Quantum Computing for the Quantum Curious

2.0 What Is a Qubit?

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Classical Bit

Classical bits are the fundamental units of information in classical computing. They can take one of two values, 0 or 1.

Computer hardware interprets a bit of 1 as an electrical current flowing through a wire, and a bit of 0 as the absence of current. This binary representation is the basis for all operations in classical computers.

Quantum Bit (Qubit)

Quantum bits, or qubits, are similar to bits but can exist in a superposition of states.

Qubits can be in the basis state $|0\rangle$, $|1\rangle$, or any quantum superposition of these states.

2.1 Dirac Bra-Ket Notation

Dirac notation, or bra-ket notation, is a shorthand used in quantum mechanics to describe quantum states. Here's the basic idea:

- **Ket notation:** A quantum state is written as a ket: $|\psi\rangle$. For example, $|0\rangle$ and $|1\rangle$ represent the two basis states of a qubit.

- **Bra notation:** The conjugate transpose of a ket is called a bra: $\langle\psi|$. It's used for inner products and measurements in quantum mechanics.

Fig. 2.2 The state of Schrödinger's cat expressed in bra-ket notation

2.1 Dirac Bra-Ket Notation (Orthogonality)

In quantum mechanics, two quantum states are said to be orthogonal if their inner product is zero.

$$\langle 0|1\rangle=0$$

Orthogonality means that the states are completely different and distinguishable from one another.

If two states are not orthogonal, there is some overlap between them, and a measurement could give ambiguous results.

2.1 Dirac Bra-Ket Notation

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Born Rule: Amplitudes are very important because they give us the probability of finding the particle in that specific state when performing a measurement. The probability of measuring the particle in state $|0\rangle$ is $|\alpha|^2$, and the probability of measuring the particle in state $|1\rangle$ is $|\beta|^2$

Since the total probability of observing all the states of the quantum system must add up to 100%, the amplitudes must obey this rule:

$$|\alpha|^2 + |\beta|^2 = 1$$

This is called a normalization rule. The coefficients α and β can always be rescaled by some factor to normalize the quantum state.

2.1 Dirac Bra-Ket Notation (Computational Basis Rules)

- **Orthogonality:** Two vectors are orthogonal if their inner product is zero. They do not need to be normalized. $\langle 0|1\rangle=0$
- **normalization rule:** total probability of observing all the states of the quantum system must add up to 100%

- **Orthonormality:** Two vectors are orthonormal if they are both orthogonal and have a norm (or length) of 1.

In quantum computing, the basis states for qubits are typically **orthonormal** because they need to be both distinguishable (orthogonal) and normalized to ensure that probability calculations work correctly.

2.1 Dirac Bra-Ket Notation

Examples:

1. The quantum state of a spinning coin can be written as a superposition of heads and tails. Using heads as $|1\rangle$ and tails as $|0\rangle$, the quantum state of the coin is

$$|\text{coin}\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right). \tag{2.3}$$

What is the probability of getting heads?

The amplitude of $|1\rangle$ is $\beta = 1/\sqrt{2}$, so $|\beta|^2 = \left(1/\sqrt{2}\right)^2 = 1/2$. So the probability is 0.5, or 50%.

2.1 Dirac Bra-Ket Notation

Examples:

2. A weighted coin has twice the probability of landing on heads vs. tails. What is the state of the coin in "ket" notation?

$$\begin{aligned}
P_{\text{heads}} + P_{\text{tails}} &= 1 \quad \text{(Normalization Condition)} \\
P_{\text{heads}} &= 2P_{\text{tails}} \quad \text{(Statement in Example)} \\
\rightarrow P_{\text{tails}} &= \frac{1}{3} = \alpha^2 \\
\rightarrow P_{\text{heads}} &= \frac{2}{3} = \beta^2 \\
\rightarrow \alpha &= \sqrt{\frac{1}{3}},\ \beta = \sqrt{\frac{2}{3}} \\
\rightarrow |\text{coin}\rangle &= \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle.
\end{aligned} \tag{2.4}$$

2.2 Matrix Representation

In matrix representation, a qubit is written as a two-dimensional vector where the amplitudes are the components of the vector

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The states $|0\rangle$ and $|1\rangle$ are usually represented as

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Experimentally, a qubit's state can be changed through some physical action such as applying an electromagnetic laser or passing it through an optical device. Changing a qubit's state through a physical action mathematically are represented by **unitary operations**, which are matrices that preserve the total probability.

2.2 Matrix Representation

Unitary Operations

- A unitary operation is a mathematical operation that changes the state of a qubit while preserving the total probability.

- Unitary matrix U changes qubit state: $|\psi'\rangle = U|\psi\rangle$

- A matrix U is unitary if the matrix product of U and its conjugate transpose $U^\dagger$ (called U-dagger) multiply to give the identity matrix: $UU^\dagger = U^\dagger U = I$

2.2 Matrix Representation

Examples: Unitary Operations

1. What is the conjugate transpose of the following matrix?

$$A = \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix}$$

First Step: The conjugate of a complex number is found by switching the sign of the imaginary part.
Second Step: we transpose the conjugated matrix. Transposing a matrix switches rows with columns.

$$A^{\dagger} = \begin{pmatrix} 1 & 1 \\ -i & -i \end{pmatrix}$$

2.2 Matrix Representation

Examples: Unitary Operations

2. Is the above matrix A unitary?

$$AA^{\dagger} = \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & -i \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Multiplying A by its conjugate transpose does not produce the identity matrix, so A is not unitary.

2.3 Bloch Sphere

Examples: Unitary Operations

3. What is the result of applying the unitary operator X onto a $|0\rangle$ state qubit?

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle.$$

The X matrix changes the $|0\rangle$ qubit state to the $|1\rangle$ qubit state.

2.3 Bloch Sphere

The Bloch sphere is a useful way to visualize the state of a qubit. It represents the state of a qubit as a point on the surface of a unit sphere.

- The north pole of the sphere represents the state $|0\rangle$, and the south pole represents the state $|1\rangle$.
- Any point on the surface of the sphere represents a superposition of $|0\rangle$ and $|1\rangle$.

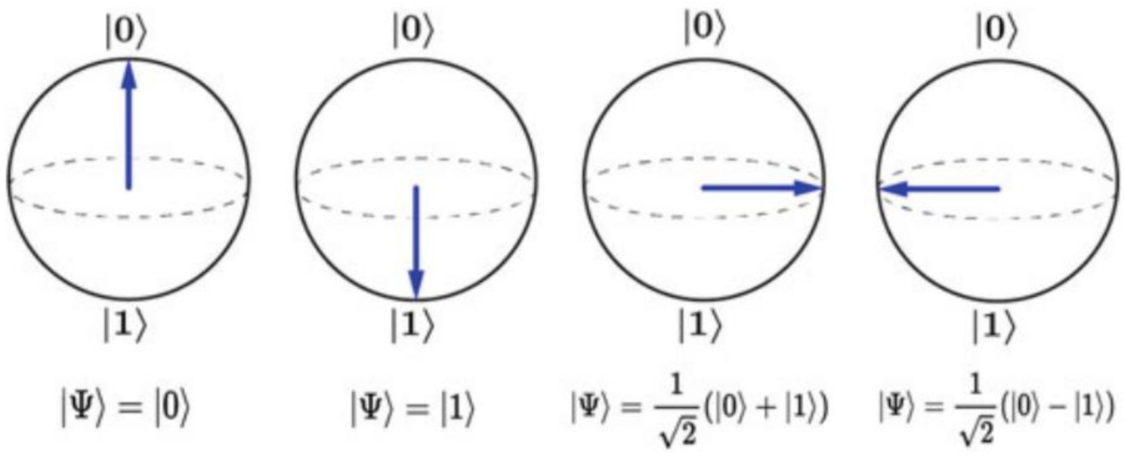


Fig. 2.3 The state of a qubit is represented by an arrow on the Bloch sphere

2.3 Bloch Sphere (Mathematically)

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

θ: The polar angle (latitude), ranging from 0 to π.
ϕ: The azimuthal angle (longitude), ranging from 0 to 2π.

This parameterization ensures that any qubit state lies on the surface of the unit sphere.

- $|0\rangle$ corresponds to the north pole of the sphere, i.e., θ=0.
- $|1\rangle$ corresponds to the south pole of the sphere, i.e., θ=π.
- Any point on the sphere's surface corresponds to a valid qubit state, representing a superposition of $|0\rangle$ and $|1\rangle$.

2.3 Bloch Sphere

- When the state of the qubit is changed, the arrow rotates to a different position on the sphere.
- One analogy is to think of the qubit like Schrödinger’s cat traveling the globe shown in Fig. 2.4. When the cat is at the North Pole, it will definitely be alive. When the cat is at the South Pole, it will definitely be dead. As long as the cat’s state is not measured, it can be anywhere else on the globe in a superposition state of alive and dead.
- As coders of the quantum computer, it is our job to manipulate the state of the qubit which gives the cat instructions on how to move around the globe.

Question 1 Schrödinger’s cat is determined to be alive. What location on the Earth in Fig. 2.4 could the cat have been before the quantum measurement?
(a) Russia (b) Australia (c) North Pole (d) all of the above

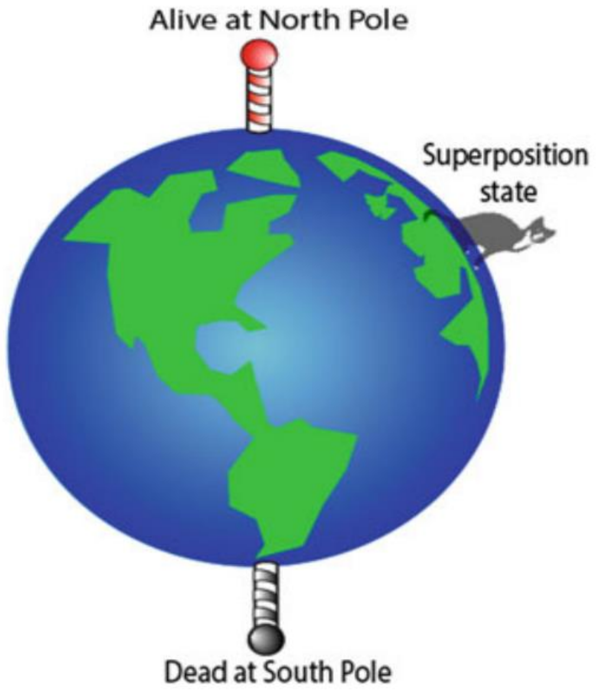


Fig. 2.4 A cartoon of the Bloch sphere depicted as the Earth, and the state of Schrödinger’s cat represented as a location on Earth

2.4 Big Ideas:

1. A qubit can be in a superposition of $|0\rangle$ and $|1\rangle$. states. The Bloch sphere can be used to visually represent a single qubit.

2. 2. A qubit can be written in terms of amplitudes. Each squared amplitude corresponds to the probability of measuring the qubit in $|0\rangle$ or $|1\rangle$.

3. 3. A physical change to a qubit mathematically corresponds to unitary matrices which multiply the qubit amplitudes

2.5 Class Activity

Visualization of Different Quantum Bases

Before you start make sure to install qiskit: `pip install qiskit`

Import necessary components

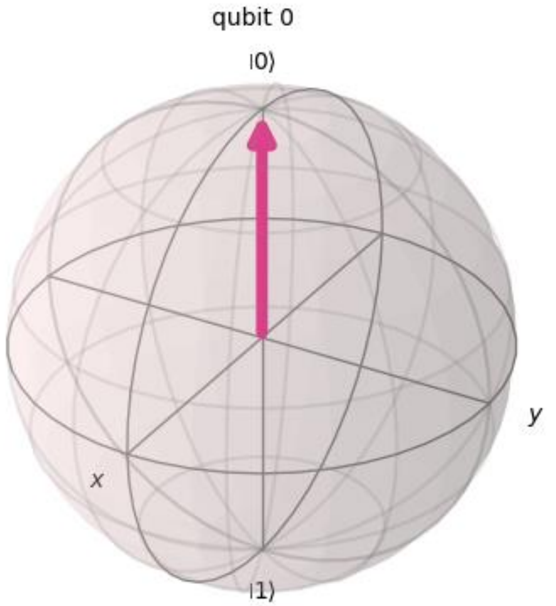
```
from qiskit import QuantumCircuit
from qiskit.visualization import plot_bloch_multivector
from qiskit.quantum_info import Statevector
```

2.5 Class Activity

Visualization of Different Quantum Bases

Visualize the Z-basis

```
# Create a quantum circuit with one qubit
qc_z = QuantumCircuit(1)

# Initial state (|0⟩)
state_z = Statevector.from_instruction(qc_z)

# Visualize the state on the Bloch sphere
plot_bloch_multivector(state_z)
```

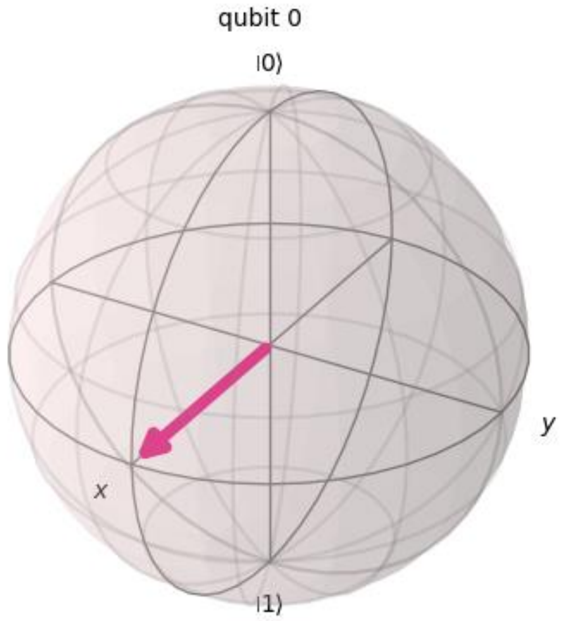


2.5 Class Activity

Visualization of Different Quantum Bases

Visualize the X-basis

```
# Apply a Hadamard gate to put the qubit in the |+⟩ state
qc_x = QuantumCircuit(1)
qc_x.h(0)
state_x = Statevector.from_instruction(qc_x)

# Visualize the state on the Bloch sphere
plot_bloch_multivector(state_x)
```

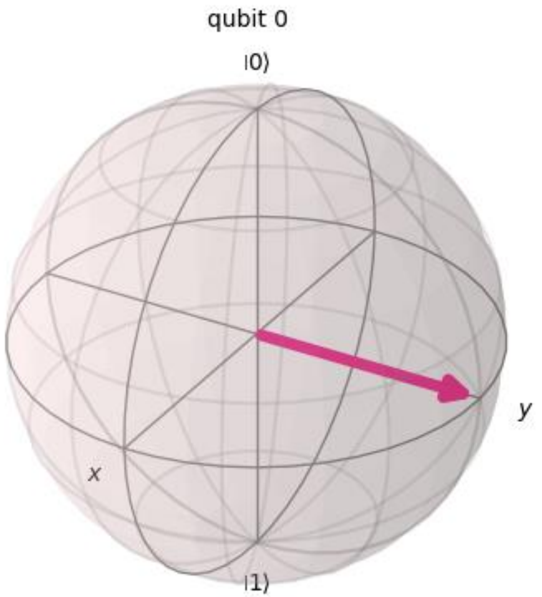


2.5 Class Activity

Visualization of Different Quantum Bases

Visualize the Y-basis

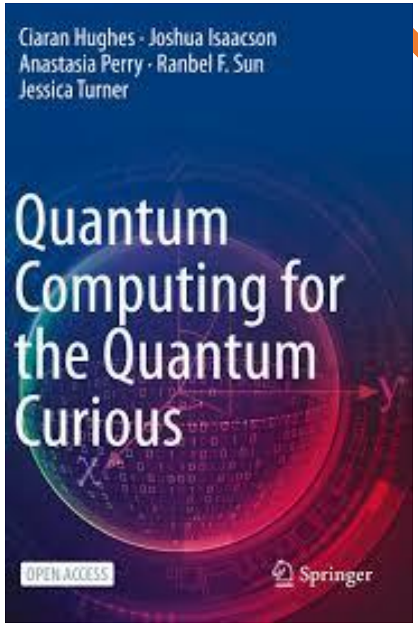
```
# Apply a Y-basis preparation using an S and H gate
combination
qc_y = QuantumCircuit(1)
qc_y.h(0)
qc_y.s(0)
state_y = Statevector.from_instruction(qc_y)

# Visualize the state on the Bloch sphere
plot_bloch_multivector(state_y)
```



Reference Book

Quantum Computing for the Quantum Curious

This open access book makes quantum computing more accessible than ever before. A fast-growing field at the intersection of physics and computer science, quantum computing promises to have revolutionary capabilities far surpassing "classical" computation. Getting a grip on the science behind the hype can be tough: at its heart lies quantum mechanics, whose enigmatic concepts can be imposing for the novice. This classroom-tested textbook uses simple language, minimal math, and plenty of examples to explain the three key principles behind quantum computers: superposition, quantum measurement, and entanglement.



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