In the name of Allah

Amirkabir University of Technology
(Tehran Polytechnic)
Industrial Engineering Department

Course Title:
Engineering Economics

4. Nominal and Effective Interest Rates

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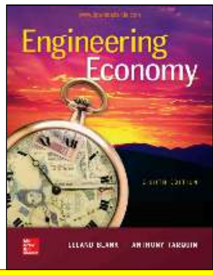
Learning Stage 1: The Fundamentals

- Chapter 1
  - Foundations of Engineering Economy
- Chapter 2
  - Factors: How Time and Interest Affect Money
- Chapter 3
  - Combining Factors and Spreadsheet Functions
- Chapter 4
  - Nominal and Effective Interest Rates



Chapter 4 of EE (BT) book 8th edition

LEARNING OUTCOMES

- Purpose:
  - Make computations for interest rates and cash flows that are on a time basis other than a year.

1. Understand interest rate statements
2. Use formula for effective interest rates
3. Determine interest rate for any time period
4. Determine payment period (PP) and compounding period (CP) for equivalence calculations
5. Make calculations for single cash flows
6. Make calculations for series and gradient cash flows with PP ≥ CP
7. Perform equivalence calculations when PP < CP
8. Use interest rate formula for continuous compounding
9. Make calculations for varying interest rates

٣ Engineering Economics

Interest Rate Statements

- The terms 'nominal' and 'effective' enter into consideration when the interest period is less than one year.
- New time-based definitions to understand and remember
  - Interest period (t)
    - Period of time over which interest is expressed. E.g., 1% per month.
  - Compounding period (CP)
    - Shortest time unit over which interest is charged or earned. E.g., 10% per year compounded monthly.
  - Compounding frequency (m)
    - Number of times compounding occurs within the interest period t. E.g., at i = 10% per year, compounded monthly, interest would be compounded 12 times during the one year interest period.

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Understanding Interest Rate Terminology

- A nominal interest rate (r) is obtained by
  - multiplying an interest rate that is expressed over a short time period by the number of compounding periods in a longer time period:
    - i.e., r = interest rate per period x number of compounding periods
    - e.g., If i = 1% per month, nominal rate per year: r = (1)(12) = 12% per year
- Effective interest rates (i) take compounding into account
  - can be obtained via a formula to be discussed later

IMPORTANT: Nominal interest rates are essentially simple interest rates. Therefore, they can never be used in interest formulas.

Effective rates must always be used hereafter in all interest formulas.

More About Interest Rate Terminology

- There are 3 general ways to express interest rates:

Sample Interest Rate Statements	Comment
(1) i = 2% per month i = 12% per year	When no compounding period is given, rate is effective
(2) i = 10% per year, comp'd semiannually i = 3% per quarter, comp'd monthly	When compounding period is given and it is not the same as interest period, it is nominal
(3) i = effective 9.4%/year, comp'd semiannually i = effective 4% per quarter, comp'd monthly	When compounding period is given and rate is specified as effective, rate is effective over stated period

Effective Annual Interest Rates (1)

- Nominal rates are converted into effective annual rates via

$$i_a = (1 + i)^m - 1$$

where i_a = effective annual interest rate
i = effective rate for one compounding period
m = number times interest is compounded per year

- Example:
  - For a nominal interest rate of 12% per year, determine the nominal and effective rates per year for (a) quarterly, and (b) monthly compounding

Solution: (a) Nominal r / year = 12% per year
Nominal r / quarter = 12/4 = 3.0% per quarter
Effective i / year = $(1 + 0.03)^4 - 1$ = 12.55% per year

(b) Nominal r /month = 12/12 = 1.0% per month
Effective i / year = $(1 + 0.01)^{12} - 1$ = 12.68% per year

Effective Annual Interest Rates (2)

- Nominal rates can be converted into effective rates for any time period:

$$i = (1 + r/m)^m - 1$$

where i = effective interest rate for any time period
r = nominal rate for same time period as i
m = no. times interest is comp'd in period specified for i

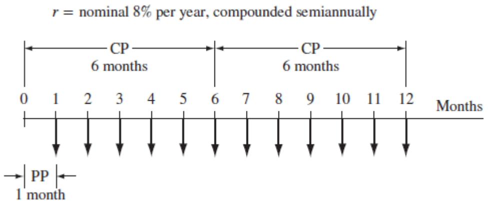
- Example:
  - For an interest rate of 1.2% per month, determine the nominal and effective rates (a) per quarter, and (b) per year

Solution: (a) Nominal r / quarter = (1.2)(3) = 3.6% per quarter
Effective i / quarter = $(1 + 0.036/3)^3 - 1$ = 3.64% per quarter

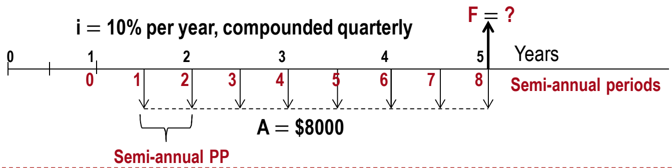
(b) Nominal i /year = (1.2)(12) = 14.4% per year
Effective i / year = $(1 + 0.144 / 12)^{12} - 1$ = 15.39% per year

Equivalence Relations: PP and CP

- Payment Period (PP): Length of time between cash flows
  - In the diagram below, the compounding period (CP) is semiannual and the payment period (PP) is monthly



  - Here, the CP is quarterly and the payment period (PP) is semiannual



Single Amounts with PP > CP

- For problems involving single amounts,
  - The payment period (PP) is usually longer than the compounding period (CP).
  - For these problems, there are an infinite number of i and n combinations that can be used, with only two restrictions:
    - (1) The i must be an effective interest rate, and
    - (2) The time units on n must be the same as those of i
- There are two equally correct ways to determine i and n
  - Method 1: Determine effective interest rate over the compounding period CP, and set n equal to the number of compounding periods between P and F
  - Method 2: Determine the effective interest rate for any time period t, and set n equal to the total number of those same time periods.

Example: Single Amounts with PP ≥ CP

- How much money will be in an account in 5 years if \$10,000 is deposited now at an interest rate of 1% per month?
  - Use different interest rates: (a) monthly, (b) quarterly, (c) yearly.

(a) For monthly rate, 1% is effective [n = (5 years) × (12 CP per year) = 60]

$F = 10{,}000(F/P, \underbrace{1\%}_{\text{effective i per month}}, \underbrace{60}_{\text{months}}) = \$18{,}167$

(b) For a quarterly rate, effective i/quarter = $(1 + 0.03/3)^3 - 1 = 3.03\%$

$F = 10{,}000(F/P, \underbrace{3.03\%}_{\text{effective i per quarter}}, \underbrace{20}_{\text{quarters}}) = \$18{,}167$

(c) For an annual rate, effective i/year = $(1 + 0.12/12)^{12} - 1 = 12.683\%$

$F = 10{,}000(F/P, \underbrace{12.683\%}_{\text{effective i per year}}, \underbrace{5}_{\text{years}}) = \$18{,}167$

i and n must always have same time units

Series with PP ≥ CP

- For series cash flows, first step is to determine relationship between PP and CP
- When PP ≥ CP, the only procedure (2 steps) that can be used:
  (1) Find effective i per PP
  e.g., if PP is in quarters, must find effective i/quarter

  (2) Determine n, the number of A values involved
  e.g., quarterly payments for 6 years yields n = 4 × 6 = 24

Note: Procedure when PP < CP is discussed later

Example: Series with PP ≥ CP

- How much money will be accumulated in 10 years from a deposit of $500 every 6 months if the interest rate is 1% per month?

Solution: First, find relationship between PP and CP
PP = six months, CP = one month; Therefore, PP > CP

Since PP > CP, find effective i per PP of six months

Step 1. i /6 months = $(1 + 0.06/6)^6 - 1 = 6.15\%$

Next, determine n (number of 6-month periods)

Step 2: n = 10(2) = 20 six month periods

Finally, set up equation and solve for F

F = 500(F/A,6.15%,20) = $18,692 (by factor or spreadsheet)

Series with PP < CP

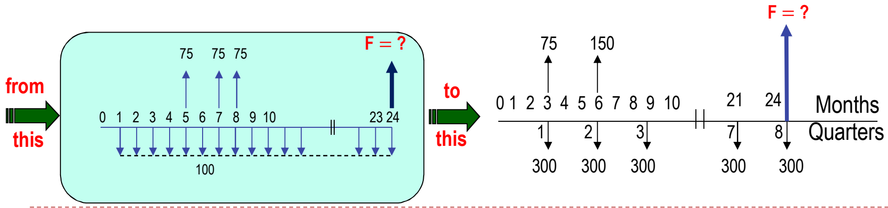
- There are two policies:
  - (1) interperiod cash flows earn no interest (most common)
  - (2) interperiod cash flows earn compound interest
  - For policy (1),
    - positive cash flows are moved to beginning of the interest period in which they occur and
    - negative cash flows are moved to the end of the interest period
  - For policy (2),
    - cash flows are not moved and equivalent P, F, and A values are determined using the effective interest rate per payment period

Note: The condition of PP < CP with no interperiod interest is the only situation in which the actual cash flow diagram is changed

Example: Series with PP < CP

- A person deposits $100 per month into a savings account for 2 years.
  - If $75 is withdrawn in months 5, 7 and 8 (in addition to the deposits), construct the cash flow diagram to determine how much will be in the account after 2 years at i = 6% per year, compounded quarterly. Assume there is no interperiod interest.

Solution: Since PP < CP with no interperiod interest, the cash flow diagram must be changed using quarters as the time periods



Example: Series with PP < CP

- Solution for the previous slide example:
  - No interperiod interest earned

Solution by hand:
F = -300(F/A, 1.5%, 8)+150(F/P, 1.5%, 6)+75(F/P, 1.5%, 7) = -2282.6

Solution by Excel:
= FV(1.5%,8,-300) + FV(1.5%,6,,150) + FV(1.5%,7,,75) = 2282.6

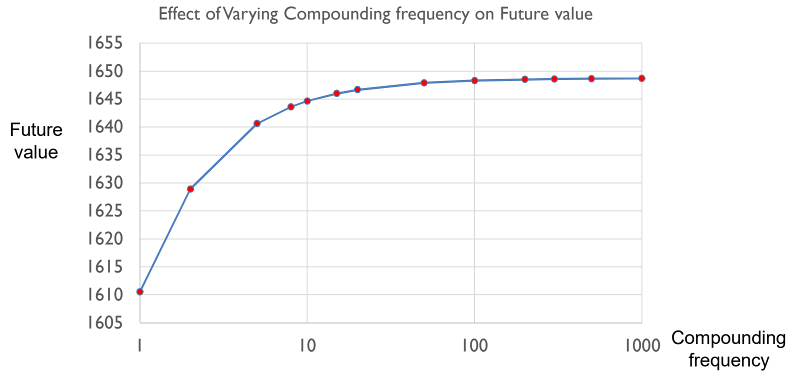
- Solution for the previous slide example:
  - There are interperiod interests

m = 1/3, Effective i/ month = $(1+0.015)^{1/3}-1$ = 0.4975% per month
Solution by hand:
F = -100(F/A, 0.4975%, 24)+75(F/P, 0.4975%, 19)+75(F/P, 0.4975%, 17)+75(F/P, 0.4975%, 16) = -2297.2

Solution by Excel:
= FV(0.4975%,24,-100)+FV(0.4975%,19,,75)+FV(0.4975%,17,,75)+FV(0.4975%,16,,75) = 2297.2

Effect of Varying CP on future value

P	1000
r	0.1
n	5
m	1
F	1610.51
F=Pe^(rn)	1648.72



Continuous Compounding

- When the interest period is infinitely small, interest is compounded continuously. Therefore, PP > CP and m increases.
  - Take limit as $m \rightarrow \infty$ to find the effective interest rate equation
  - $\mathbf{i = e^r - 1}$

- Example: If a person deposits \$500 into an account every 3 months at an interest rate of 6% per year, compounded continuously,
  - how much will be in the account at the end of 5 years?

Solution: Payment Period: PP = 3 months

Nominal rate per three months: r = 6%/4 = 1.50%

Effective rate per 3 months: $i = e^{0.015} - 1 = 1.51\%$

$\mathbf{F = 500(F/A, 1.51\%, 20) = \$11,573}$

Formulas for Continuous Compounding

Unknown	Known	Formula	Factor
P	F	$e^{-r.n}$	$(P/F,r,n)^{\infty}$
F	A	$\frac{e^{r.n}-1}{e^{r}-1}$	$(F/A,r,n)^{\infty}$
P	A	$\frac{e^{r.n}-1}{e^{r.n}(e^{r}-1)}$	$(P/A,r,n)^{\infty}$
P	G	$\frac{e^{r.n}-1-n(e^{r}-1)}{e^{r.n}(e^{r}-1)^{2}}$	$(P/G,r,n)^{\infty}$
A	G	$\frac{1}{e^{r}-1}-\frac{1}{e^{r.n}-1}$	$(A/G,r,n)^{\infty}$

Varying Rates

- When interest rates vary over time,
  - use the interest rates associated with their respective time periods to find P or F.

- Example: Find the PW of $2500 deposits in years 1 through 8
  - if the interest rate is 7% per year for the first five years and 10% per year thereafter.

Solution: P = 2,500(P/A,7%,5) + 2,500(P/A,10%,3)(P/F,7%,5) = $14,683

- An equivalent annual worth value can be obtained by
  - replacing each cash flow amount with 'A' and setting the equation equal to the calculated P or F value

14,683 = A(P/A,7%,5) + A(P/A,10%,3)(P/F,7%,5) → A = $2500 per year

Example: Varying rates

- Determine present worth in time 0 and future worth at the end of year 5 for the following cash flow.

Year	1	2	3	4	5
Cash flow	10000	10000	20000	20000	30000
Annual Interest rate	7%	8%	9%	10%	11%

- $P_0 =$ 10000/(1.07) +
  10000/(1.07)(1.08) +
  20000/(1.07)(1.08)(1.09) +
  20000/(1.07)(1.08)(1.09)(1.1) +
  30000/(1.07)(1.08)(1.09)(1.1)(1.11) = **67818**
- $F_5 =$ 30000 +
  20000(1.11) +
  20000(1.1)(1.11) +
  10000(1.09)(1.1)(1.11) +
  10000(1.08)(1.09)(1.1)(1.11) = **104302.5**

Example: Varying rates

- Determine the annual value for the last slide example, having the calculated PW and FW.
  - Using the PW
    - **67818** = A/(1.07) + A/(1.07)(1.08) + A/(1.07)(1.08)(1.09) + A/(1.07)(1.08)(1.09)(1.1) + A/(1.07)(1.08)(1.09)(1.1)(1.11)
    - 67818 = 3.966 A → A = 17099.85
  - Using the FW
    - **104302.5** = A + A(1.11) + A(1.1)(1.11) + A(1.09)(1.1)(1.11) + A(1.08)(1.09)(1.1)(1.11)
    - 104302.5 = 6.099A → A = 17101.57

How to use NPV function in Excel

- Two common ways to calculate the real NPV in Excel
  - 1. Use the NPV function, but leave out the initial cost.
    - If the initial cost is entered as a negative number, it will be added
  - 2. Use the NPV function and include the initial cost in the range of cash flows.
    - In this case, the "NPV" will be in period -1 so we must bring it forward one period in time. So, multiply the result by (1 + i), where i is the per period discount rate.

	K	L	M	N	O	P	Q	R	S
18	intrest rate	10%							
19		cash flow	discounted cash flow						
20	0	-100	-100.00						
21	1	50	45.45		92.98	>=NPV(L18,L20:L24)			wrong value
22	2	60	49.59		102.27	>=NPV(L18,L21:L24)+L20			correct value
23	3	70	52.59		102.27	>=NPV(L18,L20:L24)*(1+L18)			correct value
24	4	80	54.64						
25	npv using formula		102.27	correct value					
26									

Summary of Important Points (1)

- Must understand:
  - interest period, compounding period, compounding frequency, and payment period
- Always use effective rates in interest formulas
  - $\mathbf{i = (1 + r/m)^m - 1}$
- Interest rates are stated different ways; must know how to get effective rates
- For single amounts,
  - make sure units on i and n are the same

Summary of Important Points (2)

- For uniform series with PP $\geq$ CP,
  - find effective i over PP
- For uniform series with PP $<$ CP and no interperiod interest,
  - move cash flows to match compounding period
- For continuous compounding,
  - use $i = e^r - 1$ to get effective rate
- For varying rates,
  - use stated i values for respective time periods