Calculating the Young's modulus of a Phosphor Bronze guitar string

Research Question

What is the Young's Modulus of a D'Addario Phosphor Bronze guitar string?

1 Introduction

The Young's Modulus, a value which measures the stiffness of a string [The Editors of Encyclopaedia Britannica, 2019], will be investigated in this experiment. This investigation aims to derive the value of Young's modulus by measuring the wavelength and frequency of the wave produced, and using the data gathered from the experiment in order to calculate the value.

I picked to investigate the Young's Modulus because I have always had a passion in music, composing music and playing a multitude of instruments for nearly half my life, so naturally, I was interested in figuring out the physics behind one of my instruments - the guitar.

2 Theory

After some further research regarding the physics of the guitar, I found out about Young's modulus and found a set of equations that allowed me to calculate it experimentally.

Additionally, the information I found about standing waves was instrumental to the result of my experiment. As I knew the effective length, the distance between the two furthest nodes, of the string, I was able to find the wavelength of the string. Moreover, as the microphone used in this experiment was not very good at recording lower frequency sounds, the second harmonic, a standing wave with three nodes and two antinodes, was recorded instead. This meant that the wavelength of the string would be equal to the length of the string in the calculations done to find Young's modulus.

Moreover, with some thinking behind the physics of my guitar, I realized that while the wave velocity stays relatively constant for the first harmonic of each separate string, the pitch, which directly relates to the frequency of the string, changes drastically between them. As a result, I came to the conclusion that not only was the wave velocity related to the frequency and the wavelength, but it was also related to either of the density of the string or the thickness of the string.

After researching, I found the equation $v = \sqrt{\dfrac{F_T}{\mu}}$, which relates the velocity to tension force and the mass per unit length of the string. Another equation used to calculate wave velocity is $v = F\lambda$. These two equations were then set to be equal to each other, allowing an equation to be formed to find Young's Modulus.

2.1 Rearranging of Equations

The equation for Young's Modulus [The Editors of Encyclopaedia Britannica, 2019] is

$$E = \frac{F_T L_0}{A \Delta L}$$

In this equation, E refers to Young's Modulus, F_T refers to the tensional force, L_0 refers to the total length of the string, A refers to the cross-sectional area of the string, and ΔL refers to the change in the length of the string.

If we rearrange it so that we have F_T on one side, we get the following equation:

$$F_T = \frac{EA\Delta L}{L_0}$$

Then, using the two following equations for wave velocity,

$$v = f\lambda$$
$$v = \sqrt{\frac{F_T}{\mu}}$$

where f refers to frequency of the wave, λ refers to the wavelength, and μ refers to the mass per unit length of the string, we can then set the two velocity equations equal to each other, creating the new equation

$$f\lambda = \sqrt{\frac{F_T}{\mu}}$$

Afterwards, substituting the equation including Young's modulus for F_T into the equation above, we get the following equation:

$$f\lambda = \sqrt{\frac{EA\Delta L}{\mu L_0}}$$

Then, rearranging the equation such that we have frequency on one side, we get the equation

$$f = \sqrt{\frac{EA\Delta L}{\lambda^2 \mu L_0}}$$

Both sides are then squared in order to form a linear relationship between f and ΔL, so the equation then becomes the following:

$$f^2 = \frac{EA}{\lambda^2 \mu L_0}\Delta L$$

If f^2 is plotted on the y axis and ΔL is plotted on the x axis, we can see that there is a linear relationship in the form $y = mx(+c)$ From this, we can tell that the gradient m will be equal to the following:

$$\frac{EA}{\lambda^2 \mu L_0}$$

From this, we can find Young's modulus by rearranging the equation such that E is on one side. The equation then becomes the following:

$$E = \frac{m\lambda^2 \mu L_0}{A}$$

which is the equation that we can use to find Young's Modulus.

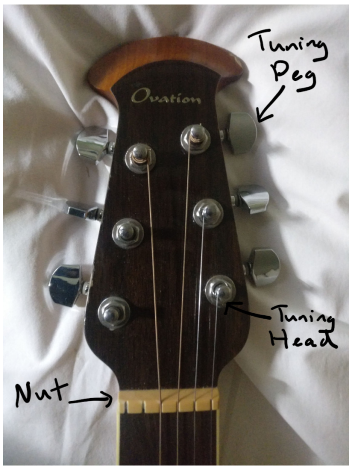
2.2 Prediction

As the material of the guitar string is Phosphor Bronze, the result that I should get from this experiment should be similar to the Young's modulus for Phosphor Bronze, which is around 120 GPa [The Editors of AZO Materials, 2012], as long as the string is made from Phosphor Bronze instead of an alloy of Phosphor Bronze.

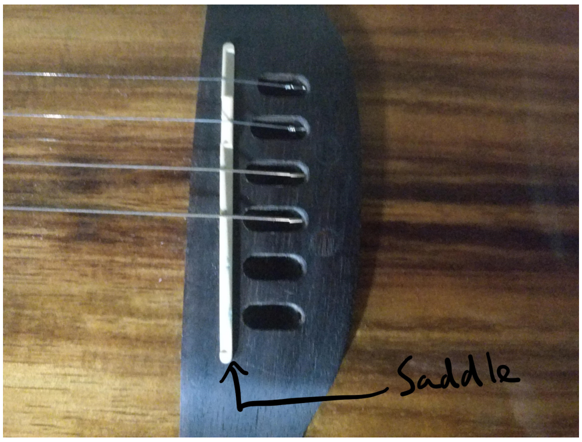
3 Language

Throughout this article, the following terms will be used to describe parts of the guitar.

Figure 1: Guitar anatomy.



(a) Locations of the nut, head and peg of the guitar



(b) Location of the saddle of the guitar

4 Method

Most of the method for this experiment comes from the textbook [Polak et al., 2018] listed in the references.

4.1 Variables

These following variables were used in my experiment in order to plot the graph to calculate my value of Young's Modulus of the phosphor bronze guitar string.

Table 1: The Independent and Dependent variables used in this experiment

Type	**Variable**	**Method of Measurement**
Independent Variable	The change in length of the guitar string (m)	To change the length of the guitar string, the radius of the tuning head was measured using Vernier Callipers (Section 4.2) in order to figure out the change in the length of the string as the tuning peg was adjusted. This was used instead of measuring using a ruler as the change in length of the string with a quarter turn was at a magnitude not suitable for a ruler to measure. 10 different data points with three repeats for each point were recorded in order to have enough data points to have a more precise representation of the gradient of the curve to calculate Young's Modulus.
Dependent Variable	The frequency of the note produced (Hz)	The dependent variable was measured by using the program Audacity to record the sound of the guitar string when plucked, and then using the 'Plot Spectrum' feature, which converts a section of audio into a graph which shows showing the peak frequencies as well as the values of said peaks, in order to calculate the frequency of the note produced. I measured the second harmonic of the sound produced by the guitar as the background noise was overlapping with the pitches of the first harmonic for the lower pitches, so those values were not used. The wavelength in the calculation was then changed to reflect the choice of second harmonic rather than first harmonic.

Table 2: The Control Variables in this experiment

Control Variable	**Level of impact** (1=lowest, 5=highest)	**Method of Minimization**
Background noise	**2:** Background noise may affect the audio signal recorded by Audacity, and thus may interfere with the peak frequencies, making the recorded value of the frequency when the string is struck less accurate.	The experiment will be carried out in a place where the background noise remains constant with few fluctuations in order for the values obtained from the experiment to be more precise. Additionally, as the background noise recorded is of lower pitch, the frequency of the second harmonic will be used instead of the first harmonic.
Same guitar used	**4:** The same guitar should be used through all these experiments as the wavelength of the standing wave needs to remain the same through the experiment, as well as the length of the whole string.	The same guitar was used through the experiment with no changes.
Same guitar string used	**5:** The same guitar string must be used for each trial as using a different guitar string may lead to unwanted fluctuations in frequency produced, as well as a different cross-sectional area, which are both variables in the equation derived to calculate Young's Modulus.	The same string was used through the experiment with no changes.

4.2 Measuring the Radius of the Tuning Head

Measuring the change of length of the string directly on the guitar is too hard and is too small to accurately do with a ruler, so I had to find an alternate method to find ΔL.

When researching [Polak et al., 2018], I found out that ΔL could instead be found by measuring the diameter of the tuning head with a Vernier Calliper, which resulted in a value of 4.0 mm. This value was then used to calculate that the circumference of the tuning head was equal to 12.6 mm. With the circumference now known, I then found that it took 15 whole turns of the tuning head in order to make the peg head finish a complete rotation, so I then calculated that the change in the length of the string with a quarter turn of the tuning head changes the length of the string by $\frac{1}{60} \times 12.6$ mm, which equates to 0.209 mm.

4.3 Finding the Cross-sectional Area of the String

As each guitar string has a different thickness, There would need to be a way to measure the thickness of the string. While a micrometer could be used to calculate the thickness of the string by measuring at different points and different orientations and taking the average, the values for the thickness of the strings were written on the guitar string packaging, so those values were used instead. The uncertainties of measurements from the factory were assumed to be negligible. In this case, the diameter of the string was measured to be 1.07 mm for the A string, which was the value used to calculate the cross-sectional area of the guitar string.

However, if the cross-sectional area of the string is not known, a Calliper can be used instead, however the uncertainties will be larger.

4.4 Measuring the Effective Length of the String

In order to find the wavelength produced, I needed to find out the effective length of the string from the nut of the guitar to the saddle of the guitar. For this, I used a soft ruler[1] in order to allow the ruler to stick tighter to the string. I measured that the distance between the nut to the saddle was 0.645 m.

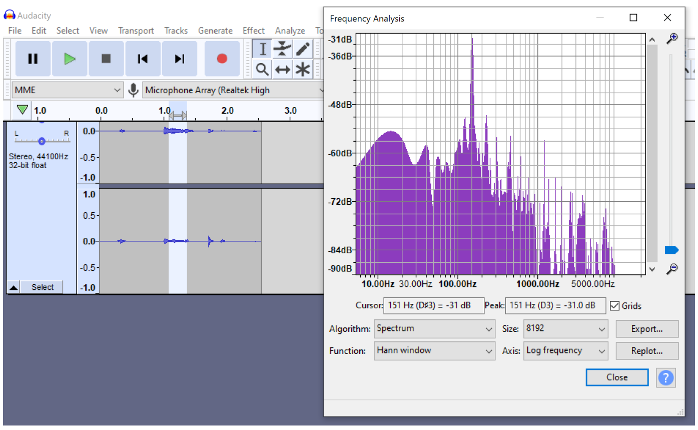
4.5 Measuring the Frequency of the Note Produced

This was the stage where I changed the length of the string to find out the frequency produced in order to plot them together to find the value of the gradient.

For this step, I had the guitar set at a tension where the string was not too loose for the string to hit the guitar board, but not too tight that the string would snap during the middle of my experiment.

I used the program Audacity[2] in order to calculate the frequency of the note produced as it has an option to plot a spectrum of the most common frequencies. This, in conjunction with the accepted standard frequencies for musical tones can be used to find the frequency of the note produced.

Figure 2: A screenshot of the process of measuring the frequency of a sound produced using Audacity.



For every length of string, I pressed record on Audacity, and struck the string 3 times, muting the string in between each trial. For each trial, I analyzed the spectrum created[3], and selected the frequency of the second harmonic. This was then placed in a table and averaged out to get a more precise value of the frequency of the note produced by the string at 10 different lengths. Every time I wanted to change the length of the string, I would tighten the string by adjusting the tuning head by a quarter turn.

4.6 Measurement of the Wavelength Produced

As the sound of the second harmonic was measured in this experiment, the wavelength produced would be equal to that of the length of the string from the saddle to the nut due to the properties of standing waves. Therefore, the wavelength produced will also be equal to 0.825 m.

[1] A flat tape measure that is able to be bent
[2] https://www.audacityteam.org/
[3] The sample size was changed from the default setting to 8192, everything else was left at the default settings.

4.7 Measurement of the Young's Modulus of the String

Using the equation,

$$E = \frac{m\lambda^2 \mu L_0}{A}$$

Young's Modulus can be found by substituting the values acquired from the above calculations, making sure that everything is done in standard units such as m^2 for A. The value for m can be found by plotting f^2 against ΔL, and finding the gradient for the line of best fit.

4.7.1 Technique

It is important to consistently strike the guitar string with a small enough force to prevent the string from hitting the fingerboard, but not too small to the point where the microphone is not able to record the sound from the guitar string.

4.8 Safety

As this experiments deals with the tightening of guitar strings to tensions that they are not expected to be at, it is advised to wear safety goggles to protect your eyes from the strings in the case that the string snaps into your eyes.

5 Data

5.1 Table of Results

Below is the results table that I collected from my experiment. The individual test values have been omitted for the sake of clarity. In this table, Quarter Turns refers to the amount of quarter turns done from the initial length, ΔL refers to the change in length of the string from the initial length, and f_A refers to the frequency of the note produced.

Table 3: Data gathered from the experiment

Quarter Turns	ΔL (m) $\pm$ 8 %	f_A (Hz) $\pm$ 1 Hz
0	0	229
1	2.10×10^{-4}	252
2	4.20×10^{-4}	274
3	6.30×10^{-4}	293
4	8.40×10^{-4}	310
5	1.05×10^{-3}	327
6	1.26×10^{-3}	344
7	1.47×10^{-3}	357
8	1.68×10^{-3}	366
9	1.89×10^{-3}	377
10	2.10×10^{-3}	389

In this case, the length from the saddle to the nut remains constant when the tuning head is adjusted. However, as this is done in a manner which stretches the string to increase tension, there will end up with more string coiled around the tuning head, so as the number of quarter turns increases, ΔL increases with it.

5.2 Uncertainties

As all of the measurements are used inside a fraction, all of the uncertainties need to be processed as a percentage uncertainty.

5.2.1 Measurement of the Frequency of the Note Produced

I used Audacity to measure the frequency produced. As the plucking of the guitar string is done by hand, as well as potential interference from background noise, the uncertainty value was estimated to be ± 1 Hz.

As f^2 is graphed, which requires multiplying frequency by itself, a percentage uncertainties is used. These uncertainties are different for every value as the absolute uncertainties stay the same for f, while the value for f in Table 3 changes. Below is a sample of the table of results for f^2 along with the percentage uncertainties.

Table 4: A sample of the table of results for f^2

Quarter Turns	**f^2 (Hz, 3 sf)**	**Uncertainty (%)**
0	52400	0.9
1	63500	0.8
2	75100	0.7
3	85800	0.7
4	96100	0.6
5	107000	0.6

5.2.2 Measurement of the Total Length of the String

I used a meter ruler to measure the total length of the string. However, as the string was bent at the end and could not have been straightened, the total length of the string was estimated to be 83 cm $\pm$ 1 cm due to the bent string at the end.

Figure 3: A picture of the bent string at the end, making the uncertainty for the length of the guitar larger.

5.2.3 Measurement of the Length of the Standing Wave

For this, the measurement taken initially was 64.5 cm with an uncertainty of 0.5cm, but I then searched on Ovation's website [Ovation Guitars, 2020] for the specifications of the guitar, and the value of the scale of the guitar was measured to be 64.3 cm.[4] As this was measured by the manufacturer of the guitar, the uncertainty is assumed to be negligible in this case. The second harmonic was measured in this experiment, so the length of the standing wave is equal to the length of the scale of the guitar.

5.2.4 Measurement of the Mass of the String

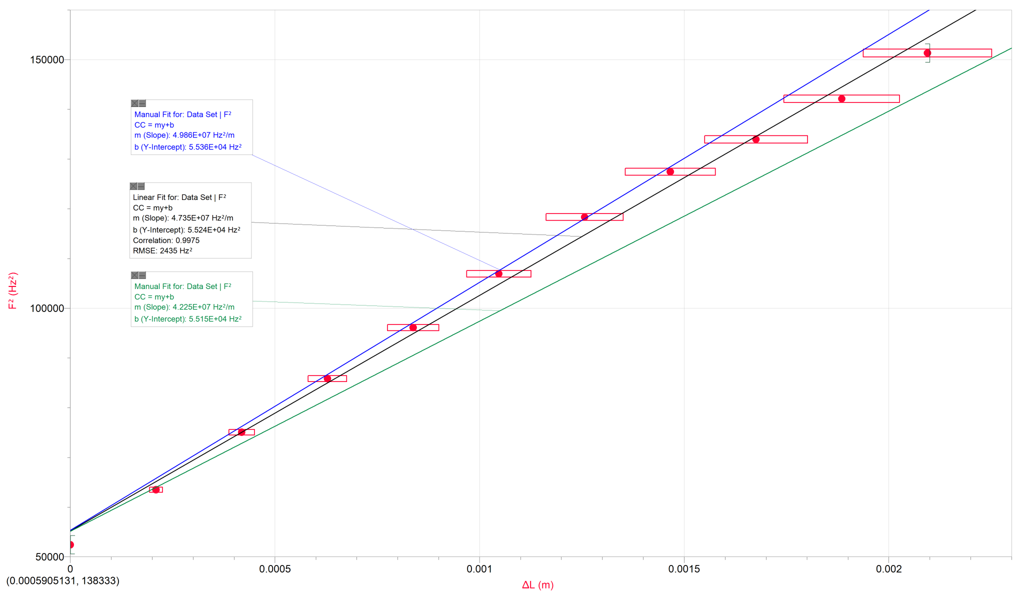
An electronic balance was used for this in order to measure the mass of the string. As the string was too long to place on the scale by itself, I used a clip to hold the string together, and took the mass of the clip away to get the total mass of the string, which was measured to be 5.74 g $\pm$ 0.01 g due to the level of accuracy on the scale.

5.2.5 Measurement of the Diameter of the Tuning Head

Even though a Vernier Calliper was used in this experiment to measure the diameter of the tuning head, making the potential uncertainty negligible for this experiment, ceteris paribus, the diameter was not consistent, meaning that depending on where you measured the diameter, the value of Young's modulus would change. Therefore, the value for the diameter of the tuning head was decided to be 4.0 mm $\pm$ 0.3 mm due to the difference in the length from the middle of the tuning head to the outside of the tuning head.

5.3 Graph

Figure 4: The scatter graph produced from the data from the experiment.



[4]http://www.ovationguitars.com/guitars-six-strings/CS24-4, Length can be found by clicking Specifications, then looking at the Scale portion.

While the y-intercept is not 0 due to the string not starting from its maximum length of 83 cm, it does not matter in this case for calculating Young's Modulus as the equation does not include the y-intercept.

For clarity, the values of the different gradients are be placed in the table below, as well as the value of Young's Modulus that it gives.

Table 5: Results from the graph

Category	**Value of gradient** (Hz^2m^{-1})	**Value of Young's Modulus** (GPa)
Upper Bound	4.99×10^7	1.33×10^{11}
Linear Regression	4.74×10^7	1.26×10^{11}
Lower Bound	4.23×10^7	1.12×10^{11}

Figure 5: An example of the calculation for Young's Modulus using the gradient from the line of best fit generated by Logger Pro (Linear Regression).

$$E = \frac{m\lambda^2\mu L_0}{A}$$
$$E = \frac{(4.74 \times 10^7) \times (0.643)^2 \times (6.96 \times 10^{-3}) \times 0.83}{8.99 \times 10^{-7}}$$
$$E = 1.26 \times 10^{11} \text{ (Pa)}$$

6 Conclusion

6.1 Analysis of Results

The Young's Modulus for the guitar string was calculated experimentally to be 1.26×10^{11} GPa, while the accepted value [The Editors of AZO Materials, 2012] for Young's Modulus for Phosphor Bronze is 1.2×10^{11} GPa. In this case, the value acquired experimentally was close to the Young's Modulus for Phosphor Bronze, and falls within the upper and lower bounds for Young's Modulus, thus, we can come to a conclusion that the D'Addario guitar string used in this experiment is made primarily from Phosphor Bronze.

However, as seen from the graph, the uncertainty range is quite large, and this is due to the inconsistent diameter of the tuning head. With a more consistently shaped tuning head, the uncertainties would be smaller and therefore the confidence in the results of the experiment is higher.

6.2 Evaluation

There were certain areas that could have been improved in the experiment.

6.2.1 Systematic Errors Affecting Accuracy

Tuning head of guitar: The tuning heads on the guitar did not have the same circumference, as each of them had a dip in the middle, as well as being oval-shaped rather than circle shaped. This would have affected ΔL as each quarter turn would not have changed the length of the string by the same amount each time. This can be shown in Figure 4 with the fluctuation of the points in both directions from the line of best fit. As a result of this, the percentage uncertainties were quite significant at ± 8%, which further affected the uncertainties of the results as ΔL rose. This had a significant effect to the data collected as seen with the error bars in figure 4, especially nearer to the top right corner of the graph. To help mitigate the issue, a guitar with a circular tuning peg with a consistent diameter should have been used, which would in theory make the data gathered lay closer to the line of best fit.

Length of the string of the guitar: As seen in Figure 3, one of the ends of the string was bent, and thus, the measurement for the total length of the string had to be estimated, as mentioned in section 5.2.2, which meant that the calculations done for Young's modulus in 5 would have a larger uncertainty. However, as the uncertainty was estimated to be ±1 cm, while the total length of the string was measured to be 83 cm, the percentage uncertainty was 1.2%, so the impact on the uncertainty of the length of the string is low.

Rust on the string: The string used was not completely new, and thus some rust accumulated on the string, potentially affecting the mass of the string. However, this should have a low impact on the calculated result for Young's Modulus, as the extra rust, caused by the oxidation of the string, would change the uncertainty of the mass of the string by less than 0.01 % due to the extremely low value of the atomic mass of Oxygen, as well as the low surface area of the string for rust to form.

Limited data points: While there were 11 unique data points collected as shown in Table 3, Figure 4's results fluctuated as shown in the *Tuning head of guitar* section above. If there were more results, the impact of the fluctuation in the graph would be less significant to Logger Pro's calculation for the gradient of the graph. This has a moderate significance to the gradient calculated by Logger Pro, as all points are taken into account. With more data points, Logger Pro would be able to calculate a more accurate value for the Young's Modulus.

Use of the second harmonic: In this experiment, the second harmonic was used instead of the first harmonic due to the limitations of the microphone used. This should have no significance to the data recorded, due to the wavelength being adjusted accordingly when using the equation derived above.

6.2.2 Random Errors Affecting Precision

Measuring the frequency of the note produced: The audio for the experiment was recorded outdoors, meaning that the recorded frequency could have been influenced by outdoor noise, and would give a less accurate value for each data point. However, this should only have a small effect on the recorded frequencies as the position of the guitar was close enough to the microphone to the point where the background noise should have a negligible impact on the frequency spectrum generated from Audacity. Even though the impact of this on the final result should not be significant, the experiment could have been done in a sound-proof room to minimize the impact of external factors.

Precision of the electronic balance: The electronic balance only recorded to the nearest hundredth of a gram, which might have slightly shifted the calculated result for Young's Modulus. This caused a percentage uncertainty of 0.2 %, which ultimately had a negligible effect on the final result calculated for Young's Modulus.

6.3 Further Investigation

The Young's modulus could be tested for all six strings instead of just one. As the four lower pitched strings are all made from phosphor bronze, the values for those strings could be used in order to take an average of the different values of the Young's Modulus. The remaining two strings are not made of phosphor bronze, but it would be interesting to find out the value for the Young's modulus of the remaining two strings, and comparing them to the value gained from phosphor bronze. This will allow us to see the possible materials that the two highest pitched strings on the guitar are made of.

Additionally, another method to calculate Young's Modulus could have been used to cross-check the results to see if they matched. For example, an experiment involving masses to stretch the string could have been used to find a value for Young's Modulus with the same string as the experiment above. The results could then be compared with each other to form a more reliable conclusion of the Young's Modulus of the string.

References

[Ovation Guitars, 2020] Ovation Guitars (2020). CS24-4. Visited on January 20th, 2020. Date of Publication unknown.

[Polak et al., 2018] Polak, R., Davenport, A., Fischer, A., and Rafferty, J. (2018). Determining young's modulus by measuring guitar string frequency. *The Physics Teacher*, 56(2):122–123.

[The Editors of AZO Materials, 2012] The Editors of AZO Materials (2012). Phosphor bronze – copper alloy uns c50500. Visited on January 20, 2020.

[The Editors of Encyclopaedia Britannica, 2019] The Editors of Encyclopaedia Britannica (2019). Young's modulus. Visited on January 20, 2020.

7 Appendix

1. The guitar string used in this experiment is the A string of the D'Addario EJ16-3D Phosphor Bronze acoustic guitar string pack.

2. The guitar used for the experiment was an Ovation Celebrity CS24-4

Table 6: Full table of results for f^2

Quarter Turns	f^2 **(Hz, 3 sf)**	**Uncertainty (%)**
0	52400	0.9
1	63500	0.8
2	75100	0.7
3	85800	0.7
4	96100	0.6
5	107000	0.6
6	118000	0.6
5	127000	0.6
5	134000	0.5
5	142000	0.5
5	151000	0.5

Figure 6: The graph produced from the experiment scaled from the origin.