



Single Neuron Modeling: Conductance-Based Models

Xiaoyu Chen

CONTENTS



- 01 | Neuronal structure, resting potential, and equivalent circuits
- 02 | Cable Theory & passive conduction
- 03 | Action potential & active transport
- 04 | The Hodgkin-Huxley Model
- 05 | Summary



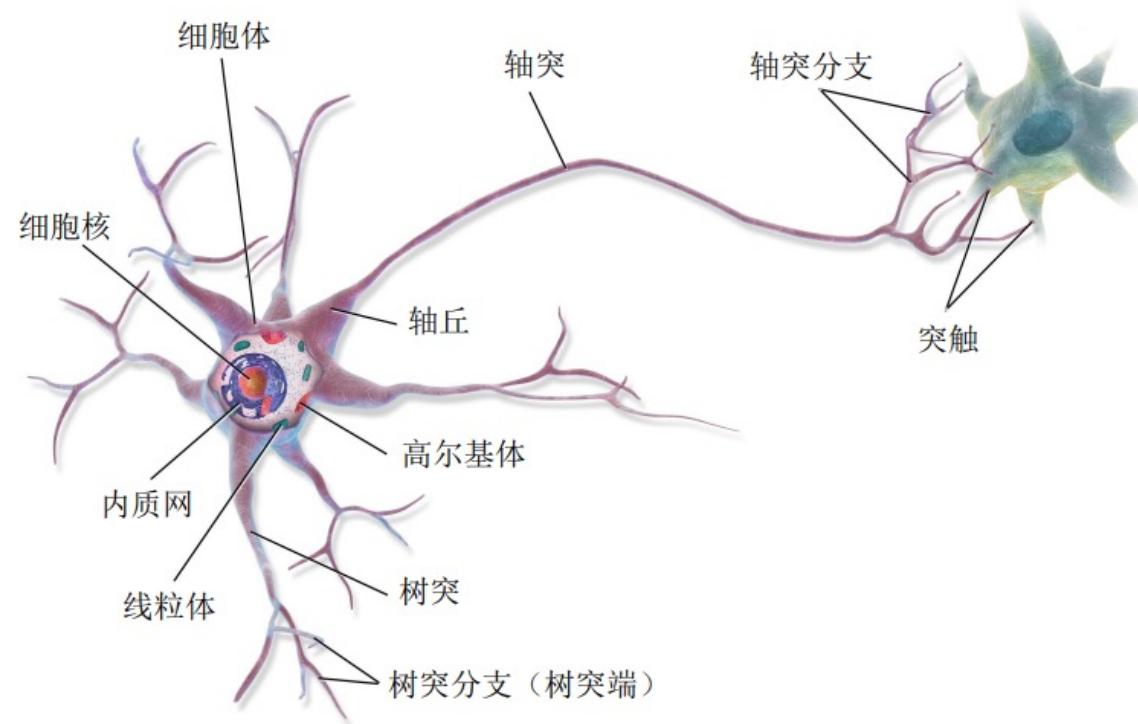
01

Neuronal structure, resting potential, and
equivalent circuits

Neuronal structure

Components of a neuron:

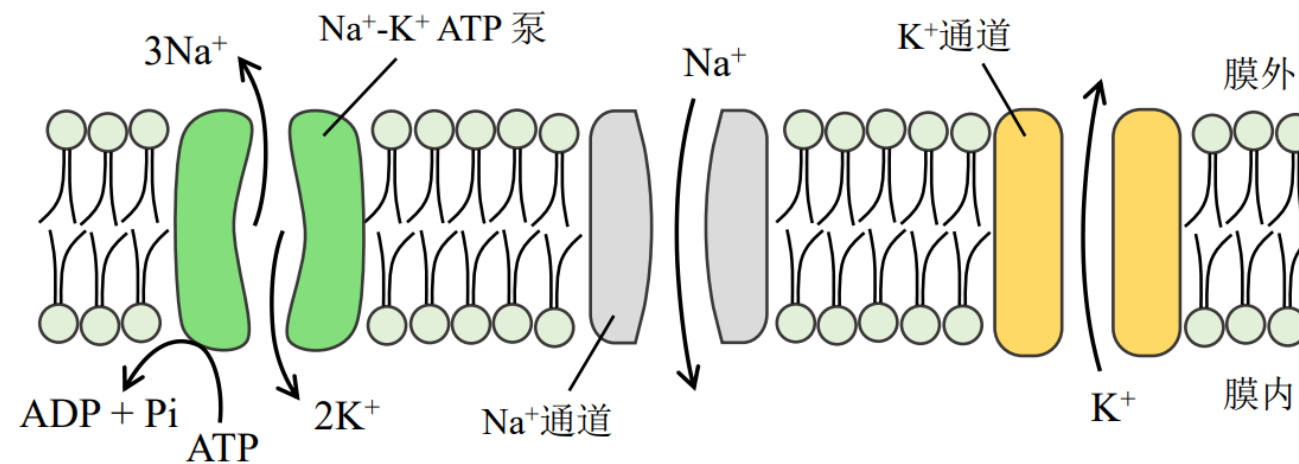
- Cell body/soma
- Axon
- Dendrites
- Synapses



Resting potential

Transport proteins for ions in neuron cell membranes:

- Ion channels: Na^+ channels, K^+ channels, ... (gated/non-gated)
- Ion pumps: the Na^+ - K^+ pump



Resting potential

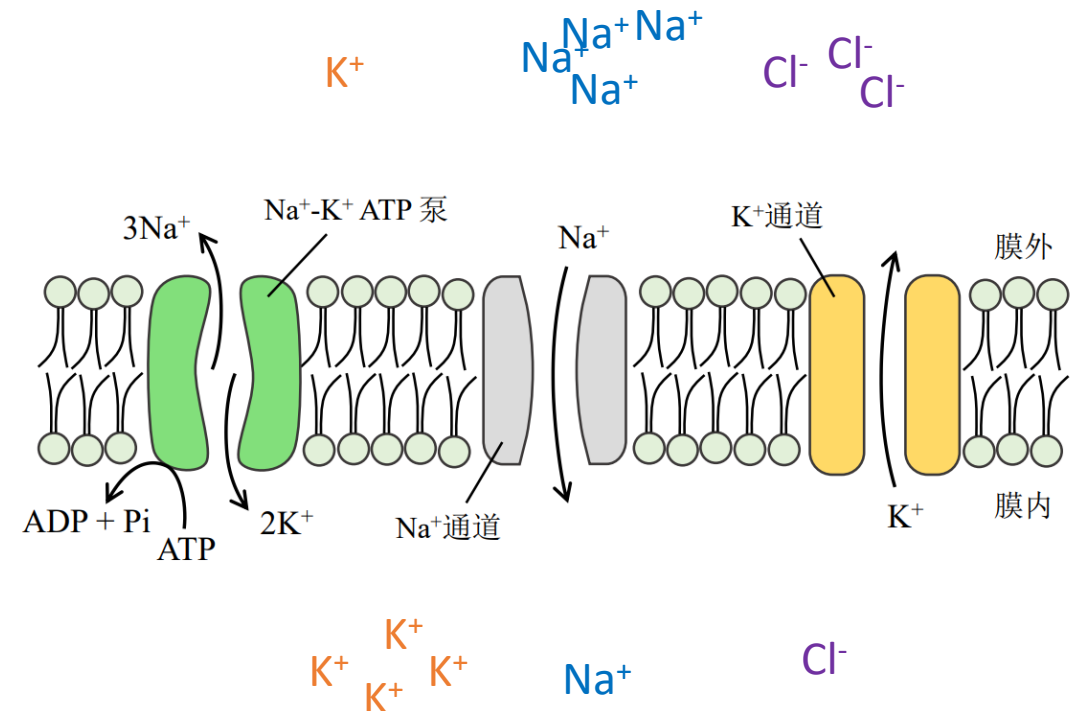
- Ion concentration difference → chemical gradient → electrical gradient

- Nernst Equation:

$$E = \frac{RT}{zF} \ln \frac{[\text{ion}]_{\text{out}}}{[\text{ion}]_{\text{in}}}$$

- Goldman-Hodgkin-Katz (GHK) Equation:

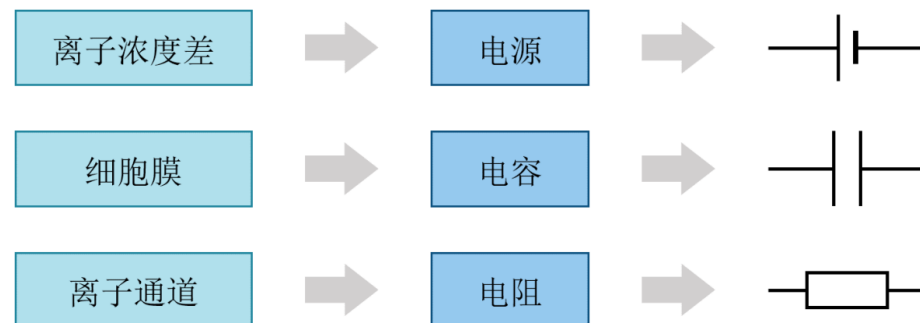
$$V_m = \frac{RT}{F} \ln \left(\frac{P_{\text{Na}}[\text{Na}^+]_{\text{out}} + P_{\text{K}}[\text{K}^+]_{\text{out}} + P_{\text{Cl}}[\text{Cl}^-]_{\text{in}}}{P_{\text{Na}}[\text{Na}^+]_{\text{in}} + P_{\text{K}}[\text{K}^+]_{\text{in}} + P_{\text{Cl}}[\text{Cl}^-]_{\text{out}}} \right)$$



Equivalent circuits

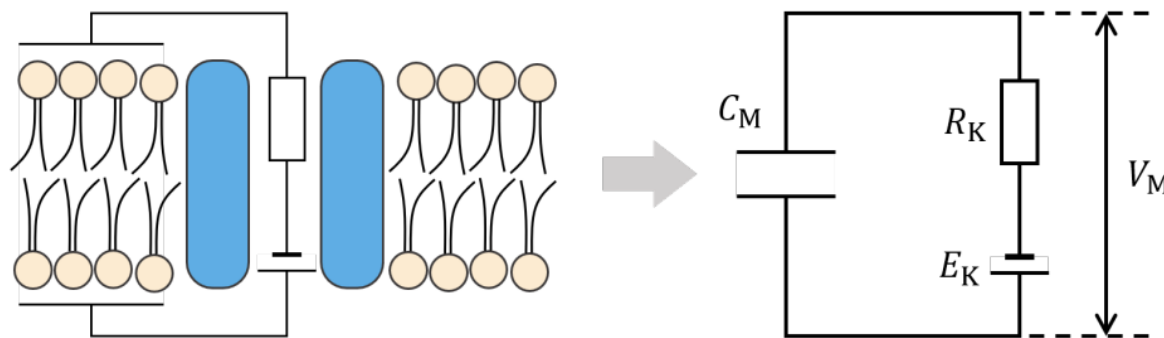
Components of an equivalent circuit:

- Battery
- Capacitor
- Resistor



Considering the potassium channel **ONLY**:

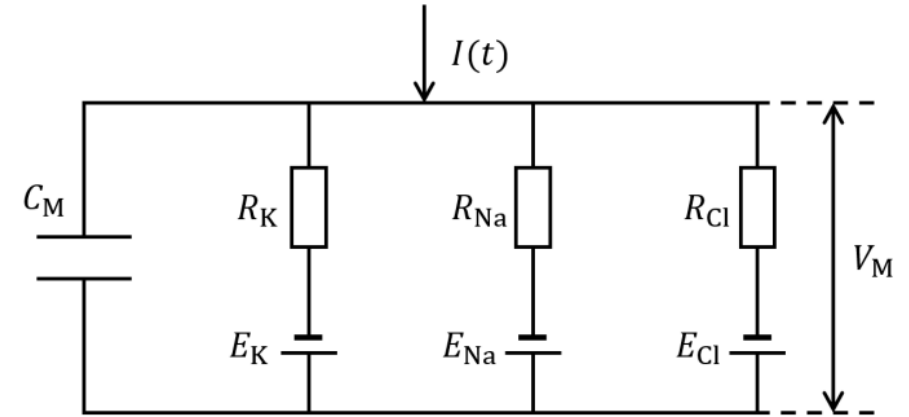
$$0 = I_{\text{cap}} + I_K = c_M \frac{dV_M}{dt} + \frac{V_M - E_K}{R_K},$$
$$c_M \frac{dV_M}{dt} = -\frac{V_M - E_K}{R_K} = -g_K (V_M - E_K).$$



Equivalent circuits

Considering the Na^+ , K^+ , and Cl^- channels and the external current $I(t)$:

$$\frac{I(t)}{A} = c_M \frac{dV_M}{dt} + i_{\text{ion}}$$
$$\Rightarrow c_M \frac{dV_M}{dt} = -g_{\text{Cl}}(V_M - E_{\text{Cl}}) - g_{\text{K}}(V_M - E_{\text{K}}) - g_{\text{Na}}(V_M - E_{\text{Na}}) + \frac{I(t)}{A}$$



Steady-state membrane potential given a constant current input I :

$$\Rightarrow c_M \frac{dV_M}{dt} = -(g_{\text{Cl}} + g_{\text{K}} + g_{\text{Na}})V_M + g_{\text{Cl}}E_{\text{Cl}} + g_{\text{K}}E_{\text{K}} + g_{\text{Na}}E_{\text{Na}} + \frac{I(t)}{A}$$

$$V_{ss} = \frac{g_{\text{Cl}}E_{\text{Cl}} + g_{\text{K}}E_{\text{K}} + g_{\text{Na}}E_{\text{Na}} + I/A}{g_{\text{Cl}} + g_{\text{K}} + g_{\text{Na}}} \xrightarrow{I=0} V_{ss, I=0} = E_R = \frac{g_{\text{Cl}}E_{\text{Cl}} + g_{\text{K}}E_{\text{K}} + g_{\text{Na}}E_{\text{Na}}}{g_{\text{Cl}} + g_{\text{K}} + g_{\text{Na}}}$$



北京大学
PEKING UNIVERSITY



02

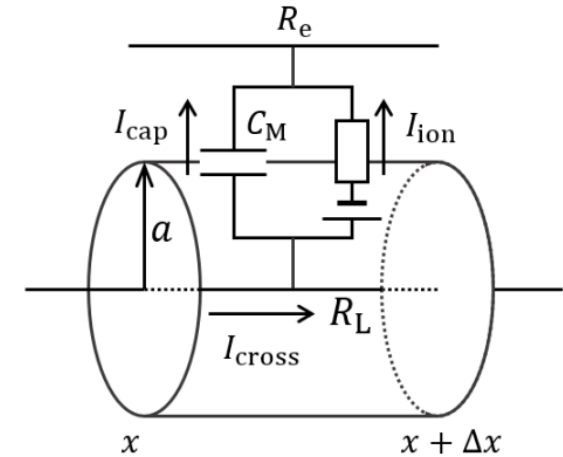
Cable Theory & passive conduction

Cable Theory

How electrical signals are transmitted along a single neuron (an axon)?

Considering the axon as a long cylindrical cable:

$$I_{\text{cross}}(x, t) = I_{\text{cross}}(x + \Delta x, t) + I_{\text{ion}}(x, t) + I_{\text{cap}}(x, t)$$



$$V(x + \Delta x, t) - V(x, t) = -I_{\text{cross}}(x, t)R_L = -I_{\text{cross}}(x, t)\frac{\Delta x}{\pi a^2}\rho_L$$

$$I_{\text{cross}}(x, t) = -\frac{\pi a^2}{\rho_L} \frac{\partial V(x, t)}{\partial x}$$

$$I_{\text{ion}} = (2\pi a \Delta x) i_{\text{ion}}$$

$$I_{\text{cap}}(x, t) = (2\pi a \Delta x) c_M \frac{\partial V(x, t)}{\partial t}$$

$$(2\pi a \Delta x) c_M \frac{\partial V(x, t)}{\partial t} + (2\pi a \Delta x) i_{\text{ion}} = \frac{\pi a^2}{\rho_L} \frac{\partial V(x + \Delta x, t)}{\partial x} - \frac{\pi a^2}{\rho_L} \frac{\partial V(x, t)}{\partial x}$$

$$c_M \frac{\partial V(x, t)}{\partial t} = \frac{a}{2\rho_L} \frac{\partial^2 V(x, t)}{\partial x^2} - i_{\text{ion}}$$

Cable Equation

Cable Theory

$$c_M \frac{\partial V(x, t)}{\partial t} = \frac{a}{2\rho_L} \frac{\partial^2 V(x, t)}{\partial x^2} - i_{\text{ion}}$$



Cable Equation

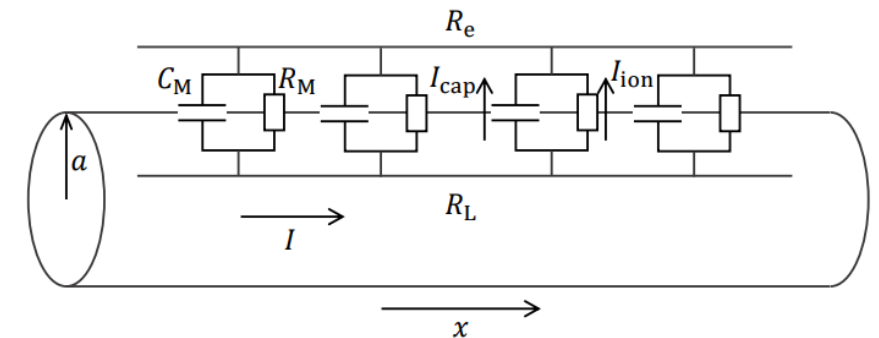
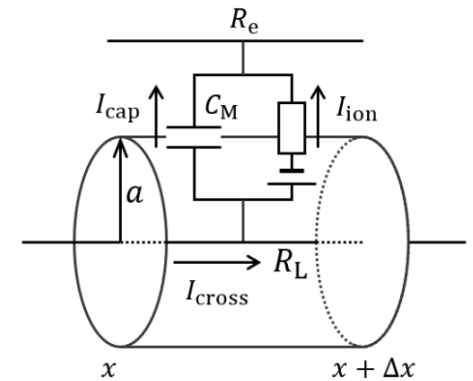
Passive conduction: ion currents are caused by leaky channels exclusively

$$i_{\text{ion}} = V(x, t)/r_M \quad \Rightarrow \quad c_M \frac{\partial V(x, t)}{\partial t} = \frac{a}{2\rho_L} \frac{\partial^2 V(x, t)}{\partial x^2} - \frac{V(x, t)}{r_M}$$

$$\tau \frac{\partial V(x, t)}{\partial t} = \lambda^2 \frac{\partial^2 V(x, t)}{\partial x^2} - V(x, t) \quad \lambda = \sqrt{0.5ar_M/\rho_L}$$

If a constant external current is applied to $x = 0$
the steady-state membrane potential $V_{ss}(x)$ is

$$\lambda^2 \frac{d^2 V_{ss}(x)}{dx^2} - V_{ss}(x) = 0 \quad \xrightarrow{I_{\text{cross}}(0, t) = I_0} \quad V_{ss}(x) = \frac{\lambda \rho_L}{\pi a^2} I_0 e^{-x/\lambda}$$



Cable Theory

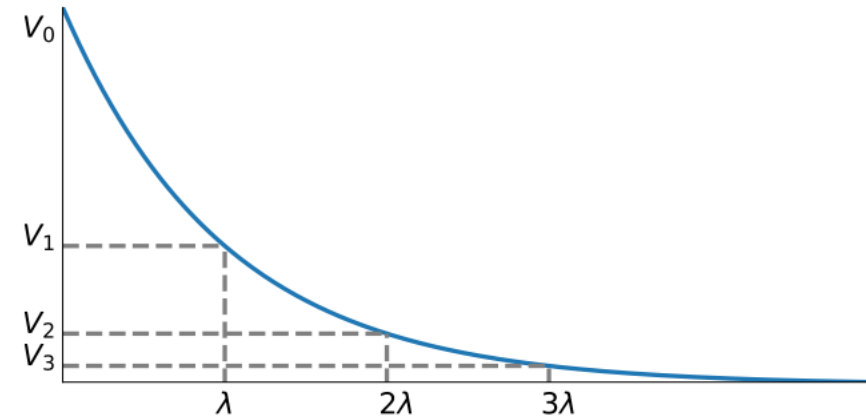
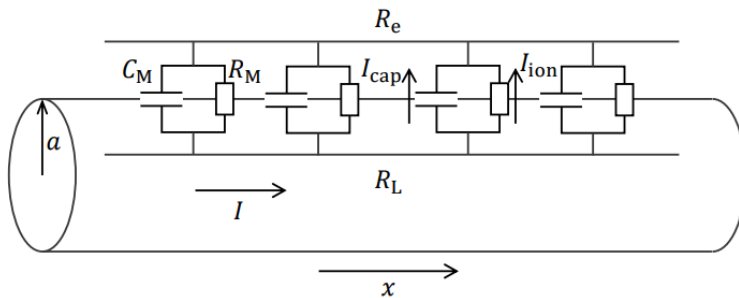
$$c_M \frac{\partial V(x, t)}{\partial t} = \frac{a}{2\rho_L} \frac{\partial^2 V(x, t)}{\partial x^2} - i_{\text{ion}}$$



Cable Equation

Passive conduction: ion currents are caused by leaky channels exclusively

$$V_{\text{ss}}(x) = \frac{\lambda \rho_L}{\pi a^2} I_0 e^{-x/\lambda}$$





03

Action potential & active transport

Action potential

How to transmit electrical signals with less or no decay?

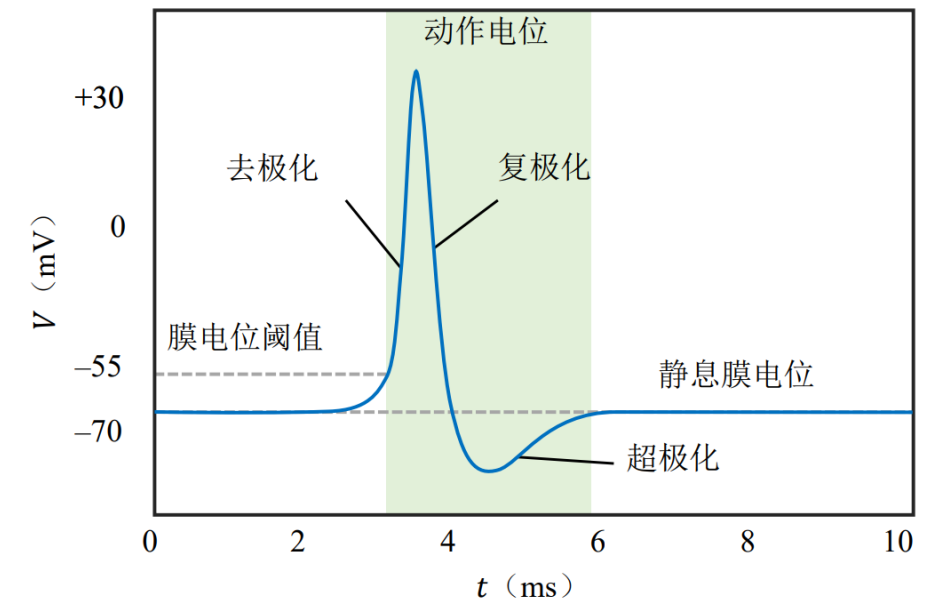
Steps of an action potential:

- Depolarization
- Repolarization
- Hyperpolarization
- Resting

Characteristics:

- All-or-none
- Fixed shape
- Active electrical property

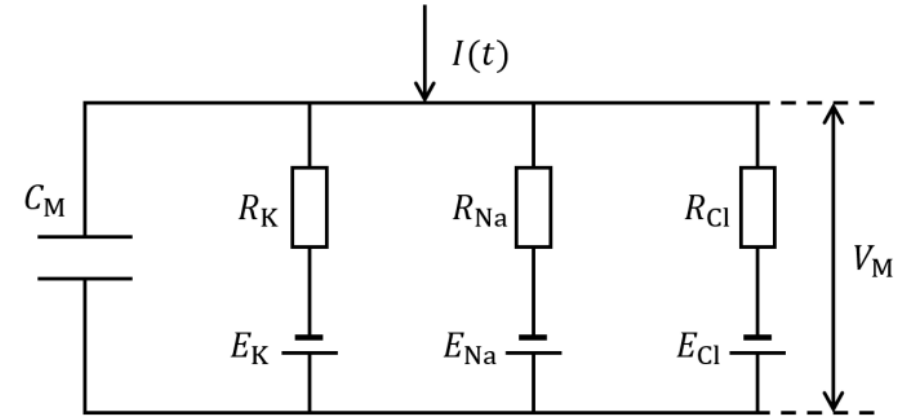
ACTION POTENTIAL



Action potential

How to simulate an action potential?

$$\frac{I(t)}{A} = c_M \frac{dV_M}{dt} + i_{\text{ion}}$$
$$\Rightarrow c_M \frac{dV_M}{dt} = -g_{\text{Cl}}(V_M - E_{\text{Cl}}) - g_{\text{K}}(V_M - E_{\text{K}}) - g_{\text{Na}}(V_M - E_{\text{Na}}) + \frac{I(t)}{A}$$



Mechanism: **voltage-gated ion channels**

$$g_{\text{Na}} \longrightarrow g_{\text{Na}}(V)$$

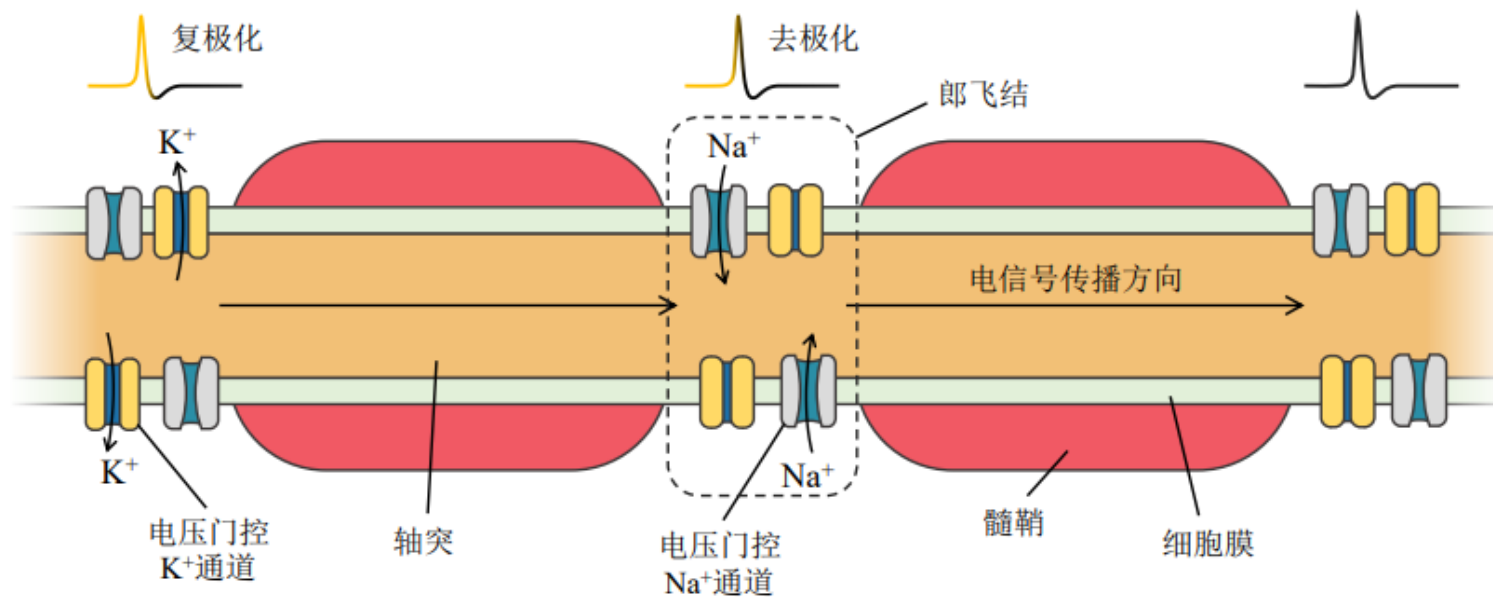
$$g_{\text{K}} \longrightarrow g_{\text{K}}(V)$$

$$g_{\text{Cl}} \longrightarrow g_{\text{Cl}}(V)$$

How would the conductance change with voltage?

Nodes of Ranvier

Saltatory conduction with a much higher speed and less energy consumption





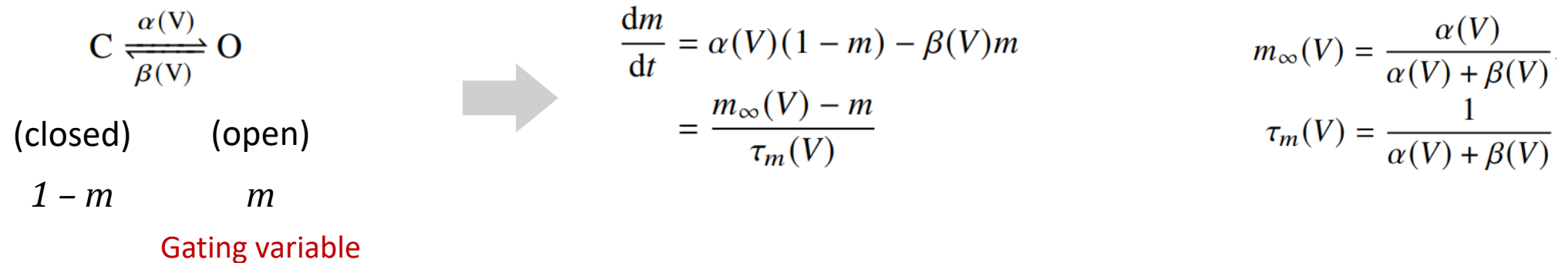
04

The Hodgkin-Huxley (HH) Model

Modeling of ion channels

Modeling of each ion channel: $g_m = \bar{g}_m m^x$

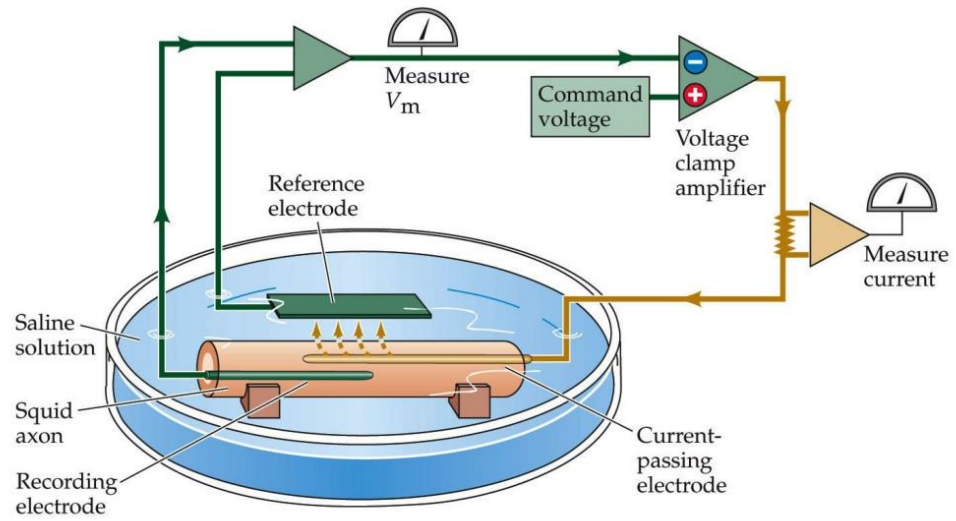
Modeling of each ion gate:



If V is constant: $m(t) = m_\infty(V) + (m_0 - m_\infty(V))e^{-t/\tau_m(V)}$

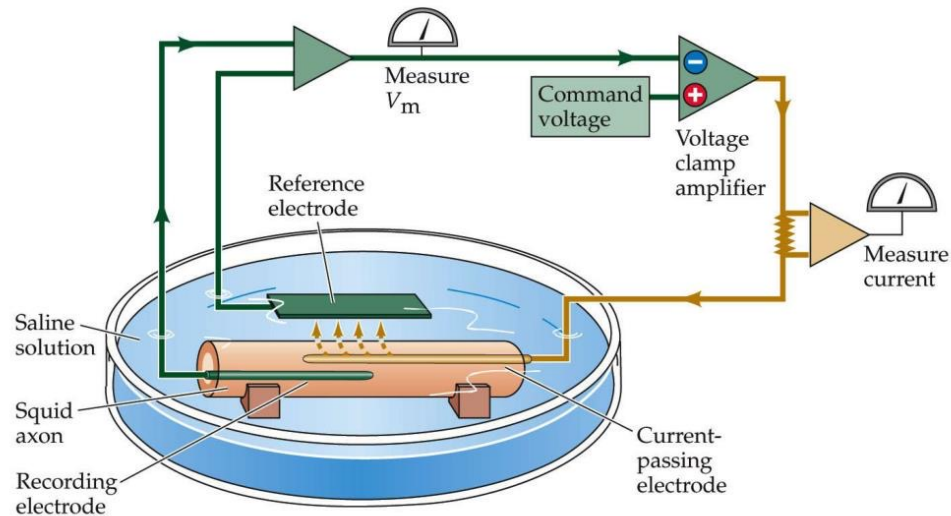
How to measure $m_\infty(V)$ and $\tau_m(V)$?

Voltage clamp



Voltage clamp

$$\frac{I(t)}{A} = c_M \frac{dV_M}{dt} + i_{\text{ion}}$$
$$\Rightarrow c_M \frac{dV_M}{dt} = -g_{\text{Cl}}(V_M - E_{\text{Cl}}) - g_{\text{K}}(V_M - E_{\text{K}}) - g_{\text{Na}}(V_M - E_{\text{Na}}) + \frac{I(t)}{A}$$



- The membrane potential is kept constant
- The current from capacitors is excluded
- Currents must come from leaky/voltage-gated ion channels

$$I_{\text{cap}} = c \frac{dV}{dt} = 0$$

$$I_{\text{fb}} = i_{\text{ion}} = g_{\text{Na}}(V - E_{\text{Na}}) + g_{\text{K}}(V - E_{\text{K}}) + g_{\text{L}}(V - E_{\text{L}})$$

1. Leaky channels

Hyperpolarization → the sodium and potassium channels are closed

$$I_{\text{fb}} = g_{\text{Na}}(V - E_{\text{Na}}) + g_{\text{K}}(V - E_{\text{K}}) + g_{\text{L}}(V - E_{\text{L}})$$



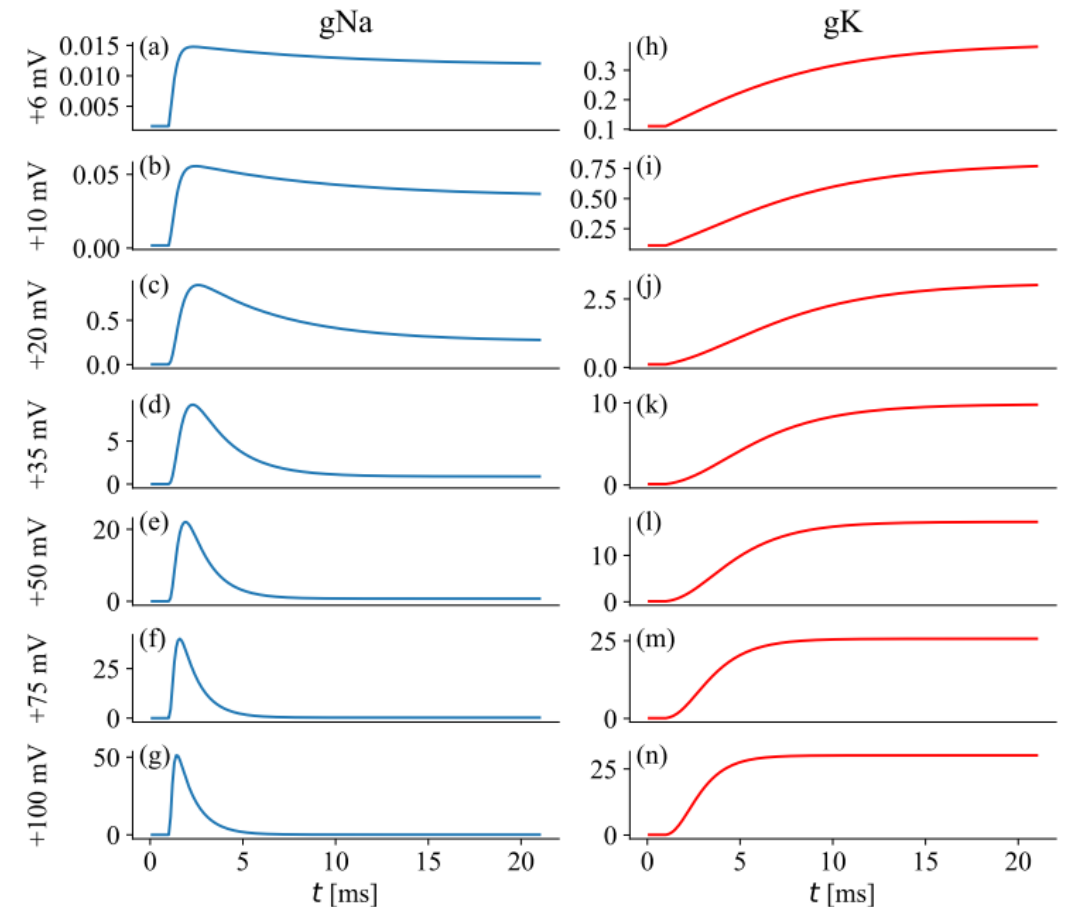
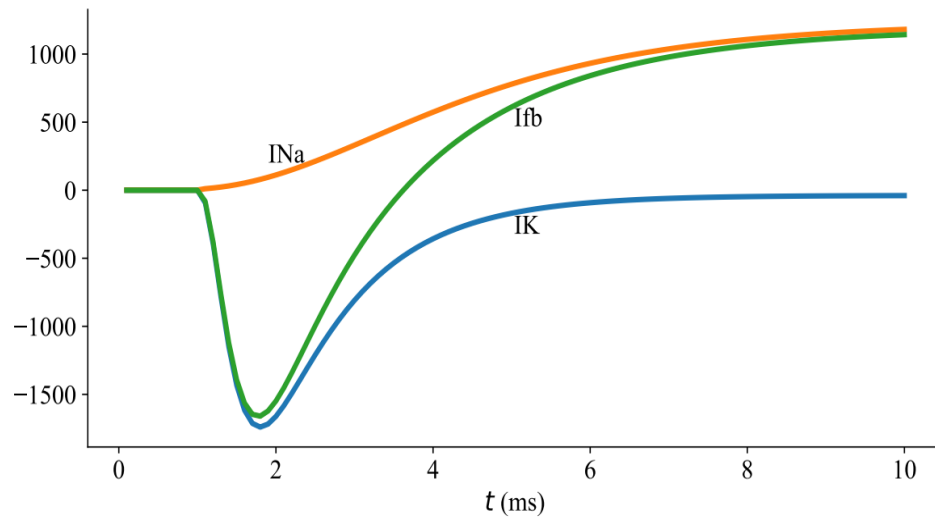
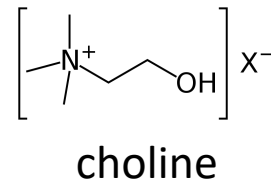
$$I_{\text{fb}} = g_{\text{L}}(V - E_{\text{L}})$$

$$g_{\text{L}} = 0.3 \text{ mS/cm}^2, E_{\text{L}} = -54.4 \text{ mV}$$

2. Potassium and sodium channels

Potassium channels: Use choline to eliminate the inward current of Na^+

Na^+ current: $I_{\text{fb}} - I_{\text{K}}$



2. Potassium and sodium channels

Potassium channels

- Resting state (gate closed)
- Activated state (gate open)



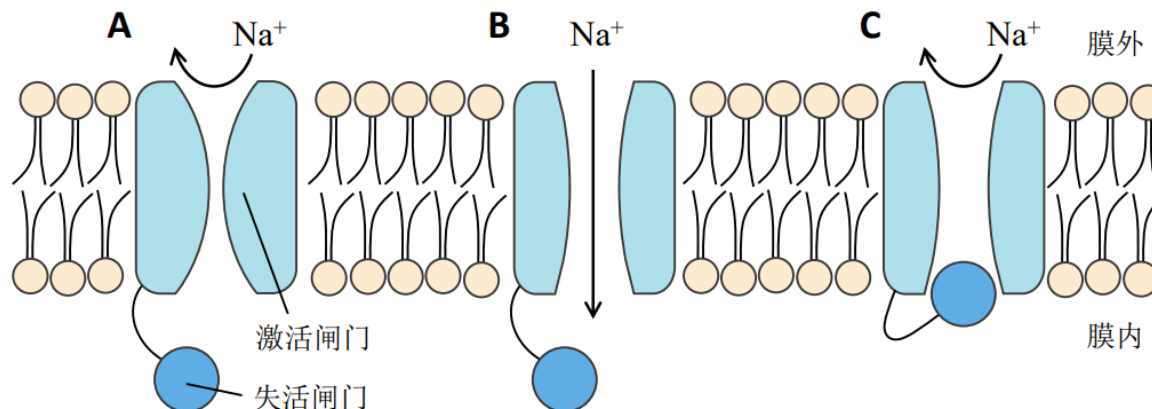
Activation gate: $g_K = \bar{g}_K n^x$

Sodium channels

- Resting state (gate closed)
- Activated state (gate open)
- Inactivated state (gate blocked)



Activation gate + inactivation gate: $g_{Na} = \bar{g}_{Na} m^3 h$



The gates of sodium channels

2. Potassium and sodium channels

Modeling of each ion gate:

$$g_K = \bar{g}_K n^x$$

$$g_{Na} = \bar{g}_{Na} m^3 h$$

$$\frac{dn}{dt} = \alpha_n(V)(1 - n) - \beta_n(V)n$$

$$\frac{dm}{dt} = \alpha_m(V)(1 - m) - \beta_m(V)m$$

$$\frac{dh}{dt} = \alpha_h(V)(1 - h) - \beta_h(V)h$$

$$\begin{aligned}\frac{dm}{dt} &= \alpha(V)(1 - m) - \beta(V)m \\ &= \frac{m_\infty(V) - m}{\tau_m(V)}\end{aligned}$$

$$\begin{aligned}m_\infty(V) &= \frac{\alpha(V)}{\alpha(V) + \beta(V)} \\ \tau_m(V) &= \frac{1}{\alpha(V) + \beta(V)}\end{aligned}$$

$$m(t) = m_\infty(V) + (m_0 - m_\infty(V))e^{-t/\tau_m(V)}$$

The Hodgkin-Huxley (HH) Model



$$c_M \frac{dV_M}{dt} = -g_{Cl}(V_M - E_{Cl}) - g_K(V_M - E_K) - g_{Na}(V_M - E_{Na}) + \frac{I(t)}{A}$$

$$\left\{ \begin{array}{l} c \frac{dV}{dt} = -\bar{g}_{Na} m^3 h (V - E_{Na}) - \bar{g}_K n^4 (V - E_K) - \bar{g}_L (V - E_L) + I_{ext}, \\ \frac{dn}{dt} = \phi [\alpha_n(V)(1 - n) - \beta_n(V)n] \\ \frac{dm}{dt} = \phi [\alpha_m(V)(1 - m) - \beta_m(V)m], \\ \frac{dh}{dt} = \phi [\alpha_h(V)(1 - h) - \beta_h(V)h], \end{array} \right.$$

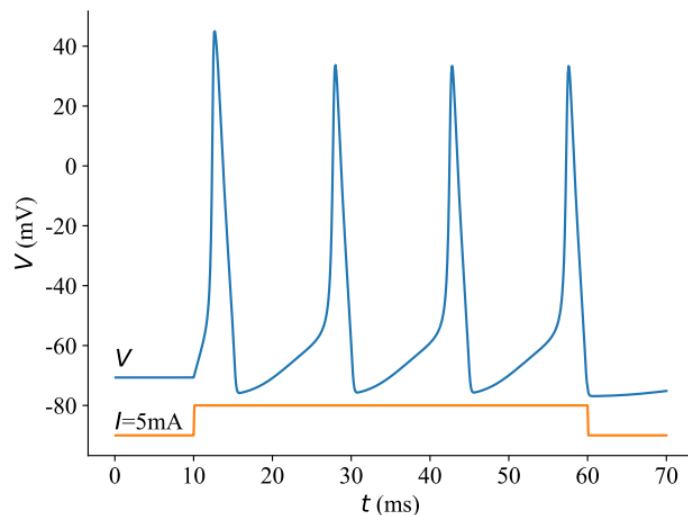
$$\alpha_n(V) = \frac{0.01(V + 55)}{1 - \exp\left(-\frac{V+55}{10}\right)}, \quad \beta_n(V) = 0.125 \exp\left(-\frac{V + 65}{80}\right),$$

$$\alpha_h(V) = 0.07 \exp\left(-\frac{V + 65}{20}\right), \quad \beta_h(V) = \frac{1}{\left(\exp\left(-\frac{V+35}{10}\right) + 1\right)},$$

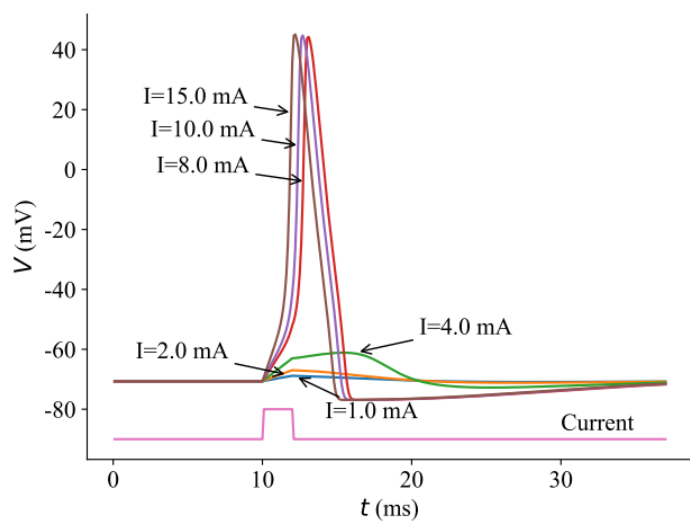
$$\alpha_m(V) = \frac{0.1(V + 40)}{1 - \exp(-(V + 40)/10)}, \quad \beta_m(V) = 4 \exp(-(V + 65)/18).$$

$$\phi = Q_{10}^{(T - T_{base})/10}$$

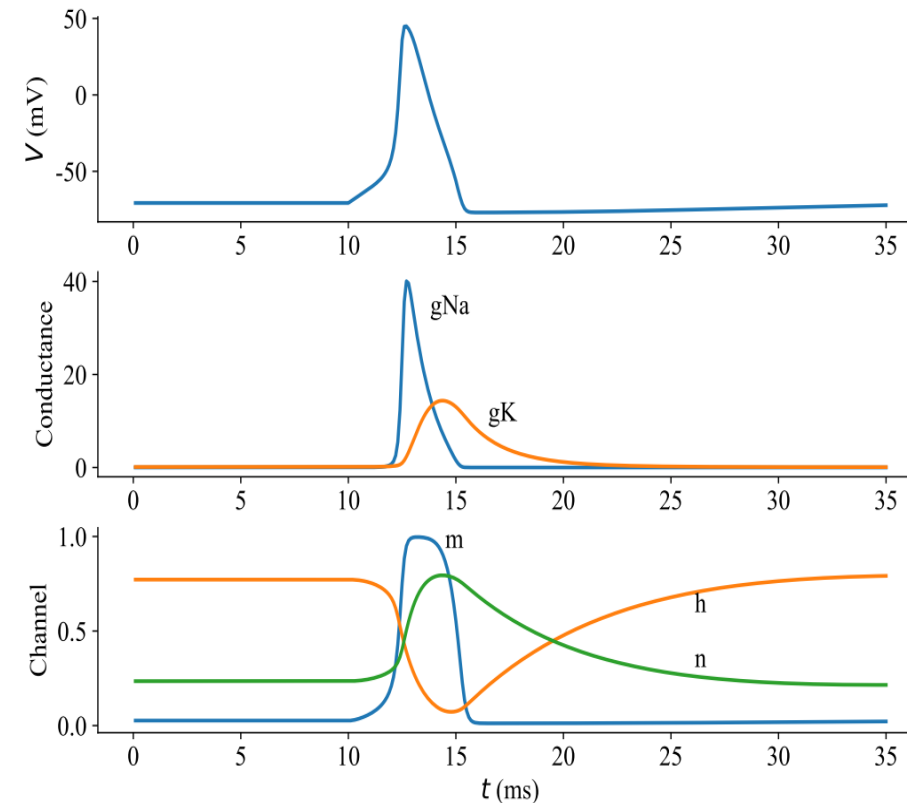
The Hodgkin-Huxley (HH) Model



Response to a constant input



All-or-none characteristics



Change of ion channel conductance and gating variables

How to fit each gating variable?

Fitting n :

$$g_K = \bar{g}_K n^x$$

$$m(t) = m_\infty(V) + (m_0 - m_\infty(V))e^{-t/\tau_m(V)}$$

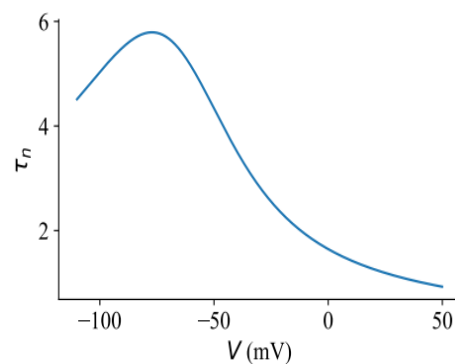
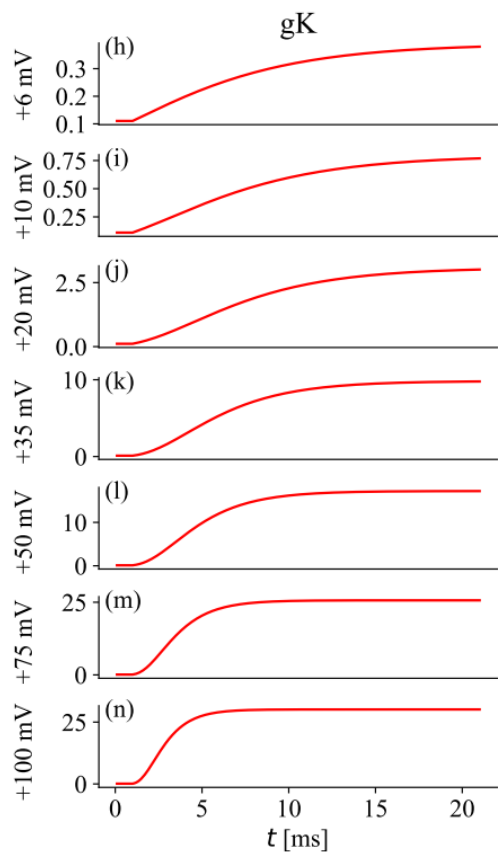


$$g_K(V, t) = \bar{g}_K \left[n_\infty(V) - (n_\infty(V) - n_0(V))e^{-\frac{t}{\tau_n(V)}} \right]^x$$

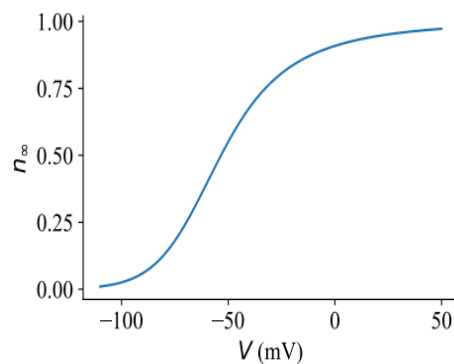


$$g_{K\infty} = \bar{g}_K n_\infty^x, g_{K0} = \bar{g}_K n_0^x$$

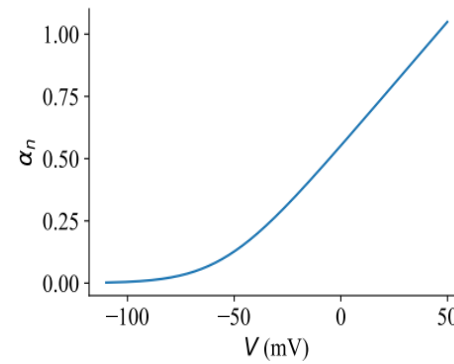
$$g_K(V, t) = \left[g_{K\infty}^{1/x} - (g_{K\infty}^{1/x} - g_{K0}^{1/x})e^{-\frac{t}{\tau_n(V)}} \right]^x$$



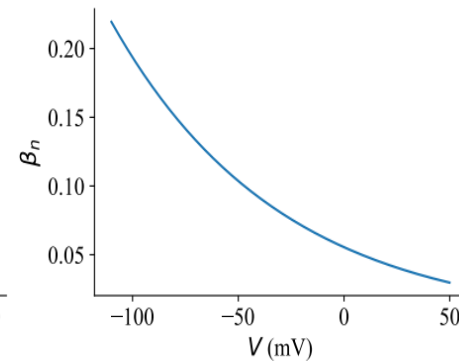
(a) $\tau_n(V)$ 随电压 V 的变化曲线



(b) $n_\infty(V)$ 随电压 V 的变化曲线



(c) $\alpha_n(V)$ 随电压 V 的变化曲线



(d) $\beta_n(V)$ 随电压 V 的变化曲线

How to fit each gating variable?

Fitting h :

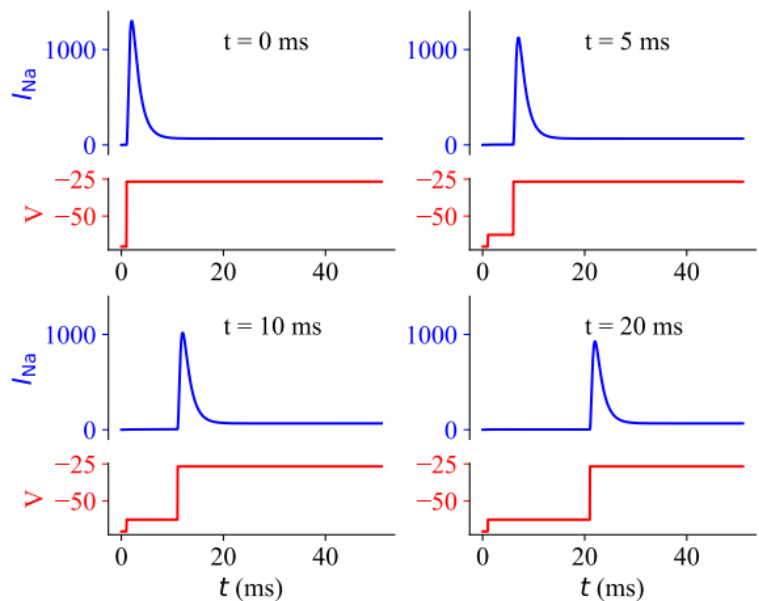
$$m(t) = m_{\infty}(V) - (m_{\infty}(V) - m_0) \exp(-t/\tau_m(V))$$

$$h(t) = h_{\infty}(V) - (h_{\infty}(V) - h_0) \exp(-t/\tau_h(V))$$



$$g_{\text{Na}}(t) = \bar{g}_{\text{Na}} m(t)^3 h(t)$$

$$= \bar{g}_{\text{Na}} \left[m_{\infty}(V) - (m_{\infty}(V) - m_0) e^{-\frac{t}{\tau_m(V)}} \right]^3 \left[h_{\infty}(V) - (h_{\infty}(V) - h_0) e^{-\frac{t}{\tau_h(V)}} \right]$$



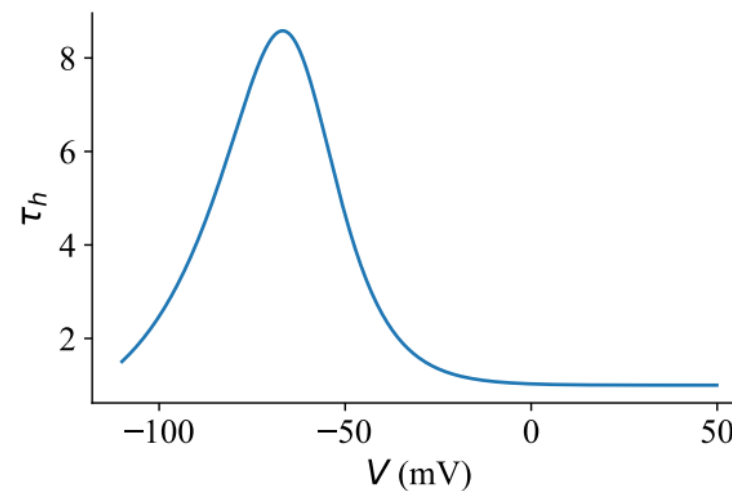
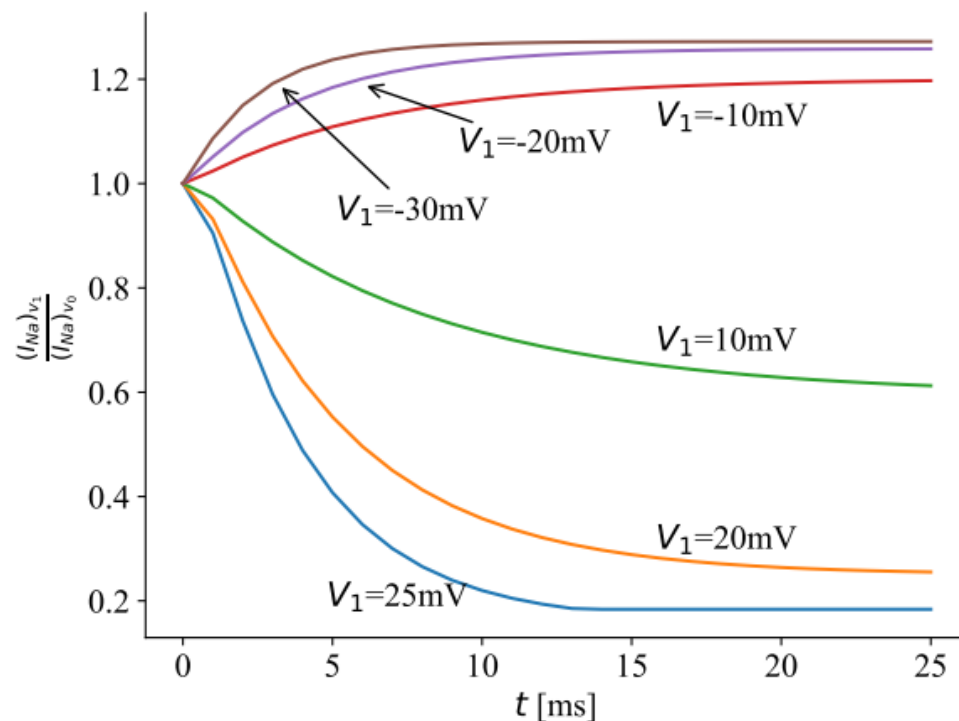
$$\begin{aligned} \frac{I_{\text{Na}}(\Delta V, t)}{I_{\text{Na}}(\Delta V, 0)} &= \frac{h^{\text{cond}}(V_r + \Delta V, t)}{h^{\text{cond}}(V_r + \Delta V, 0)} \\ &= \frac{h_{\infty}(V_r + \Delta V) - (h_{\infty}(V_r + \Delta V) - h_0) \exp(-t/\tau_h(V_r + \Delta V))}{h_{\infty}(V_r + \Delta V) - (h_{\infty}(V_r + \Delta V) - h_0) \exp(-0/\tau_h(V_r + \Delta V))} \\ &= y - (y - 1) \exp(-t/\tau_h(V_r + \Delta V)) \end{aligned}$$

$$y = \frac{h_{\infty}(V_r + \Delta V)}{h_0}$$

How to fit each gating variable?

Fitting h :

$$\frac{I_{\text{Na}}(\Delta V, t)}{I_{\text{Na}}(\Delta V, 0)} = y - (y - 1) \exp(-t/\tau_h(V_r + \Delta V)). \quad \rightarrow \quad \tau_h \quad \checkmark$$



(a) $\tau_h(V)$ 随电压 V 的变化曲线

How to fit each gating variable?

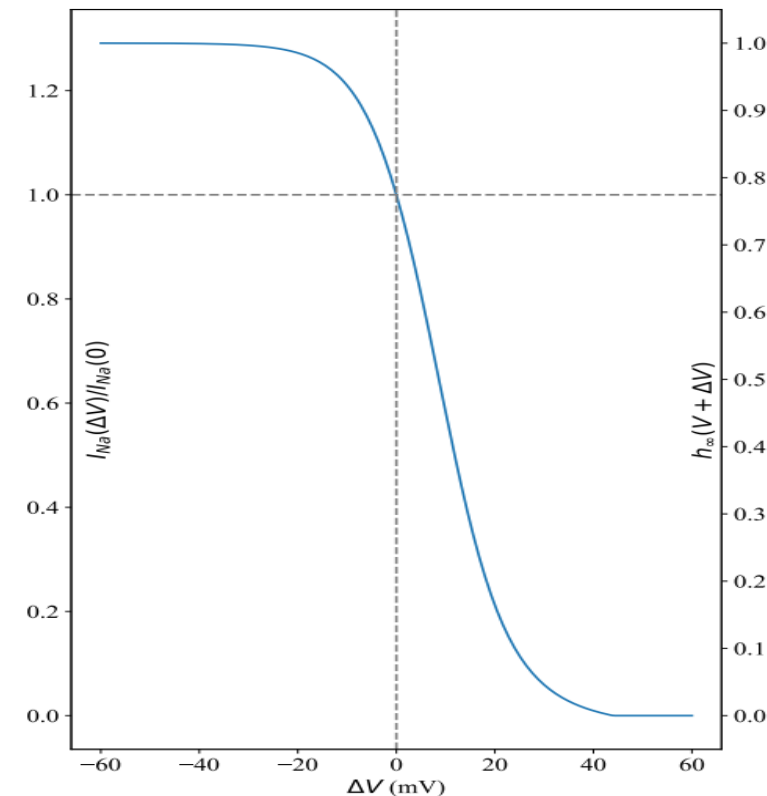
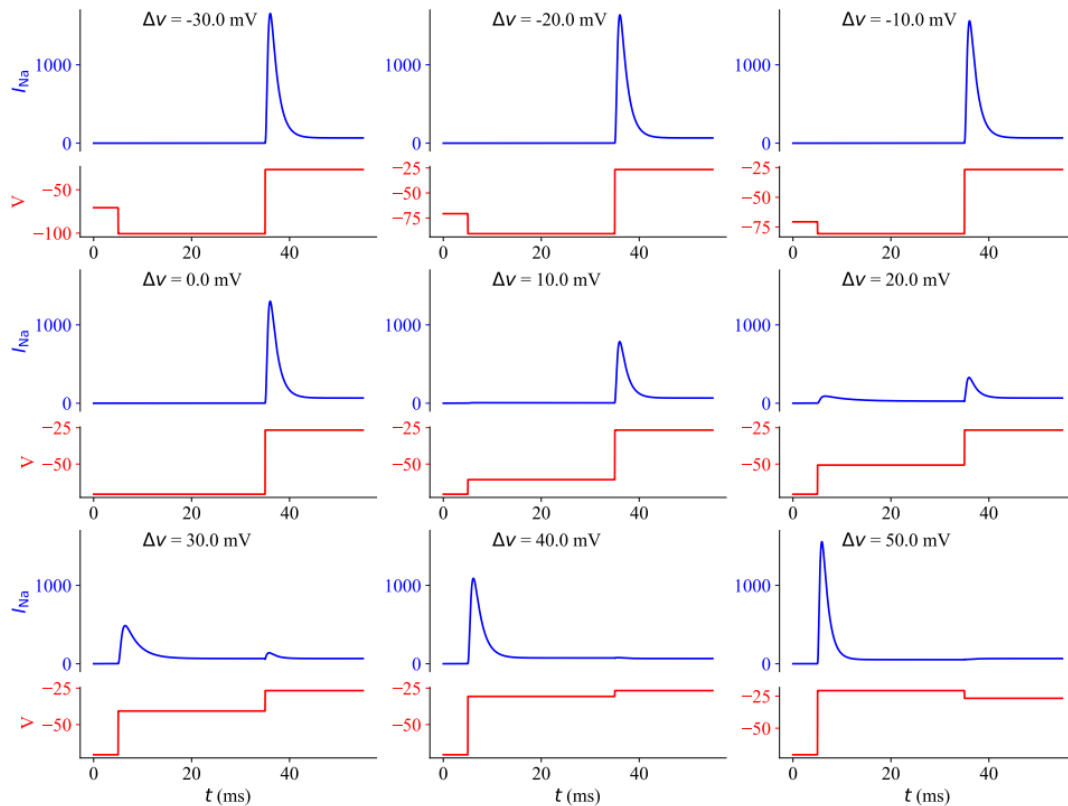
Fitting h :

$h_{\infty}(V)$?

Let t to be large enough so that variable h has reached its steady state

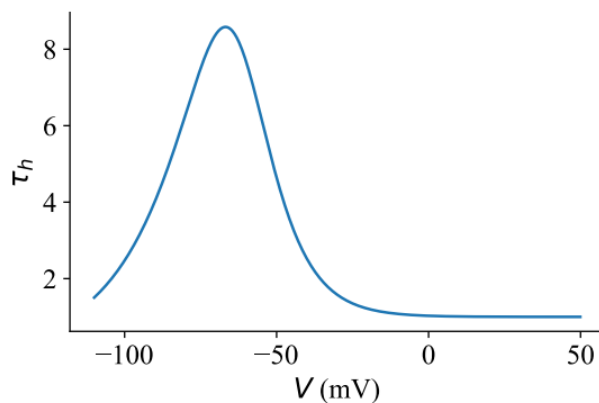
$$\frac{I_{\text{Na}}(\Delta V, t)}{I_{\text{Na}}(\Delta V, 0)} = \frac{h^{\text{cond}}(V_r + \Delta V, t)}{h^{\text{cond}}(V_r + \Delta V, 0)} = \frac{h_{\infty}(V_r + \Delta V)}{h_0} = \frac{h_{\infty}(V_r + \Delta V)}{h_{\infty}(V_r)}$$

$h_{\infty}(V)$ ✓

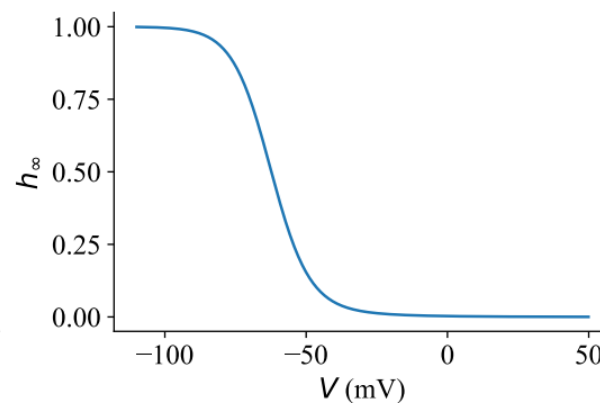


How to fit each gating variable?

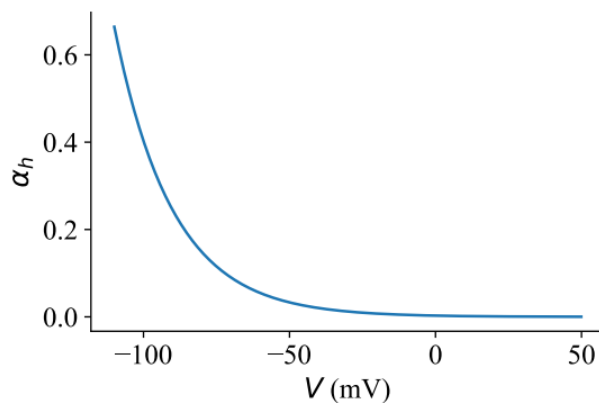
Fitting h :



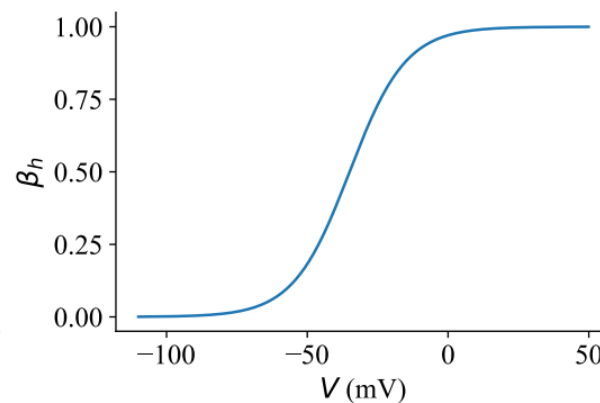
(a) $\tau_h(V)$ 随电压 V 的变化曲线



(b) $h_\infty(V)$ 随电压 V 的变化曲线



(c) $\alpha_h(V)$ 随电压 V 的变化曲线



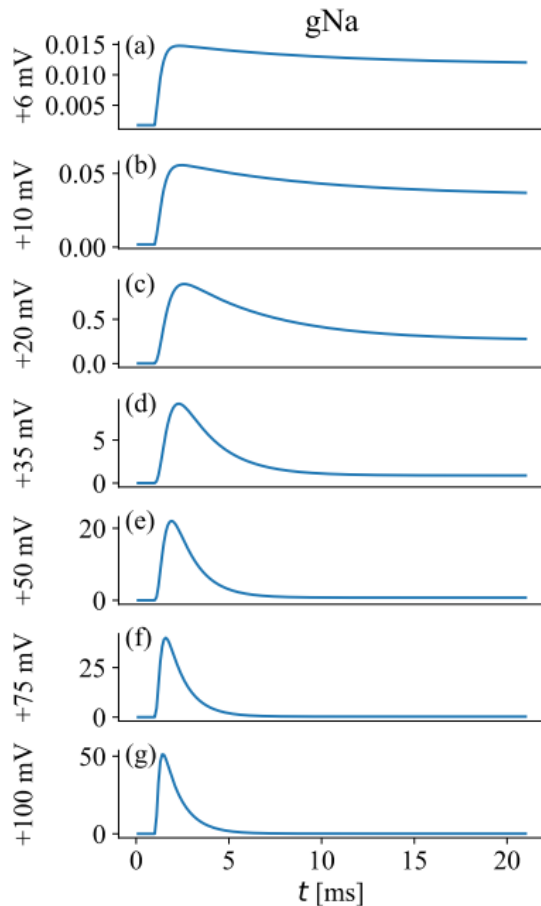
(d) $\beta_h(V)$ 随电压 V 的变化曲线

How to fit each gating variable?

Fitting m :

1. $g_{\text{Na}}(V_r) \approx 0 \Rightarrow m_0 \approx 0$

2. At high depolarization voltage: $h_{\infty} \approx 0$



$$g_{\text{Na}}(t) = \bar{g}_{\text{Na}} m(t)^3 h(t)$$

$$= \bar{g}_{\text{Na}} \left[m_{\infty}(V) - (m_{\infty}(V) - m_0) e^{-\frac{t}{\tau_m(V)}} \right]^3 \left[h_{\infty}(V) - (h_{\infty}(V) - h_0) e^{-\frac{t}{\tau_h(V)}} \right]$$

$$= g'_{\text{Na}}(V) [1 - \exp(-t/\tau_m(V))]^3 \exp(-t/\tau_h(V)) \quad g'_{\text{Na}}(V) = \bar{g}_{\text{Na}} m_{\infty}(V)^3 h_0$$

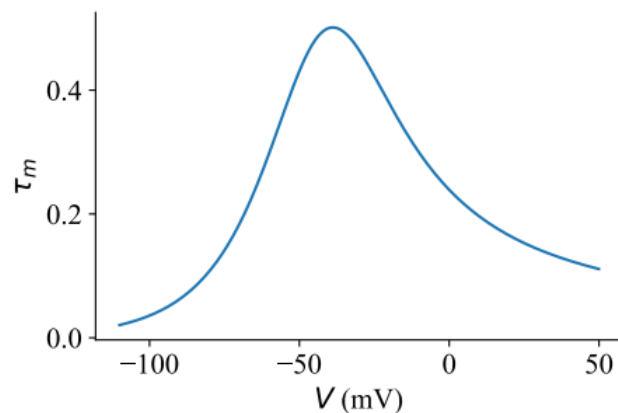
$$\longrightarrow g'_{\text{Na}}(V) \checkmark \quad \tau_m(V) \checkmark$$

3. High depolarization voltage: $m_{\infty} \approx 1 \quad \longrightarrow \bar{g}_{\text{Na}} \approx g'_{\text{Na}}/h_0$

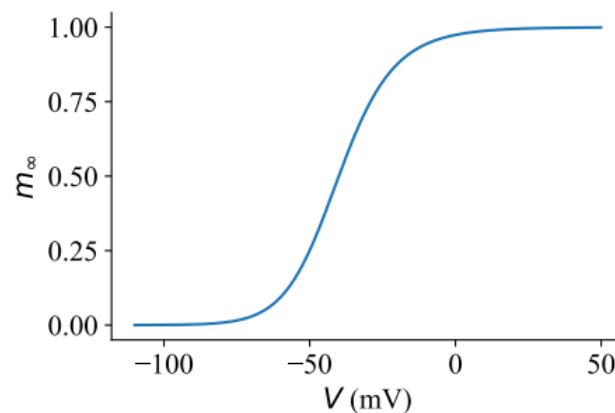
$$\longrightarrow m_{\infty} = \sqrt[3]{g_{\text{Na}}/(\bar{g}_{\text{Na}} h_0)} \quad \checkmark$$

How to fit each gating variable?

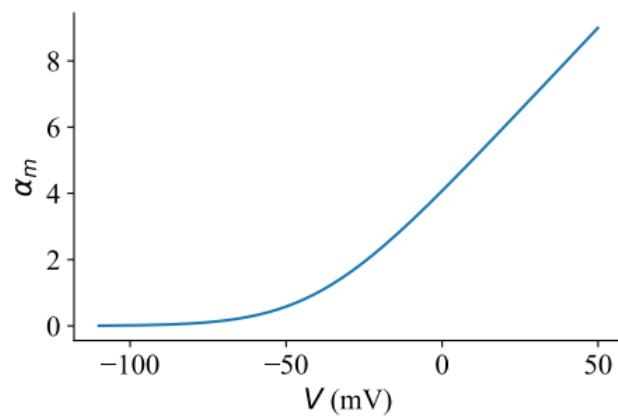
Fitting m :



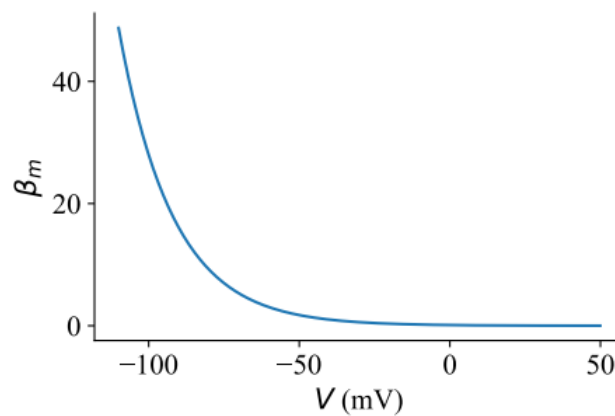
(a) $\tau_m(V)$ 随电压 V 的变化曲线



(b) $m_\infty(V)$ 随电压 V 的变化曲线



(c) $\alpha_m(V)$ 随电压 V 的变化曲线



(d) $\beta_m(V)$ 随电压 V 的变化曲线



北京大学
PEKING UNIVERSITY



05

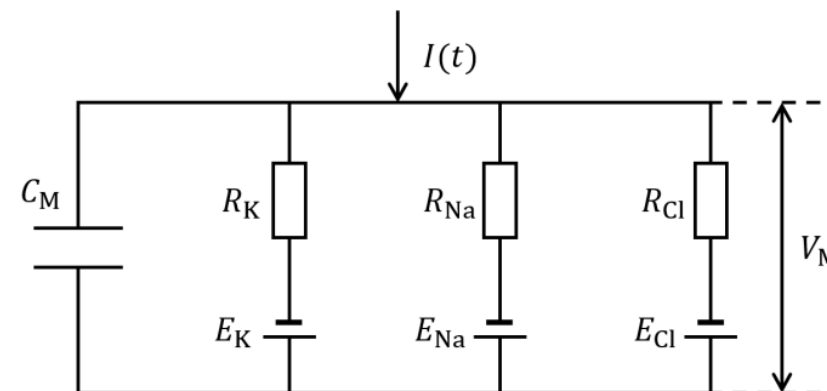
Summary

Summary

- Equivalent circuits:

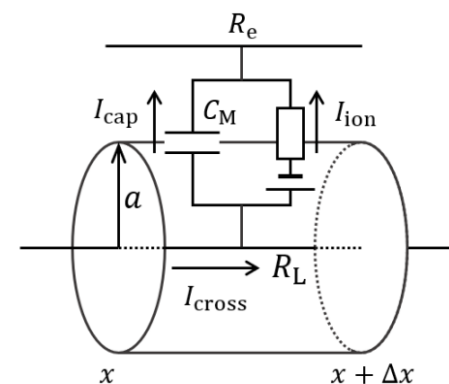
$$\frac{I(t)}{A} = c_M \frac{dV_M}{dt} + i_{\text{ion}}$$

$$\Rightarrow c_M \frac{dV_M}{dt} = -g_{\text{Cl}}(V_M - E_{\text{Cl}}) - g_{\text{K}}(V_M - E_{\text{K}}) - g_{\text{Na}}(V_M - E_{\text{Na}}) + \frac{I(t)}{A}$$



- Cable theory:
$$c_M \frac{\partial V(x, t)}{\partial t} = \frac{a}{2\rho_L} \frac{\partial^2 V(x, t)}{\partial x^2} - i_{\text{ion}}$$

- Passive conductance



Summary

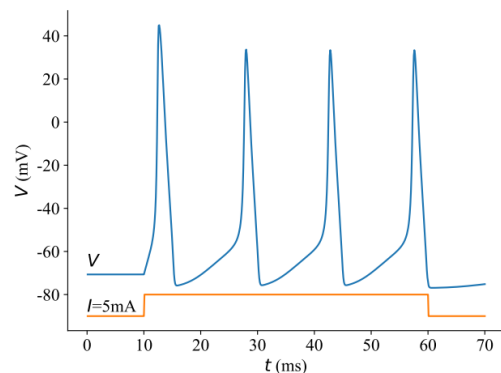
- Action potential

Mechanism: voltage-gated ion channels

$$g_{\text{Na}} \longrightarrow g_{\text{Na}}(V)$$

$$g_{\text{K}} \longrightarrow g_{\text{K}}(V)$$

- The Hodgkin-Huxley Model
- Voltage clamp



$$C \frac{dV}{dt} = -\bar{g}_{\text{Na}} m^3 h (V - E_{\text{Na}}) - \bar{g}_{\text{K}} n^4 (V - E_{\text{K}}) - \bar{g}_{\text{L}} (V - E_{\text{L}}) + I_{\text{ext}},$$

$$\frac{dn}{dt} = \phi [\alpha_n(V)(1 - n) - \beta_n(V)n]$$

$$\frac{dm}{dt} = \phi [\alpha_m(V)(1 - m) - \beta_m(V)m],$$

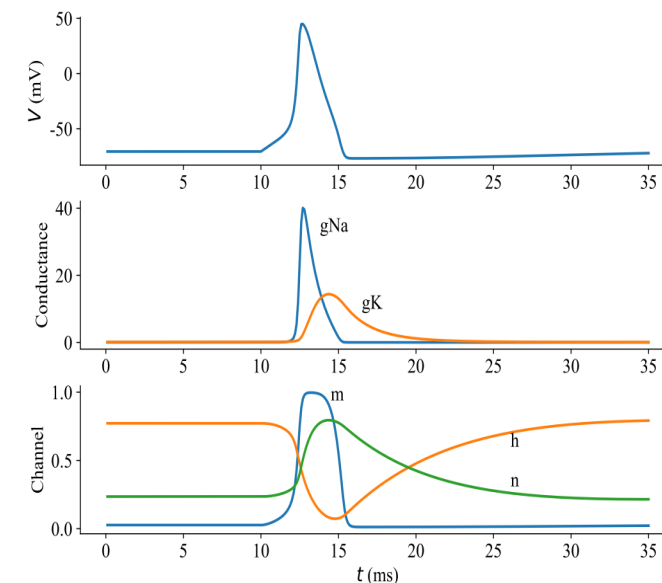
$$\frac{dh}{dt} = \phi [\alpha_h(V)(1 - h) - \beta_h(V)h],$$

$$\alpha_n(V) = \frac{0.01(V + 55)}{1 - \exp\left(-\frac{V + 55}{10}\right)}, \quad \beta_n(V) = 0.125 \exp\left(-\frac{V + 65}{80}\right),$$

$$\alpha_h(V) = 0.07 \exp\left(-\frac{V + 65}{20}\right), \quad \beta_h(V) = \frac{1}{\left(\exp\left(-\frac{V + 35}{10}\right) + 1\right)},$$

$$\alpha_m(V) = \frac{0.1(V + 40)}{1 - \exp(-(V + 40)/10)}, \quad \beta_m(V) = 4 \exp(-(V + 65)/18).$$

$$\phi = Q_{10}^{(T - T_{\text{base}})/10}$$



Summary

What are the advantages and disadvantages of the HH model?

THANK YOU

Aug. 24, 2023

