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INELASTIC CONSTITUTIVE RELATIONS FOR SOLIDS: AN INTERNAL-VARIABLE THEORY AND ITS APPLICATION TO METAL PLASTICITY

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SUMMARY

THIS PAPER is a study of the theoretical foundations of constitutive relations at finite strain for a class of solids exhibiting inelasticity as a consequence of specific structural rearrangements, on the microscale, of constituent elements of material. Metals deforming plastically through dislocation motion are of this class and form the primary application of the theory. The development is in terms of a general internal-variable thermodynamic formalism for description of the microstructural rearrangements, and it is shown how metal plasticity may be so characterized.

The principal result is in the normality structure which is shown to arise in macroscopic constitutive laws when each of the local microstructural rearrangements proceeds at a rate governed by its associated thermodynamic force. This provides a theoretical framework for time-dependent inelastic behavior in terms of a "flow potential", and reduces to statements on normality of strain increments to yield surfaces in the time-independent case. Conventional characterizations of the stress-state dependence of metallic slip are noted to be in accord with this concept of associated forces governing rates, so that the resulting normality structure may be considered directly applicable to metal plasticity.

1. INTRODUCTION

THIS paper is concerned with the 'essential structure' (in the sense of HILL (1967)) of inelastic constitutive relations for a class of solid materials at finite strain. Both time-dependent and time-independent behavior are included. The limitation is to cases for which the inelastic behavior of interest arises as a consequence of specific structural rearrangements, on the microscale, of constituent elements of a material. The most notable case is perhaps that of the plastic behavior of metals arising as a consequence of slip rearrangements of crystallographic planes through dislocation motion. Inelastic behaviors arising from twinning in crystals, grain-boundary sliding and stress-induced phase transformations also fall within the classification. The intent is to provide a general theoretical framework for material behavior of this type, the emphasis being on determining those results which rely on features common to a wide class of inelastic behaviors rather than on specific and detailed features of any one particular case.

Primary among results of this kind is the unifying normality structure which is demonstrated in macroscopic constitutive laws for a specific class of micro-structural deformation mechanisms and which depends only on the most essential feature of this class: Namely, that the rate of progression of any local micro-structural rearrangement

within the material is dependent on the current stress state only through the thermodynamic force conjugate to the extent of that rearrangement. (When applied to metal plasticity this statement simply embodies the conventional notion that slip on a given system within a single-crystalline element is stress-state dependent only through the local resolved shear stress on that system or, at a more fundamental level of modeling, that the velocity of a given segment of dislocation line is stress-state dependent only through the glide force per unit length of that line.) When the kinetics of the structural rearrangements is thus characterized, it is shown that there exists a scalar potential function of the macroscopic stress-state at each instant in the history of deformation such that the inelastic part of the strain rate is given by its derivatives on corresponding stress components. This is for the general case of a time-dependent material; when the behavior may be adequately idealized as time-independent the theory reduces to a normality requirement of inelastic strain-increments with respect to a yield surface in stress space. These results are general in being completely independent of the microstructural complexity or heterogeneity of a particular material, requiring only that local rates and associated thermodynamic forces be related in the manner postulated. Hence, in the case of metal plasticity, if slip is so characterized at the microstructural level, then the resulting macroscopic normality structure is applicable no matter how complex the structure of the polycrystal or composite material in which this mechanism takes place.

It is most convenient to develop the theory in terms which are general enough to encompass relevant material behaviors but at the same time so simple that the essential content is kept clear. Hence, the formalism is developed in Section 2 in terms of a finite number of discrete scalar internal-variables $\xi_1, \xi_2, \dots, \xi_n$, each of which is assumed to characterize the extent of some local structural rearrangement which takes place at one of n different sites within a given sample of material. No actual inelastic behavior is likely to be so simply characterized. However, this is sufficient to lay bare the structure of the theory, and, as will be evident when the formalism is applied to metal plasticity in Section 3, it is a simple matter to adequately extend the work so that results apply to cases requiring a field-like internal-variable description of, say, the slipped state within a metal polycrystal. Indeed, all the basic results are derived in Section 2, with Section 3 being concerned to show how the internal-variable description and the consequent results are adapted to the particular mechanism of metallic slip.

The discrete internal-variable theory provides a generalization of the work of KESTIN and RICE (1970) to finite deformations. (Their work was in turn an extension of earlier studies on viscoelasticity by BIOT (1954) and MEIXNER (1954).) This approach views inelastic deformation of a given sample of material of the type considered under macroscopically homogeneous strain and temperature as a sequence of constrained equilibrium states: The state of the material sample at any given time in the deformation history is taken to be fully characterized by corresponding values of the strain and temperature and the collection of internal variables which mark the extent of microstructural rearrangement within the sample. Further, the relations between these and other state properties, such as the stresses, are taken to be the same as if an imaginary equilibrium state were created by fixing the internal variables at their current values through imposition of appropriate constraints. Hence, the formalism of equilibrium thermodynamics is adopted for processes, and using the point-function properties of thermodynamic potentials (especially reciprocity), we are able to relate changes in the

extent of any one local structural rearrangement to corresponding changes in the macroscopic stress or strain state. The result is embodied in (13) and (14) and this important relation allows an immediate demonstration of the unifying normality structure when rates and forces are related as discussed above. Section 2 also contains a discussion of thermodynamic restrictions on constitutive laws and of the relation of internal-variables of the local structural type considered here to the more usually intended internal-variable of an averaging type.

Similar results on normality for constitutive laws in metal plasticity have been derived on mechanical grounds by MANDEL (1964) and HILL (1967) for the classical time-independent idealization and by RICE (1970) for time-dependent plasticity as in transient creep and rate-sensitive yielding. These studies similarly sought overall statements on the mechanics of polycrystal or composite elastic-plastic systems deforming by slip in each grain or sub-element on the assumption that associated shear stresses govern slip. They represent a progression from the early rigid-plastic study by BISHOP and HILL (1951) on polycrystal plasticity and incorporate elastic effects in a linearized fashion appropriate to infinitesimal strain. The present study, particularly in Section 3, provides a wider framework for these results and at the same time rigorously generalizes them to finite strain. In so doing, the relation of these studies to the present thermodynamic framework is made apparent, in that the mechanical model of an elastic body capable of internal rearrangement by slip is readily treated in a precisely analogous fashion based on point-function properties of its total strain energy. Indeed, the isothermal version of the present results may be interpreted in this way upon identification of the free energy of (9) as the strain energy of the mechanical model. (In this connection, I have learned from Dr. R. Hill that results on time-independent plasticity, apparently inclusive of those given here, have been obtained by him in a finite strain extension of studies in the spirit of HILL (1967).)

1.1. Notation and definitions

We shall let bold-face lower and upper case letters denote, respectively, first- and second-order tensors having components referred to a fixed cartesian system x_1, x_2, x_3 . Dots and double dots will denote matrix products having equivalents in indicial notation (with summation convention) given by the following typical examples:

$$\mathbf{A} \cdot \mathbf{a} = [A_{ij} a_j], \quad \mathbf{A} \cdot \mathbf{B} = [A_{ij} B_{jk}], \quad \mathbf{A} : \mathbf{B} = A_{ij} B_{ji};$$

also,

$$\frac{\partial \phi}{\partial \mathbf{E}} = [\partial \phi / \partial E_{ij}], \quad \frac{\partial^2 \phi}{\partial \mathbf{E} \partial \mathbf{E}} : \mathbf{A} = [(\partial^2 \phi / \partial E_{ij} \partial E_{kl}) A_{lk}].$$

The superscript T on a tensor denotes its transpose; the superscript -1 its matrix inverse. \mathbf{I} denotes the unit matrix; $|\mathbf{A}|$ is the determinant of \mathbf{A} .

To describe a finite deformation, let \mathbf{x} denote position vectors of material points in some reference state, and let $\mathbf{z} = \mathbf{z}(\mathbf{x}, t)$ denote positions at some later time t . Let \mathbf{F} be the deformation gradient,

$$d\mathbf{z} = \mathbf{F} \cdot d\mathbf{x}, \quad \mathbf{F} = \partial \mathbf{z} / \partial \mathbf{x}, \quad (1)$$

and define the Lagrange (or material) strain tensor as

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}). \quad (2)$$

\mathbf{T} will denote the Cauchy (or true) stress, and the Kirchhoff stress \mathbf{S} (symmetric) is defined as

$$\mathbf{S} = |\mathbf{F}| \mathbf{F}^{-1} \cdot \mathbf{T} \cdot \mathbf{F}^{-1T}. \quad (3)$$

The rate of stress working per unit current volume is

$$\mathbf{T} : (\partial \dot{\mathbf{z}} / \partial \mathbf{z}) = (\mathbf{F}^{-1} \cdot \mathbf{T}) : \dot{\mathbf{F}} = |\mathbf{F}|^{-1} \mathbf{S} : \dot{\mathbf{E}}. \quad (4)$$

To study constitutive behavior, we shall consider a macroscopic sample of material, presumed statistically homogeneous and sufficient in size so that its mechanical response is representative of similarly chosen samples. Boundary conditions will be set so as to simulate macroscopically homogeneous deformation (HILL, 1967, and RICE, 1970). Hence, to the neglect of strain gradient and like effects, conclusions drawn on mechanical behavior of the sample represent constitutive relations for the material. Let V and A denote the volume and bounding surface of the sample, respectively, V° and A° denoting the same in some reference state at a temperature θ_0 , for which no forces act on the sample boundary. Macroscopically homogeneous deformation is simulated by prescribing locations \mathbf{z} of surface points, originally having locations \mathbf{x} , as

$$\mathbf{z} = \mathbf{F} \cdot \mathbf{x}, \text{ for } \mathbf{x} \text{ on } A^\circ, \quad (5)$$

where \mathbf{F} is spatially constant and obviously to be interpreted as the macroscopic deformation gradient. \mathbf{F} is readily shown to be the average over V° of the (generally) inhomogeneous deformation gradient in the material sample. A macroscopic strain \mathbf{E} is defined in terms of \mathbf{F} by (2). In view of the interpretation (4) of \mathbf{S} as the work conjugate, on a reference volume basis, of \mathbf{E} , we define a macroscopic Kirchhoff stress \mathbf{S} as the symmetric tensor satisfying the work equality

$$\mathbf{S} : \delta \mathbf{E} = \frac{1}{V^\circ} \int_A \mathbf{t} \cdot \delta \mathbf{z} \, dA, \quad (6)$$

for arbitrary $\delta \mathbf{z} = \delta \mathbf{F} \cdot \mathbf{x}$, where \mathbf{t} is the Cauchy surface stress vector (assumed to constitute a self-equilibrating load field on A : zero body-force and acceleration). This equation may be solved as

$$\mathbf{S} = |\mathbf{F}| \mathbf{F}^{-1} \cdot \left\{ \frac{1}{V^\circ} \int_A \mathbf{t} \mathbf{z} \, dA \right\} \cdot \mathbf{F}^{-1T}. \quad (7)$$

It may be shown that bracketed quantity in (7) is the volume average of the Cauchy stress, so that in this volume average sense (3) is also satisfied by the macroscopic Kirchhoff stress. Nothing in the subsequent analysis will change if \mathbf{E} is chosen as some other symmetric material strain tensor, provided that the associated stress \mathbf{S} is chosen in the work sense of (4) and (6).

2. INTERNAL VARIABLES

2.1 Characterization of constrained equilibrium states

The macroscopic inelastic behaviors under study are considered a consequence of internal rearrangements of constituent parts of the material sample. Here, for simplicity in illustrating the structure of the theory, we assume that a discrete set of scalar internal variables $\xi_1, \xi_2, \dots, \xi_n$ (collectively ξ) characterizes the state of internal rearrangement, leaving the physical identification of these variables (in a specific case) to Section 3. The total number n is left unspecified. However, since each variable characterizes a specific structural rearrangement at a site within the material sample, n will

increase or decrease in approximate proportion to the size V° of the sample considered.

In general, each internal variable will change with time in some manner determined by local conditions at its site within the sample. However, this *kinetic* aspect of the problem is not yet to be considered. Rather, we deal here with the characterization of constrained equilibrium states. That is, we adopt the point of view that in principle, if not in practice, the internal variables could be held at any definite set of values by imposition of appropriate constraints, with the material sample attaining an equilibrium state corresponding to a prescribed stress \mathbf{S} or strain \mathbf{E} and temperature θ . Thus, \mathbf{S} or \mathbf{E} and θ , together with ξ , are thermodynamic state variables.

If various equilibrium states are considered, each corresponding to the same set of values for the internal variables, then neighboring states are related by the usual laws of thermo-elasticity. Hence, since $V^\circ \mathbf{S} : \delta \mathbf{E}$ is the work increment of surface forces on the material sample, (6), we will have

$$V^\circ \mathbf{S} : \delta \mathbf{E} + \theta \delta(V^\circ \eta) = \delta(V^\circ u) \quad \text{or} \quad \mathbf{S} : \delta \mathbf{E} + \theta \delta \eta = \delta u,$$

where $V^\circ \eta$ and $V^\circ u$ are, respectively, the entropy and internal energy of the material sample. More generally, if neighboring constrained equilibrium states corresponding to different sets of internal variables are considered, we must write

$$V^\circ \mathbf{S} : \delta \mathbf{E} - f_\alpha \delta \xi_\alpha + \theta \delta(V^\circ \eta) = \delta(V^\circ u). \quad (8)$$

Equation (8) serves to define the thermodynamic forces f_1, f_2, \dots, f_n (collectively \mathbf{f}) acting on the internal variables. (In the case of metallic slip, Section 3, a mechanical interpretation will be established for the thermodynamic forces.)

For our following considerations, it is convenient to introduce the specific *free energy* ϕ and its Legendre transform ψ with respect to strain:

$$\phi = \phi(\mathbf{E}, \theta, \xi) = u - \theta \eta, \quad \psi = \psi(\mathbf{S}, \theta, \xi) = \mathbf{E} : \frac{\partial \phi}{\partial \mathbf{E}} - \phi. \quad (9)$$

We shall refer to ψ as the *complementary energy*. Thus, (8) becomes

$$\mathbf{E} : \delta \mathbf{S} + \frac{1}{V^\circ} f_\alpha \delta \xi_\alpha + \eta \delta \theta = \delta \psi. \quad (10)$$

Note that since the number of necessary internal variables increases, on the average, in proportion to the volume of the material sample considered, the term $f_\alpha \delta \xi_\alpha / V^\circ$ may be interpreted as the volume average of work increments on internal variables.

Equation (10) leads to the usual thermo-elastic constitutive structure when the internal variables are viewed as fixed:

$$\mathbf{E} = \frac{\partial \psi(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}}$$

and, from (9),

$$\mathbf{S} = \frac{\partial \phi(\mathbf{E}, \theta, \xi)}{\partial \mathbf{E}}. \quad (11)$$

In addition, the forces associated with the internal variables are

$$f_\alpha = V^\circ \frac{\partial \psi(\mathbf{S}, \theta, \xi)}{\partial \xi_\alpha} = -V^\circ \frac{\partial \phi(\mathbf{E}, \theta, \xi)}{\partial \xi_\alpha}. \quad (12)$$

By equating cross-derivatives of ψ , these lead to the Maxwell relation

$$\frac{\partial \mathbf{E}(\mathbf{S}, \theta, \xi)}{\partial \xi_\alpha} = \frac{1}{V^\circ} \frac{\partial f_\alpha(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}}. \quad (13)$$

Equation (13) will serve as a central equation in the development of the inelastic constitutive theory in Sub-section 2.2. Its employment will be kinematical, in that it relates variations in the internal variables to corresponding variations in the macroscopic strain.

Let $\delta \mathbf{E}$ be the difference in strain between neighboring constrained equilibrium states, differing by $\delta \mathbf{S}$, $\delta \theta$, $\delta \xi$. An inelastic or *plastic* portion $(\delta \mathbf{E})^p$ of the strain difference is *defined* as that part which would result from the change in internal variables if stress and temperature were held fixed:

$$(\delta \mathbf{E})^p \equiv \frac{\partial \mathbf{E}(\mathbf{S}, \theta, \xi)}{\partial \xi_\alpha} \delta \xi_\alpha = \frac{1}{V^\circ} \frac{\partial f_\alpha(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}} \delta \xi_\alpha. \quad (14)$$

Similarly, an *elastic* (or thermoelastic) portion $(\delta \mathbf{E})^e$ is defined as that which would result from the change in stress and temperature, if the internal variables were held fixed. Thus,

$$\left. \begin{aligned} \delta \mathbf{E} &= (\delta \mathbf{E})^e + (\delta \mathbf{E})^p, \\ \text{where from (11),} \quad (\delta \mathbf{E})^e &= \frac{\partial^2 \psi}{\partial \mathbf{S} \partial \mathbf{S}} : \delta \mathbf{S} + \frac{\partial^2 \psi}{\partial \mathbf{S} \partial \theta} \delta \theta. \end{aligned} \right\} \quad (15)$$

It should be noted that the physical dimension change of the material sample, corresponding to the inelastic strain increment $(\delta \mathbf{E})^p$, is *not* independent of the particular strain measure represented by the symbol \mathbf{E} . This is because the increment is taken at constant \mathbf{S} , and the change (if any) in forces applied to the boundary of the material sample so as to keep \mathbf{S} constant will, of course, depend on the stress measure, and this will bear different relations to the boundary forces according to the adopted strain measure.

The actual concern of this study is, of course, not with equilibrium states but with time-dependent deformation processes imposed upon the material sample. In this regard the viewpoint of the classical theory of irreversible processes (e.g. DE GROOT and MAZUR (1962)) is adopted here. Specifically, it is assumed that macroscopically homogeneous deformation processes may be suitably approximated as sequences of constrained equilibrium states, each fully characterized by, say, values for \mathbf{E} , θ , ξ at the corresponding instant. Equivalently, validity of all the preceding relations is assumed during processes. Thus, for example, during a process,

$$\left. \begin{aligned} \dot{\mathbf{E}} &= (\dot{\mathbf{E}})^e + (\dot{\mathbf{E}})^p, \\ \text{where} \quad (\dot{\mathbf{E}})^p &= \frac{1}{V^\circ} \frac{\partial f_\alpha(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}} \dot{\xi}_\alpha, \end{aligned} \right\} \quad (16)$$

from (14), with an analogous expression in terms of $\dot{\mathbf{S}}$ and $\dot{\theta}$ for the elastic portion of the strain rate, it being understood that all coefficients of the rate terms are to be evaluated from equilibrium relations.

It should be noted that effects of local inertia terms, temperature gradients, etc. within the material sample (say, as a consequence of local fields due to dislocation

motion in the case of metal plasticity) have not been considered. Their incorporation would lead to a fuller characterization of state during processes, but at the expense of great complications which are best neglected in this first treatment.

Sub-sections 2.2 to 2.4 are concerned with a normality structure in macroscopic inelastic constitutive relations. This results from the assumption of a specific but appealing class of kinetic relations between the internal variables and their associated forces. The general thermodynamic argument is taken up again later in this section.

2.2 A flow potential for the inelastic strain-rate

A unifying normality structure emerges in macroscopic constitutive relations when the following class of kinetic relations is adopted: At any given temperature and pattern of internal rearrangement within the material sample, the rate at which any specific structural rearrangement occurs is fully determined by the thermodynamic force associated with that rearrangement. That is,

$$\dot{\xi}_\beta \text{ is a function of } f_\beta, \theta, \xi \text{ (for } \beta = 1, 2, \dots, n\text{).} \quad (17)$$

Hence, the current temperature and pattern of internal rearrangement may enter the kinetic equations as parameters, but the influence of the macroscopic stress state on a given structural rearrangement appears only through the fact that the associated force is dependent on stress. This is clearly not the most general class of kinetic equations. However, as will be seen in Section 3, it does embody conventional notions in metal plasticity that the associated shear stress governs slip on a given crystallographic system or, at the discrete dislocation level, that the force on a given segment of dislocation line governs its motion. Note that the inelastic behavior considered here is time-dependent in that a finite rate of change of each internal variable, and hence a finite inelastic strain-rate $(\dot{\mathbf{E}})^p$, is associated with each state of stress, temperature, and pattern of internal rearrangement. The time-independent idealization of inelasticity is considered separately in Sub-section 2.3.

The kinetic equations may be recast in the form

$$\dot{\xi}_\beta = \frac{\partial}{\partial f_\beta} \int_{\mathbf{f}} \dot{\xi}_\alpha(\mathbf{f}, \theta, \xi) df_\alpha, \quad (18)$$

where the integral is carried out at fixed values of θ and ξ , and defines a point function of \mathbf{f} since each term in the integrand is an exact differential. Now let us recognize that the thermodynamic forces may be viewed as functions of the macroscopic stress \mathbf{S} , θ , and ξ , and define

$$\Omega(\mathbf{S}, \theta, \xi) = \frac{1}{V^\circ} \int_0^{\mathbf{f}(\mathbf{S}, \theta, \xi)} \dot{\xi}_\alpha(\mathbf{f}, \theta, \xi) df_\alpha. \quad (19)$$

The stress derivatives of Ω are

$$\frac{\partial \Omega(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}} = \frac{1}{V^\circ} \dot{\xi}_\alpha(\mathbf{f}, \theta, \xi) \frac{\partial f_\alpha(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}},$$

and the right-hand side of this equation has been shown (see (14) and (16)) to equal the inelastic strain-rate.

Hence, we see that the general form of (17) for the kinetic relations implies a remarkable unifying feature for macroscopic constitutive relations, in that we have demonstrated the existence of a 'flow potential' Ω such that

$$(\dot{\mathbf{E}})^p = \frac{\partial \Omega(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}}. \quad (20)$$

Geometrically, the inelastic portion of the strain-rate vector lies normal to surfaces of constant flow potential in stress space.

Adding the elastic portion of the strain rate, as given in terms of the complementary energy by (15), the complete constitutive form is

$$\dot{\mathbf{E}} = \frac{\partial^2 \psi}{\partial \mathbf{S} \partial \mathbf{S}} : \dot{\mathbf{S}} + \frac{\partial^2 \psi}{\partial \mathbf{S} \partial \theta} \dot{\theta} + \frac{\partial \Omega}{\partial \mathbf{S}}, \quad (21)$$

where ψ and Ω are functions of \mathbf{S} , θ , and ξ . Alternatively, having established the primary result, we may suppress reference to internal variables. From (17), ξ is some functional over prior time of \mathbf{S} and θ , so that ψ and Ω may be viewed as functions of the current stress and temperature and as functionals, over prior time, of stress and temperature history. With this viewpoint, the derivatives in (21) are to be taken with respect to current values of \mathbf{S} and θ , regarding the prior history of both as fixed.

The inverted form of the constitutive rate law is readily shown to be

$$\dot{\mathbf{S}} = \frac{\partial^2 \phi}{\partial \mathbf{E} \partial \mathbf{E}} : \mathbf{E} + \frac{\partial^2 \phi}{\partial \mathbf{E} \partial \theta} \dot{\theta} - \frac{\partial \Omega}{\partial \mathbf{E}}, \quad (22)$$

where here the free energy ϕ and the flow potential Ω are viewed as functions of \mathbf{E} , θ , and ξ (or alternatively as functions of current \mathbf{E} and θ , and as functionals, over prior time, of their history). Of course, the first two terms of (22) represent the stress increment (*actually*, rate) associated with increments in strain and temperature at fixed values of the internal variables.

2.3 Time-independent behavior and normality to yield surfaces

Time-independent inelastic behavior (as in the classical elastic-plastic idealization for metals, neglecting creep and rate effects) may be formulated as a limiting case of the above time-dependent formulation. Following the discussion of RICE (1970) for small-strain metal plasticity, we consider materials for which there exists, at each instant in the course of deformation, a range of stress space over which the flow potential is nearly constant, and outside of which enormous gradients occur in the flow potential. The time-independent idealization results from representing the flow potential in a singular limit in which we replace, in the above wording, 'nearly constant' by 'constant' and 'enormous gradients' by 'an unbounded gradient at the limit of the range'. The boundary of the corresponding range in stress space is what is commonly referred to as the current yield-surface. That is, the yield surface may be viewed as a singular clustering of surfaces of constant flow potential. Normality of the inelastic strain-rate to flow potential surfaces results, in the singular limit, to normality of inelastic strain-increments to the current yield surface (at least at smooth points).

The time-independent idealization of inelasticity may be discussed directly, rather than as a limiting case of the time-dependent theory, by adopting HILL's (1967)

formulation to the present internal-variable model. The time-independent analog of (17) is that for $\beta = 1, 2, \dots, n$, viz.

$$\dot{\xi}_\beta = 0 \text{ if } f_\beta^L < f_\beta < f_\beta^U; \quad f_\beta = f_\beta^U \text{ if } \dot{\xi}_\beta > 0; \quad f_\beta = f_\beta^L \text{ if } \dot{\xi}_\beta < 0, \quad (23)$$

where f_β^L and f_β^U are functions of θ and ξ which mark the lower and upper limits to the range of forces which are incapable of inducing a corresponding structural rearrangement. Here, each f_β is regarded as a function of \mathbf{S} , θ , and ξ . A *yield surface* in stress space, corresponding to a temperature θ and pattern of internal rearrangement ξ , is defined as the locus of stress states which produce equality in at least one, and violate none, of the n inequalities

$$f_\beta^L(\theta, \xi) \leq f_\beta(\mathbf{S}, \theta, \xi) \leq f_\beta^U(\theta, \xi). \quad (24)$$

That is, the yield surface is the envelope of the $2n$ surfaces in stress space corresponding to each of the limiting yield equalities for the internal forces. The stress state at any instant during the course of an inelastic deformation must, of course, lie on the yield surface.

It is clear that $\partial f_\beta / \partial \mathbf{S}$, when evaluated for any stress state on the surface $f_\beta = f_\beta^U$, is a 'vector' lying in the outward normal direction to the surface at that stress state. Similarly, $-\partial f_\beta / \partial \mathbf{S}$ is an outward normal on the surface $f_\beta = f_\beta^L$. (Here, the 'outward' region of stress space is that for which the yield inequality on f_β is violated.) Since (see equations (14) and (16))

$$(\dot{\mathbf{E}})^p = \frac{1}{V^\circ} \frac{\partial f_\alpha(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}} \dot{\xi}_\alpha, \quad (25)$$

the inelastic strain-rate is resolvable into portions which may be represented as outward normal vectors to each of the surfaces representing limiting yield equalities for the active internal variables.

Thus, if the yield surface is smooth (has a unique outward normal) at some stress state achieved in a program of inelastic deformation, the inelastic strain-rate takes the direction of the outward normal to the yield surface at that stress state. More generally, it must be expected (HILL, 1967) that the yield surface will contain a pointed vertex at the current stress point. This vertex corresponds to the envelope of surfaces of the form $f_\beta = f_\beta^U$ (or f_β^L) passing through the current stress point, there being one surface for each variable which is active or potentially active under the current stress state. The inelastic strain-rate must then take a direction within Hill's cone of limiting outward normals.

The rates $\dot{\xi}$ associated with stress and temperature rates $\dot{\mathbf{S}}$ and $\dot{\theta}$ imposed on the material sample are obtained by consistency conditions. Suppose, for example, that for a particular variable we have $f_\beta(\mathbf{S}, \theta, \xi) = f_\beta^U(\theta, \xi)$ when $\dot{\mathbf{S}}$ and $\dot{\theta}$ are imposed. If the variable remains active (i.e. if $\dot{\xi}_\beta \neq 0$) the equality must remain and therefore

$$\frac{\partial f_\beta}{\partial \mathbf{S}} : \dot{\mathbf{S}} + \frac{\partial f_\beta}{\partial \theta} \dot{\theta} + \frac{\partial f_\beta}{\partial \xi_\alpha} \dot{\xi}_\alpha = \frac{\partial f_\beta^U}{\partial \theta} \dot{\theta} + \frac{\partial f_\beta^U}{\partial \xi_\alpha} \dot{\xi}_\alpha. \quad (26)$$

There is an analogous equation for each active variable and we assume that the equations are, in general, uniquely solvable, not only for $\dot{\xi}$ under the assumption of a given set of active variables, but also for the set of variables which are indeed active in accord with (23) and (24). The complete constitutive law between $\dot{\mathbf{E}}$, $\dot{\mathbf{S}}$, and $\dot{\theta}$ is then as in

(21), with the term $\partial\Omega/\partial\mathbf{S}$ replaced by the expression for $(\dot{\mathbf{E}})^p$ in (25), and with the above consistency relations employed to express $\dot{\xi}$ as a homogeneous function of degree one in $\dot{\mathbf{S}}$ and $\dot{\theta}$.

2.4 Internal rearrangements not describable through explicit state-variables

The principal results developed so far, pertaining to the flow potential and normality structure of macroscopic constitutive relations, do not, in fact, require the existence of a set of explicit thermodynamic state variables ξ . Rather, as will be shown here, we need only the notion that a specification of stress or strain, temperature, and the current pattern of internal rearrangement (including the path history by which it was achieved) of a material sample is sufficient to characterize the state, in that definite values can be assigned to the free energy, entropy, etc. for any such specification. For example, we may wish to assume that there exists an explicit set of variables $\xi_1, \xi_2, \dots, \xi_n$ which characterize the internal rearrangement, but that these are not state variables in the sense that thermodynamic state functions are not point functions of the ξ 's, but instead depend on their path history. That is, we may assume that $f_\alpha \delta\xi_\alpha$, at fixed \mathbf{E} and θ , is *not* an exact differential. More generally, we do not have to assume that any explicit set of variables is available to characterize the internal rearrangement. Instead, we need only to assume that we can uniquely characterize any infinitesimal change in the pattern of internal rearrangement. (For example, anticipating the studies of Section 3, the *continuum slip-model* for a crystal characterizes the state in terms of the plastic shear strains on each of the slip systems, but it would be naive to assume that the state achieved is independent of the path by which a given set of shears is attained. Indeed, not even the shape of a deformed crystal is, at finite strain, independent of the order in which the various plastic shears are applied. Also, for the *discrete dislocation slip-model*, it does not appear that any set of variables, even of a field type, can characterize the internal rearrangement due to dislocation motion, especially when dislocation generation and annihilation is considered. One can, however, uniquely characterize all possible infinitesimal changes in the position of each dislocation line.)

These possibilities may be included, keeping within the simplicity of the present discrete variable description, by assuming that at any given pattern of internal rearrangement of the material sample, a set of discrete scalar infinitesimals $\delta\xi_1, \delta\xi_2, \dots, \delta\xi_n$ characterizes all possible infinitesimal variations in internal arrangement. It is not necessary to assume that any variable ξ_α exists such that $\delta\xi_\alpha$ represents an infinitesimal change in that variable, although the treatment may include this as a special case. Equation (8) is adopted, under this more general interpretation of the symbols $\delta\xi_1, \delta\xi_2, \dots$, as defining the forces f_1, f_2, \dots associated with each of the infinitesimals. We shall work with its transformed version in terms of the complementary energy ψ ,

$$\mathbf{E} : \delta\mathbf{S} + \frac{1}{V^\circ} f_\alpha \delta\xi_\alpha + \eta \delta\theta = \delta\psi,$$

where now ψ is viewed as an explicit function of \mathbf{S} and θ , which depends additionally on the pattern of internal rearrangement (denoted symbolically by the letters *PIR*, so that we write $\psi = \psi(\mathbf{S}, \theta, \text{PIR})$). This latter dependence is assumed to be such that $\delta\psi$ is linear in $\delta\xi_1, \delta\xi_2, \dots$, which is the same as assuming that each of the forces is independent of the $\delta\xi$'s: $\mathbf{f} = \mathbf{f}(\mathbf{S}, \theta, \text{PIR})$.

We have in analogy to (11),

$$\mathbf{E} = \frac{\partial \psi(\mathbf{S}, \theta, PIR)}{\partial \mathbf{S}}. \quad (27)$$

Now let δPIR denote symbolically the change in the pattern of internal rearrangement resulting from imposition of the $\delta \xi$'s. The associated change in strain, when \mathbf{S} and θ are viewed as fixed, has previously been defined as the inelastic portion of a strain increment. Thus,

$$\left. \begin{aligned} (\delta \mathbf{E})^p &\equiv \mathbf{E}(\mathbf{S}, \theta, PIR + \delta PIR) - \mathbf{E}(\mathbf{S}, \theta, PIR) \\ &= \frac{\partial}{\partial \mathbf{S}} [\psi(\mathbf{S}, \theta, PIR + \delta PIR) - \psi(\mathbf{S}, \theta, PIR)] \\ &= \frac{\partial}{\partial \mathbf{S}} \left[\frac{1}{V^\circ} f_\alpha(\mathbf{S}, \theta, PIR) \delta \xi_\alpha \right] = \frac{1}{V^\circ} \frac{\partial f_\alpha(\mathbf{S}, \theta, PIR)}{\partial \mathbf{S}} \delta \xi_\alpha \end{aligned} \right\} \quad (28)$$

This important result shows that (14) retains validity (with the simple replacement of ξ by PIR) even when there are no explicit thermodynamic state variables to characterize the internal rearrangement. Since (14) was the key element in demonstrating the flow potential representation for $(\dot{\mathbf{E}})^p$ for time-dependent behavior, and the normality structure in the time-independent case, it is seen that these results carry over at once to the present formulation. We need only replace ξ by PIR in the equations of Sub-sections 2.2 and 2.3, and interpret $\dot{\xi}_\alpha$ as $\delta \xi_\alpha / \delta t$ where δt is an infinitesimal time increment during which variations $\delta \xi_1, \delta \xi_2, \dots$ occur. For example, if in the time-dependent case, we assume that each $\dot{\xi}_\beta (\equiv \delta \xi_\beta / \delta t)$ is dependent on the state of stress only through the associated force f_β , then

$$\Omega(\mathbf{S}, \theta, PIR) = \frac{1}{V^\circ} \int_0^{\mathbf{f}(\mathbf{S}, \theta, PIR)} \dot{\xi}_\alpha(\mathbf{f}, \theta, PIR) df_\alpha \quad (29)$$

(the integral on \mathbf{f} being carried out at fixed θ and PIR) is the flow potential in that we may demonstrate directly from (28) that $\partial \Omega / \partial \mathbf{S} = (\dot{\mathbf{E}})^p$.

2.5 Thermodynamic restrictions

Let us suppose that our material sample is taken from one constrained equilibrium state to another, denoted by (1) and (2), by an irreversible process extending from times $t^{(1)}$ to $t^{(2)}$. Suppose heat is supplied to the surface of the sample at a rate q per unit area, positive inward, at temperature θ during the process, and let \mathbf{t} be the surface stress vector during the process. Then, the first and second laws of thermodynamics, in classical form dealing exclusively with the comparison of equilibrium states, are

$$\left. \begin{aligned} \int_{t^{(1)}}^{t^{(2)}} \left\{ \int_A \mathbf{t} \cdot \dot{\mathbf{z}} dA + \int_A q dA \right\} dt &= V^\circ [u^{(2)} - u^{(1)}], \\ \int_{t^{(1)}}^{t^{(2)}} \left\{ \int_A (q/\theta) dA \right\} dt &\leq V^\circ [\eta^{(2)} - \eta^{(1)}], \end{aligned} \right\}$$

where the time integrals follow the actual irreversible process. Define $V^\circ Q = \int_A q dA$ so that Q is the total heat supply rate per unit reference volume. Following the classical approach to irreversible processes for a (macroscopically) homogeneous system with internal variables (e.g. KESTIN and RICE (1970)), we view all actual

processes as sequences of constrained equilibrium states. The above equations then apply for any choice of times $t^{(1)}$ and $t^{(2)}$ during a process. Also, the surface work-rate term may be written as $V^\circ \mathbf{S} : \dot{\mathbf{E}}$, where \mathbf{S} is the stress as defined by (11) for constrained equilibrium states, and the temperature θ is taken as spatially uniform in the material sample. Hence, the assumed generalization of the laws of equilibrium thermodynamics to actual homogeneous processes is

$$\mathbf{S} : \dot{\mathbf{E}} + Q = \dot{u}, \quad Q \leq \theta \dot{\eta}. \quad (30)$$

Let $\sigma (\geq 0)$ be the entropy production rate, defined by $Q + \theta \sigma = \theta \dot{\eta}$. Then, from (8) and (30),

$$\left. \begin{aligned} Q &= \theta \dot{\eta} - \frac{1}{V^\circ} f_\alpha \dot{\xi}_\alpha, \\ \sigma &= \frac{1}{\theta V^\circ} f_\alpha \dot{\xi}_\alpha \geq 0. \end{aligned} \right\} \quad (31)$$

Thus, the second-law requirement is that the kinetic relations (19) involve a non-negative work-rate of the associated forces on the internal variables. Alternatively, if (12) is taken for f_α , we have

$$\dot{\xi}_\alpha \frac{\partial \phi(\mathbf{E}, \theta, \xi)}{\partial \xi_\alpha} \leq 0. \quad (32)$$

That is, the rates of internal rearrangement actually occurring during a process must be such that the free energy would decrease if strain and temperature were held fixed at current values.

2.6 Transformation from specific structural variables to averaging variables

The restrictions on kinetic equations implied in (31) and (32) parallel those in other internal-variable treatments of inelasticity (e.g. DE GROOT and MAZUR (1962), COLEMAN and GURTIN (1967)), at least when these are specialized to homogeneous deformation. The primary difference is in the interpretation of the internal variables. Here, we have thought of each variable as describing a specific structural rearrangement occurring at a local site within the material sample. Since the number of sites which must be characterized increases in proportion to the size of the material sample considered, so also does the number of necessary variables. In contrast, the more usual viewpoint would have the internal variables represent averaged measures of the structural rearrangements taking place at the many operative sites within the material sample. The number of such averaging variables required for a suitable characterization bears no relation, of course, to the size of the material sample.

The utility of the specific structural-variables considered here is in the general normality structure of constitutive laws which can be uncovered when our understanding of the structural rearrangements permits us to assume that the rate at which each takes place is governed solely by its associated thermodynamic force. Such an assumption does seem warranted, for example, in the case of plastic deformation through slip in metals (Section 3). However, once this normality structure is established, we no longer need to consider the dependence of the flow potential (or yield-surface location, in the time-independent case) on all the specific structural-variables

ξ_1, \dots, ξ_n . Instead, as suggested after (21), we may regard the ξ -dependence of Ω as equivalent to a memory functional dependence of Ω on prior stress and temperature history. Alternatively, the specific structural-variables may be replaced by internal variables of the averaging type.

An illustration of the different viewpoints. Consider a metal which deforms plastically as a consequence of dislocation motion. The description of state in terms of specific structural-variables is then equivalent to a description in terms of the location of all the discrete dislocation lines in a macroscopic sample of the metal. Such a description obviously cannot be pursued in detail. It is useful, however, if conclusions of a general nature (such as the existence of a flow potential) can be drawn from it. Once this is done, further specialization of the form of the constitutive laws must follow simpler lines: PIPKIN and RIVLIN (1964), for example, suggest memory functional representations in metal plasticity. Alternatively, KRÖNER (1963) suggests that various statistical moments of the dislocation distribution may serve as internal variables. Kröner's suggestion is, of course, that of a description in terms of averaging variables.

Let us assume, then, that a certain set of averaging variables $\zeta_1, \zeta_2, \dots, \zeta_k$ (collectively ζ) may be found such that, to a suitable approximation, we may consider these as internal state variables which, together with \mathbf{S} or \mathbf{E} and θ , fully determine the energy and entropy densities (u, ϕ, ψ, η) of the material. We consider these variables to be given by expressions of the type

$$\zeta_\mu = \zeta_\mu(\xi_1, \xi_2, \dots, \xi_n; V^\circ); \quad \mu = 1, 2, \dots, k, \quad (33)$$

where the inclusion of V° is meant to indicate that these are volume average expressions of a type whose value may be assumed to be independent of n and V° when both are sufficiently large. Naturally, we shall wish to assume that the number of averaging variables, k , necessary for an accurate description is small, whereas the number of specific structural-variables, n , is to be considered enormous for any macroscopic volume V° .

A set of generalized forces g_1, \dots, g_k conjugate to the averaging variables may be defined through writing, in analogy to (8),

$$\mathbf{S} : \delta \mathbf{E} - g_\mu \delta \zeta_\mu + \theta \delta \eta = \delta u \quad (34)$$

for all neighboring states of constrained equilibrium. Thus, proceeding in parallel to the discussion in terms of the specific structural variables, one may write, for example,

$$\begin{aligned} \mathbf{E} &= \partial \psi(\mathbf{S}, \theta, \zeta) / \partial \mathbf{S}, & g_\mu &= \partial \psi(\mathbf{S}, \theta, \zeta) / \partial \zeta_\mu, \\ (\delta \mathbf{E})^\mu &\equiv \delta \zeta_\mu \partial \mathbf{E}(\mathbf{S}, \theta, \zeta) / \partial \zeta_\mu = \delta \zeta_\mu \partial g_\mu(\mathbf{S}, \theta, \zeta) / \partial \mathbf{S}. \end{aligned} \quad (35)$$

Also, on treating processes as sequences of constrained equilibrium states, the entropy production inequality reads

$$\sigma = \frac{1}{\theta} g_\mu \dot{\zeta}_\mu \geq 0, \quad (36)$$

and so restricts kinetic equations for the averaging variables.

Let us now consider what is implied, at the level of a description in terms of averaging variables, when the kinetic equations in terms of the specific structural-

variables have the form of (17). Comparing (34) with (8), it is seen that

$$g_\mu \delta \zeta_\mu \text{ replaces } \frac{1}{V^\circ} f_\alpha \delta \xi_\alpha.$$

We interpret our assumption that the state is characterized to sufficient accuracy by the chosen set of averaging variables as implying that f_α may be written to sufficient accuracy by requiring equality of these two work-terms for all $\delta \xi$. Hence,

$$f_\alpha = f_\alpha(\mathbf{g}, \xi) = V^\circ g_\mu \partial \zeta_\mu(\xi, V^\circ) / \partial \xi_\alpha, \quad (37)$$

and one may write the rate of change of the averaging variables as

$$\dot{\zeta}_\mu = \dot{\xi}_\alpha \frac{\partial \zeta_\mu(\xi, V^\circ)}{\partial \xi_\alpha} = \frac{1}{V^\circ} \dot{\xi}_\alpha \frac{\partial f_\alpha(\mathbf{g}, \xi)}{\partial g_\mu}. \quad (38)$$

This implies that $\dot{\zeta}$ is derivable from a potential of \mathbf{g} , for $\dot{\xi}_\alpha df_\alpha$ (at fixed θ and ξ) is an exact differential. Indeed, the potential is simply the flow potential of (19) written as a function of \mathbf{g} , θ , and ξ ,

$$\Omega(\mathbf{g}, \theta, \xi) = \frac{1}{V^\circ} \int_0^{\mathbf{f}(\mathbf{g}, \xi)} \dot{\xi}_\alpha(\mathbf{f}, \theta, \xi) df_\alpha \quad (39)$$

(the integral again being carried out at fixed θ and ξ), for then we have

$$\dot{\zeta}_\mu = \frac{\partial \Omega(\mathbf{g}, \theta, \xi)}{\partial g_\mu}. \quad (40)$$

Given the interpretation of the averaging variables, we may wish to assume that the ξ -dependence of Ω may be expressed to sufficient accuracy by writing $\Omega(\mathbf{g}, \theta, \xi)$, although the most general case would have the ξ -dependence replaced by a memory functional of prior \mathbf{g} and θ history.

From (35) and (40) we have

$$(\dot{\mathbf{E}})^p = \dot{\zeta}_\mu \frac{\partial g_\mu(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}} = \frac{\partial \Omega(\mathbf{g}, \theta, \xi)}{\partial g_\mu} \cdot \frac{\partial g_\mu(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}} = \frac{\partial \Omega(\mathbf{S}, \theta, \xi)}{\partial \mathbf{S}}, \quad (41)$$

which shows how the flow potential representation arises within the present formalism. Also, as may be expected from (40), the corresponding time-independent treatment leads (cf. (38) and (25)) to the conclusion that $\dot{\zeta}$ lies in the outward normal direction to the yield surface in \mathbf{g} -space at a smooth point, and within the limiting normal directions at a vertex. Hence, we see that $\dot{\zeta}$, \mathbf{g} and $(\dot{\mathbf{E}})^p$, \mathbf{S} have precisely parallel relations in terms of the normality structure of constitutive laws. No essential changes result when the dependence on ξ is replaced by one on *PIR*.

Last, it is worthy of note that in the simplest linear theory, Ω may be taken as a quadratic function of the g 's so that (40) results in

$$\dot{\zeta}_\mu = L_{\mu\lambda} g_\lambda, \text{ where } L_{\mu\lambda} = L_{\lambda\mu}. \quad (42)$$

But \mathbf{g} and $\dot{\zeta}$ are the conjugate forces and fluxes in the entropy-production expression (36), and if the various assumptions underlying Onsager's theory (see DE GROOT and MAZUR (1962)) are to be considered applicable here, the symmetry of $L_{\mu\lambda}$ would be expected on different grounds. The detailed relation between these points of view is unclear. We can, however, see that reciprocity will be a general consequence when the dissipation during a macroscopic process is describable, at the specific structural-variable level, as being due to a collection of linear dissipators, each characterized by a scalar rate which is dependent only on the associated force.

3. APPLICATION TO METAL PLASTICITY

3.1 *Background*

The results of Section 2 on the flow potential and associated normality structure of constitutive laws are applicable to metal plasticity if the plastic behavior of interest may be suitably represented through internal structural variables, and if the rate at which each structural rearrangement proceeds is governed by the corresponding thermodynamic force. Taking crystalline slip to be the mechanism of plastic deformation, we shall see in this section that an internal variable-description is possible. This involves a generalization of the discrete variable formulation to field-like descriptions of slip, but no change of substance, and is pursued for two slip-models: a *continuum slip-model* and a *discrete dislocation slip-model*.

The assumption that each slip rearrangement is governed by its associated thermodynamic force is shown, for the most common circumstances of small elastic dimension changes, to agree with the Schmid-law characterization of the stress dependence of slip (in terms of the resolved shear stress on a slip system). Hence, within the accuracy of the Schmid characterization in such cases, the flow potential and associated normality structure developed in Section 2 are applicable to metal plasticity. Indeed, this structure was first developed on the direct assumption of either negligible or small and linear elastic effects by BISHOP and HILL (1951), MANDEL (1964), HILL (1967) and RICE (1970). For those circumstances in which elastic dimension changes are not small, the assumption of the governance of slip processes by the associated thermodynamic forces provides one possible and appealing generalization of the Schmid law. However, the proper characterization of the stress dependence of slip in such cases is unresolved at present.

3.2 *Continuum slip-model*

This model ignores the discrete dislocation substructure in a metal crystal, considering instead that plastic deformation occurs in the form of smooth (in the usual continuum sense) shearing deformations on certain planes of a crystal and in certain permissible directions on each such plane. The allowed planes and directions constitute the 'slip systems' of the crystal. This continuum slip-model was employed in a rigid-plastic context by BISHOP and HILL (1951). HILL (1966, 1967) has discussed the corresponding characterization of slip in the elastic-plastic case on the assumption of small elastic effects. Here, a description of slip is developed for cases of finite elastic as well as plastic effects. The procedure employs a kinematical decomposition of deformations analogous to that employed by LEE (1969).

We consider a homogeneous single crystal which is free of stress and at a reference temperature θ_0 . Let α be an index denoting its slip systems. Orthogonal unit vectors $\mathbf{n}^{(\alpha)}$ and $\mathbf{s}^{(\alpha)}$ will denote the normal to the slip plane and slip direction of a given system. We are interested in characterizing the structural rearrangement of the crystal into other stress-free states at θ_0 as a consequence of homogeneous simple shears of amounts $\gamma^{(\alpha)}$ on the various slip-systems. These are assumed to leave the volume of the crystal and the relative orientation of crystallographic directions unaltered. It is well to keep a clear distinction between the kinematics and kinetics of the problem. In particular, it is inconsequential at present that stresses would have to be applied to the crystal to actually produce such shears. Nor does it matter that after a given set of

shears is produced, an actual unloading of the crystal to the stress-free state might result in further slip deformation of a Bauschinger or time-dependent strain recovery type. The point of view is that one may consider the structural rearrangements and the applied stress or strains as independently prescribable quantities (to be related only when the kinetic relations are considered).

Let \mathbf{x} denote initial coordinates of material points and \mathbf{y} their coordinates after structural rearrangement of the crystal by the shears. \mathbf{F}^p will denote the transformation in this rearrangement so that, apart from a rigid translation,

$$\mathbf{y} = \mathbf{F}^p \cdot \mathbf{x}. \quad (43)$$

We choose the rotational part of \mathbf{F}^p so that all crystallographic directions remain fixed in space. If the crystal is rearranged by simple shears of amount $\delta\gamma^{(\alpha)}$, we have

$$\delta\mathbf{y} = \sum_{\alpha} \delta\gamma^{(\alpha)} \mathbf{s}^{(\alpha)} [\mathbf{n}^{(\alpha)} \cdot \mathbf{y}], \quad \text{and thus} \quad \delta\mathbf{F}^p = \left[\sum_{\alpha} \delta\gamma^{(\alpha)} \mathbf{s}^{(\alpha)} \mathbf{n}^{(\alpha)} \right] \cdot \mathbf{F}^p. \quad (44)$$

Note that \mathbf{F}^p is not generally a point function of the shears, but depends instead on their sequence of application.

As remarked, to effect changes in the shears it will be necessary to apply stresses to the crystal and this may occur at a temperature other than θ_0 . Presuming homogeneous deformation, coordinates \mathbf{z} of material points in a stressed state at temperature θ may be written

$$\mathbf{z} = \mathbf{F} \cdot \mathbf{x}, \quad \text{where} \quad \mathbf{F} = \hat{\mathbf{F}} \cdot \mathbf{F}^p \quad \text{and} \quad \mathbf{z} = \hat{\mathbf{F}} \cdot \mathbf{y}, \quad (45)$$

so that $\hat{\mathbf{F}}$ is the deformation gradient from the structurally rearranged state of the crystal to its current state. $\hat{\mathbf{F}}$ includes the effects of rigid rotations of crystallographic directions, elastic stressing from an unstressed state at the same shears, and temperature change from θ_0 . We shall call it the *thermoelastic deformation gradient*. Also, in LEE's (1969) terminology, \mathbf{F}^p characterizes the *intermediate reference state* resulting if the crystal is returned to an unstressed state at temperature θ_0 with the shears $\gamma^{(\alpha)}$ held fixed at their current values.

More generally, if the crystal is inhomogeneously deformed so that the magnitudes of the shears $\gamma^{(\alpha)}$ vary from point to point, we may define \mathbf{F}^p at each point of the crystal through the integration of (44) at that point. \mathbf{F}^p then need not be compatible. Rather, it is the total displacement gradient $\hat{\mathbf{F}} \cdot \mathbf{F}^p$, appearing as

$$d\mathbf{z} = \hat{\mathbf{F}} \cdot \mathbf{F}^p \cdot d\mathbf{x}, \quad (46)$$

which must be derivable from a displacement field. Of course, if an element of small size (compared to characteristic lengths in the gradients of the shears) is cut from the crystal, freed of stress, and returned to its reference temperature and orientation, all while the shears are held fixed, then \mathbf{F}^p as so defined will be its deformation gradient. This idea will form the point of departure for the discrete dislocation model of slip, \mathbf{F}^p then having a Dirac delta-function form with singularity on the portion of slip plane swept over by a dislocation line.

Now let us follow the internal-variable formulation of Section 2: We consider a macroscopic sample of polycrystalline or other composite material having volume V^0 in a load-free reference state at θ_0 . Each grain or sub-element is assumed to have a single-crystal structure capable of slip in the sense described above. The sample is subjected to boundary conditions simulating macroscopically homogeneous deformation, and \mathbf{S} and \mathbf{E} are the macroscopic stress and strain as in (5), (2) and (6). Our

viewpoint is that for the characterization of constrained equilibrium states, we may independently prescribe the strain \mathbf{E} , temperature θ , and shears $\gamma^{(\alpha)}$ (the latter point-by-point in each grain or sub-element, for a collection of slip systems appropriate to that sub-element). The shears play the role of variables describing the pattern of internal rearrangement of the material sample. It is not necessary to consider them as internal state variables (nor could we, for not even \mathbf{F}^p of (44) is fully determined until their sequence of application is given). Rather, the viewpoint is as in Sub-section 2.4. A set of thermodynamic shear stresses $\tau^{(\alpha)}$ are defined for each operative slip system at each point of the material sample by writing, in analogy to (8),

$$V^{\circ} \mathbf{S} : \delta \mathbf{E} - \int_{V^{\circ}} \sum_{\alpha} \tau^{(\alpha)} \delta \gamma^{(\alpha)} dV^{\circ} + \theta \delta(V^{\circ} \eta) = \delta(V^{\circ} u) \quad (47)$$

for neighboring constrained equilibrium states. Later, we shall see the mechanical interpretation of these stresses.

Results may now be read directly from Section 2: The inelastic portion of any general strain increment is (see equations (14) and (28))

$$(\delta \mathbf{E})^p = \frac{1}{V^{\circ}} \int_{V^{\circ}} \sum_{\alpha} \frac{\partial \tau^{(\alpha)}(\mathbf{S}, \theta, \gamma)}{\partial \mathbf{S}} \delta \gamma^{(\alpha)} dV^{\circ}, \quad (48)$$

where γ is meant to indicate a dependence on the current slipped-state throughout the material sample, it being understood that $\tau^{(\alpha)}$ and other thermodynamic variables are not simple point-functions of the shears. Hence, if for each operative slip-system at each point of the sample we have

$$\dot{\gamma}^{(\alpha)} \text{ is a function of } \tau^{(\alpha)}, \theta, \gamma, \quad (49)$$

then a flow potential Ω exists such that $(\dot{\mathbf{E}})^p = \partial \Omega / \partial \mathbf{S}$ with (cf. (29))

$$\Omega(\mathbf{S}, \theta, \gamma) = \frac{1}{V^{\circ}} \int_{V^{\circ}} \sum_{\alpha} \left\{ \int_0^{\tau^{(\alpha)}(\mathbf{S}, \theta, \gamma)} \dot{\gamma}^{(\alpha)}(\tau^{(\alpha)}, \theta, \gamma) d\tau^{(\alpha)} \right\} dV^{\circ}. \quad (50)$$

In like fashion, the results of Section 2 on complete constitutive rate laws, the time-independent case, thermodynamic restrictions, and the use of averaging variables are all directly applicable here. For the normality structure, the key assumption is expressed in (49): slip at a point on a given system is influenced by the local state of stress only through the thermodynamic shear stress for that system.

3.3 Mechanical interpretation of thermodynamic shear stresses

To interpret the stresses $\tau^{(\alpha)}$ in mechanical terms, let us begin by rewriting (47) in the form

$$\int_A \mathbf{t} \cdot \delta \mathbf{z} dA - \int_{V^{\circ}} \sum_{\alpha} \tau^{(\alpha)} \delta \gamma^{(\alpha)} dV^{\circ} + \theta \int_{V^{\circ}} \delta \eta dV^{\circ} = \int_{V^{\circ}} \delta u dV^{\circ}, \quad (51)$$

where (6) has been employed and where η and u have been re-defined as the entropy and internal energy densities per unit volume of the material sample as measured in its macroscopically stress-free reference state at temperature θ_0 (and hence reflect the heterogeneity of the polycrystal or composite). If we assume that an equation of type (51) holds for all sub-regions of the sample, then

$$|\mathbf{F}| \mathbf{T} : (\delta \mathbf{F} \cdot \mathbf{F}^{-1}) - \sum_{\alpha} \tau^{(\alpha)} \delta \gamma^{(\alpha)} + \theta \delta \eta = \delta u, \quad (52)$$

where \mathbf{F} is the deformation gradient at the point of the material sample for which the equation is written and \mathbf{T} is the local Cauchy stress tensor. It must be noted that while

the macroscopic stress is zero, there may be a residual stress field within the grains or sub-elements of the sample in its initial reference state. Hence, the thermoelastic deformation gradient $\hat{\mathbf{F}}$ is initially different from the unit tensor. If we denote its value by $\hat{\mathbf{F}}^0$ at the point under consideration, then

$$\mathbf{F} = \hat{\mathbf{F}} \cdot \mathbf{F}^p \cdot (\hat{\mathbf{F}}^0)^{-1}, \quad (53)$$

where $\hat{\mathbf{F}}$ and \mathbf{F}^p are defined as in Sub-section 3.2, the latter being based on an integration of (44) for any fixed orientation of the crystal directions in a stress-free crystal at θ_0 . (Note that a point of the macroscopic material sample is a point of one of the single crystal grains or sub-elements of the sample.)

With (44) and (53) the increment of stress working may be written as

$$|\mathbf{F}| \mathbf{T} : (\delta \mathbf{F} \cdot \mathbf{F}^{-1}) = |\hat{\mathbf{F}}^0|^{-1} \left\{ |\hat{\mathbf{F}}| (\hat{\mathbf{F}}^{-1} \cdot \mathbf{T}) : \delta \hat{\mathbf{F}} + \sum_{\alpha} \mathbf{n}^{(\alpha)} \cdot [|\hat{\mathbf{F}}| (\hat{\mathbf{F}}^{-1} \cdot \mathbf{T})] \cdot (\hat{\mathbf{F}} \cdot \mathbf{s}^{(\alpha)}) \delta \gamma^{(\alpha)} \right\}. \quad (54)$$

But $|\hat{\mathbf{F}}| (\hat{\mathbf{F}}^{-1} \cdot \mathbf{T})$ is the nominal stress tensor based on the intermediate reference-state (that is, on the state characterized by the rearrangement matrix \mathbf{F}^p , obtainable by relief of local stresses and return of temperature to θ_0 while the shears are held fixed). Hence, $\mathbf{n}^{(\alpha)}$ dotted with this stress tensor is the force acting on slip plane α per unit of area as measured in that intermediate reference-state. Also, $\hat{\mathbf{F}} \cdot \mathbf{s}^{(\alpha)}$ is a vector which coincides with the direction taken by the crystallographic vector $\mathbf{s}^{(\alpha)}$ in the current deformed state of the crystal, and which is of length $\hat{\lambda}^{(\alpha)}$, where $\hat{\lambda}^{(\alpha)}$ is the stretch ratio for the α slip-direction in deformation from the intermediate reference-state to the current state. Thus, the stress work increment may be written as

$$|\mathbf{F}| \mathbf{T} : (\delta \mathbf{F} \cdot \mathbf{F}^{-1}) = |\hat{\mathbf{F}}^0|^{-1} \left\{ |\hat{\mathbf{F}}| (\hat{\mathbf{F}}^{-1} \cdot \mathbf{T}) : \delta \hat{\mathbf{F}} + \sum_{\alpha} \hat{\pi}^{(\alpha)} \hat{\lambda}^{(\alpha)} \delta \gamma^{(\alpha)} \right\}, \quad (55)$$

where $\hat{\pi}^{(\alpha)}$ is the nominal shear stress acting on system α (i.e. shear force in the slip direction per unit of area in the intermediate reference state).

We may therefore rewrite (52) as

$$|\hat{\mathbf{F}}^0|^{-1} |\hat{\mathbf{F}}| (\hat{\mathbf{F}}^{-1} \cdot \mathbf{T}) : \delta \hat{\mathbf{F}} + \sum_{\alpha} (|\hat{\mathbf{F}}^0|^{-1} \hat{\pi}^{(\alpha)} \hat{\lambda}^{(\alpha)} - \tau^{(\alpha)}) \delta \gamma^{(\alpha)} - \eta \delta \theta = \delta \phi, \quad (56)$$

where $\phi = u - \theta \eta$ is the free-energy density. Let us consider ϕ as a function of $\hat{\mathbf{F}}$, θ , and the γ 's, it being understood that ϕ need not be a point function of the latter, but only that its increments take the form

$$\delta \phi(\hat{\mathbf{F}}, \theta, \gamma) = \frac{\partial \phi(\hat{\mathbf{F}}, \theta, \gamma)}{\partial \hat{\mathbf{F}}} : \delta \hat{\mathbf{F}} + \frac{\partial \phi(\hat{\mathbf{F}}, \theta, \gamma)}{\partial \theta} \delta \theta + \sum_{\alpha} p^{(\alpha)}(\hat{\mathbf{F}}, \theta, \gamma) \delta \gamma^{(\alpha)}, \quad (57)$$

where $p^{(\alpha)}$ is the change in free energy per unit of shear rearrangement $\gamma^{(\alpha)}$, when θ and the thermoelastic deformation gradient are held fixed. Hence,

$$\mathbf{T} = |\hat{\mathbf{F}}^0| |\hat{\mathbf{F}}|^{-1} \hat{\mathbf{F}} \cdot \frac{\partial \phi(\hat{\mathbf{F}}, \theta, \gamma)}{\partial \hat{\mathbf{F}}}, \quad \text{and} \quad \tau^{(\alpha)} = |\hat{\mathbf{F}}^0|^{-1} \hat{\pi}^{(\alpha)} \hat{\lambda}^{(\alpha)} - p^{(\alpha)}(\hat{\mathbf{F}}, \theta, \gamma). \quad (58)$$

(It is clear that $|\hat{\mathbf{F}}^0|$ arises here only because $\tau^{(\alpha)}$, η and u were taken on a unit volume basis in the initial reference-state of the material sample. It would not appear if instead these quantities were based on a unit volume in the incompatible stress-free reference state defined at each point of the sample.)

The simplest interpretation of the thermodynamic shear stresses arises when we assume that the thermoelastic properties of a crystal are unaffected by the shear rearrangements: That is, \mathbf{T} is determined solely by $\hat{\mathbf{F}}$ and θ , independently of the γ 's.

Thus, $\partial\phi/\partial\hat{\mathbf{F}}$ is independent of γ and we may write

$$\phi = \phi^*(\hat{\mathbf{F}}, \theta) + \phi^{**}(\theta, \gamma), \text{ and therefore } p^{(\alpha)} = p^{(\alpha)}(\theta, \gamma). \quad (59)$$

If $\phi^*(\mathbf{I}, \theta_0)$ is chosen to be zero, then ϕ^{**} may be interpreted as the residual free-energy density in the intermediate reference-state. Hence, in this case the thermodynamic stresses are

$$\tau^{(\alpha)} = |\hat{\mathbf{F}}^0|^{-1} \hat{\pi}^{(\alpha)} \hat{\lambda}^{(\alpha)} - p^{(\alpha)}(\theta, \gamma), \quad (60)$$

and the assumption that the thermodynamic stresses govern slip is, in mechanical terms, an assumption that the stress state in a crystal influences slip only through the product of the nominal shear stress on the slip system and the elastic stretch ratio in the slip direction. Of course, when elastic deformations are small, $\lambda \approx 1$ and a nominal shear stress differs little from a true shear stress or some other kind, so that this agrees with the (imprecise) Schmid law that the resolved shear stress governs slip.

It is of interest to note that the entropy-production inequality takes the form

$$\sum_{\alpha} \tau^{(\alpha)} \dot{\gamma}^{(\alpha)} \geq 0, \text{ or } |\hat{\mathbf{F}}^0|^{-1} \sum_{\alpha} \hat{\pi}^{(\alpha)} \hat{\lambda}^{(\alpha)} \dot{\gamma}^{(\alpha)} \geq \sum_{\alpha} p^{(\alpha)} \dot{\gamma}^{(\alpha)}, \quad (61)$$

so that thermodynamic considerations do not require that the shear stress on a slip system and the shear rate on that system be of like sign (i.e. Bauschinger and strain recovery effects are not ruled out by the model). For isothermal deformation at temperature θ_0 in a crystal with elastic properties unaffected by slip, this reads that the work rate involving the nominal shear stresses must exceed the rate of storage of the residual free-energy $\phi^{**}(\theta_0, \gamma)$.

3.4 Discrete dislocation slip-model

This model considers plasticity to be due to the motion of discrete dislocation lines on planes belonging to the permissible slip systems of a crystal. The dislocations are viewed as singular curves in an elastic continuum which bound planar regions translated one side relative to the other by characteristic lattice spacings. As such, the model takes no explicit account of the atomistically discrete structure of the lattice and its effects on the detailed kinematics of dislocation motion. RICE (1970) has discussed some difficulties of the model, having to do primarily with the closeness of the 'sequence of constrained equilibrium states' approximation for actual processes of dislocation motion, and these remarks apply here as well.

Since finite deformations are to be considered, it is necessary to characterize the position of a dislocation line and dislocation motion in terms which reflect position relative to the lattice, and which are invariant to distortions of the lattice by elastic stressing or temperature change. Let us consider a single crystal which is, in some initial reference-state, free of dislocations, free of applied stresses, and at a temperature θ_0 . We let $\mathbf{b}^{(\alpha)}$ denote the Burger's vectors as measured from lattice spacings in this *unstressed* reference-state and $\mathbf{n}^{(\alpha)}$ the unit normals to slip planes for the various slip-systems of the crystal, the vectors being defined for some arbitrary but fixed orientation of the crystal. Suppose a dislocation is introduced by cutting and translating along a certain region of one of the planes belonging to system α . We shall let S denote the area of the cut and L its bounding curve, where both are measured-off in the *initial reference-state* of the crystal existing before the cut and translation (hence, S is a flat planar region which has $\mathbf{n}^{(\alpha)}$ as a normal and contains the direction of $\mathbf{b}^{(\alpha)}$). After introducing the dislocation, and possibly applying external stresses and changing the

temperature, locations \mathbf{z} of material points having positions \mathbf{x} in the initial reference-state are related as in (46) with

$$\mathbf{F}^p = \mathbf{I} + \mathbf{b}^{(\alpha)} \mathbf{n}^{(\alpha)} \delta_D(\mathbf{n}^{(\alpha)} \cdot \mathbf{x}) H_S[\mathbf{x} - \mathbf{n}^{(\alpha)}(\mathbf{n}^{(\alpha)} \cdot \mathbf{x})]. \quad (62)$$

Here, we have imagined the origin of coordinates to be located so that $\mathbf{x} = 0$ is a point of S , δ_D is the Dirac delta function, and H_S is unity or zero according to whether its argument is or is not a point of S . Note that if \mathbf{x}^- and \mathbf{x}^+ denote lower and upper limits (according to the direction of $\mathbf{n}^{(\alpha)}$) to a point on S , then this results in

$$\int_{\mathbf{x}^-}^{\mathbf{x}^+} \mathbf{F}^p \cdot d\mathbf{x} = \mathbf{b}^{(\alpha)}.$$

This is consistent with the interpretation of \mathbf{F}^p given after (46), for if portions of the dislocated crystal are cut out in such a way as to be free of stress, and are returned to the temperature θ_0 and to the standard orientation, then material points will have locations \mathbf{y} where $d\mathbf{y} = \mathbf{F}^p \cdot d\mathbf{x}$.

To characterize neighboring dislocation positions on the same slip plane, suppose the slipped area is changed to S' , having bounding curve L' , where each point of L' is obtained by proceeding an infinitesimal perpendicular distance δs from a point of L . Here, δs , like L , L' , S , and S' , is measured-off in the initial reference-state and is hence invariant to thermoelastic deformations. Forces q per unit invariant length of dislocation line are now defined so that $-\int_L q \delta s \, dL$ is the change in free energy of the crystal for arbitrary variations δs in the dislocation position, when the temperature and position of the boundary of the crystal are held fixed.

These ideas are extended at once to a crystal, polycrystal, or composite material sample containing many dislocation lines on the several slip-systems within the individual sub-elements. In particular, we define a force per unit length of dislocation line by writing in analogy to (8) and (47)

$$V^\circ \mathbf{S} : \delta \mathbf{E} - \int_L q \delta s \, dL + \theta \delta(V^\circ \eta) = \delta(V^\circ u) \quad (63)$$

for all neighboring constrained equilibrium states. Here, the integral is understood to be taken over all dislocation lines in the material sample, dL is a thermoelastically invariant measure of arc length along the dislocations, and δs an invariant measure of perpendicular advance of the dislocation lines in their slip planes. These measures are defined by the following conceptual operations: For dL , we scribe a line in the slip plane at a fixed small perpendicular distance ε from the dislocation line, cut out a small surrounding tube about this line (of radius smaller than ε) such that the material of the tube is stress-free, reduce the temperature to θ_0 , and take dL as the corresponding arc length along the scribe line when $\varepsilon \rightarrow 0$. Similarly, for δs we cut out elements of material so as to include the portion of slip plane traversed by the dislocation but take care not to enclose the dislocation core, so that an invariant advance relative to the lattice may be defined when the elements are freed of stress and reduced in temperature to θ_0 . Of course, these definitions are in accord with the meanings of dL and δs established above for the single crystal containing a single dislocation.

Let us consider q as a function of \mathbf{S} , θ , and various thermoelastically invariant features of the slipped state, which we need not enumerate here but denote collectively by γ . Then, the inelastic portion of a general strain increment, or equivalently the strain increment due to changing dislocation positions at fixed stress and temperature,

is (see (14) and (28))

$$(\delta \mathbf{E})^p = \frac{1}{V^\circ} \int_L \frac{\partial q(\mathbf{S}, \theta, \gamma)}{\partial \mathbf{S}} \delta s dL. \quad (64)$$

Thus, if we assume that for any given pattern of dislocation lines the invariant lattice velocity $v \equiv \delta s / \delta t$ at a point along a dislocation is dependent on stress only through the associated force per unit length at that point, or

$$v \text{ is a function of } q, \theta, \gamma, \quad (65)$$

then a flow potential Ω exists such that $(\dot{\mathbf{E}})^p = \partial \Omega / \partial \mathbf{S}$. Here, by (29), the flow potential is

$$\Omega(\mathbf{S}, \theta, \gamma) = \frac{1}{V^\circ} \int_L \left\{ \int_0^{q(\mathbf{S}, \theta, \gamma)} v(q, \theta, \gamma) dq \right\} dL. \quad (66)$$

It is, of course, common to assume that the force on a dislocation line governs its motion. As with the continuum slip-model, the statement is imprecise in that it tacitly assumes negligible elastic effects. Our present considerations again provide a precise statement and a possible generalization to cases when elastic effects are not small.

4. CONCLUSION

Here, our concern has been exclusively with the theoretical foundations of inelastic constitutive laws for solids. This includes the general internal-variable thermodynamical framework which has been developed, and also the unifying normality structure which has been shown to result when each microstructural rearrangement proceeds at a rate governed by its associated force. Section 3 has shown how metal plasticity is treated within this framework, and it has been noted that conventional characterizations of metallic slip are such that the normality structure may be considered applicable to this case. That section has thus provided a new and wider framework, together with a rigorous extension to finite strain, of previous work on constitutive laws in metal plasticity by BISHOP and HILL (1951), MANDEL (1964), HILL (1967), and RICE (1970).

The crux of the argument leading to normality lies with (14) in the general formulation of Section 2, and with the corresponding relations (48) and (64) for the two metallic slip-models of Section 3. These directly relate microstructural rearrangements at sites within the material to inelastic variations in its macroscopic strain. When rates and associated forces are related as above, the existence of a flow potential for time-dependent behavior, or of yield surface normality in the time-independent case, is an immediate consequence.

There is, of course, much related work done and to be done toward obtaining specific functional representations of memory effects, and toward developing tractable models incorporating slip mechanisms on the microscale. The effort is far too large to summarize here, although specific note may be made of the promising results from numerical analysis of 'self-consistent' models by HUTCHINSON (1964, 1970) for time-independent behavior and by BROWN (1970) for elevated temperature creep. It would be of interest to develop the relation of such models to the use of averaging internal variables as discussed at the end of Section 2, and also to seek their extension to the discrete dislocation slip-model.

Last, we note the relation of the present theory to a quite distinct and entirely macroscopic approach to metal plasticity, based on studies by DRUCKER (1951) and IL'YUSHIN (1961), in which cyclic-work inequalities of various types are employed to characterize material response in the time-independent case. HILL (1968) has examined such inequalities with special reference to their suitability at finite strain. He concludes that Il'yushin's postulate of non-negative stress work in a strain cycle leads, when stated in our present notation, to normality of $(\delta\mathbf{E})^p$ to the yield surface in \mathbf{S} -space at smooth points. We have concluded the same, and it is interesting to see the relation between the two approaches: This is most simply done by reverting to the discrete internal variable notation of Section 2. Consider isothermal deformation at a temperature θ_0 and let $\phi(\mathbf{E}, \xi) = u - \theta_0 \eta$ be the free-energy density so that (8) may be written as

$$\mathbf{S} : d\mathbf{E} = d\phi(\mathbf{E}, \xi) + \frac{1}{V^0} f_a(\mathbf{E}, \xi) d\xi_a, \quad (67)$$

Now let us assume that the material is taken through a strain cycle, beginning at \mathbf{E}^0, ξ^0 and ending at \mathbf{E}^f, ξ^f . By (12),

$$\phi(\mathbf{E}^f, \xi^f) - \phi(\mathbf{E}^0, \xi^0) = \int_{\xi^0}^{\xi^f} \frac{\partial \phi(\mathbf{E}^0, \xi)}{\partial \xi_a} d\xi_a = - \frac{1}{V^0} \int_{\xi^0}^{\xi^f} f_a(\mathbf{E}^0, \xi) d\xi_a.$$

Thus, the work during the cycle from \mathbf{E}^0 to \mathbf{E}^f may be expressed as

$$\int_{\mathbf{E}^0}^{\mathbf{E}^f} \mathbf{S} : d\mathbf{E} = \frac{1}{V^0} \int_{\xi^0}^{\xi^f} [f_a(\mathbf{E}, \xi) - f_a(\mathbf{E}^0, \xi)] d\xi_a, \quad (68)$$

where it is to be understood that each integral follows the actual path of deformation. If the strain cycle involves only infinitesimal changes in ξ and hence also in plastic portions of the strains, and if increments in each internal variable are related to the associated forces as in (23), then it is clear that the right-hand side of (68) is non-negative at least to first order in $\xi^f - \xi^0$. Thus, for strain cycles of this type, we see that the Il'yushin postulate is implied by the internal-variable theory developed in Section 2. (It should be noted that the proof of normality based on Il'yushin's postulate (e.g. HILL (1968)) requires its validity only for cycles of this type.)

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