Quantum Computing for the Quantum Curious

6.0 Quantum Gates

6.0 Objectives

- Understand what quantum gates do and how they are different from classical gates.
- Learn the functions of single-qubit gates like X, Y, Z, and Hadamard.
- Understand multi-qubit gates like CNOT, Toffoli, and controlled gates.

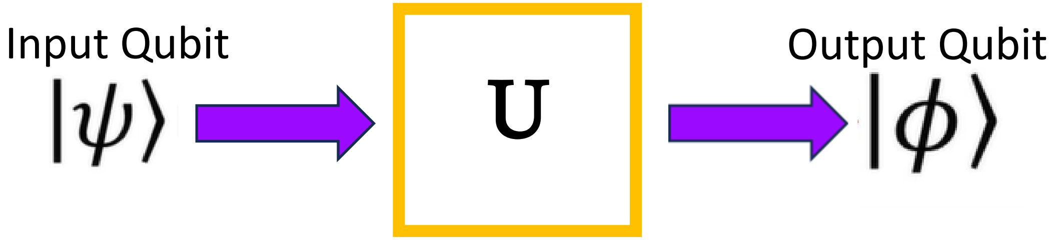
Key Terms: single qubit gates, Quantum Circuit, Multi-Qubit Gates

6.1 Quantum Gates

Quantum gates are the fundamental building blocks of quantum computers, enabling them to perform logical operations and process quantum information

What they do

Quantum gates manipulate the state of qubits, which are quantum bits that can exist in superpositions of 0 and 1



$$U|\psi\rangle = |\phi\rangle$$

6.2 Single Qubit Gates

- **X, Y, Z Gates:** Quantum versions of classical logic gates
- **Hadamard Gate:** Creates superposition ($|0\rangle \rightarrow |+\rangle$, $|1\rangle \rightarrow |-\rangle$)
- **Phase Gates (S, T):** Change the phase of qubits

6.2 X-Gate (Pauli-X Gate):

- Acts as a quantum NOT gate, flipping the state of a qubit.
- Matrix Representation:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- When applied to a qubit, it swaps |0⟩ with |1⟩ and vice versa.
- Example: Applying X to |ψ⟩ = α β: $X|\psi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$
- This transforms |ψ⟩ = α|0⟩ + β|1⟩ into β|0⟩ + α|1⟩

6.2 Y-Gate (Pauli-Y Gate)

- Introduces a phase of i while flipping the qubit.
- Matrix Representation:

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- Example: Applying Y to |ψ⟩:

$$Y|\psi\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i\beta \\ i\alpha \end{bmatrix}$$

- Resulting in $|\psi\rangle = -i\beta|0\rangle + i\alpha|1\rangle$

6.2 Z-Gate (Pauli-Z Gate):

- Acts as a phase flip, leaving |0⟩ unchanged and flipping the sign of |1⟩.
- Matrix Representation:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Example: Applying Z to |ψ⟩:

$$Z|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

- Resulting in |ψ⟩ = α|0⟩ - β|1⟩

6.2 Hadamard Gate (H-Gate):

- Creates superposition from a basis state.
- Matrix Representation:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Example:

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

6.2 The Phase Gates (S and T Gates)

- **S-Gate (Phase Gate):**
- Adds a phase of $\frac{\pi}{2}$ to the $|1\rangle$ state.
- Matrix Representation:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

- Example: Applying S to $|\psi\rangle$:

$$S|\psi\rangle = \alpha|0\rangle + e^{i(\theta+\pi/2)}\beta|1\rangle$$

- **T-Gate (π/8 Gate):**
- Adds a phase of $\frac{\pi}{4}$ to the $|1\rangle$ state.
- Matrix Representation:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

- Example: Applying T to $|\psi\rangle$:

$$T|\psi\rangle = \alpha|0\rangle + e^{i(\theta+\pi/4)}\beta|1\rangle$$

The conjugates of these gates ($S^\dagger$ and $T^\dagger$) reverse the phase shifts

6.3 Multiple Qubits

A multi-qubit system is represented as the tensor product of individual qubits. For example, for two qubits $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$:

$$|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \otimes \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix}$$

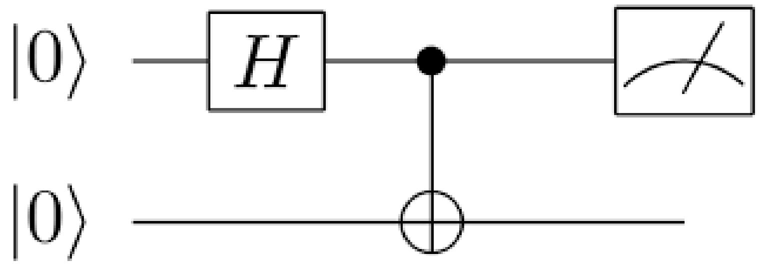
Example: Expanding the state $(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$

$$\alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

6.3 Quantum Circuits

Quantum circuits represent sequences of quantum gates applied to qubits.

The input state (usually |0⟩, |1⟩, or a superposition) passes through gates to produce a final state |ψ⟩.



6.3 Multi-Qubit Gates

- **CNOT Gate (Controlled-NOT Gate):**Flips (applies NOT to) the target qubit if the control qubit is $|1\rangle$. If the control qubit is $|0\rangle$, the target qubit remains unchanged.

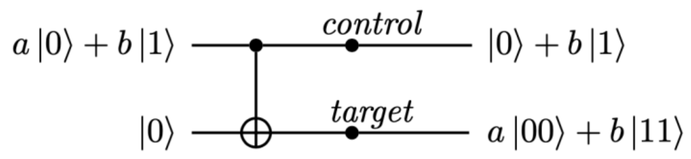
- **Toffoli Gate (CCNOT Gate):**Flips the target qubit if both control qubits are $|1\rangle$. If either control qubit is $|0\rangle$, the target qubit remains unchanged.

- **Controlled Gates:** Applies a specific quantum operation (e.g., X, Y, Z) on the target qubit, but only if the control qubit is $|1\rangle$. Otherwise, the operation does not occur.

6.3 CNOT Gate (Controlled-NOT Gate)

- **CNOT Gate (Controlled-NOT Gate):** The Controlled-NOT (CNOT) gate is a two-qubit gate that operates based on the state of a control qubit
- **Matrix Representation**:

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



Control	Target	Result
$\|0\rangle$	$\|0\rangle$	$\|00\rangle$
$\|0\rangle$	$\|1\rangle$	$\|01\rangle$
$\|1\rangle$	$\|0\rangle$	$\|11\rangle$
$\|1\rangle$	$\|1\rangle$	$\|10\rangle$

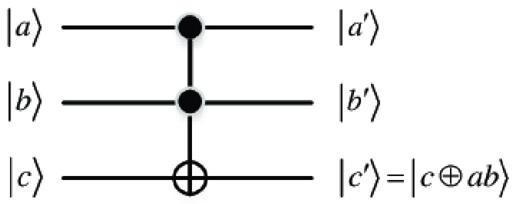
- Functionality:
  - If the control qubit is $|0\rangle$, the target qubit remains the same: $|00\rangle \rightarrow |00\rangle, |01\rangle \rightarrow |01\rangle$
  - If the control qubit is $|1\rangle$, the target qubit is flipped: $|10\rangle \rightarrow |11\rangle$, $|11\rangle \rightarrow |10\rangle$

6.3 Toffoli Gate (CCNOT Gate)

- The Toffoli gate, also known as the Controlled-Controlled-NOT (CCNOT) gate, is a three-qubit gate. The target qubit flips only if both control qubits are in the $|1\rangle$ state.
- **Matrix Representation**:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Inputs			Ouputs		
a	b	c	a'	b'	c'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0



- **Functionality:**
  - If both control qubits are $|1\rangle$, the target qubit flips.
  - If either or both control qubits are $|0\rangle$, the target qubit remains unchanged.

6.3 Controlled Gates

- Controlled gates are a class of quantum gates where the operation on a target qubit is conditional upon the state of a control qubit.

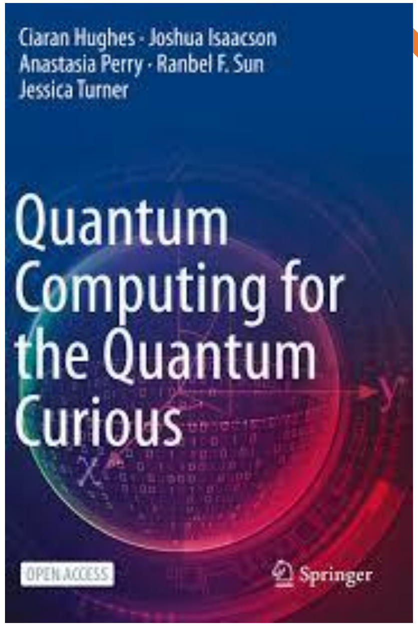
- The most common example is the CNOT gate, but controlled versions of other single-qubit gates (e.g., X, Y, Z, H) can also be created.

- General Representation: A controlled gate applies a quantum operation U on the target qubit if the control qubit is $|1\rangle$. Otherwise, the target qubit remains unchanged.

- **Functionality:**
  - They enable the construction of more complex quantum circuits by making operations conditional.

Reference Book

Quantum Computing for the Quantum Curious



This open access book makes quantum computing more accessible than ever before. A fast-growing field at the intersection of physics and computer science, quantum computing promises to have revolutionary capabilities far surpassing “classical” computation. Getting a grip on the science behind the hype can be tough: at its heart lies quantum mechanics, whose enigmatic concepts can be imposing for the novice. This classroom-tested textbook uses simple language, minimal math, and plenty of examples to explain the three key principles behind quantum computers: superposition, quantum measurement, and entanglement.

Hughes, C., Isaacson, J., Perry, A., Sun, R. F., & Turner, J. (2021). Quantum computing for the quantum curious. In Springer eBooks. https://doi.org/10.1007/978-3-030-61601-4

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