Summary of Key Concepts

The Math of Quantum: Multiple Qubits

Week of January 21, 2024

Key Terms

Key Term	Definition
Inner Product	The inner product, or dot product, calculates the amount of overlap between two states.
Tensor Product	The tensor product combines quantum states to represent multiple systems (ex: qubits) at once. The tensor product works by describing all possible combinations.

Lecture

Learning Objectives

1. Recognize that the tensor product is a mathematical tool to use multi-qubit states and gates.
2. Recognize what non-separable states are.
3. Recognize how large vectors and gates need to be to represent a given number of qubits.

Key Ideas

1. The **tensor product** combines quantum states or gates to represent multiple systems (ex: qubits) at once, by describing all possible combinations.

$$\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$\text{Gate 1} \otimes \text{Gate 0} = \begin{bmatrix} a \cdot \text{Gate 0} & b \cdot \text{Gate 0} \\ c \cdot \text{Gate 0} & d \cdot \text{Gate 0} \end{bmatrix}$$

2. **Non-separable** or **entangled states** are a purely quantum phenomenon in which it is only possible to describe the state of all qubits together as one quantum object. Mathematically, they *cannot* be made by a single tensor product of states.

Separable States	Non-Separable States
$\vert\mathbf{01}\rangle = \vert\mathbf{0}\rangle \otimes \vert\mathbf{1}\rangle$	$\frac{1}{\sqrt{2}}(\vert\mathbf{00}\rangle + \vert\mathbf{11}\rangle) \neq \vert\mathbf{a}\rangle \otimes \vert\mathbf{b}\rangle$
$\vert\mathbf{0}+\rangle = \vert\mathbf{0}\rangle \otimes \vert+\rangle$	$\frac{1}{\sqrt{2}}(\vert\mathbf{01}\rangle + \vert\mathbf{10}\rangle) \neq \vert\mathbf{a}\rangle \otimes \vert\mathbf{b}\rangle$
$\vert-+\rangle = \vert-\rangle \otimes \vert+\rangle$	$\frac{1}{\sqrt{2}}(\vert\mathbf{000}\rangle + \vert\mathbf{111}\rangle) \neq \vert\mathbf{a}\rangle \otimes \vert\mathbf{b}\rangle \otimes \vert\mathbf{c}\rangle$
$\frac{1}{\sqrt{2}}(\vert\mathbf{10}\rangle + \vert\mathbf{11}\rangle) = \vert\mathbf{1}\rangle \otimes \vert+\rangle$	

3. All quantum states are on a spectrum of entanglement from unentangled to maximally entangled. Famous examples of maximally entangled states include:

Bell States (EPR Pairs)
$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$
$\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$
$\frac{1}{\sqrt{2}}(\lvert 01\rangle - \lvert 10\rangle)$
$\frac{1}{\sqrt{2}}(\lvert 01\rangle + \lvert 10\rangle)$

GHZ States
$\frac{1}{\sqrt{2}}(\lvert 000\rangle + \lvert 111\rangle)$
$\frac{1}{\sqrt{n}}(\lvert 00...0\rangle + \lvert 11...1\rangle)$

4. For every qubit we want to describe, we must account for 2 times the possibilities. This means to describe *n* qubits:

$$\begin{array}{c} \xleftrightarrow{\quad 2^n \quad} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \end{array} \Big\updownarrow 2^n \qquad \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Big\updownarrow 2^n$$

5. Because we need to perform an exponential (at least 2^n) number of operations mathematically, it becomes extremely inefficient to simulate circuits classically instead of running them on a quantum computer directly.

Lab

Learning Objectives

1.

Key Ideas

1. The inner product is a measurement of how much two vectors “overlap”.
   a. An inner product of 0 means no overlap.
   b. An inner product of 1 means total overlap (they are the same vector).
   c. An inner product between 0 and 1 means partial overlap (the closer the value is to 1, the more overlap there is).
2. The tensor product allows us to represent multi-qubit systems using vectors.
3. The tensor product shows the probability of each qubit combination being measured.
   a. This is a result of the fact that the tensor product of two normalized vectors is normalized, so Born’s Rule still applies.
4. Tensor products are only useful for two unrelated vectors. Entangled state probabilities cannot be calculated using tensor products.