Single Neuron Modeling: Simplified Models

Xiaoyu Chen

The Hodgkin-Huxley Model:

$$c\frac{dV}{dt} = -\bar{g}_{Na}m^3h\,(V - E_{Na}) - \bar{g}_K n^4\,(V - E_K) - \bar{g}_L\,(V - E_L) + I_{ext},$$

$$\frac{dn}{dt} = \phi\,[\alpha_n(V)(1 - n) - \beta_n(V)n]$$

$$\frac{dm}{dt} = \phi\,[\alpha_m(V)(1 - m) - \beta_m(V)m],$$

$$\frac{dh}{dt} = \phi\,[\alpha_h(V)(1 - h) - \beta_h(V)h],$$

$$\alpha_n(V) = \frac{0.01(V + 55)}{1 - \exp\left(-\frac{V+55}{10}\right)}, \quad \beta_n(V) = 0.125\exp\left(-\frac{V+65}{80}\right),$$
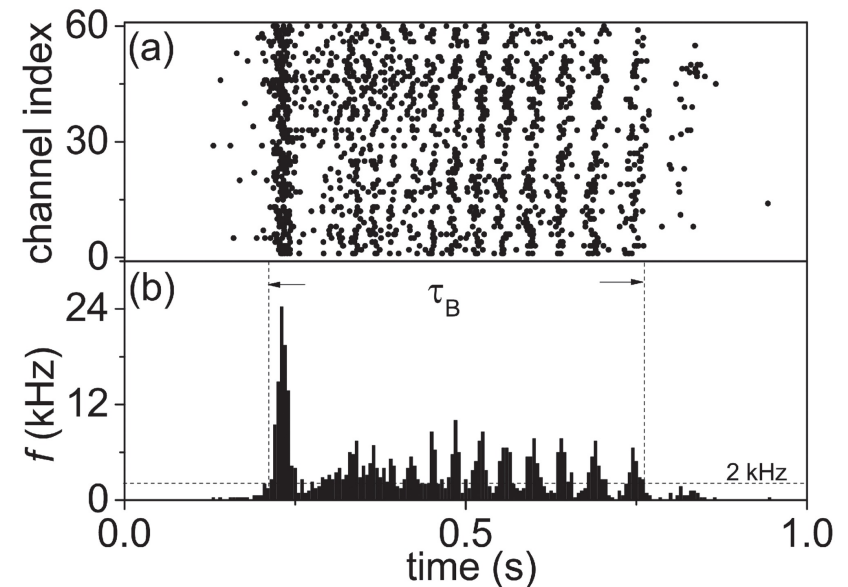
$$\alpha_h(V) = 0.07\exp\left(-\frac{V+65}{20}\right), \quad \beta_h(V) = \frac{1}{\left(\exp\left(-\frac{V+35}{10}\right) + 1\right)},$$

$$\alpha_m(V) = \frac{0.1\,(V + 40)}{1 - \exp\left(-(V + 40)/10\right)}, \quad \beta_m(V) = 4\exp\left(-(V + 65)/18\right).$$

$$\phi = Q_{10}^{(T - T_{base})/10}$$

Weakness:

computationally expensive

Huang YT, et al. PLoS One. 2017

CONTENTS

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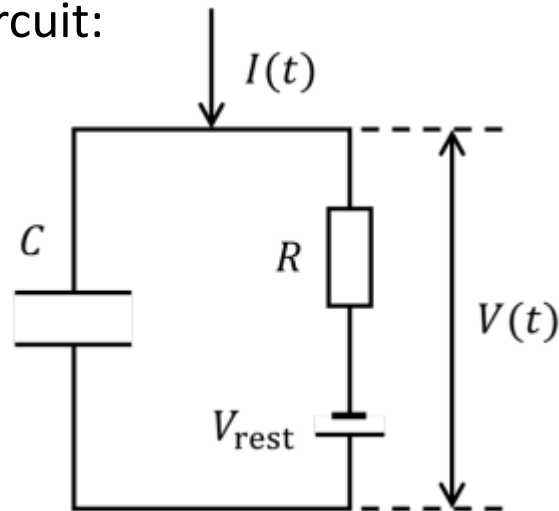## The Leaky Integrate-and-Fire (LIF) Neuron Model

The LIF neuron model

$$\tau \frac{dV}{dt} = -(V - V_{\text{rest}}) + RI(t)$$

$$\text{if } V > V_{\text{th}}, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$

↑

Refractory period
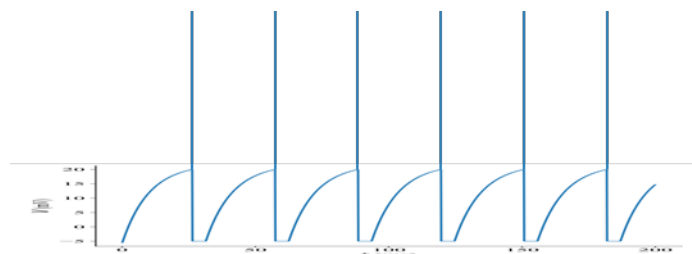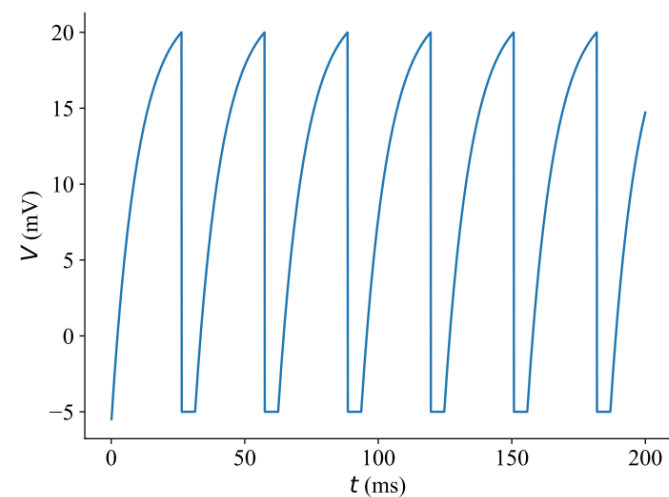
Equivalent circuit:

Comparing to the HH model:

$$c \frac{dV}{dt} = -\bar{g}_{\text{Na}} m^3 h (V - E_{\text{Na}}) - \bar{g}_{\text{K}} n^4 (V - E_{\text{K}}) - \bar{g}_{\text{L}} (V - E_{\text{L}}) + I_{\text{ext}},$$
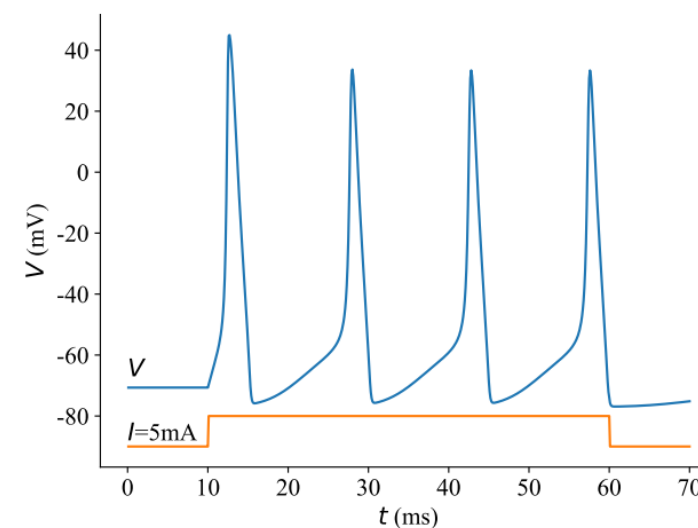
The LIF neuron model

Given a constant current input:

$$\tau \frac{dV}{dt} = -(V - V_{rest}) + RI(t)$$

$$\text{if } V > V_{th}, \quad V \leftarrow V_{reset} \text{ last } t_{ref}$$

Comparing to the HH model:

The dynamic features of the LIF model

$$\tau \frac{dV}{dt} = -(V - V_{\text{rest}}) + RI(t)$$

$$\text{if } V > V_{\text{th}}, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$

General solution (constant input):

$$V(t) = V_{\text{reset}} + RI_c(1 - e^{-\frac{t-t_0}{\tau}})$$

Firing frequency:

$$T = -\tau \ln\left(1 - \frac{V_{\text{th}} - V_{\text{rest}}}{RI_c}\right)$$

$$f = \frac{1}{T + t_{\text{ref}}} = \frac{1}{t_{\text{ref}} - \tau \ln\left(1 - \frac{V_{\text{th}} - V_{\text{rest}}}{RI_c}\right)}$$

Rheobase current (minimal current):

$$I_\theta = \frac{V_{\text{th}} - V_{\text{reset}}}{R}$$

Strengths & weaknesses of the LIF model

$$\tau \frac{dV}{dt} = -(V - V_{rest}) + RI(t)$$

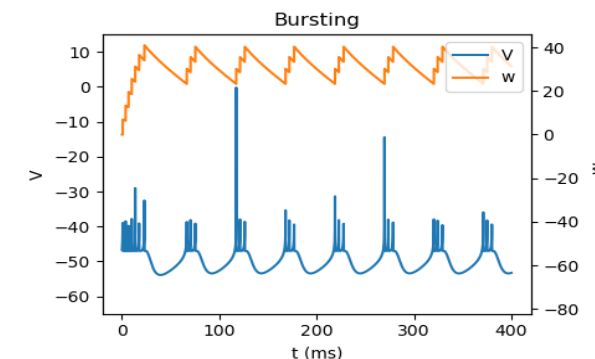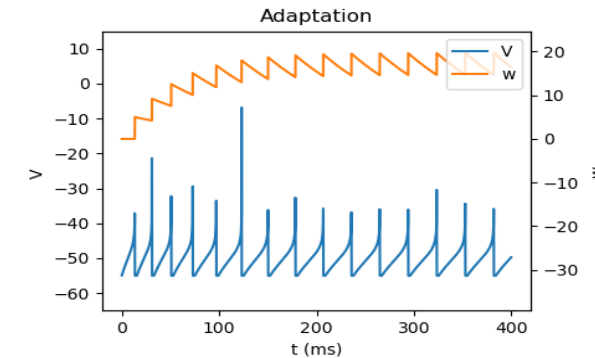$$\text{if } V > V_{th}, \quad V \leftarrow V_{reset} \text{ last } t_{ref}$$
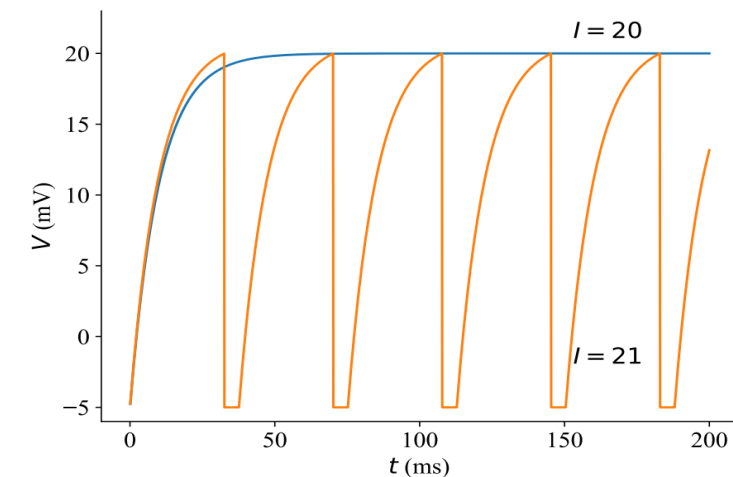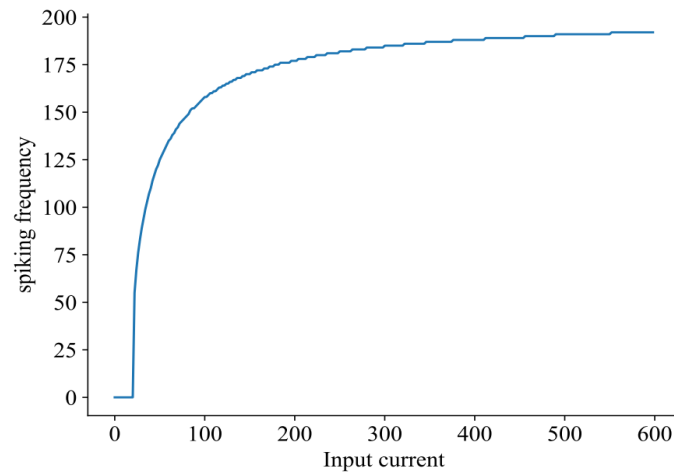
Strengths:

- Simple, high simulation efficiency

- Intuitive

- Fits well the subthreshold membrane potential

Weaknesses:

- The shape of action potentials is over-simplified

- Has no memory of the spiking history
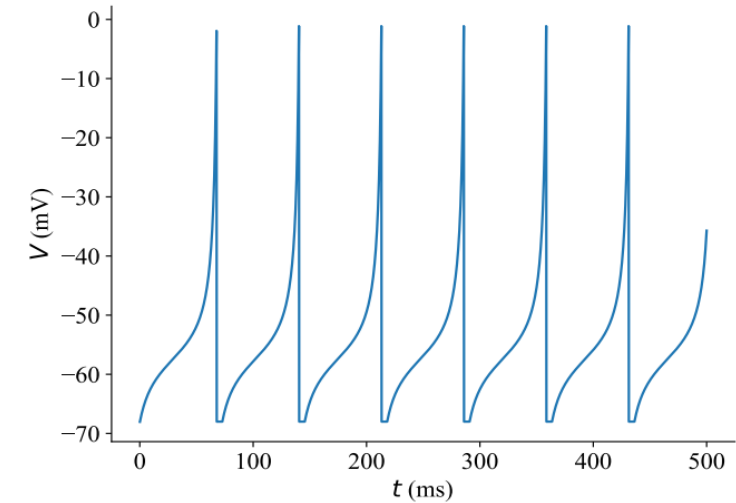
- Cannot reproduce diverse firing patterns

Other Univariate neuron models

- The Quadratic Integrate-and-Fire (QIF) model:

$$\tau \frac{dV}{dt} = a_0 (V - V_{rest})(V - V_c) + RI(t)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{reset} \text{ last } t_{ref}$$

- The Theta neuron model:

$$\frac{d\theta}{dt} = 1 - \cos\theta + (1 + \cos\theta)(\beta + I(t))$$
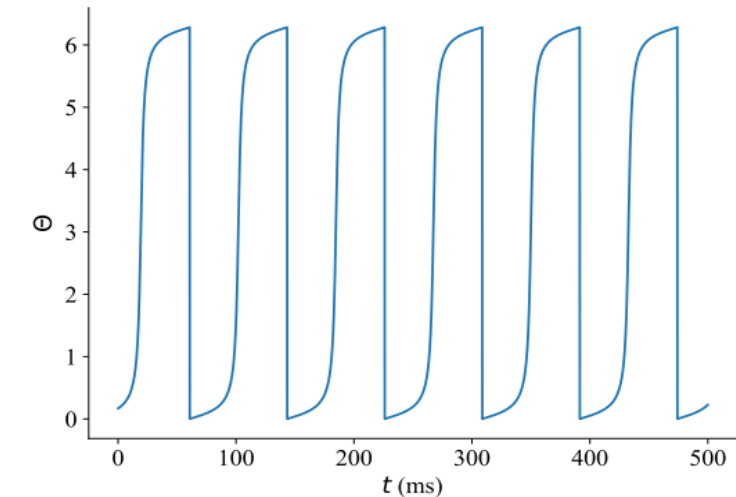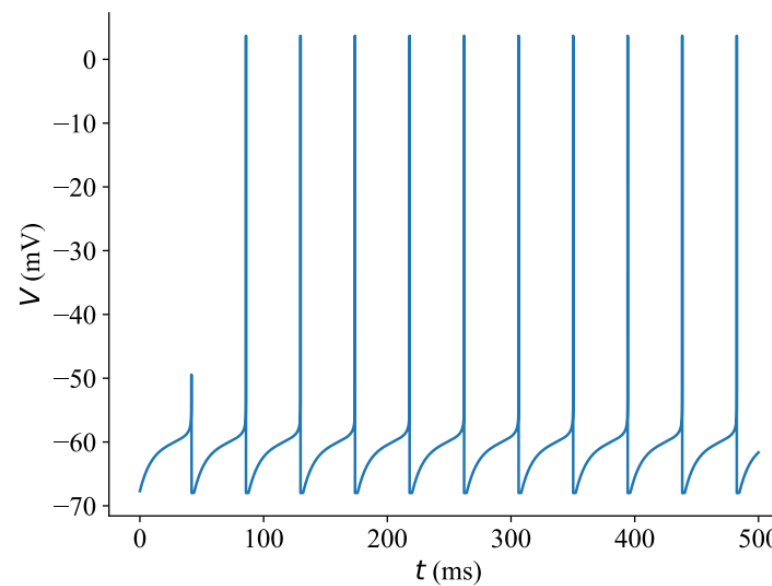
Other Univariate neuron models

- The Exponential Integrate-and-Fire (ExpIF) model:

$$\tau \frac{dV}{dt} = -(V - V_{\text{rest}}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} + RI(t)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$
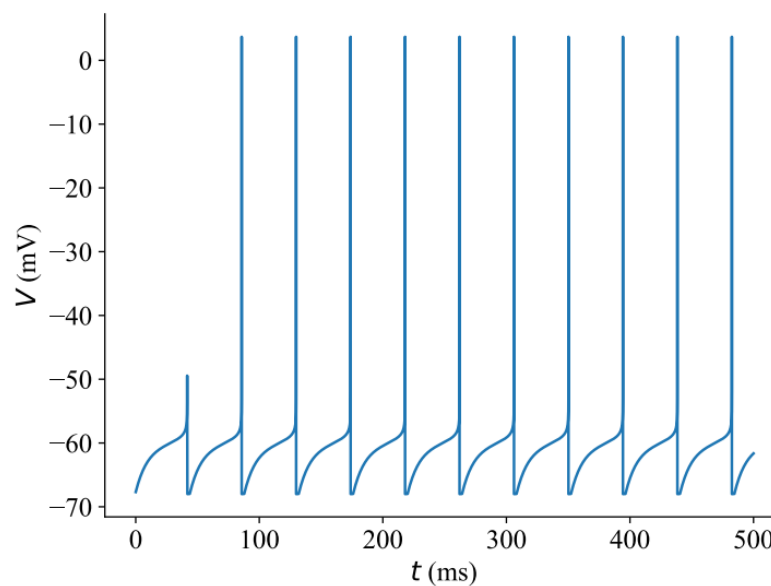
02

The Adaptive Exponential Integrate-and-Fire (AdEx) Neuron Model

The Exponential Integrate-and-Fire (ExpIF) neuron model

$$\tau \frac{dV}{dt} = -(V - V_{\text{rest}}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} + RI(t)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$

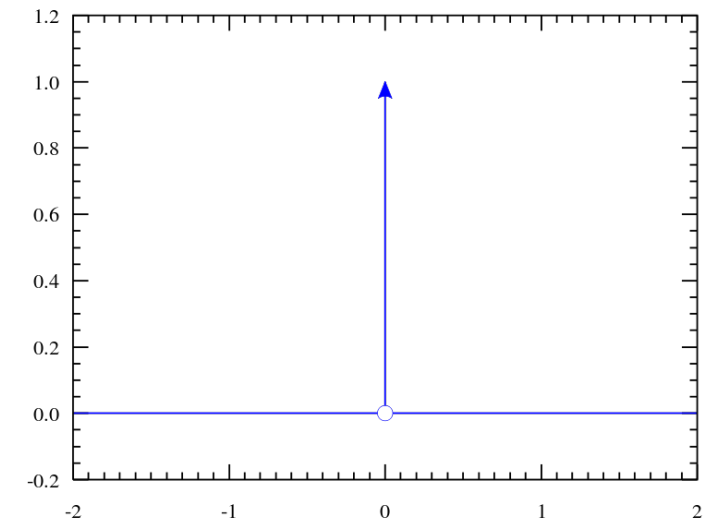Given a constant current input:

The AdEx neuron model

Two variables:

- V: membrane potential

- w: adaptation variable

$$\tau_m \frac{dV}{dt} = -(V - V_{\text{rest}}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{\text{rest}}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$

- A larger w suppresses V from increasing

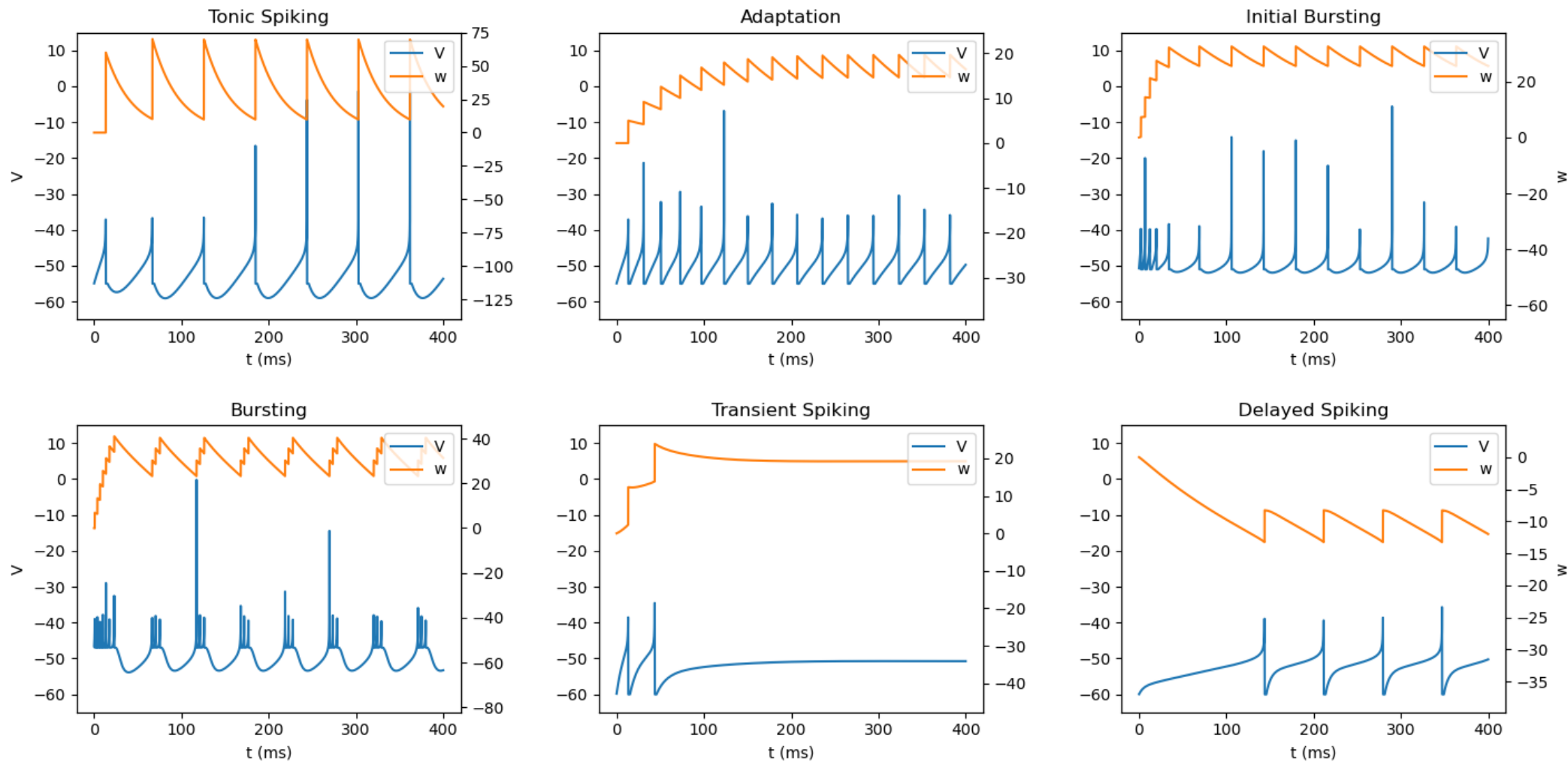- w decays exponentially while having a sudden increase when the neuron fires

δ function

Dirac delta function - Wikipedia

The AdEx neuron model

Firing patterns of the AdEx model:
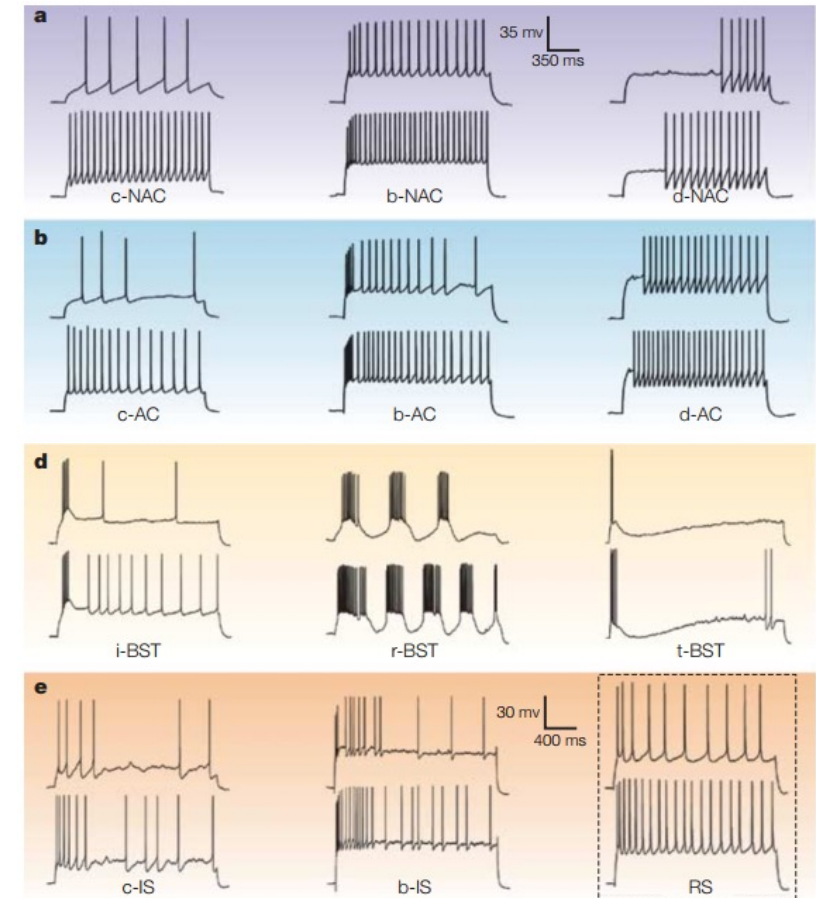
The AdEx neuron model

Categorization of firing patterns

According to the steady-state firing time intervals:

- Tonic/regular spiking

- Adapting

- Bursting

- Irregular spiking

According to the initial-state features:

- Tonic/classic spiking

- Initial burst
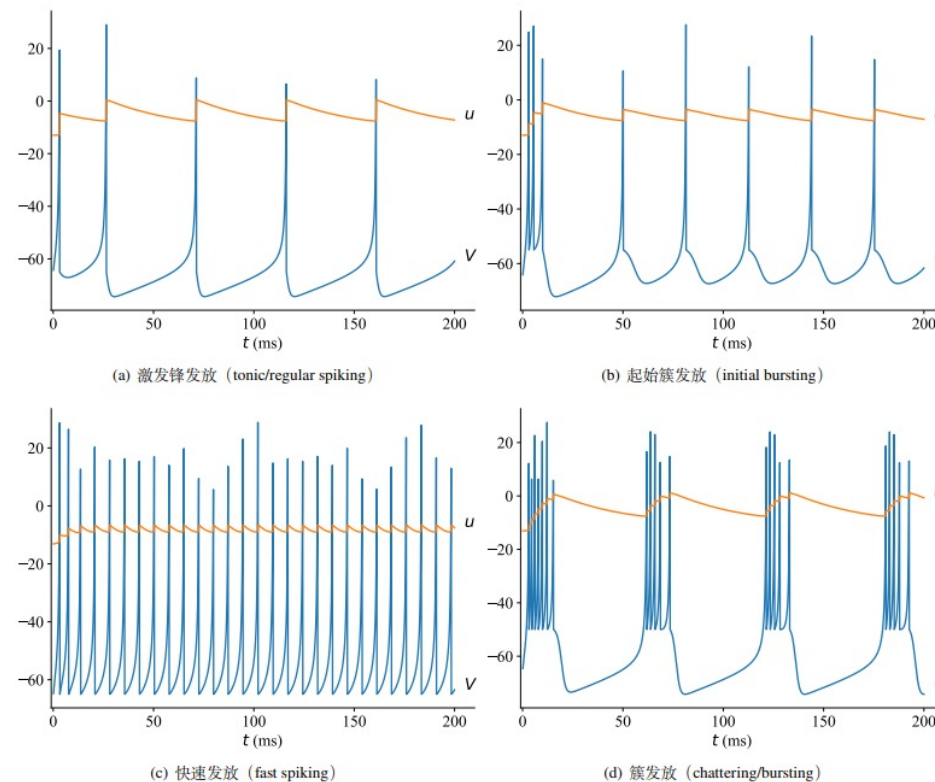
- Delayed spiking

Markram H, et al. Nat Rev Neurosci. 2004

Other multivariate neuron models

- The Izhikevich model:

$$\frac{dV}{dt} = 0.04V^2 + 5V + 140 - u + I$$

$$\frac{du}{dt} = a(bV - u)$$

$$\text{if } V > \theta, \quad V \leftarrow c, \ u \leftarrow u + d \ \text{ last } t_{ref}$$

(a) 激发锋发放（tonic/regular spiking）

(b) 起始簇发放（initial bursting）

(c) 快速发放（fast spiking）
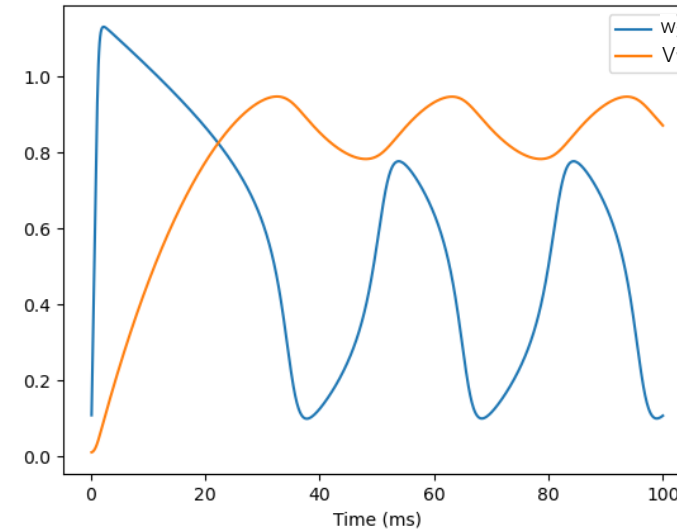
(d) 簇发放（chattering/bursting）

Other multivariate neuron models

- The FitzHugh–Nagumo (FHN) model

$$\dot{v} = v - \frac{v^3}{3} - w + RI_{\text{ext}}$$

$$\tau \dot{w} = v + a - bw.$$
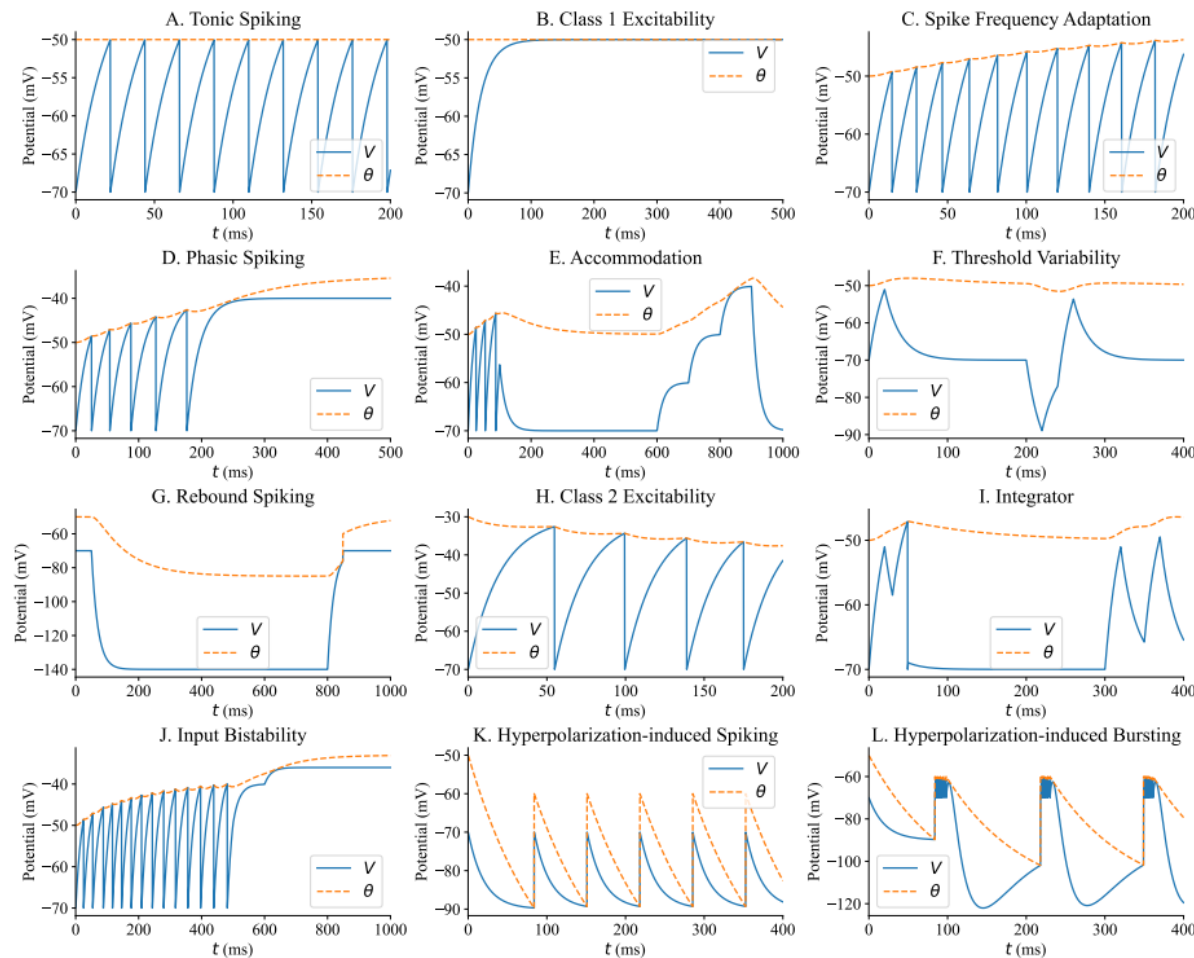
Other multivariate neuron models

- The Generalized Integrate-and-Fire (GIF) model:

$$\tau \frac{dV}{dt} = -(V - V_{\text{rest}}) + R \sum_j I_j + RI$$

$$\frac{d\Theta}{dt} = a(V - V_{\text{rest}}) - b(\Theta - \Theta_\infty)$$

$$\frac{dI_j}{dt} = -k_j I_j, \quad j = 1, 2, \dots, n$$

$$\text{if } V > \Theta, \quad I_j \leftarrow R_j I_j + A_j, \; V \leftarrow V_{\text{reset}}, \; \Theta \leftarrow max(\Theta_{\text{reset}}, \Theta)$$

03

Dynamic analysis: phase-plane analysis
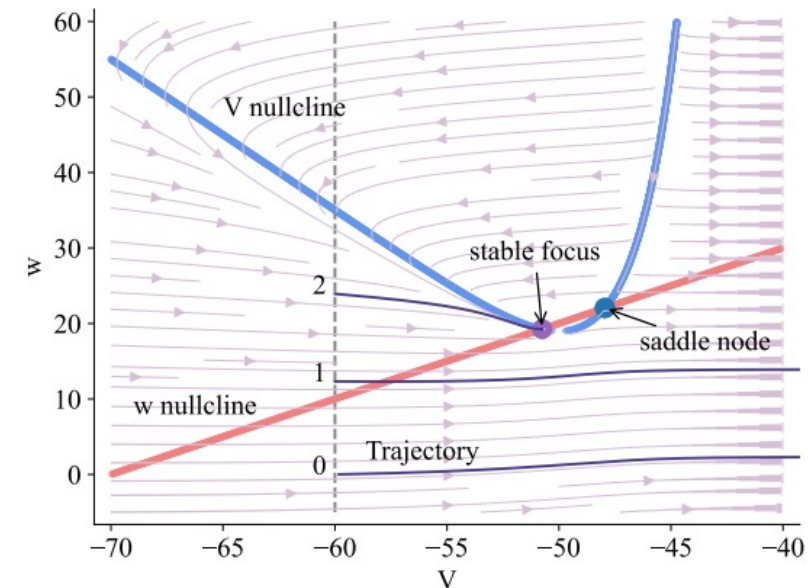
Phase plane analysis

Analyzes the behavior of a dynamical system with (usually two) variables described by ordinary differential equations

Elements:

- Nullclines: $dV/dt = 0; \; dw/dt = 0$

- Fixed points: $dV/dt = 0 \text{ and } dw/dt = 0$
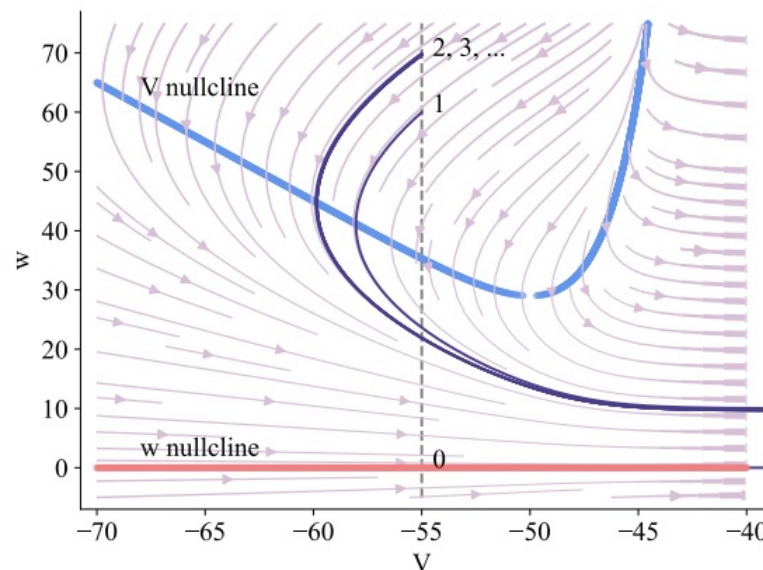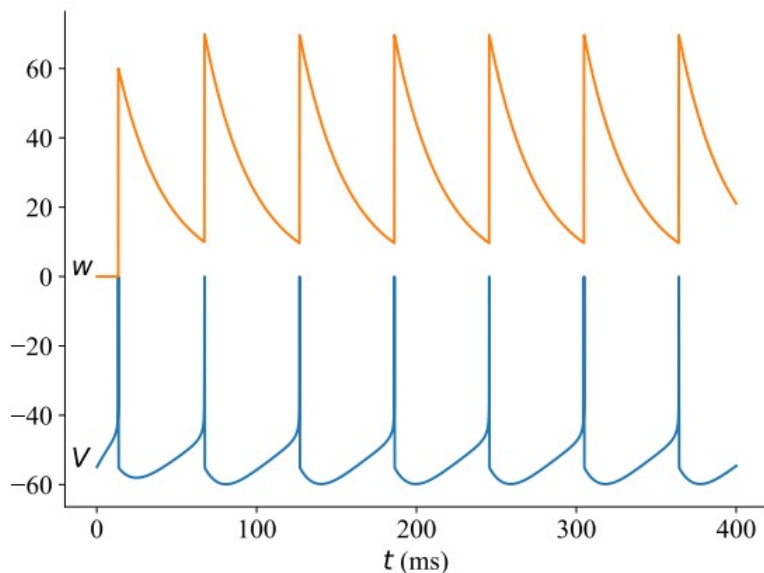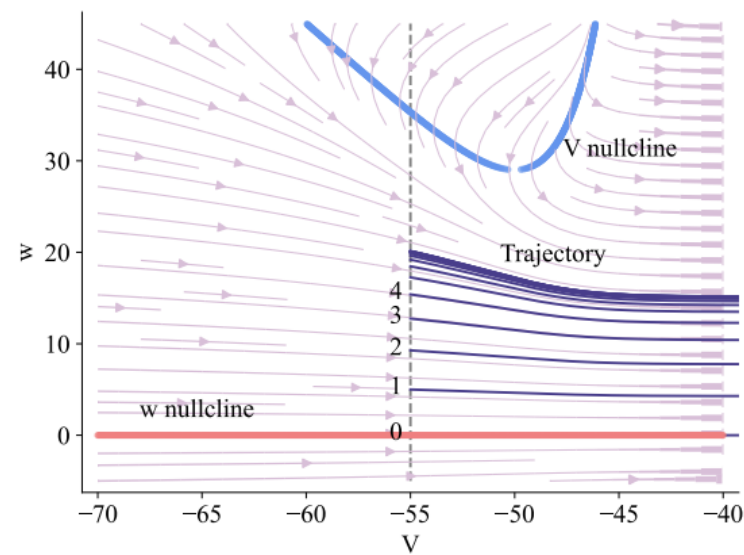
- The vector field

- The trajectory of variables

$$\tau_m \frac{dV}{dt} = -(V - V_{\text{rest}}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{\text{rest}}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \; \text{ last } \; t_{\text{ref}}$$

Phase plane analysis for the AdEx neuron model

1. Tonic spiking

$$\tau_m \frac{dV}{dt} = -(V - V_{rest}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{rest}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{reset} \text{ last } t_{ref}$$

表 3.1: AdEx 模型各种发放形式对应的参数

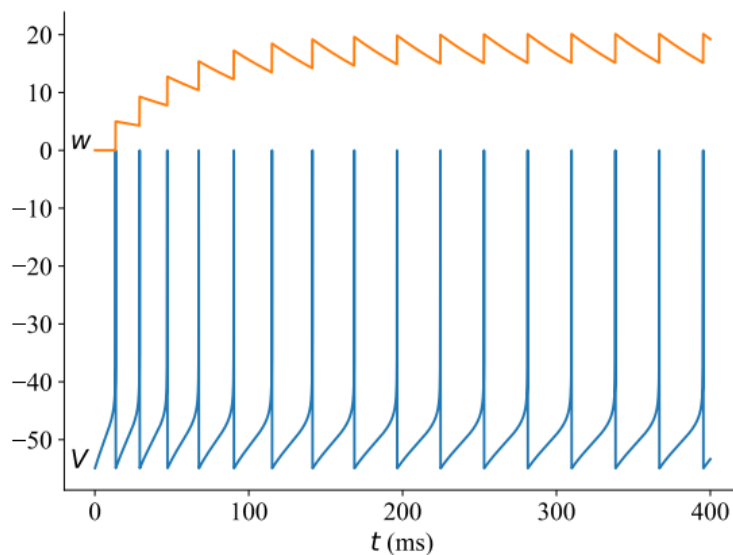| 发放形式	τ	τ_w	a	b	V_{reset}	I
激发锋发放	20	30	0	60	-55	65
适应	20	100	0	5	-55	65
起始簇发放	5	100	0.5	7	-51	65
簇发放	5	100	-0.5	7	-47	65
瞬时锋发放	10	100	1	10	-60	55
延迟发放	5	100	-1	5	-60	25

Phase plane analysis for the AdEx neuron model

2. Adaptation

$$\tau_m \frac{dV}{dt} = -(V - V_{rest}) + \Delta_T e^{\frac{V-V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{rest}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

if $V > \theta$, $V \leftarrow V_{reset}$ last t_{ref}

表 3.1: AdEx 模型各种发放形式对应的参数

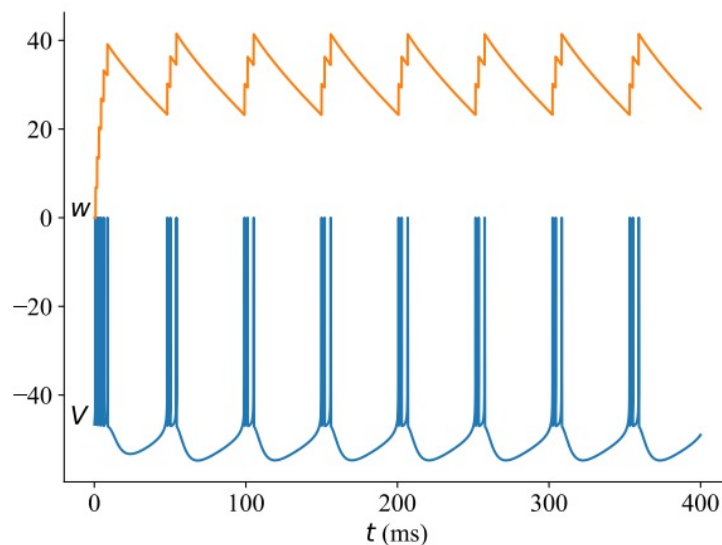| 发放形式	τ	τ_w	a	b	V_{reset}	I
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3. Bursting

$$\tau_m \frac{dV}{dt} = -(V - V_{rest}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

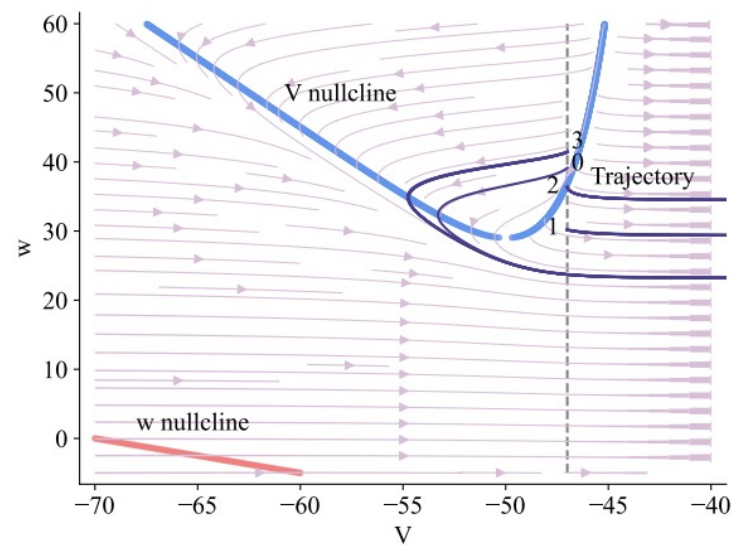$$\tau_w \frac{dw}{dt} = a(V - V_{rest}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

if $V > \theta$, $V \leftarrow V_{reset}$ last t_{ref}

表 3.1: AdEx 模型各种发放形式对应的参数

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延迟发放	5	100	-1	5	-60	25

(a) 发放模式

(b) 相平面分析
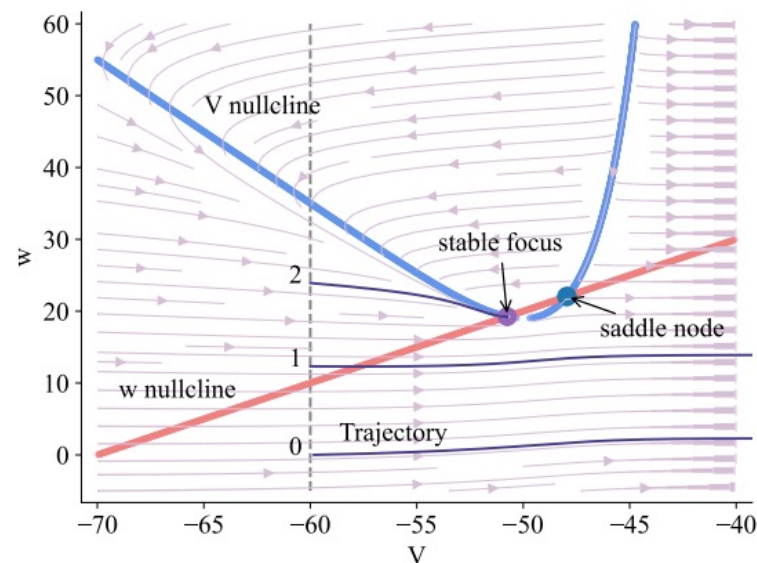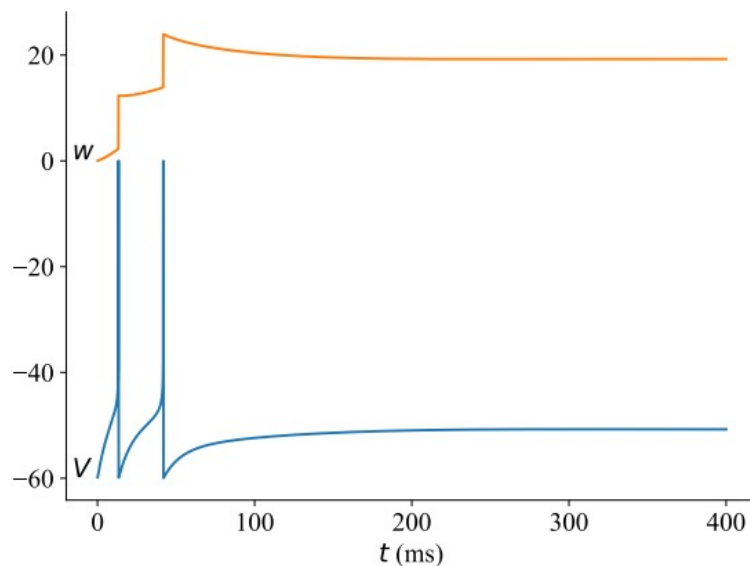
Phase plane analysis for the AdEx neuron model

4. Transient spiking

$$\tau_m \frac{dV}{dt} = -(V - V_{rest}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{rest}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

if $V > \theta$, $V \leftarrow V_{reset}$ last t_{ref}

表 3.1: AdEx 模型各种发放形式对应的参数

| 发放形式	τ	τ_w	a	b	V_{reset}	I
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04

Dynamic analysis: bifurcation analysis

Bifurcation analysis

Quantitative analysis of the existence and the properties of fixed points in a dynamical system with a changing parameter

$$\tau_m \frac{dV}{dt} = -(V - V_{\text{rest}}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{\text{rest}}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$

Elements:

- Lines of fixed points

- Stability properties of fixed points
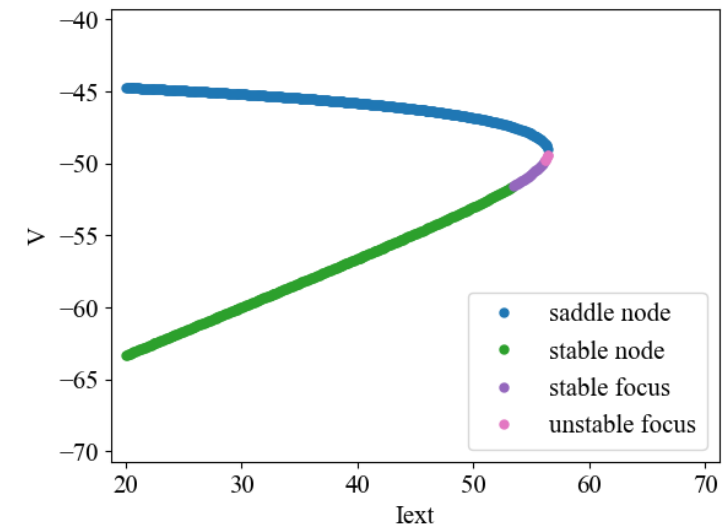
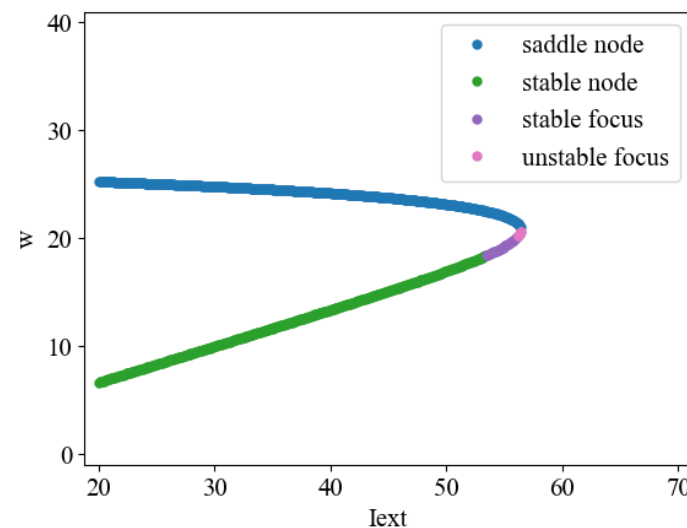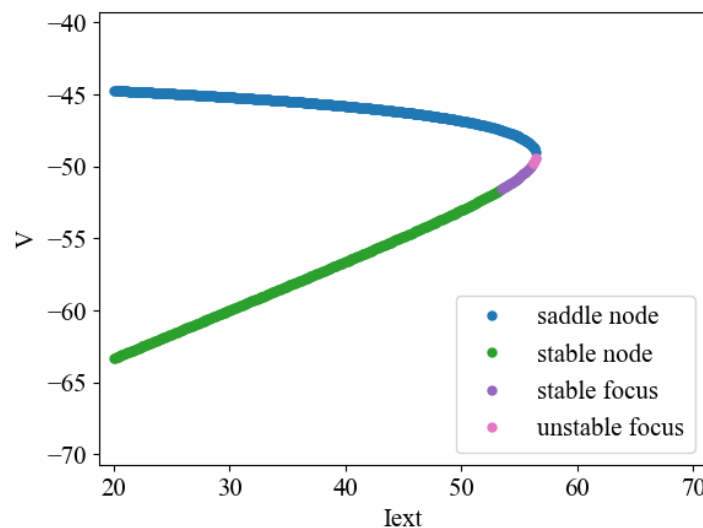# Bifurcation analysis for the AdEx neuron model

bifurcation analysis for 2 variables

Variables: V and w

Parameters: I_{ext}

$$\tau_m \frac{dV}{dt} = -(V - V_{\text{rest}}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{\text{rest}}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

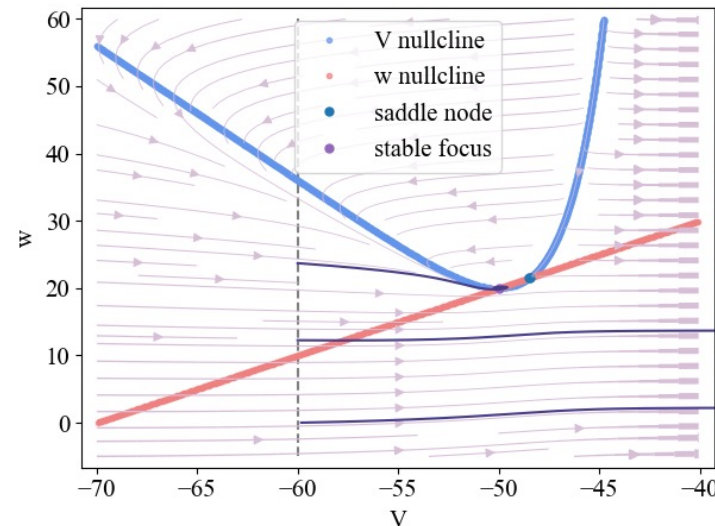$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$

Bifurcation analysis for the AdEx neuron model

Subjects: two variables (V and w)

$$\tau_m \frac{dV}{dt} = -(V - V_{\text{rest}}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{\text{rest}}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

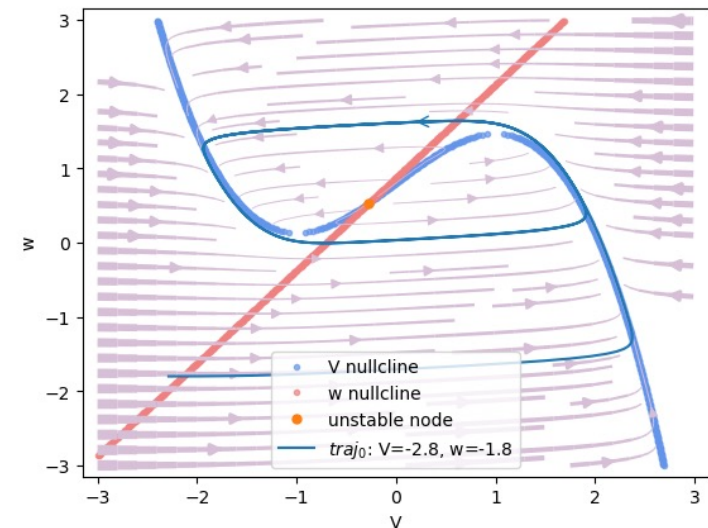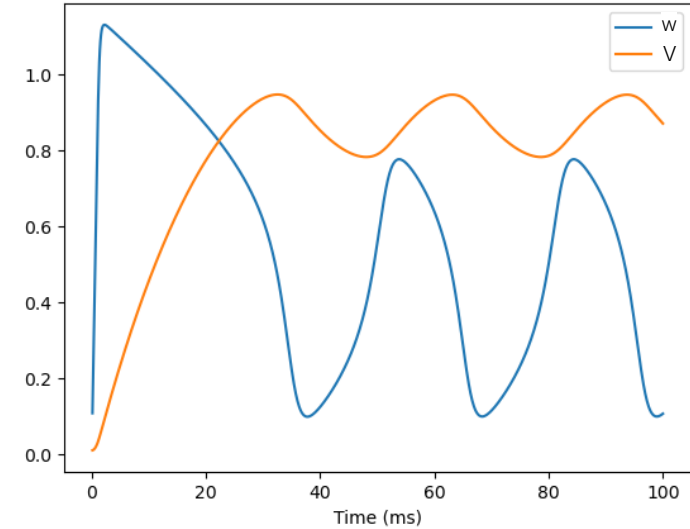$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$
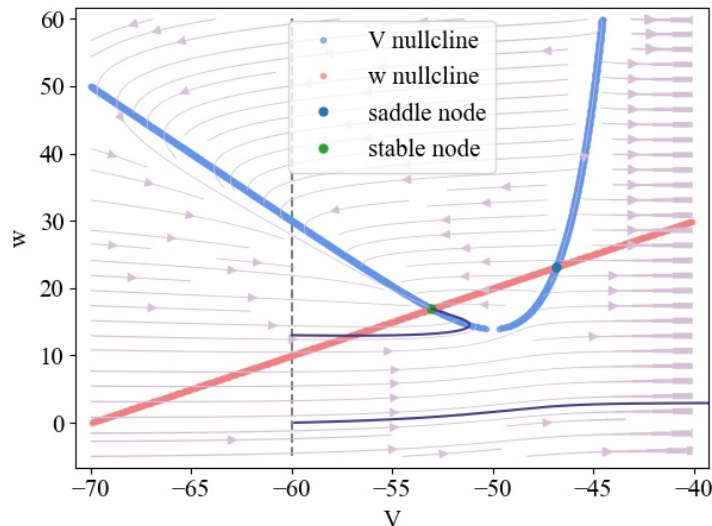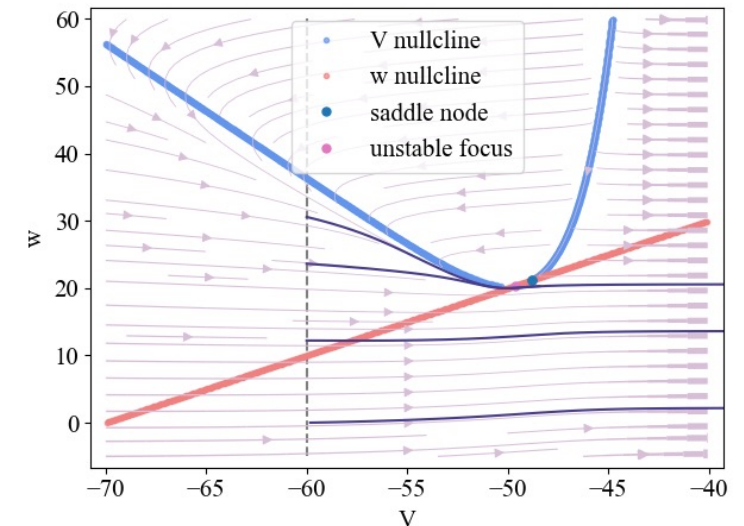
Extended: The limit cycle

The FitzHugh–Nagumo (FHN) model

$$\dot{v} = v - \frac{v^3}{3} - w + RI_{\text{ext}}$$

$$\tau\dot{w} = v + a - bw.$$

This dynamical system, in certain conditions, exhibits a cyclic pattern of variable changes which can be visualized as a closed trajectory in the phase plane.
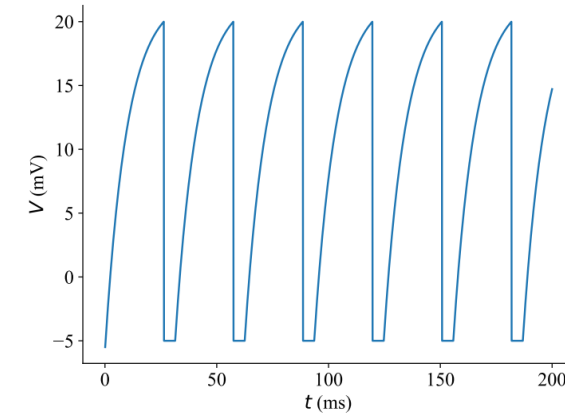
05
Summary

Strengths & weaknesses of the LIF model

- The Leaky Integrate-and-Fire (LIF) Neuron Model

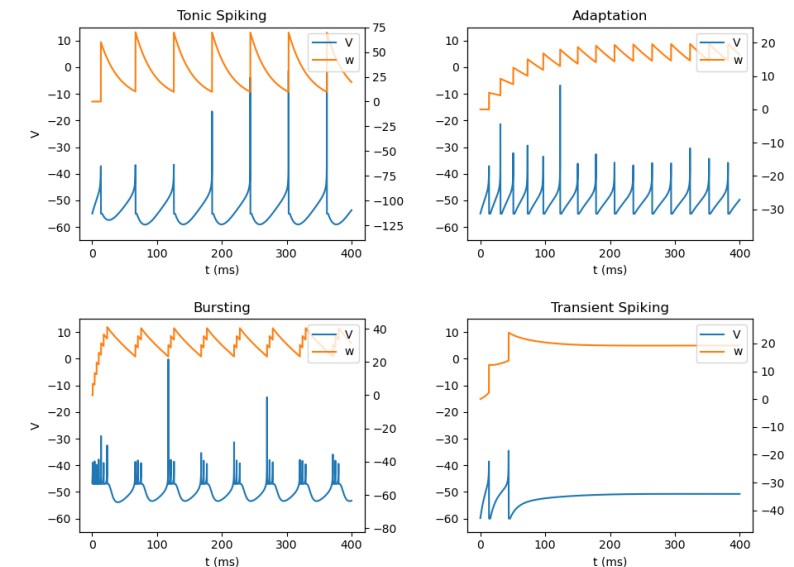$$\tau \frac{dV}{dt} = -(V - V_{\text{rest}}) + RI(t)$$

$$\text{if } V > V_{\text{th}}, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$

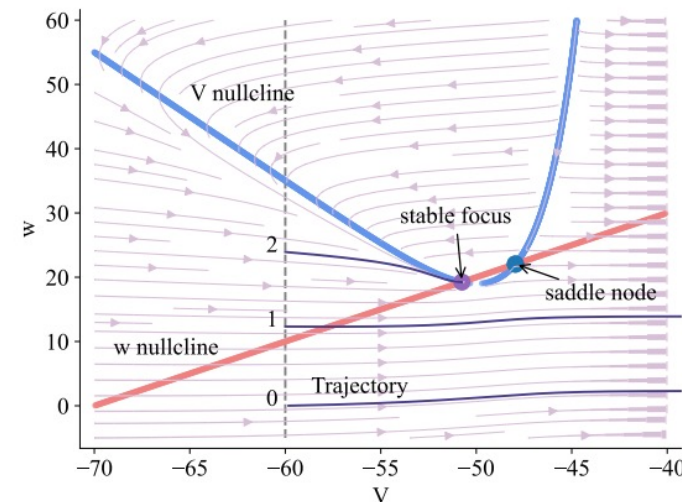- The Adaptive Exponential Integrate-and-Fire (AdEx) Neuron Model

$$\tau_m \frac{dV}{dt} = -(V - V_{\text{rest}}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{\text{rest}}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

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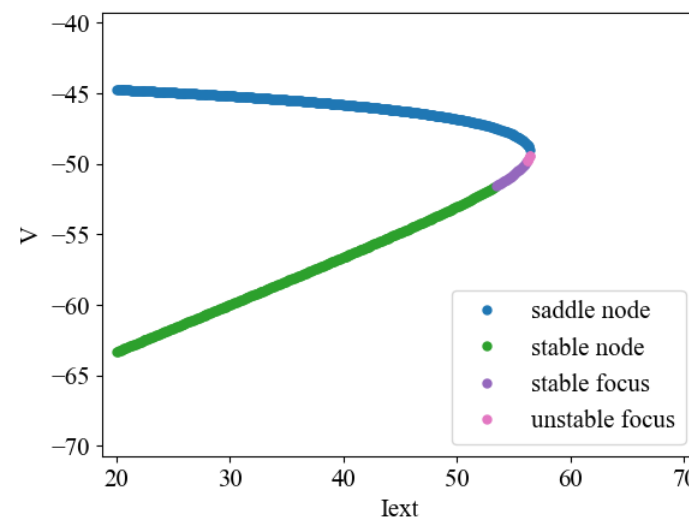
Strengths & weaknesses of the LIF model

- Dynamic analysis: phase-plane analysis

- Dynamic analysis: bifurcation analysis

THANK YOU

Aug. 25, 2023