A Framework for Probabilistic Numerical Evaluation of Sensor Networks: A Case Study of a Localization Protocol

PIERRE LEONE and JOSE ROLIM
University of Geneva
PAUL ALBUQUERQUE
Ecole d'Ingénieurs de Genève
and
CHRISTIAN MAZZA
University of Geneva

In this paper we show how to use stochastic estimation methods to investigate topological properties of sensor networks as well as the behavior of dynamical processes on these networks. The framework is particularly important to study problems for which no theoretical results are known, or cannot be directly applied in practice, for instance, when only asymptotic results are available. We also interpret Russo's formula in the context of sensor networks and thus obtain practical information on their reliability. As a case study, we analyze a localization protocol for wireless sensor networks and validate our approach by numerical experiments. Finally, we mention three applications of our approach: estimating the number of pivotal sensors in a real network, minimizing the number of such sensors for robustness purposes during the network design and estimating the distance between successive localized positions for mobile sensor networks.

Categories and Subject Descriptors: C.2.4 [**Distributed Systems**]

General Terms: Algorithms, Reliability

Additional Key Words and Phrases: Sensor networks, stochastic recursive estimation, localization process and reliability

1. INTRODUCTION

Recent technological improvements have favored the development of small, inexpensive, low-power, multifunctional sensor nodes. These tiny sensor nodes

This work has been partially funded by European project FET CRESCCO IST-2001-33135.
Authors' addresses: Pierre Leone and Jose Rolim, Computer Science Department, University of Geneva, 1211 Geneva 4, Switzerland; email: [jose.rolim, pierre.leone]@cui.unige.ch. Paul Albuquerque, Ecole d'Ingénieurs de Genève, LII, HES-SO, 1202 Geneva, Switzerland; email: paul.albuquerque@eig.unige.ch. Christian Mazza, Mathematics Department, University of Geneva, 1211 Geneva 4, Switzerland; email: christian.mazza@math.unige.ch.

are the numerous sensing, data processing, and communicating components of an extended network: a wireless sensor network [Akyildiz et al. 2002]. There are several types of sensors (e.g., thermal, visual, infrared, acoustic, or radar), which are able to monitor their vicinity (e.g., measure temperatures, noise levels, or vehicular movements). Wireless sensor networks can thus help monitor and control physical environments. Indeed, usually densely scattered sensors collaborate to perform some specific action through collection and distribution of data. The collected data must be routed back through a multihop infrastructureless architecture to a distinguished node which can then communicate to a task manager via the Internet or by satellite. In most cases, the environment to be monitored does not have an existing infrastructure either for energy or communication and is often very rough. This influences the design of a sensor network with respect to fault tolerance, scalability, production costs, sensor network topology, hardware constraints, transmission media, computational capacity, and power consumption. The constraints are often because of the inexpensive nature and ad hoc methods of deployment of the sensors. As a matter of fact, the major concerns are to extend the lifetime and robustness of sensor networks. Particular care must be given to energy-efficient routing, localization algorithms, and system design. To this end, energy consumption for processing and communication must be optimized. It is also required that sensor networks display some form of self-organization for both communication and positioning in order to adapt to changing connectivity (e.g., addition or failure of nodes) as well as changing environmental conditions.

Application areas include military, environment, health and chemical processing. In particular, wireless sensor networks can perform biosystem analysis or surveillance of seismic activity through area monitoring. They can, for example, help to identify the type, concentration, and location of pollutants in air or water. In such settings, sensors monitoring a given part of the entire area should know their location in order to provide information on *what* happens together with *where* it happens. Toward this goal, sensors could be equipped with GPS systems. However, this solution would have two main drawbacks: the cost and the energy consumption [Srivastava 2002a, 2002b]. Indeed, for large-scale sensor networks the individual price of GPS systems becomes an important factor in the network design. Moreover, the energy requirement would shorten the lifespan of the network. Therefore, many strategies have been developed to ensure localization of the entire set of sensors while minimizing the number of GPS systems involved in the localization process. Surveys on the existent strategies can be found in Bulusu et al. [2000] and Rao et al. [2003] where they are classified and analyzed and in Diaz et al. [2002, 2003] in which the authors make an empirical analysis of different protocols, in particular, for localization. With respect to complexity, the localization problem turns out to be *NP*-hard [Aspnes et al. 2004]. It can be worthwhile to use stochastic estimation methods to obtain numerical information about the process. In relation to this [Chatzigiannakis et al. 2003, 2004, 2005] make strong use of randomization and stochastic processes.

In this paper, we show how to use stochastic estimation methods, first introduced in Robbins and Monro [1951], to investigate the topological properties of

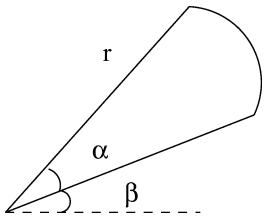

Fig. 1. Sensor with radius of emission r, angle of emission α, and orientation of the emission β.

sensor networks as well as the behavior of dynamical processes on these networks. The framework is particularly important to study problems for which no theoretical results are known or cannot be directly applied in practice, for instance, when only asymptotic results are available. The framework is also useful, for example, for network design or optimization of network parameters. Moreover, we interpret the well-known Russo formula [Chayes and Chayes 1987; Russo 1981] in the context of sensor networks and we thus obtain practical information on the network reliability. In Burkholder [1956] and Venter [1967] suitable numerical estimation methods are presented and implemented for our practical analysis. Concretely, our framework is applied to the problem of localization in wireless sensor networks.

The paper is organized as follows. In the next section, we introduce a model of sensor networks and a localization protocol. Section 3 presents the probabilistic model, theoretical results concerning the probability of success of the localization protocol, and stochastic estimation methods to be implemented. Numerical experiments are then performed and discussed in Section 5. We conclude in Section 6 with three possible applications of the framework developed previously.

2. MODEL OF SENSOR NETWORKS AND THE LOCALIZATION PROTOCOL

We consider sensors scattered over a unit square region $[0, 1]^2$ with a uniform distribution. A given sensor can communicate with all the sensors located in a region corresponding to a sector of a disk (see Figure 1). The motivation for considering such a geometry for the transmission of the signal comes from optical networks [Diaz et al. 2003] as well as from the development of smart antennas for radio transmission. If radio waves are chosen as the mean of transmission of the information, the problem of interferences among simultaneous emissions has to be faced. A way of addressing this is to choose a shape for the antenna emission pattern, which minimizes the probability of interferences, provided a solution exists! Results in the field of percolation [Booth et al. 2003] show that the shape of the antenna emission pattern has an influence on the percolation probability and, quite surprisingly, that the circle seems to be the worst shape. In this paper interferences are not taken into account and we assume that transmission is supported by protocols able to recover from lost data. The radius of emission r, as well as the angle of emission α, are kept constant through time and are the same for all sensors. The orientation β is also kept constant. It is drawn from a uniform distribution on the circle ($[0, 2\pi[$). Notice

that the assumption that the angle of emission is constant with time and is not essential. Actually our analysis remains valid even if the angles change independently. This is also the case if the sensors move in such a way that they remain uniformly distributed on the square region. The localization protocol whose properties we investigate is the following. The total number of sensors is fixed equal to n out of which a subset, henceforth referred to as *localized* sensors, is equipped with GPS systems. For each sensor, this occurs independently with probability p. Localized sensors make their positions known to connected sensors. A sensor, which receives three positions, is able to locate itself, for example, by triangulation using distance estimates. Then, a sensor, which has just computed its position, sends it out so that the localization process goes on until no new sensor becomes localized. We say that the process is *successful* if the percentage of sensors that are eventually localized, is greater than a fixed threshold.

3. PROBABILISTIC MODEL

We begin by describing the probabilistic model for the point process which accounts for the uniform scattering of the sensors over the square area $[0, 1]^2$ and the random selection of an orientation for information transmission. Recall that n denotes the total number of scattered sensors. A point configuration $V = \{(x_1, y_1, \theta_1), \ldots, (x_n, y_n, \theta_n)\}$ of n sensors is an unordered sequence of triplets (x_i, y_i, θ_i), where (x_i, y_i) are coordinates and θ_i an orientation angle for the transmission of sensor i. The coordinates and the orientations, are random variables independently and uniformly distributed in $[0, 1]$, respectively, in the circle $[0, 2\pi[$. Let S_i be the disk sector described in Figure 1, with origin (x_i, y_i). We define the following incidence relation on V: there is an oriented edge from sensor X_i located at (x_i, y_i) to sensor X_j located at (x_j, y_j) if $(x_j, y_j) \in S_i$. Clearly, an edge indicates that sensor X_i can transmit data to sensor X_j.

Definition 3.1. We denote by Λ the entire set of sector graphs (V, A) where A is the set of edges associated to the point configuration V.

The space Λ can be endowed with a probability structure. For our purposes, we do not need to exhibit how this probability space is defined and, hence, the construction is omitted and we refer to Meester and Roy [1996] for similar constructions.

Let us formulate the localization process probabilistically. We first obtain a sector graph in Λ resulting from the scattering of sensors in $[0, 1]^2$. A random orientation for the emission is assigned to each sensor, which, in turn, receives a GPS system with probability p. Thus p is roughly the density of localized sensors among the total set of sensors at the beginning of the localization process. We denote by η or η_p, to stress the dependence on p, a possible assignment of GPS, where $\eta(i) = 1$ if sensor i is localized and 0 otherwise. The set of all GPS assignments, denoted by Ω, is a probability space with the power set $\mathcal{P}(\Omega)$ as $\sigma-$algebra and the product probability measure. The process of localization starts following the protocol described at the end of the previous section and we consider whether this process is successful or not. Thus, we have defined a

Bernoulli random variable $L : \Lambda \times \Omega \to \{0, 1\}$, which depends on the parameter p, i.e., $L = 1$ with probability $m(p) := P_p(\{L = 1\})$. The primary task in the analysis of the chances of success for the localization process is to obtain a plot of the function $m(p)$. Theoretical results about this function can then be derived to support the effectiveness of our numerical methods.

THEOREM 3.2. *The function $m(p)$ is polynomial in p.*

PROOF. To a given sector graph we associate its adjacency matrix. Denoting by *Adj* the set of adjacency matrices there is then a measurable map $\Lambda \to Adj$, which defines a probability measure on the set *Adj*. The random variable $L : \Lambda \times \Omega \to \{0, 1\}$ can actually be seen as a random variable $L : Adj \times \Omega \to \{0, 1\}$. We thus keep the same notation for both random variables. Since *Adj* and Ω are finite sets and since each element belonging to $Adj \times \Omega$ have polynomial probability in p, all events belonging to $\mathcal{P}(Adj \times \Omega)$ are polynomial in p. □

Definition 3.3. We define a partial order on Ω by setting $\eta \leq \eta'$ if $\eta(i) \leq \eta'(i), \forall i$.

Definition 3.4. An event $B \subset \Omega$ is said to be increasing if its indicator function, I_B is increasing, namely, $I_B(\eta) \leq I_B(\eta')$ whenever $\eta \leq \eta'$.

PROPOSITION 3.5. *Given a sector graph $g \in \Lambda$, the event $\{L = 1 \mid g\}$ is increasing.*

PROOF. The event $\{L = 1 \mid g\}$ depends only on the GPS assignment vector η and it is clear that adding a localized sensor can do nothing, but increases the indicator function of $\{L = 1 \mid g\}$. Formal arguments based on coupling can be found in Grimmett [1999]. □

Definition 3.6. Given a sector graph g, a sensor i is called *pivotal* if $\eta \in \{L = 1 \mid g\}$ and $\eta' \notin \{L = 1 \mid g\}$, where η' is obtained from the assignment η by changing $\eta(i) = 1$ to $\eta(i) = 0$.

We must emphasize the following fact: given a sector graph, a pivotal sensor can neither fail nor be removed without compromising the success of the localization algorithm. Hence, the number of pivotal sensors gives crucial information about the reliability of the localization process.

In the situation where the sector graph is fixed (we consider the event $\{L = 1 \mid g\}$), there is a relationship between the expected number of pivotal sensors and the derivative of the function $P_p(L = 1 \mid g)$ with respect to the parameter p. This relationship results from the application of Russo's formula, which is a classical tool in percolation theory [Grimmett 1999]. The proof of this result is not reproduced here (see Grimmett [1999]). However, we stress that it uses the fact that the graph is kept fixed and the the number of sensors is finite.

THEOREM 3.7 (RUSSO'S FORMULA [RUSSO 1981]). *The probability function $P_p(L = 1 \mid g)$ is differentiable with respect to p and satisfies*

$$\frac{d}{dp} P_p(\{L = 1 \mid g\}) = E_p(\textit{number of pivotal sensor for } \{L = 1 \mid g\}) \qquad (1)$$

This result can be extended to our framework.

THEOREM 3.8. *The derivative of the function* $m(p) := P_p(\{L = 1\})$ *satisfies*

$$m'(p) := \frac{d}{dp} P_p(\{L = 1\}) = E_p(\textit{number of pivotal sensor for } \{L = 1\}) \quad (2)$$

Moreover, the function $P_p(\{L = 1\})$ *is increasing with* p.

PROOF. By proposition 3.5, the event $\{L = 1 \mid g\}$ is increasing. Hence, we can apply Russo's formula [Chayes and Chayes 1987; Grimmett 1999; Meester and Roy 1996; Russo 1981]. In our context, the formula states

$$\frac{d}{dp} P_p(\{L = 1 \mid g\}) = E_p(\text{number of pivotal sensor for} \{L = 1 \mid g\}) \quad (3)$$

Taking the expectation of Eq. (3), we get, at once, the right-hand side of Eq. (2). The probability $P_p(\{L = 1 \mid g\})$ being polynomial in p the expectation and the derivative commute to obtain the left-hand side of Eq. (2). The right-hand side of Eq. (2) is positive, implying that the function $P_p(\{L = 1\})$ is increasing. □

4. RECURSIVE ESTIMATION AND CONFIDENCE INTERVAL

Our numerical analysis is based on stochastic estimation methods, first introduced in Robbins and Monro [1951]. Research in this field is synthesized in M. B. Nevel'son [1973]. In Leone et al. [2005], the authors already applied a similar procedure to implement a solution to the energy-balanced data propagation in wireless sensor networks.

Initially these procedures were introduced to solve nonlinear equations of the form

$$R(x) = 0$$

where x is a random variable. For instance, one can consider that we are sampling a random variable x whose probability law depends on a tunable parameter p. The function R might be the expectation of the random variable x, i.e., $R(x) = E_p(x)$ and we are looking for the value p^* of the parameter such that $E_{p^*}(x) = 0$. Basically, these stochastic estimation methods are applicable to determine the critical parameter p^*, provided that $E_p(x)$ is increasing with respect to p in the vicinity of p^*. Hence, the scope of application of the methods is very large. However, the main drawback of these procedures is their relatively slow rate of convergence, which is of order $\mathcal{O}(\#\text{step}^{-1/2})$.

In this section, we do not consider the general theory of stochastic estimation methods, but tailor their use to our particular framework. We assume that the total number of sensors n, as well as the radius of emission r and the angle of emission α, are fixed. We choose an initial value for the parameter p_1, which is the probability that a sensor is equipped with a GPS system. We draw a sample sector graph and a configuration of sensors, which are localized accordingly to the value of the parameter p_1. We then start the localization process, which yields a sampling of the random variable $L : \Lambda \times \Omega \to \{0, 1\}$ whose probability distribution is parametrized by p_1. We denote by $L(p_1)$ the outcome of the

localization process. The parameter p_1 is updated with the rule [Robbins and Monro 1951]

$$p_{j+1} = p_j + \frac{1}{j}(m - L(p_j)) \tag{4}$$

with $m \in [0, 1]$ kept fixed. Generally, if the parametrized probability function $m(p) := P_p(\{L = 1\})$ satisfies some regularity conditions [Blum 1954; M. B. Nevel'son 1973; Robbins and Monro 1951], then it is known that $p_j \to p_m$, where p_m is such that $m(p_m) = m$. Such conditions are fulfilled in our framework as it appears in the proof of Theorem 4.1. Hence, by repeating this procedure with different values of y, we get a set of points $(p(y), y)$ of the curve $p \mapsto m(p)$.

THEOREM 4.1. *The sequence $(p_j)_{j\geq 1}$ recursively defined by Eq.* (4) *converges almost surely to p^* such that $m(p^*) = y$.*

A way of proving the almost sure convergence, is to associate a supermartingale to the sequence $\{p_j\}_{j\geq 1}$ whose convergence implies the convergence of the p_j toward p^*. Here, we choose to prove the convergence in probability. The result is weaker, but the proof is easier and allows to understand the convergence of the numerical scheme (4).

PROOF. We first notice that since the event $\{L = 1\}$ is increasing, $P_p(\{L = 1\})$ is increasing with p. If we denote by p^* the value of the parameter such that $m(p^*) = y$, then

$$(p - p^*)(y - m(p)) < 0 \tag{5}$$

Using Eq. (4) we have

$$\begin{aligned} E((p_{j+1} - p^*)^2) \;=\; & E((p_j - p^*)^2) + \frac{2}{j}E((p_j - p^*)(y - L(p_j)) \\ & + \frac{1}{j^2}E((y - L(p_j))^2) \end{aligned}$$

which can be repeatedly applied so as to obtain

$$\begin{aligned} E((p_{j+1} - p^*)^2) = E((p_1 - p^*)^2) + \sum_{k=1}^{j} \frac{2}{k}E((p_k - p^*)(y - L(p_k)) \\ + \sum_{k=1}^{j} \frac{1}{k^2}E((y - L(p_k))^2) \end{aligned} \tag{6}$$

Since $E((y - L(p_k))^2)$ is bounded the last sum converges as $j \to \infty$, because of the factor $\frac{1}{k^2}$ in the sum and is positive. Moreover,

$$E((p_k - p^*)(y - L(p_k)) = E(E((p_k - p^*)(y - L(p_k)|p_k)) \tag{7}$$

$$= E(\underbrace{(p_k - p^*)(y - m(p_k))}_{<0}) \leq 0 \tag{8}$$

Since $E((p_{j+1} - p^*)) \geq 0$ and Eq. (8), it should be that $E(\|(p_k - p^*)(y - m(p_k))\|) \to 0$ as $k \to \infty$. Otherwise, the second term of the right-hand side

of Eq. (4) will tend to $-\infty$, which contradicts $E((p_{j+1} - p^*)^2) \geq 0$. Markov's inequality implies that $P(\|(p_k - p^*)(y - m(p_k))\| > \epsilon) \to 0$ as $k \to \infty$. To conclude, we notice that since the function $m(p)$ is increasing, we have $\|p_k - p^*\| > C \Rightarrow \|y - m(p_k)\| > C'$ and, hence, $P(\|p_k - p^*\| > \epsilon) \to 0$, i.e., the convergence in probability. □

The derivative of $m(p)$ must also be estimated, since it is related to the expected number of pivotal sensors [Venter 1967; Burkholder 1956; M. B. Nevel'son 1973]. The implementation requires to simulate the localization process twice per random sector graph with different values of the parameter p.

THEOREM 4.2 (VENTER 1967). *The intertwined numerical scheme recursively defined by*

$$a_j = \frac{1}{j} \sum_{k=1}^{j} \frac{L(p_j + j^{-\gamma}) - L(p_j - j^{-\gamma})}{2j^{-\gamma}} \tag{9}$$

and

$$p_{j+1} = p_j - \left(\frac{1}{j} + \frac{1}{\sqrt{j}}\right) \frac{1}{2a_n}(L(p_j + j^{-\gamma}) + L(p_j - j^{-\gamma})) \tag{10}$$

where $0 < \gamma \leq 0.5$ *converges almost surely for any initial value* p_1. *The sequence* p_j *converges to the value* p^*, *which satisfies* $m(p^*) = y$ *and* $a_j \to m'(p^*)$ *with* $m'(p)$ *the derivative of* $m(p)$ *at* p.

In Theorems 4.1 and 4.2, the almost sure convergence of the numerical scheme is asserted. Actually, stronger results [Chung 1954; Sacks 1958; M. B. Nevel'son 1973; Venter 1967] assert that the probability distribution of $\sqrt{n}(p_j - p^*)$ converges to a normal distribution as $j \to \infty$. These results are central limit theorems for recursive approximation procedures and are the basis for the efficient construction of confidence intervals. The strategy proposed in Hsieh and Glynn [2002] is to run multiple replicas of the approximation procedures, which lead to multiple simultaneous estimation of the parameter. We denote by $\{p_j^i\}$ the resulting multiple sequences, where $i = 1, \ldots, m$ (m is fixed and is the number of replicas that are run). For each step two estimators are defined

$$p(m, j) = \frac{1}{m} \sum_{i=1}^{m} p_j^i \tag{11}$$

$$V(m, j) = \frac{1}{m-1} \sum_{i=1}^{m} \left(p_j^i - p(m, j)\right)^2 \tag{12}$$

which are, respectively, the sample mean and the sample variance. To establish a $1 - \delta$ interval of confidence, we select the value z such that $P(F_{(1,m-1)} \leq z) = 1 - \delta$, where $F_{(1,m-1)}$ is a F distributed random variable with $(1, m-1)$ degrees of freedom (see Grimmett and Stirzaker [2001]). Then, we construct the interval of confidence

$$\Lambda_{mj}(z) = \left\{ p \,\middle|\, \frac{(p(m, j) - p)^2}{V(m, j)} \leq z \right\} \tag{13}$$

The following result is proved in Hsieh and Glynn [2002].

THEOREM 4.3. *If a central limit theorem is valid for the approximation procedures,* $E((m(p^*) - y)^2)$ *is nonsingular and* $m \geq 2$*, then*

$$P(p^* \in \Lambda_{mj}(z)) \to 1 - \delta$$

as $j \to \infty$.

5. NUMERICAL SIMULATIONS

We propose a numerical scaling analysis of the localization process by plotting the probability of success $m(p)$ and its derivative $m'(p)$ for various antenna radiation patterns and different density of sensors. We performed four sets of experiments. For each set of experiments the number of iterations (i.e., the number of elements in the sequence $\{p_j\}$ according to formula (4)) is chosen in such a way that the relative error is smaller than 5% in the confidence interval, which is 95%, i.e., $\delta = 0.05$ in Theorem 4.3. We ran three replicas of the approximation procedures and the mean of the three values are plotted in the figures (see formula 11). We also checked that the three results obtained each belong to the confidence interval. The performance of the different configurations (number of sensors, radius, and angle of emission) are directly compared by the probability of success of the localization algorithm. However, in sensor networks, energy consumption is a key factor to minimize. Hence, the energy consumption induced by different configurations must also be compared. Concerning the distance of emission, it is usually admitted that power consumption increases with the square of the radius of emission r^2. (More precisely, the exponent may change, basically from 2 to 4, depending on external conditions). Concerning the angle of emission, the radiated power in the directed beam depends on the geometry of the antenna. We here assume, as a first approximation, that the radiated power p_r depends linearly on the consumed power p_c,

$$p_r = g p_c$$

where g is the gain of the antenna. We will assume for simplicity that, if the angle of emission is divided by a given factor, then the consumed energy is divided by the same factor. Hence, the energy consumption is proportional to the area of the antenna radiation pattern.

In the first set of experiments, we numerically analyze the impact of the radius of emission on the chances of success of the localization algorithm in different situations (see Figures 2–4). For the computations, we keep the angle of emission and the density of sensors ($n = 50$) constant, while the radius of emission varies $r = 0.1, 0.3$, and 0.7. Thus, the energy consumption increases by a factor 9 and 49 with respect to the case where $r = 0.1$. The number of iterations is 25,000. In Figure 2 the angle of emission is $\alpha = 0.3$. We observe that increasing the radius of emission r does not significantly improve the chances of success of the localization algorithm. However, as α grows, we observe that the impact on the performances of the localization algorithm of increasing r is stronger. Indeed, in Figure 3 the emission angle is $\alpha = 1.0$ and the performances are better with a radius of emission $r = 0.7$. However, no significant difference is observed between the performances of the algorithm with $r = 0.1$ and $r = 0.3$,

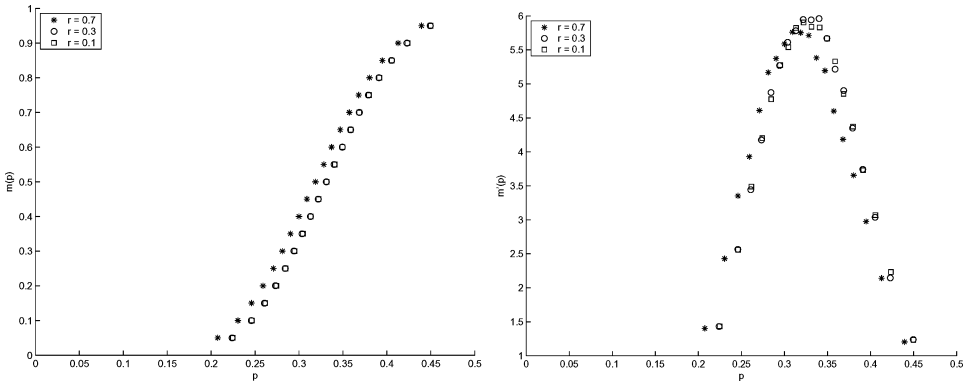

Fig. 2. Graphs of the regression function $m(p)$ with $n = 50$ sensors (left) and $m'(p)$ (right) with $\alpha = 0.3$ and $r = 0.1, 0.3$, and 0.7.

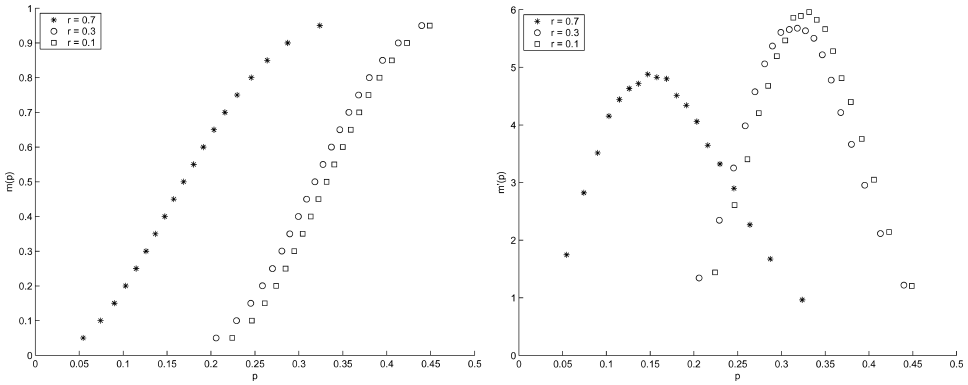

Fig. 3. Graphs of the regression function $m(p)$ with $n = 50$ sensors (left) and $m'(p)$ (right) with $\alpha = 1.0$ and $r = 0.1, 0.3$, and 0.7.

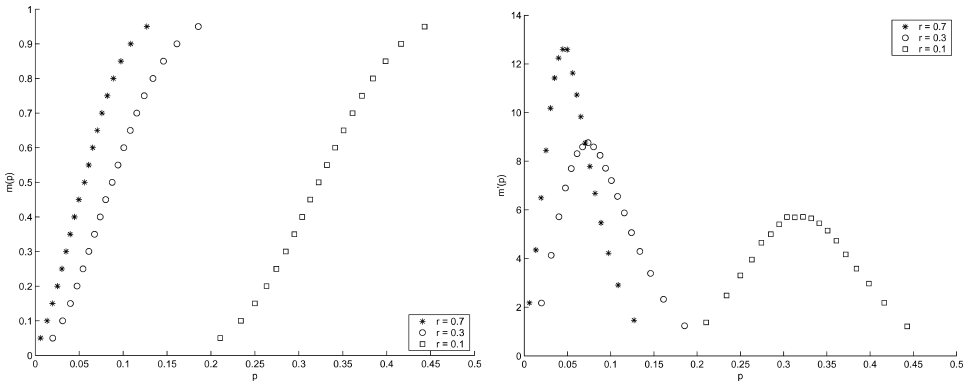

Fig. 4. Graphs of the regression function $m(p)$ with $n = 50$ sensors (left) and $m'(p)$ (right) with $\alpha = 2\pi$ and $r = 0.1, 0.3$, and 0.7.

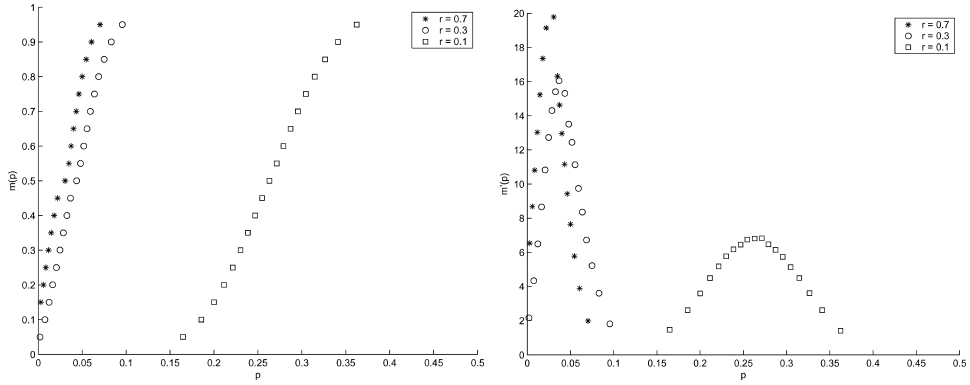

Fig. 5. Graphs of the regression function $m(p)$ with $n = 100$ sensors (left) and $m'(p)$ (right) with $\alpha = 2\pi$ and $r = 0.1, 0.3$, and 0.7.

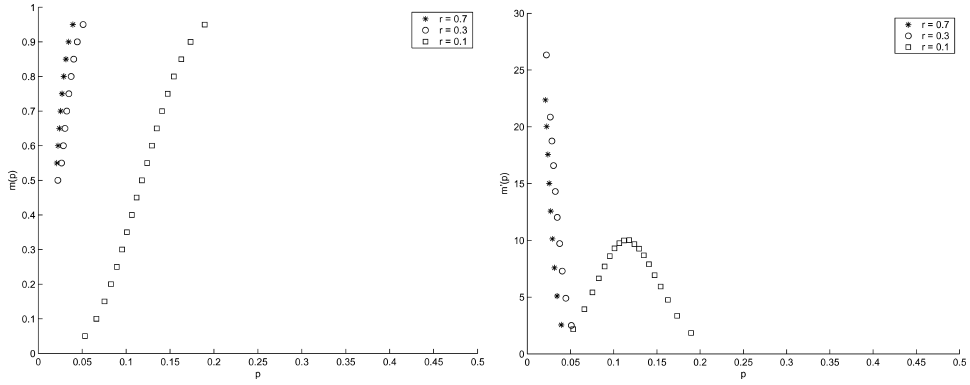

Fig. 6. Graphs of the regression function $m(p)$ with $n = 200$ sensors (left) and $m'(p)$ (right) with $\alpha = 2\pi$ and $r = 0.1, 0.3$, and 0.7.

although the difference in energy consumption is important (nine times more). The angle of emission must still be increased to observe a notable improvement for $r = 0.3$. This can be observed on Figure 4. From an algorithmic point of view, one may say that the impact of adapting the radius of emission in order to fulfill a successful localization algorithm changes significantly with the angle of emission.

In the second set of experiments, we consider the impact of the density (number) of sensors on the localization algorithm. We fix the angle of emission $\alpha = 2\pi$ and vary the radius of emission $r = 0.1, 0.3$, and 0.7 with $n = 50, 100$, and 200 sensors. Depending on the density $n = 50, 100$, and 200, the number of iterations is, respectively, 25,000, 45,000 and 75,000 in order to obtain results with a relative error less than 5%. Results are plotted in Figures 4–6. As expected, we observe that the chances of success increase with the density. However, it is interesting to notice that, as the density increases, the maximum number of pivotal sensors also increases significantly. This shows that the transition between the 0 and the 100% chance of successful localization becomes sharper.

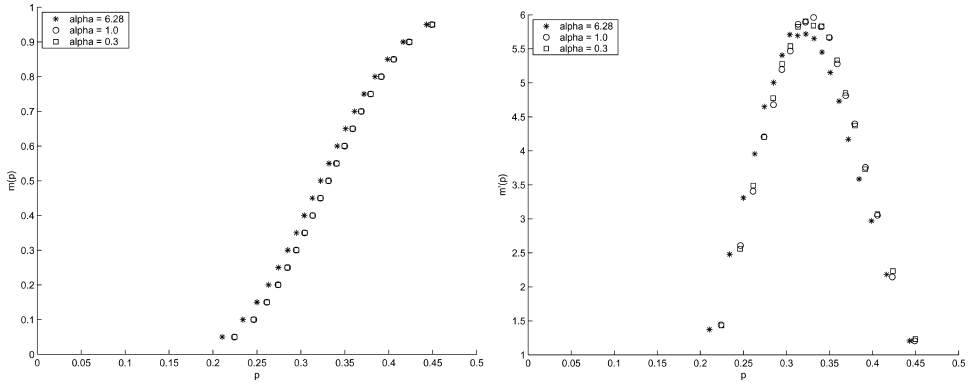

Fig. 7. Graphs of the regression function $m(p)$ with $n = 50$ sensors (left) and $m'(p)$ (right) with $\alpha = 0.3, 1.0,$ and 2π and $r = 0.1$.

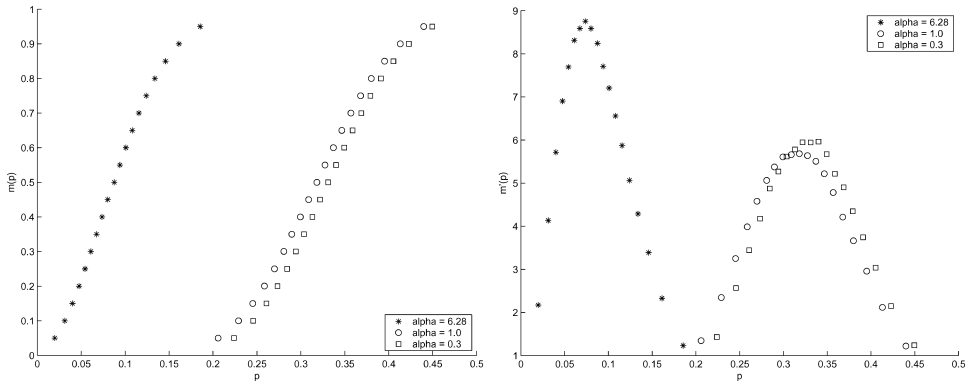

Fig. 8. Graphs of the regression function $m(p)$ with $n = 50$ sensors (left) and $m'(p)$ (right) with $\alpha = 0.3, 1.0,$ and 2π and $r = 0.3$.

In this transition region, the configurations are more sensitive to failure as the density increases.

In the third set of experiments, we consider the impact of increasing the angle of emission. We consider angles $\alpha = 0.3, 1.0, 2\pi$ for various values $r = 0.1, 0.3,$ and 0.7 of the emission radius. Compared to the results of the first set of experiments, we observe a similar quantitative and qualitative behavior. Increasing the angle of emission has no significant impact, while the radius of emission is small (see Figure 7), and the impact increases with the radius of emission (see Figures 8 and 9). Looking at the results of the first set of experiments, we may conjecture that an adaptive algorithm to ensure localization should adapt the angle of emission first. Roughly, the impact of multiplying the radius by a given factor is the same as multiplying the angle of emission by the same factor, but the latter amounts to consuming much less energy. More systematic experiments must be performed to accurately determine how to adapt the angle and the radius of emission in order to achieve localization

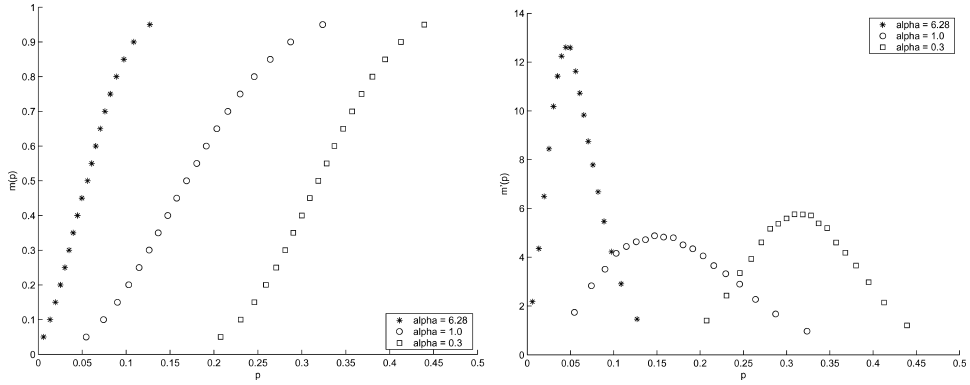

Fig. 9. Graphs of the regression function $m(p)$ with $n = 50$ sensors (left) and $m'(p)$ (right) with $\alpha = 0.3, 1.0,$ and 2π and $r = 0.7$.

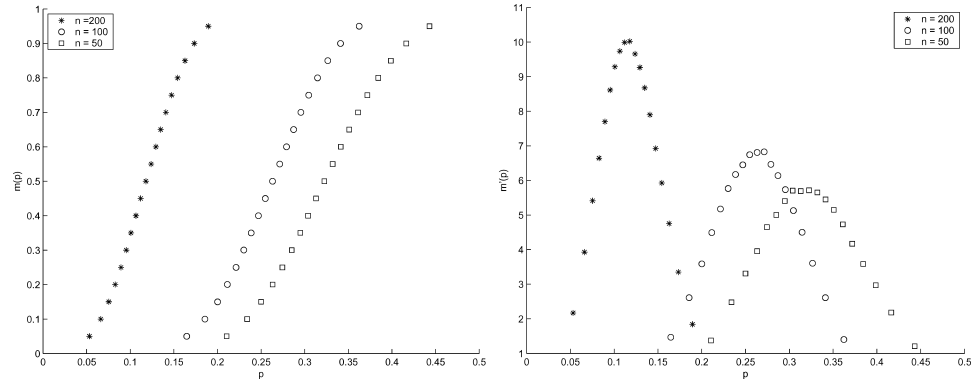

Fig. 10. Graphs of the regression function $m(p)$ (left) and $m'(p)$ (right) with $\alpha = 2\pi$ and $r = 0.1$ and $n = 50, 100,$ and 200 (top to bottom).

and minimize the energy consumed. This would require much more effort and is postponed to further work. The numerical experiments presented here aim mainly at illustrating the potential of stochastic estimation methods. They also raise interesting questions and suggest tracks for further research.

The last set of experiments deals with the impact of the density of sensors on the success of the localization process. We successively choose $n = 50, 100,$ and 200 with radius of emission $r = 0.1$ and angle of emission $\alpha = 2\pi$, kept constant. Numerical results are presented in Figure 10. We again observe that the transition between the 0 and the nearly 100% chance of successful localization becomes sharper as the number of sensors increases.

6. APPLICATIONS AND CONCLUSION

In this section, we describe some practical applications of the framework introduced in this paper. The first application is concerned with the problem of estimating the instability coefficient, hence, the expected number of pivotal sensors, in a given network. The second application is oriented toward network

design. We assume that we know the total number of sensors to be scattered in a given region, as well as their characteristics. We look for a placement of the sensors in the region, which minimizes the expected number of pivotal sensors in order to make the network more robust to failures. The last application deals with mobility in sensor networks.

6.1 Networks Robustness Estimation

We consider a given network with n sensors, k of them being localized with a GPS system and we assume that the process of localization as described at the end of Section 1 is successful. The goal is to estimate the number of pivotal sensors. If we assume that the probability that a sensor is localized is p, the expected total number of localized sensors X is $E(X) = np$ and we search for the probability that $X = k$. We fix $\epsilon < 1$ the order of certainty in the following analysis. Using Chernoff's bound [Motawni and Raghavan 1995], we know that with

$$\delta(np, \epsilon) = \sqrt{\frac{4\ln(1/\epsilon)}{np}} \qquad 0 < \delta < 1 \tag{14}$$

we have

$$Pr((1-\delta)np < X < (1+\delta)np) < e^{-np\delta^2/2} \tag{15}$$

Clearly, the right-hand side of Eq. (15) has to be made as small as possible. We rewrite the expression on the left side replacing δ with the expression (14) to obtain

$$\left(1 - \sqrt{\frac{4\ln(1/\epsilon)}{np}}\right) np < k < \left(1 + \sqrt{\frac{4\ln(1/\epsilon)}{np}}\right) \tag{16}$$

Given that we have fixed the order of uncertainty ϵ, we obtain

$$\frac{2\ln(1/\epsilon) + k - 2\sqrt{\ln(1/\epsilon)^2 + k\ln(1/\epsilon)}}{n} < p \ldots \tag{17}$$

$$\ldots < \frac{2\ln(1/\epsilon) + k + 2\sqrt{\ln(1/\epsilon)^2 + k\ln(1/\epsilon)}}{n} \tag{18}$$

For example, assuming $n = 50$ and $k = 5$, we get $\epsilon = 0.1$

$$0.28 < p < 0.36 \tag{19}$$

If we assume that the radius of emission is 0.1 and the angle of emission $\alpha = 1.0$, we can look at Figure 3 to determine that the expected number of pivotal sensors lies in the interval [5, 6].

6.2 Network Design

Assume we are given a fixed number of sensors n, with specified characteristics that we would like to use to monitor a given area. The goal is to place the sensors in such a way that a given process will succeed and ensure that the placement maximizes the robustness of the network. Although, the process we

consider is evidently not the localization process, it could be a flooding process, for example. We keep in mind the situation of the previous section as an illustration. We assume that we computed a figure similar to the one in Figure 3 for the particular process under investigation. Consider, for example, the top-right graph in Figure 3. We would like to exhibit a configuration of sensors, which minimizes the expected number of pivotal sensors, i.e., maximizes the robustness of the network. This can be done using the same type of probabilistic method by choosing a value of the parameter p, which minimizes the expected number of pivotal elements and obtaining sample points through simulations. For this application, it would be useful to estimate the variance of the expected number of pivotal sensors in order to bound the real number of pivotal elements.

6.3 Mobile Sensor Networks

Consider a sensor network with mobile sensors monitoring a given area. We assume that for the reasons already discussed in the introduction, not every sensor is equipped with a GPS system. This implies that when the sensors want to send data, a localization process has to be initiated to localize every sensor. Once the sensors are localized, data can be transmitted. The sensors, which do not manage to compute their position, cannot transmit their data, since the data is meaningless without the location of the sensor. Thus, it is of interest to have a probabilistic estimate of the maximal or typical distance a sensor travels from its initial position to the next localized one. Notice that, at this stage, different scenarios are possible. For instance, only a particular sensor may want to send data and needs to be localized. Such scenarios can also be analyzed within our framework. However, we still continue to deal with the situation where every sensor need to be localized at a given time, mainly because it is coherent with the numerical experiments we have done and with others applications described in this paper.

A possible approach to deal with the above problem is to assume that the motion of the sensors is described as a continuous application $X : \mathcal{R} \to \Lambda$ in the set of random sector graphs. For each time t, the localization process on the random sector graph $X(t)$ can be successful or not and we denote this random variable $S(X(t)) \in \{0, 1\}$. Intuitively, it is clear that $S(X(t))$ is locally constant and transition times are discrete. Hence, the evolution of the success process $S(X(t))$ is formally described as a discrete event dynamical system. Assume that the process $X(t)$ possesses limit laws, which allows us to predict that $\lim_{t\to\infty} P(S(X(t)) = 1) = \nu$ and $\lim_{t\to\infty} P(S(X(t)) = 0) = 1 - \nu$. The limit ν can generally be inferred from numerical simulations, as described in Section 6.1. For instance, the process describing the motion $X(t)$ is a suitable Markov chain. Thus, for τ large enough and $t > \tau$, we have $P(S(X(t)) = 1) \approx \nu$. We fix t_0 with $S(X(t_0)) = 1$ and, hence,

$$P(S(X_{t_0+\tau}) = 0, \ldots, S(X_{t_0+(l-1)\tau}) = 0, S(X_{t_0+l\tau} = 1) \approx (1-\nu)^{l-1}\nu$$

If we know that the maximal or typical distance covered by a sensor during an interval of time τ is s, the maximal or typical distance covered by a sensor

between successive localized positions can be written ξs, with ξ a geometric random variable with parameter ν.

6.4 Conclusion

The numerical experiments performed in this paper show that this probabilistic framework can be applied for numerically handling difficult problems. A complete numerical analysis of the protocol would require more systematic numerical computations and multiparameter stochastic estimations implemented. However, our aim in this paper was to build the framework and to validate our approach. The listed applications, introduced in the previous section, demonstrate the potential of the method and this list is far from being exhaustive.

REFERENCES

AKYILDIZ, I., SU, W., SANKARASUBRAMANIAM, Y., AND CAYIRCI, E. 2002. Wireless sensor networks: a survey. *Computer Networks 38*, 4, 393–422.

ASPNES, J., GOLDENBERG, D., AND YANG, Y. 2004. On the computational complexity of sensor network localization. In *Proc. of the 1st Int'l Workshop on Algorithmic Aspects of Wireless Sensor Networks (ALGOSENSORS 2004)*. Lecture Notes in Computer Science, vol. 3121. Turku, Finland. Springer Verlag, New York. 32–44.

BLUM, J. 1954. Approximation methods which converge with probability one. *Ann. Math. Statist. 25*, 2, 382–386.

BOOTH, L., BRUCK, J., FRANCESCHETTI, M., AND MEESTER, R. 2003. Covering algorithms, continuum percolation and the geometry of wireless networks. *Annals of Applied Probability 13*, 2, 722–741.

BULUSU, N., HEIDEMANN, J., AND ESTRIN, D. 2000. Gps-less low cost outdoor localization for very small devices. *IEEE Personal Communications Magazine 7*, 5 (Oct.), 28–34.

BURKHOLDER, D. 1956. On a class of stochastic approximation processes. *Ann. Math. Statist 27*, 1044–1059.

CHATZIGIANNAKIS, I., DIMITRIOU, T., M. MAVRONICOLAS, S. N., AND SPIRAKIS, P. 2003. A comparative study of protocols for efficient data propagation in smart dust networks. *Parallel Processing Letters 13*, 4 (Dec.), 615–627.

CHATZIGIANNAKIS, I., DIMITRIOU, T., NIKOLETSEAS, S., AND SPIRAKIS, P. 2004. A probabilistic algorithm for efficient and robust data propagation in smart dust networks. In *Proc. of the 5th European Wireless Conf. on Mobile and Wireless Systems Beyond 3G*. 344–350.

CHATZIGIANNAKIS, I., NIKOLETSEAS, S., AND SPIRAKIS, P. 2005. Efficient and robust protocols for local detection and propagation in smart dust networks. *Mobile Network and Applications 10*, 1–2, 133–149.

CHAYES, J. AND CHAYES, L. 1987. The mean-field bound for the order of parameter of bernoulli percolation. In *Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985)*. IMA Vol. Math. Appl., vol. 8. Springer, New York. 49–71.

CHUNG, K. 1954. On a stochastic approximation method. *Ann. Math. Statist. 25*, 463–483.

DIAZ, J., PETIT, J., AND SERNA, M. 2002. A random raph model for optical networks of sensors. Tech. Rep. LSI-02-72-R, Universitat Politecnica Catalunya.

DIAZ, J., PETIT, J., AND SERNA, M. 2003. Evaluation of basic protocols for optical smart dust networks. In *Proc. of the 2nd Int'l Workshop on Experimental and Efficient Algorithms (WEA 2003)*. Lecture Notes in Computer Science, vol. 2647. Ascona, Switzerland. Springer Verlag, New York.

GRIMMETT, G. 1999. *Percolation*, 2nd ed. A Series of Comprehensive Studies in Mathematics, vol. 321. Springer Verlag, New York.

GRIMMETT, G. AND STIRZAKER, D. 2001. *Probability and Random Processes*, 3rd ed. Oxford University Press, Oxford.

HSIEH, M. AND GLYNN, P. 2002. Confidence regions for stochastic approximation algorithms. In *Proc. of the 2002 Winter Simulation Conference*. 370–376.

LEONE, P., NIKOLETSEAS, S., AND ROLIM, J. 2005. An adaptative blind algorithm for energy balanced data propagation in wireless sensor networks. Submitted to DCOSS.

M. B. NEVEL'SON, R. H. 1973. Stochastic approximation and recursive estimation. In *Translations of Mathematical Monographs*. vol. 47. American Mathematical Society, Providence, R.I.

MEESTER, R. AND ROY, R. 1996. *Continuum percolation*. Cambridge University Press, Cambridge.

MOTAWNI, R. AND RAGHAVAN, P. 1995. *Randomized Algorithms*. Cambridge University Press, Cambridge.

RAO, A., RATNASAMY, S., PAPADIMITRIOU, C., SHENKER, S., AND STOICA, I. 2003. Geographic routing without location information. In *MOBICOM'03*. 96–108.

ROBBINS, H. AND MONRO, S. 1951. A stochastic approximation method. *Ann. Math. Statistics 22*, 400–407.

RUSSO, L. 1981. A note on percolation. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandt Gebiete 56*, 229–237.

SACKS, J. 1958. Asymptotic distributions of stochastic approximation procedures. *Ann. Math. Statist. 29*, 373–405.

SRIVASTAVA, M. 2002a. Part ii: Sensor node platforms & energy issues. In *Tutorial on Wireless Sensor Networks, Mobicom 2002*. Available at http://nesl.ee.ucla.edu/tutorials/mobicom02.

SRIVASTAVA, M. 2002b. Part iii: Time & space problems in sensor networks. In *Tutorial on Wireless Sensor Networks, Mobicom 2002*. Available at http://nesl.ee.ucla.edu/tutorials/mobicom02.

VENTER, J. 1967. An extension of the robbins-monro procedure. *Ann. Math. Statist. 38*, 181–190.

Received September 2005; revised January 2006; accepted February 2006