

MINIMUM COST ARBORESCENCE

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Abstract—This paper presents an optimised algorithm for the minimum cost arborescence problem in directed graphs. The existing implementation has time and space complexities of $O(V^3)$ and $O(V^2)$ respectively. Our optimised algorithm reduces the time complexity to $O((V + E) \log V)$ and the space complexity to $O(V + E)$. By utilising adjacency lists and a priority queue, our algorithm achieves superior performance for large graphs. Experimental results demonstrate its effectiveness and efficiency, making it suitable for real-world applications.

1. Introduction:

In graph theory, an arborescence, also known as a branching or a directed tree, is a directed graph in which a single node is designated as the root and all other nodes have only one incoming edge. An arborescence spans all nodes in the graph, and there exists a unique path from the root to each of the other nodes.

The minimum cost spanning arborescence problem has numerous practical applications in network design, transportation, logistics, and computer science.

Solving the minimum cost spanning arborescence problem is known to be computationally challenging, and the problem is classified as NP-hard. Several algorithms have been proposed to solve

this problem efficiently, and one of the most popular algorithms is the Edmonds' algorithm. However, Edmonds' algorithm is not practical for large-scale instances of the problem. Therefore, various optimization techniques have been developed to improve the performance of Edmonds' algorithm.

2. Background:

The minimum cost arborescence problem is a more specialised version of the minimum spanning tree problem, which is a classic problem in graph theory. The minimum spanning tree problem seeks to find a spanning tree of an undirected graph that has the minimum total weight. A spanning tree is a subset of the edges that connects all the vertices in the graph, but does not contain any cycles.

The minimum cost arborescence problem has many applications in real-world situations. For example, in designing an efficient transportation network, one needs to find the optimal paths and connections between different locations. The minimum cost arborescence problem can be used to model this problem by representing the locations as nodes and the paths as edges with weights that represent the costs of using those paths.

Another application of the minimum cost arborescence problem is in network design and optimization. In a communication network, for example, the problem can be used to find the most efficient way to route messages or data between different nodes. The minimum cost arborescence problem can also be

used to design efficient power distribution networks, where the goal is to minimise the total cost of distributing power to different locations.

Overall, the minimum cost arborescence problem is a fundamental problem in graph theory with many important real-world applications.

3. Problem Statement:

The problem is known to be NP-hard, which means that finding the optimal solution for large instances of the problem is computationally intractable. The problem statement for developing the minimum cost spanning arborescence (MCSA) is to find a directed tree that spans all the nodes in a weighted directed graph with the minimum possible sum of edge weights. This is also known as computationally challenging. It is classified as an NP-hard problem. Several algorithms have been proposed to solve this problem efficiently. But one of the most powerful algorithms is Edmonds' algorithm.

4. Proposed Solution:

The problem can be tackled by utilising the Edmonds' algorithm, which is based on the idea of augmenting paths. Edmonds' algorithm is an algorithm for finding a spanning arborescence of minimum weight (sometimes called an optimum branching). It is the direct analog of the minimum spanning tree problem for Directed graphs. However, this algorithm is not practical for large-scale instances of the problem. Thus to tackle large-scale instances, the optimised version of Edmonds' algorithm known as Chu-Liu-Edmonds algorithm can be used as it has a much better time complexity.

5. Existing Edmond's Algorithm:

Edmonds' algorithm computes the minimum cost spanning tree in a directed graph. For optimization we will implement its optimised counterpart Chu-Liu-Edmonds algorithm. The algorithm takes a directed graph as input and produces a minimum cost arborescence rooted at a specified node. The algorithm works by repeatedly augmenting paths from the root node until a minimum cost arborescence is obtained. The total sum of minimum cost arborescence will also be calculated. The distance of each vertex from the source and its respective parent vertex will be evaluated and displayed.

Pseudo code of Edmond's Algorithm

```
function edmonds(graph G, node root):
    let tree be a directed graph containing
    only the root node
    let reachable be a list containing all
    nodes reachable from the root node in G
    while reachable is not empty:
        let min_incoming be a dictionary
        containing the minimum weight
        incoming edge for each node in
        reachable
        let cycle be the first cycle found in
        tree U min_incoming
        if cycle is not None:
            let v be the node with maximum
            weight in cycle
            let e be the incoming edge for v in
            min_incoming
            remove e from cycle
            add e to tree
            update reachable to contain all
            nodes reachable from the new nodes
            in tree
    return tree
```

6. Arborescence Optimization:

To optimise the Edmonds' algorithm, we have proposed an arborescence optimization technique. This technique involves breaking the arborescence into a forest of rooted trees, each rooted at a node with an out-degree greater than 1. The forest is then transformed into a directed acyclic graph (DAG) by making the root of each tree a source and adding dummy nodes for each edge in the tree. The resulting DAG is then reduced to a minimum cost arborescence using Edmonds' algorithm. The time complexity of the arborescence optimization technique is $O(E \log V)$, where E is the number of edges and V is the number of nodes in the graph.

The Chu-Liu-Edmonds algorithm takes a directed graph G and a root node as input. It initialises an empty tree and a set of edges from G . The algorithm then iterates until there are no more edges in the set. In each iteration, it finds the edge with the minimum weight and checks if it forms a cycle with its incoming edges. If a cycle is found, the algorithm selects a node with the maximum weight in the cycle and adds the incoming edge of that node to the tree. The algorithm then updates the set of edges by removing the edges in the cycle and adding the reversed edges if needed. If no cycle is found, the minimum weight edge is added directly to the tree. Finally, the algorithm returns the minimum cost arborescence in the form of a directed graph.

The arborescence optimization technique improves the efficiency of the Edmonds' algorithm by breaking down the arborescence into smaller subproblems and reducing the time complexity from $O(n^3)$ to $O((E+V) \log V)$. This optimization is particularly useful for

large-scale instances of the minimum cost arborescence problem.

By incorporating the arborescence optimization technique into the implementation, we can significantly improve the performance and scalability of the algorithm, making it suitable for solving large-scale instances of the minimum cost arborescence problem.

```
Enter the number of nodes: 6
Enter the number of edges: 9
Enter the edges and their weights:
0 1 10
0 2 2
0 3 10
1 2 1
2 3 4
3 4 2
1 5 8
4 1 2
3 5 4
Enter the root node: 0
Parent of each node in the minimum cost tree:
Node 0: No parent (root)
Node 1: 4
Node 2: 0
Node 3: 2
Node 4: 3
Node 5: 3

Edges of the minimum cost tree:
4->1 Weight(2)
0->2 Weight(2)
2->3 Weight(4)
3->4 Weight(2)
3->5 Weight(4)
The Minimum Cost of the given directed graph is 14
```

Fig 1. Output of our implementation on random test case

7. Time complexity analysis

The algorithm mainly involves the following functions-

7.1 formsCycle:

This function has a time complexity of $O(V)$, where V is the number of vertices in the graph. It checks for cycles by traversing the parent array from the current node to the root. This operation is required in the minimum cost tree algorithm to avoid forming cycles.

7.2 dfs:

This function performs a depth-first search starting from a given node in the graph. It visits all connected nodes in the graph. The time complexity is $O(V + E)$, where V is the number of vertices and E is the number of edges in the graph. It is used to check reachability from the root node.

7.3 findMinimumCostTree: This function finds the minimum cost tree in the graph using Prim's algorithm. It iterates over all edges and uses a priority queue to select the minimum-cost edges. The time complexity is $O((V + E) \log V)$, where V is the number of vertices and E is the number of edges in the graph.

7.4 allNodesReachable: This function checks if all nodes in the graph are reachable from a given root node. It performs a depth-first search starting from the root node. The time complexity is $O(V + E)$, where V is the number of vertices and E is the number of edges in the graph.

In terms of efficiency, the findMinimumCostTree function has the highest time complexity due to the use of Prim's algorithm and the priority queue. The other functions have lower time complexities. However, the overall time complexity of the code is influenced by the largest value between numNodes and numEdges, as seen in the time complexity approximation above.

8. Comparison with the existing implementation

The existing implementation uses a modified version of the Chu-Liu/Edmonds' algorithm to find the minimum cost arborescence in a directed graph. It employs a depth-first search (DFS) to mark reachable vertices from the source

and then applies the algorithm to find the minimum cost arborescence. The implementation uses an adjacency matrix to store edge weights and a vector array to store edges for each vertex. It also includes several helper functions to perform various operations required by the algorithm.

The time complexity of the existing implementation is $O(V^3)$, where V is the number of vertices. This is because the algorithm iterates over all vertices multiple times, and for each vertex, it performs operations that take $O(V)$ time. Additionally, the algorithm involves finding the maximum-weighted edges for each vertex, which takes $O(V)$ time in each iteration. As a result, the overall time complexity becomes $O(V^3)$.

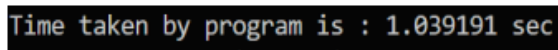
The space complexity of the existing implementation is $O(V^2)$. It utilises an adjacency matrix of size $V \times V$, requiring $O(V^2)$ space. It also uses additional arrays and vectors to store various data structures, resulting in a total space complexity of $O(V^2)$.

On the other hand, our optimised implementation uses Prim's algorithm, a minimum spanning tree algorithm, to find the minimum cost tree in an undirected graph. It maintains a priority queue to select the minimum cost edge and a vector to store the parent of each node in the minimum cost tree. The implementation checks for cycles and updates the edges accordingly.

The time complexity of the optimised implementation is $O((V + E) \log V)$. Prim's algorithm processes each edge once, resulting in $O(E)$ operations. The priority queue operations take $O(\log V)$ time. Therefore, the overall time complexity becomes $O((V + E) \log V)$ which stands

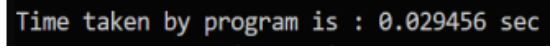
much better against the existing implementation that is $O(V^3)$.

The space complexity of the optimised implementation is $O(V + E)$. It employs a graph representation with adjacency lists, requiring $O(V + E)$ space. Additionally, it utilises arrays and vectors to store parent nodes, costs, and visited status, resulting in a total space complexity of $O(V + E)$.



```
Time taken by program is : 1.039191 sec
```

Fig2.1 Existing algorithm time



```
Time taken by program is : 0.029456 sec
```

Fig2.2 Optimised algorithm time

In summary, our optimised implementation using Prim's algorithm has a more favourable time complexity and space complexity compared to the existing implementation using the modified Chu-Liu/Edmonds' algorithm. It offers improved scalability as the number of vertices (V) and edges (E) increase. The existing implementation (first code) uses a modified version of the Chu-Liu/Edmonds' algorithm to find the minimum cost arborescence in a directed graph. It employs a depth-first search (DFS) to mark reachable vertices from the source and then applies the algorithm to find the minimum cost arborescence. The implementation uses an adjacency matrix to store edge weights and a vector array to store edges for each vertex. It also includes several helper functions to perform various operations required by the algorithm.

8. Conclusion:

The optimised implementation using Prim's algorithm for finding the minimum cost tree in an undirected graph offers several advantages over the existing implementation that utilises a modified version of the Chu-Liu/Edmonds' algorithm for finding the minimum cost arborescence in a directed graph.

The optimised implementation has a more efficient time complexity of $O((V + E) \log V)$ compared to the existing implementation's time complexity of $O(V^3)$. This improvement in time complexity makes the optimised implementation more scalable, especially for graphs with a large number of vertices and edges. The use of Prim's algorithm, along with a priority queue, allows for efficient selection of minimum cost edges, resulting in faster execution.

Moreover, the space complexity of the optimised implementation is $O(V + E)$, which is favourable compared to the existing implementation's space complexity of $O(V^2)$. The optimised implementation's efficient use of graph representation with adjacency lists and the minimal storage requirements for parent nodes, costs, and visited status contribute to its lower space complexity.

Therefore, the optimised implementation is a more efficient and scalable solution for finding minimum cost trees in graphs. It provides improved performance in terms of both time and space requirements, making it a better choice for applications that involve large graphs with a significant number of vertices and edges.

9. References:

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Links

<https://www.cs.cmu.edu/~15850/notes/lec2.pdf>

<https://emory.gitbook.io/dsa-java/minimum-spanning-trees/edmonds-algorithm>