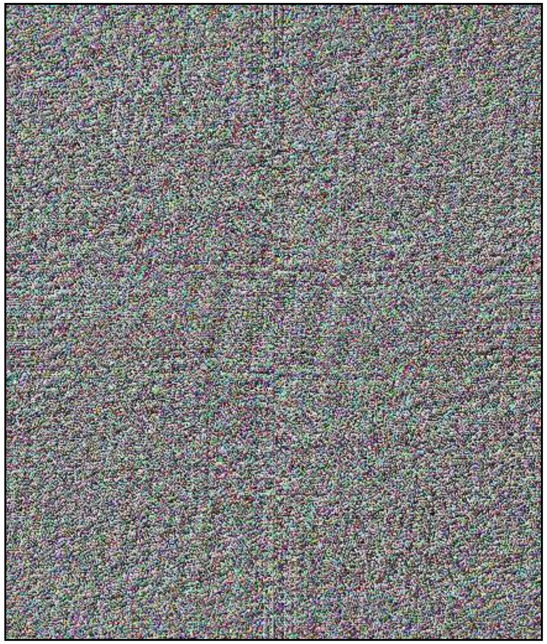
Power Spectrum and Phase of a Sharpened Image



sharpened image



power spectrum



phase

Learn more about FT on image processing

http://homepages.inf.ed.ac.uk/rbf/HIPR2/fourier.htm

Q&A