4) Mathematical Definition of a Stiff equation is such that while solving such equations numerically certain numerical solution methods become numerically unstable while solving them, unless the step size is taken extremely small. These occur in equations where some terms can lead to a rapid variation in the solution.

For some equations there may be situation that the solution curve is terribbly smooth, yet the step size has to be made considerably small to obtain a numerically correct solution. This phenomenon is called stiffness. The stiffness of a solution curve is property of the differential equation it self and such systems are called stiff systems.

For a system of equation

$$\bar{y}' = \bar{A}\bar{y} + f(x).$$

where $\bar{y}$ is the solution vector.

if $\{\lambda_i\}$ are set of eigen vectors for the Homogeneous system.

where λ_i can be complex no.s and specify the change in the system wrt independent variable x then stiffness is characterized by stiffness ratio. given by:

$$\frac{|Re(\lambda_{max})|}{|Re(\lambda_{min})|}$$

where

$|Re(\lambda_{max})| \geq |Re(\lambda)| \geq |Re(\lambda_{min})|$

There are criterion to which stiffness depends, they can be listed as:

i> when eigen values are negative and the ratio is large.

ii> when step size is decided by numerical stability rather than accuracy.

iii> when some components decay much faster than the others.

These conditions are more or less the roadmap to characterize a stiff equation.

One physical process that has high stiffness is ~~a large~~ a simple Harmonic oscillator with large velocity dependent damping.

let us define a system as;

$$\frac{d^2y}{dt^2} = -\omega^2 y - Ry'$$

where $R >> \omega^2$ (High damping).

if we vectorize this equation,

let $u = y$, $v = \frac{dy}{dt}$

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & -R \end{pmatrix}}_{A} \begin{pmatrix} u \\ v \end{pmatrix}.$$

if we do eigen analysis of A.

$$\begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -(R+\lambda) \end{vmatrix} = 0.$$

$$\Rightarrow R\lambda + \lambda^2 + \omega^2 = 0.$$

$\Rightarrow \lambda = \dfrac{-R \pm \sqrt{R^2 - 4\omega^2}}{2}$

λ_{max} $\lambda_1 = -R + \sqrt{R^2 - 4\omega^2}$

λ_{min} $\lambda_2 = -R - \sqrt{R^2 - 4\omega^2}$

~~(λ_1)~~ $\rightarrow$ If $R^2 >> 4\omega^2$ since $R >> \omega$

we can consider both eigen values are real.

~~max~~ then

$|Re(-R - \sqrt{R^2 - 4\omega^2})| \geq |Re(-R + \sqrt{R^2 - 4\omega^2})|$

so the stiffness ratio is

$$S = \frac{R + \sqrt{R^2 - 4\omega^2}}{|-R + \sqrt{R^2 - 4\omega^2}|}$$

as $R^2 >> 4\omega^2$

$$\sqrt{R^2 - 4\omega^2} = R\sqrt{1 - \left(\frac{2\omega}{R}\right)} \sim R\left(1 - \frac{\omega}{R}\right)$$

$$= R - \omega$$

~~$R + R - \omega$~~

$$\therefore S \approx \frac{R + R - \omega}{|-R + R - \omega|} = \frac{2R - \omega}{\omega}$$

$$= \frac{2R}{\omega} - 1$$

since $R >> \omega^2$

the stiffness ratio is quite high.

so we see conditions (1) and (3) are satisfied so this is a ~~phys~~ physical example giving rise to a stiff ODE system

The Numerical methods that are mostly chosen to solve stiff equation are Implicit ~~equation~~ methods like

Implicit Euler or **Backward Euler** method. Even for RK also ∃ **Implicit RK methods** that must be used for solving stiff equations.

Numpy has no ~~in~~ module to do ODE solution. But Scipy has a special function called

scipy.integrate.odeint() that can solve stiff equations. I'll choose this if I have to deal with a stiff equation.