Summary of Key Concepts

Math of Quantum: Single Qubits

Week of January 21, 2024

Key Terms

Key Term	Definition
Inner Product	The inner product, or dot product, calculates the amount of overlap between two states.
Bra-Ket Notation	A mathematical framework using kets to represent the state we're in and their "twin" bras to represent the state we're going to or measuring.
Matrix	A mathematical representation for a gate. It is made up of rows and columns which act on a vector such that the final result is a vector.

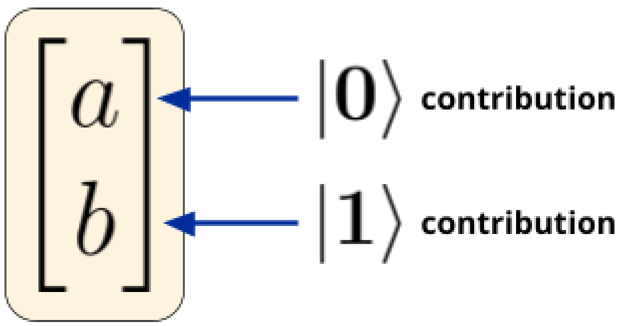
Lecture and Lab

Learning Objectives

1. Recognize what the inner product is and how to calculate it with statevectors and bra-kets.
2. Recognize how to use Born's rule for statevectors and bra-kets.
3. Recognize how to apply gates using matrices.

Key Ideas

1. **Statevectors** are mathematical objects that can represent quantum states.



2. The **inner product**, or **dot product**, calculates the amount of overlap between two states. It says how much one state contributes to another on a scale of 0 to 1. We perform this in 3 steps:
   a. Multiply the contributions from 0.
   b. Multiply the contributions from 1.
   c. Add these numbers together.

$$\left.\begin{matrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \langle 0 | 1 \rangle \end{matrix}\right\} \quad \begin{array}{r} 1 * 0 \\ + 0 * 1 \\ \hline = \quad 0 \end{array}$$

3. We can make quantum math easier by using **bra-ket notation**:

Ket	Bra
$\vert\ \rangle$	$\langle\ \vert$
Kets represent states we're in.	Bras represent states we're measuring or going to.
$\vert 0\rangle$ $\vert 1\rangle$ $\vert +\rangle$ $\vert -\rangle$	$\langle 0\vert$ $\langle 1\vert$ $\langle +\vert$ $\langle -\vert$

4. The four main states we've discussed can be represented as follows:

Vectors	Kets	Bras
$\begin{bmatrix}1\\0\end{bmatrix}$	$\vert 0\rangle$	$\langle 0\vert$
$\begin{bmatrix}0\\1\end{bmatrix}$	$\vert 1\rangle$	$\langle 1\vert$
$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\vert +\rangle$	$\langle +\vert$
$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	$\vert -\rangle$	$\langle -\vert$

5. In turn the inner product can be calculated with either vectors or bra-kets:

Vectors	Kets and Bras	Value
$\begin{bmatrix}1\\0\end{bmatrix}\bullet\begin{bmatrix}0\\1\end{bmatrix}$	$\langle 0\Vert 1\rangle$	0
$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\bullet\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\langle +\vert +\rangle$	1
$\begin{bmatrix}1\\0\end{bmatrix}\bullet\frac{1}{\sqrt{3}}\begin{bmatrix}\sqrt{2}\\1\end{bmatrix}$	$\frac{1}{\sqrt{3}}(\sqrt{2}\langle 0\vert 0\rangle + \langle 0\vert 1\rangle)$	$\frac{\sqrt{2}}{\sqrt{3}}$

6. Using the inner product, we can use **Born's rule** to determine the probability of different measurement outcomes:

$$\textbf{prob(measuring } \begin{bmatrix} a \\ b \end{bmatrix} \textbf{ in the } \begin{bmatrix} x \\ y \end{bmatrix} \textbf{ state)} = \left(\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \right)^2$$

$$\text{prob(measuring state a in state b)} = |\langle \mathbf{a} | \mathbf{b} \rangle|^{\mathbf{2}}$$

7. Gates can be represented by grids of numbers called **matrices**:

I	X	Z	H
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
I Gate (Identity)	**X Gate**	**Z Gate**	**H Gate**

8. We can apply gate matrices to vectors by taking an inner product with each row of the matrix and the whole vector:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$