A Unified View of the Equations of Motion used for Control Design of Humanoid Robots

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The Role of the Base Frame in Free-Floating Mechanical Systems and its Connection to Centroidal Dynamics

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Abstract This paper contributes towards the development of a unified standpoint on the equations of motion used for the control of free-floating mechanical systems. In particular, the contribution of the manuscript is twofold. First, we show how to write the system equations of motion for any choice of the base frame, without the need of re-applying algorithms for evaluating the mass, coriolis, and gravity matrix. A particular attention is paid to the properties associated with the mechanical systems, which are shown to be invariant with respect to the base frame choice. Secondly, we show that the so-called *centroidal dynamics* can be obtained from any expression of the equations of motion via an appropriate system state transformation. In this case, we show that the mass matrix associated with the new state is block diagonal, and the new base velocity corresponds to the so-called *average 6D velocity.*

Keywords Free-floating mechanical systems · humanoid robots · centroidal dynamics · average angular velocity.

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1 Introduction

Classical modelling techniques for multi-body mechanical systems have their roots in the fundamental principles of mechanics. For instance, Euler-Lagrange and Euler-Poincaré formalisms are among the main tools for deriving the equations of motion for systems evolving on vector spaces and Lie groups, respectively [20]. The application of these formalisms to common robotics applications assume that the underlying mechanical system is composed of a collection of rigid bodies – called links – interconnected by mechanisms – called joints – constraining the links relative motion. The obtained equations of motion, however, often depend upon the arbitrary representation of the configuration space of the multi-body system. This paper proposes a unified view of the equations of motion used for the control of free-floating systems. In particular, it investigates the role of the *base frame* choice when defining the configuration space of free-floating robots [7].

In general, the configuration space of a multi-body system composed by $n+1$ links may be characterised by the positions of all the points of the system. To this inconvenient representation, it is always preferred that consisting of the orientations and the center-of-mass positions of all the rigid bodies composing the system. When the links are interconnected by n joints, any point of the system can be expressed in terms of the *joint positions* and a *frame* attached to a link of the system. This frame is referred to as *base frame* and its choice is dictated by rule-of-thumb principles, such as attaching the base frame to the heaviest link of the considered system [35]. Then, the configuration space of the mechanical system is of dimension $n+6$, and it is is referred to as *free-floating* or *floating base*.

If one of the links has a constant position-and-orientation with respect to the inertial frame, the configuration space collapses to joint positions only, and its dimension drops to n. In this case, the system is referred to as *fixed base*, and robotic manipulators attached to ground fall into this category.

Modelling, estimation, and control of robots, however, were developed first for fixed-base manipulators, and then for floating-base systems. Consequently, floating-base systems are usually viewed as a special case of fixed base robots, and their modelling is often addressed by considering a six degree of freedom joint between the inertial and the base frame [27]. This solution has the advantage of reusing the fixed base machinery to derive the equations of motion for floating base robots. However, it does hide some properties of floating systems, and it makes the use of minimal representations for the base orientation really tempting, which in turn introduces artificial singularities for this orientation [37].

Attempts to generalize the floating-base systems' equations of motion irrespective from the base frame choice can be found in [34, Chapter 3] and in [15, Chapter 17, Section 3.6]. The main theoretical drawbacks of these work, which have been largely ignored by the humanoid control literature, is the assumption that the base frame is rigidly attached to one of the links of the system.

In this paper, we model floating systems without assuming any preferred base frame, and we show how the kinematics and dynamics of the robot arise by choosing different base frames. In particular, we show how the equations of motion associated with different *base frames* can be obtained as a nonlinear change of variables of the robot state. We also show that this change of variables can be used to express the robot dynamics as a combination of the "internal" and the "centroidal dynamics", as introduced in [27] and extended in [10]. While most of the results of this paper can be connected to classical Lie Group geometrical mechanics results, we avoid to introduce such a connection to simplify the paper reading for readers not familiar with geometrical mechanics. An extended explanation on the connection between the notation employed here and geometrical mechanics theory can be found in [33].

This paper is organized as follows. Section 2 recalls notation, kinematics, and dynamics for a single rigid body. Section 3 recalls and complements the free-floating notation and equations of motion. Section 4 explains the effects of changing the base frame on the system's kinematics and dynamics. Section 5 discusses the connection between a change of the base frame and the centroidal dynamics. Section 6 illustrates an application to humanoid robots of the content of this paper. Remarks and perspectives conclude the paper.

2 Background

This section introduces basic notation used throughout the paper. For further information and details on the notation used here, the reader is referred to [33]. Although what follows is partially based on Spatial Vector Algebra notation [7,9], several specific modifications have been introduced for the purpose of this paper.

- The set of real numbers is denoted by $\mathbb{R}$. Let u and v be two n-dimensional column vectors of real numbers, i.e. $u, v \in \mathbb{R}^n$, then their inner product is denoted as $u^\top v$, with "$\top$" the transpose operator.
- The identity matrix of dimension n is denoted by $1_n \in \mathbb{R}^{n \times n}$; the zero column vector of dimension n is denoted by $0_n \in \mathbb{R}^n$; the zero matrix of dimension $n \times m$ is denoted by $0_{n \times m} \in \mathbb{R}^{n \times m}$.
- Given a vector $w = \begin{bmatrix} x \ y \ z \end{bmatrix}^\top \in \mathbb{R}^3$, we define $w^\wedge$ (read *w hat*) as the 3×3 *skew-symmetric* matrix associated with the cross product $\times$ in $\mathbb{R}^3$, i.e. $w^\wedge v = w \times v$. Given the *skew-symmetric matrix* $W = w^\wedge$, we define $W^\vee \in \mathbb{R}^3$ (read *W vee*) as $W^\vee := w$.
- Let $\mathrm{SO}(3)$ denote the set of $\mathbb{R}^{3\times 3}$ orthogonal matrices with determinant equal to one, namely $\mathrm{SO}(3) := \{ R \in \mathbb{R}^{3\times 3} \mid R^\top R = I_3,\ \det(R) = 1 \}$.

2.1 Frame notation

We define a *frame* as a pair of a point (called *origin*) and an *orientation frame* in the 3D space [5], and we use capital letters to denote frames. Given a frame F, we denote with o_F its origin and with $[F]$ its orientation frame. Formally, we write this as $F = (o_F, [F])$. Then, we have:

A	The inertial (*A*bsolute) frame.
p	An arbitrary point.
$B[A]$	Frame with origin o_B and orientation $[A]$.
$\overline{B}$	Shorthand for $B[A]$.

2.2 Notation for coordinate vectors and transformation matrices

This section details the main symbols used in the paper for expressing kinematics and dynamics quantities. For precise meanings of these symbols, see sections next.

${}^A p \in \mathbb{R}^3$	Coordinates of p w.r.t. to A.
${}^A o_B \in \mathbb{R}^3$	Coordinates of o_B w.r.t. to A.
${}^A R_B \in \mathbb{R}^{3\times 3}$	Rotation matrix transforming a 3D vector expressed in $[B]$ into one expressed in $[A]$.
${}^A X_B = \begin{bmatrix} {}^A R_B & {}^A o_B^\wedge {}^A R_B \\ 0_{3\times 3} & {}^A R_B \end{bmatrix}$	Velocity transformation from B to A.
${}^C \mathrm{v}_{A,B} \in \mathbb{R}^6$	6D velocity of B w.r.t. to A expressed in C.
${}_B \mathrm{f} \in \mathbb{R}^6$	Coordinates of the 6D force f w.r.t. B.
${}_A X^B = \begin{bmatrix} {}^A R_B & 0_{3\times 3} \\ {}^A o_B^\wedge {}^A R_B & {}^A R_B \end{bmatrix}$	6D force transformation from B to A (${}_A X^B = {}^A X_B^{-\top}$).
${}_B I_L \in \mathbb{R}^{6\times 6}$	6D inertia matrix of body L expressed in the frame B.

Remark 1 In this paper, instead of the terms *wrench* and *twist*, we use *6D force* and *6D velocity*, respectively. This choice is made because the terms *wrench* and *twist* have precise meanings that depend on the specific author using them, and these meanings may not correspond to our cases. For example, in screw theory *wrench* is defined as a 6D vector composed of a force and a toque, with the force parallel to the torque [1, Chapter 1, Section 7]. In robotics literature [23, Chapter 2, Section 5.1], instead, it is defined as a generic 6D force, with no constraint on the direction of the linear or angular part.

2.3 Dot notation

Given a quantity such as position, velocity, etc., we define the *dot operator* $(\dot{-})$ as the total time derivative of the quantity. Hence, given a 6D velocity ${}^C\mathrm{v}$ expressed in a frame C, the symbol ${}^C\dot{\mathrm{v}}$ means

$$ {}^C\dot{\mathrm{v}} = \frac{d}{dt}\left({}^C\mathrm{v}\right). $$

While this may be obvious and pedantic, let us recall that the *dot operator* $(\dot{-})$ is defined differently in some of the robotics dynamics literature. For instance, given a 6D velocity ${}^C\mathrm{v}$ expressed in a frame C, [7, Section 2.10] defines ${}^C\dot{\mathrm{v}}$ as

$$ {}^C X_A \frac{d}{dt}({}^A\mathrm{v}), $$

where A is an arbitrary (and often hidden) inertial frame with respect to (w.r.t.) which the derivative is computed.

2.4 Frame velocity representation

The origin and orientation of a frame B w.r.t. the inertial frame A is usually represented with the homogeneous matrix ${}^A H_B \in \mathbb{R}^{4\times 4}$ transforming a position vector expressed in B to a position vector expressed in A, i.e.

$$ {}^A H_B = \begin{bmatrix} {}^A R_B & {}^A o_B \\ 0_{1\times 3} & 1 \end{bmatrix}, \quad {}^A p = {}^A R_B {}^B p + {}^A o_B. \tag{1} $$

Then, the body velocity can be represented by ${}^A \dot{H}_B$. It is more convenient, however, to express the body velocity as a $\mathbb{R}^6$ vector. Common representations of ${}^A \dot{H}_B$ as an element of $\mathbb{R}^6$ are given by:

$$ {}^A \mathrm{v}_{A,B} = \begin{bmatrix} {}^A \dot{o}_B - {}^A \omega^{\wedge}_{A,B} {}^A o_B \\ {}^A \omega_{A,B} \end{bmatrix}, \quad {}^B \mathrm{v}_{A,B} = \begin{bmatrix} {}^B R_A {}^A \dot{o}_B \\ {}^B R_A {}^A \omega_{A,B} \end{bmatrix}, \quad {}^{B[A]} \mathrm{v}_{A,B} = \begin{bmatrix} {}^A \dot{o}_B \\ {}^A \omega_{A,B} \end{bmatrix}, $$

where ${}^A \omega_{A,B} := ({}^A \dot{R}_B {}^A R_B^{\top})^{\vee}$. The above representations are usually referred to as: ${}^A \mathrm{v}_{A,B}$ the *left-trivialized* or *inertial* representation of the body velocity, ${}^B \mathrm{v}_{A,B}$ the *right-trivialized* or *body-fixed* representation, and ${}^{B[A]} \mathrm{v}_{A,B}$ the *mixed* representation (also known as *hybrid* representation [2]). The *inertial* and *body-fixed* representation are widespread in the literature of Lie group-based geometric mechanics [23] and recursive robot dynamics algorithms [7,16]. Yet, in this paper we use the *mixed* velocity representation because it is commonly used in multi-task control frameworks [31,3,24]. Furthermore the use of the *mixed* representation simplifies the computations of frames not rigidly attached to a rigid body, see subsection 4.1. It can be shown, however, that most of the results of the paper hold irrespective of the velocity representation. To avoid overloading the notation, in the rest of the document we define:

$$ \mathrm{v}_L := {}^{\overline{L}} \mathrm{v}_{A,L} = {}^{L[A]} \mathrm{v}_{A,L}, \qquad \dot{\mathrm{v}}_L := {}^{\overline{L}} \dot{\mathrm{v}}_{A,L} = {}^{L[A]} \dot{\mathrm{v}}_{A,L}. \tag{2} $$

2.5 Rigid Body Dynamics

The Newton-Euler equations of motion for a rigid body L can be expressed using the *mixed* representation of the velocity. Then, one has

$$ {}_{\overline{L}} I_L \dot{\mathrm{v}}_L + C({}^L \omega_{A,L}, {}^A R_L) \mathrm{v}_L = \mathrm{f}^x_L + {}_{\overline{L}} I_L \begin{bmatrix} {}^A g \\ 0_{3\times 1} \end{bmatrix}, \tag{3} $$

with ${}_{\overline{L}} I_L$ the time-varying 6D inertia matrix expressed in mixed coordinates, i.e.,

$$ {}_{\overline{L}} I_L = {}_{\overline{L}} X^L I_L {}^L X_{\overline{L}}, $$

${}_L I_L$ the 6D inertia expressed w.r.t. the origin of L and the orientation of A, i.e.

$$ {}_L I_L = \begin{bmatrix} m_L 1_3 & -m_L {}^L c_L^{\wedge} \\ m_L {}^L c_L^{\wedge} & \overline{I}_L \end{bmatrix}, \tag{4} $$

$m_L \in \mathbb{R}$ the mass of the body, ${}^L c_L \in \mathbb{R}^3$ the coordinates of the body center of mass c_L w.r.t. the frame L, ${}^A g \in \mathbb{R}^3$ the gravitational acceleration w.r.t. A,

$$ C({}^L \omega_{A,L}, {}^A R_L) := \begin{bmatrix} 0_{3\times 3} & m \left(({}^A R_L {}^L c_L)^{\wedge} \omega^{\wedge}_{A,L} - \omega^{\wedge}_{A,L} ({}^A R_L {}^L c_L)^{\wedge} \right) \\ 0_{3\times 3} & {}^L \omega^{\wedge}_{A,L} {}^A R_L \overline{I}_L {}^L R_A \end{bmatrix}, $$

and $\overline{I}_L \in \mathbb{R}^{3\times 3}$ the 3D inertia of the body expressed w.r.t. the orientation and origin of L, and $\mathrm{f}^x_L \in \mathbb{R}^6$ the *mixed* 6D external force applied to the body, i.e.

$$ \mathrm{f}^x_L := {}_{\overline{L}} \mathrm{f}^x_L = {}_{L[A]} \mathrm{f}^x_L. $$

3 Recalls and complements on free-floating multi-body dynamics

This section recalls notation, state definition, and equations of motion associated with *free-floating* mechanical systems. In particular, we assume that one is given with a mechanical system composed of $n+1$ rigid bodies –called *links*– interconnected by n mechanisms –called *joints*– constraining the link relative motion. The set of links is indicated by $\mathfrak{L}$, and the set of joints is indicated by $\mathfrak{J}$. We assume that each link $L \in \mathfrak{L}$ is associated with a link-fixed frame. In the sequel, we often refer to the frame attached to link L simply as L, and one of these link frames is called *base frame* B. We also assume that each joint possess only one degree-of-freedom, and the joints interconnect the links in such way that no kinematic loop is present in the structure.

3.1 State definition

Being a mechanical system, the equations of motion governing its dynamics are a second-order differential system. Hence, we have to define a *state* of the system composed of a properly defined *position* and *velocity*.

3.1.1 Free-Floating system position

The process of defining the *system position* aims at determining a set of variables from which the position of each point of the multi-body system can be retrieved in the absolute frame A. Being a composition of rigid bodies, each point of the multi-body system can be retrieved from the position-and-orientation – referred to as *pose* – of each link frame $L \in \mathfrak{L}$. Given the topology of the considered multi-body system, however, each link pose can be determined from the pose of the base frame B and the *joint configurations* (see Figure 1).

We assume that the configuration space of each joint is $\mathbb{R}$, which is the case of rotational and prismatic joints subject to joint limit constraints. We then refer to *joint positions* as vector of coordinates $s \in \mathbb{R}^n$ since they represent the *shape* of the multibody system. Contrary to [7], we assume that the *numbering* of the joints remains constant, namely, the joint serialization is independent from the base frame choice B, thus getting rid of the dependence of the joint serialization upon floating base choice [35]. The joint positions s uniquely characterize the pose of any link w.r.t. the base frame B (see Figure 1).

In light of the above, the configuration of a free-floating system is given by the *base pose* $({}^A o_B, {}^A R_B) \in \mathbb{R}^3 \times \mathrm{SO}(3)$ and the *joint* positions $s \in \mathbb{R}^n$, i.e. one can define the configuration set $\mathbb{Q}$ as follows:

$$\mathbb{Q} = \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^n, \tag{5}$$

$$q^B = ({}^A o_B, {}^A R_B, s) \in \mathbb{Q}. \tag{6}$$

The B superscript in q^B emphasizes the dependency on B of the representation of the configuration.

3.1.2 Free-Floating system velocity

In view of (6), the derivative of the robot position is given by:

$$\dot{q}^B = ({}^A\dot{o}_B, {}^A\dot{R}_B, \dot{s}).$$

As in the case of a single rigid body (see Section 2.4), it is more convenient to represent the velocity as a column vector. Using the *mixed* body velocity representation, we can define the system velocity vector $\nu^B \in \mathbb{R}^{6+n}$ as follows:

$$\nu^B = \begin{bmatrix} \mathrm{v}_B \\ \dot{s} \end{bmatrix} = \begin{bmatrix} {}^A\dot{o}_B \\ {}^A\omega_{A,B} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} {}^A\dot{o}_B \\ ({}^A\dot{R}_B {}^AR_B^\top)^\vee \\ \dot{s} \end{bmatrix} \in \mathbb{R}^{n+6}, \tag{7}$$

with $\mathrm{v}_B \in \mathbb{R}^6$ the *base* velocity, and $\dot{s}$ the *joint* velocities. Notice that also the system velocity depends upon the choice of base frame B.

3.2 Kinematics

We recall below how to relate the pose and velocity of an arbitrary link frame $L \in \mathfrak{L}$ to the Free-Floating system position and velocity.

The pose of a link L w.r.t to the inertial frame A is a function of the system position q^B (see Figure 1):

$$\begin{aligned} {}^AH_L(q^B) &: \mathbb{Q} \mapsto \mathbb{R}^3 \times \mathrm{SO}(3), && \text{(8a)} \\ {}^AH_L(q^B) &= {}^AH_B {}^BH_L(s) = \begin{bmatrix} {}^AR_B & {}^Ao_B \\ 0_{1\times3} & 1 \end{bmatrix} {}^BH_L(q). && \text{(8b)} \end{aligned}$$

The velocity of a frame L w.r.t. to the inertial frame A is the product between a robot position-dependent Jacobian matrix $J_{L,B}(q^B) \in \mathbb{R}^{6\times n+6}$ and the system velocity:

$$\begin{aligned} \mathrm{v}_L(q^B, \nu^B) &= J_{L,B}(q^B)\nu^B, && \text{(9a)} \\ J_{L,B}(q^B) &= \begin{bmatrix} {}^{\overline{L}}X_{\overline{B}} & S_{L,B}(s) \end{bmatrix}, && \text{(9b)} \\ {}^{\overline{L}}X_{\overline{B}} &= \begin{bmatrix} 1_3 & ({}^AR_L {}^Lo_B)^\wedge \\ 0_{3\times3} & 1_3 \end{bmatrix} \\ &= \begin{bmatrix} 1_3 & ({}^Ao_B - {}^Ao_L)^\wedge \\ 0_{3\times3} & 1_3 \end{bmatrix}. && \text{(9c)} \end{aligned}$$

The equations (8) and (9) represent the so-called *forward kinematics* of the link L using B as the base frame. How to compute the *forward kinematics* and the *Jacobian matrix* is out of the scope of this paper. If the base frame is a link-fixed frame, algorithms for computing such quantities can be found in [7,16,11].

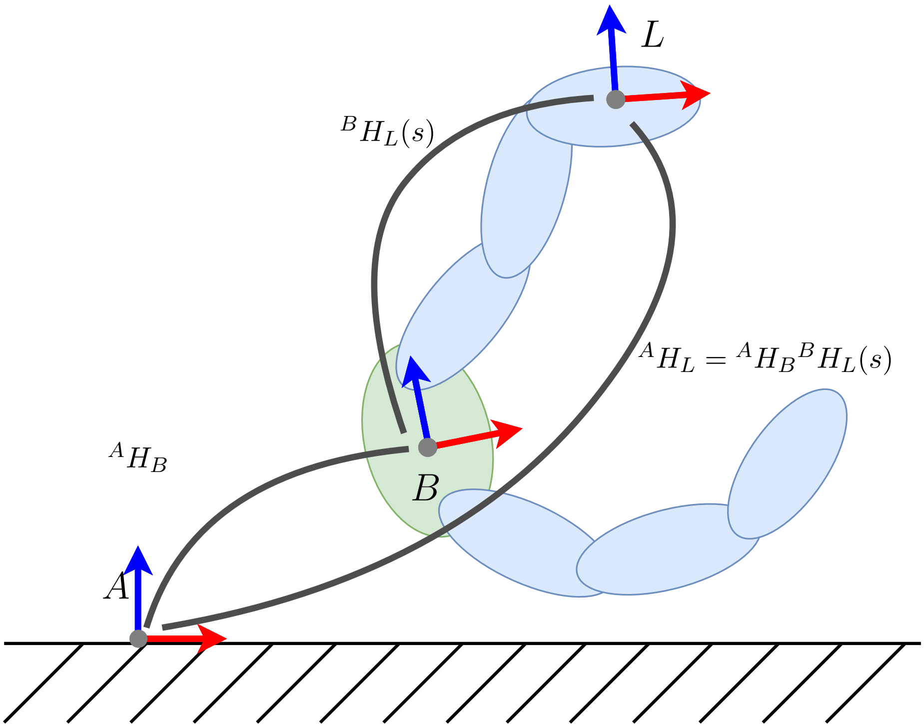

Fig. 1 The pose of a frame $^{A}H_L$ is a function of the base pose $^{A}H_B$ and the shape s of the mechanism.

3.3 Free-floating system equations of motion

Using Euler-Poincarè equations [20, Chapter 13], it is possible to write the equations of motion for a free-floating system subject to a set of external wrenches $\mathrm{f}^x_L \in \mathbb{R}^6$, one for each link in the set of links $\mathfrak{L}$. These equations write [7,15]:

$$\dot{q}^B = \left({^{A}\dot{o}_B}, {^{A}\omega^{\wedge}_{A,B}}{^{A}R_B}, \dot{s}\right) \tag{10a}$$

$$M_B(q^B)\dot{\nu}^B + C_B(q^B, \nu^B)\nu^B + G_B(q^B) = \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix} + \sum_{L\in\mathfrak{L}} J^{\top}_{L,B}\mathrm{f}^x_L, \tag{10b}$$

where $M_B(q^B) \in \mathbb{R}^{(n+6)\times(n+6)}$, $C_B(q^B, \nu^B) \in \mathbb{R}^{(n+6)\times(n+6)}$, and $G_B(q^B) \in \mathbb{R}^{(n+6)}$ denote the mass matrix, the Coriolis matrix, and the gravity generalized forces, respectively. The matrix $J_{L,B} \in \mathbb{R}^{6\times(n+6)}$ denotes the jacobian of the link L on which the external wrench acts, defined in (10). The suffix B in all these symbols highlights the dependence of the quantities upon the choice of the base frame B.

Let us now recall some important facts about the model (10b).

Lemma 1 (Kinetic Energy, [7]) *The free-floating system kinematic energy, i.e. the sum of all the kinematic energies of each link of the system, is given by:*

$$K = \frac{1}{2} \sum_{L \in \mathfrak{L}} \mathrm{v}_L^\top {}_{\overline{L}} I_L \mathrm{v}_L = \frac{1}{2} (\nu^B)^\top M_B \nu^B. \tag{11}$$

Lemma 2 (Gravity generalized forces, [7]) *The $G_B(q^B)$ gravity generalized forces of the free-floating system are given by:*

$$G_B(q^B) = -M_B(q^B) \begin{bmatrix} {}^A g \\ 0_{3\times 1} \\ 0_{n \times 1} \end{bmatrix}, \tag{12}$$

where we recall that ${}^A g \in \mathbb{R}^3$ is the gravitational acceleration vector expressed in the absolute frame, as defined in (3)*.*

Theorem 1 (Mass Matrix structure, [36]) *The mass matrix M_B can be expressed as follows:*

$$M_B = \begin{bmatrix} {}_{\overline{B}} I^C & F_B \\ F_B^\top & H_B \end{bmatrix}, \tag{13}$$

with $F_B \in \mathbb{R}^{6 \times n}$ the upper right block of the mass matrix, $H_B \in \mathbb{R}^{n \times n}$ the joint mass matrix and ${}_{\overline{B}} I^C$ the so-called locked 6D rigid body inertia *of the multi-body system, i.e.*

$${}_{\overline{B}} I^C = \sum_{L \in \mathfrak{L}} {}_{\overline{B}} X^L I_L {}^L X_{\overline{B}}. \tag{14}$$

Furthermore, the first six rows of the mass matrix define the jacobian of the articulated body momentum (i.e. the sum of the linear/angular momentum of all the bodies composing the robot), referred to as the Momentum Matrix*.*

$${}_{\overline{B}} \mathrm{h} = \begin{bmatrix} {}_{\overline{B}} I^C & F_B \end{bmatrix} \nu^B = \sum_{L \in \mathfrak{L}} {}_{\overline{B}} X^L I_L {}^L \mathrm{v}_L. \tag{15}$$

The matrix M_B is known as *Composite Rigid Body Inertia* (CRBA) of the robot in whole body control literature [27]. The matrix C_B is choosen such that the following properties hold on the model (10b) [23]:

Property 1. *The mass matrix M_B is symmetric and positive definite.*

Property 2. *The matrix $\dot{M}_B - 2C_B$ is skew symmetric.*

3.3.1 Free-floating generalized force and its power

The effects of an external wrench (i.e. a wrench acting between a robot link and the environment) and of internal wrenches (i.e. acting between two robot links) can be represented with a free-floating generalized force vector: $\phi_B = \begin{bmatrix} \mathrm{f}_B^x \\ \tau_B \end{bmatrix} \in \mathbb{R}^{n+6}$, where $\mathrm{f}_B^x \in \mathbb{R}^6$ (the *base* wrench) represents the net external force-torque exerted by the environment on the robot and expressed with the orientation of the inertial frame and at the origin of the B frame; while $\tau_B \in \mathbb{R}^n$ represents the *joint* torques. Then, the power $P \in \mathbb{R}$ transmitted by a given generalized force on the free-floating robot moving with a given velocity can be defined as follows [7]:

$$P := \phi_B^\top \nu^B. \tag{16}$$

4 The effects of changing base frame: from the system state to the equations of motion

In this section, we answer to the following questions.

Q1) Which state transformation is associated with the base frame change from the frame B to a new base frame D?

Q2) How does this transformation affect the equations of motion (10) *and its properties?*

The answer to the first question is stated below.

Lemma 3 *Assume that the base frame is changed from a frame B to a frame D. Then, the following results hold.*

1. *The system position and velocity are subject to the following transformations*

$$q^D = ({}^A H_B {}^B H_D(s), s), \tag{17a}$$

$$\nu^D = {}^D T_B \nu^B, \tag{17b}$$

where

$$^D T_B = \begin{bmatrix} {}^{\overline{D}} X_{\overline{B}} & S_{D,B} \\ 0_{n\times 6} & 1_n \end{bmatrix}, \tag{18}$$

and $\begin{bmatrix} {}^{\overline{L}} X_{\overline{B}} & S_{D,B}(s) \end{bmatrix}$ *is given by* (9).

2. *A free-floating generalized force $\phi_B \in \mathbb{R}^{n+6}$ is subject to the following transformation*

$$\phi_D = {}_D T^B \phi_B, \tag{19}$$

with

$${}_D T^B = {}^D T_B^{-\top} = \begin{bmatrix} {}_{\overline{D}} X^{\overline{B}} & 0_{6\times n} \\ -(S_{D,B})^\top {}_{\overline{D}} X^{\overline{B}} & 1_n \end{bmatrix} . \tag{20}$$

3. *The jacobian $J_{L,B} \in \mathbb{R}^{6\times n+6}$ of a link frame L is subject to the following transformation*

$$J_{L,D} = J_{L,B} {}^B T_D. \tag{21}$$

The proof is given in the Appendix. Item 1) details how the system position and velocity transform when the base frame is moved from B to D. The relationships (17) follow directly from the system kinematics (8) and (9) between the two frames. The item 2) defines the transformation of the free-floating generalized force ϕ_B. The key point to define the transformation (19)-(20) is to observe that the power P is independent from choice of the base frame, and thus invariant when moving the base frame from B to D. Then, the transformation associated with the force ϕ_B can be found. The item 3) defines how the jacobian of a link frame L changes depending on a base frame link change. The relationship (21) is a direct consequence of (9).

Remark 2 (Pure internal joint torques invariance) Let ϕ_B be a free-floating generalized force acting on the multi-body system. If ϕ_B has a base wrench equal to zero, then it is independent from the base frame in which it is expressed, i.e.

$$ {}_D T^B \begin{bmatrix} 0_{6\times 1} \\ \tau_B \end{bmatrix} = \begin{bmatrix} 0_{6\times 1} \\ \tau_D \end{bmatrix} = \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix}. $$

Remark 3 (External joint torque base dependency) Let B be a base frame rigidly attached to a link B. The effect of a pure base wrench f^x_B on a the link B is by definition the generalized force where the first six elements are given by f^x_B and joint part is equal to zero:

$$ \phi_B = \begin{bmatrix} \mathrm{f}^x_B \\ 0_{n\times 1} \end{bmatrix}. $$

The generalized force expressed w.r.t. another link-fixed base frame D is given by:

$$ \phi_D = {}_D T^B \phi_B = \begin{bmatrix} {}_{\overline{D}} X^{\overline{B}} & 0_{6\times n} \\ (S_{D,B})^\top {}_{\overline{D}} X^{\overline{B}} & 1_n \end{bmatrix} \begin{bmatrix} \mathrm{f}^x_B \\ 0_{n\times 1} \end{bmatrix} = \begin{bmatrix} {}_{\overline{D}} X^{\overline{B}} \mathrm{f}^x_B \\ (S_{D,B})^\top {}_{\overline{D}} X^{\overline{B}} \mathrm{f}^x_B \end{bmatrix}. $$

The first six elements of the transformed generalized torques represent the external wrench f^x_B, but expressed with respect to the origin of D rather then the origin of B. The last n elements are instead a form of *external joint torques*, and they are non-zero only for the joints on the path from B to D. It is important to note that contrary to the case of fixed base robots, the *external joint torques* depend on the floating base, and, consequently, they do not describe any meaningful base-invariant physical quantity.

Let us now answer to the question $Q2$).

Theorem 2 *Assume that the base frame is changed from B to a frame D. Then, the following results hold.*

1. *The equations of motions* (10) *transform into*

$$ \dot{q}^D = \left({}^A\dot{o}_D, {}^A\omega^\wedge_{A,D} {}^A R_D, \dot{s} \right) \tag{22a} $$

$$ M_D(q^D)\dot{\nu}^D + C_D(q^D, \nu^D)\nu^D + G_D(q^D) = \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix} + \sum_{L \in \mathfrak{L}} J^\top_{L,D} \mathrm{f}^x_L, \tag{22b} $$

with

$$ M_D = {}_D T^B M_B {}^B T_D, \tag{23a} $$

$$ C_D = {}_D T^B \left(M_B {}^B\dot{T}_D + C_B {}^B T_D \right), \tag{23b} $$

$$ G_D = {}_D T^B G_B, \tag{23c} $$

with ${}_D T^B$ given by (20) *and ${}^D T_B$ given by* (18) *.*

2. *The free-floating system kinetic energy is given by:*

$$ K = \frac{1}{2} (\nu^D)^\top M_D \nu^D. \tag{24} $$

3. *Assume that Property 2 holds. Then $\dot{M}_D - 2C_D$ is skew-symmetric.*

The proof is given in the Appendix. The above theorem characterizes how the equations of motion of the free-floating system transform when the base frame is moved from B to D. In particular, it states that the representation of the equations of motion (22)-(23) implies the invariance of the fundamental Properties 1 and 2 w.r.t. a base frame change. Also, the item 2) and 3) highlight that the expressions (22)-(23) are particularly useful when designing passivity-based control laws [11]. In fact, the stability proof of these control law usually assumes that the mass matrix is positive definite, and that the passivity property $\dot{M} - 2C$ holds.

4.1 Frames not rigidly attached to a link

So far, the base frame transformation was meant to be between a frame B and a frame D, and both of these frames were assumed to be attached to a physical link. This assumption, however, is not strict. As a matter of fact, we can assume that the base frame D is a frame whose origin is that of a frame E, and whose orientation that of a frame F:

$$q^{E[F]} := ({}^A o_E, {}^A R_F, s). \tag{25}$$

Then, the associated generalized robot velocity is given by:

$$\nu^{E[F]} = \begin{bmatrix} {}^A \dot{o}_E \\ {}^A \omega_{A,F} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} {}^A \dot{o}_E \\ \left({}^A \dot{R}_F {}^A R_F^\top \right)^\vee \\ \dot{s} \end{bmatrix} \tag{26}$$

The new *base* velocity reflect the different choices of the *base* position variables. The Jacobian of the velocity of a frame defined in such a compound way is simply given by the combination of the linear part (first three rows) of $J_{C,B}$, indicated with $J^l_{C,B} \in \mathbb{R}^{3\times(6+n)}$ and the angular part of (last three rows) of $J_{D,B}$, indicated with $J^a_{D,B} \in \mathbb{R}^{3\times(6+n)}$:

$$J_{E[F],B} = \begin{bmatrix} J^l_{E,B} \\ J^a_{F,B} \end{bmatrix}. \tag{27}$$

Note that the property of (27) be the simple stacking of the linear and angular Jacobians is a consequence of the use of *mixed* velocity representation, as discussed in subsection 2.4. Indeed the equivalent jacobians using the *inertial* or *body-fixed* representation are related in a more complex way to the Jacobians of frame E and F, but a complete discussion on this is out of the scope of this paper.

From (27), one obtains the transformation matrix ${}^{E[F]}T_B$ to be used in Theorem 2:

$${}^{E[F]}T_B = \begin{bmatrix} J_{E[F],B} \\ 0_{6\times n} \; 1_{n\times n} \end{bmatrix}. \tag{28}$$

5 System state transformation providing centroidal dynamics

In the humanoid robotics literature, it is widespread to control as primary task the position of the center of mass [28] and to minimize the angular momentum of the robot [26,27,4,13,19,18]. For analysis purposes, it is then convenient to include the center of mass in the state representing the robot position [28,24]. Such an inclusion can be expressed as a change of variables using the formalism discussed in the previous sections. Expressing such change of variables in the proposed formalism simplifies the computation of the system dynamics in the new state.

5.1 Recalls on centroidal dynamics quantities

To properly define the *centroidal dynamics*, it is convenient to define a frame $\overline{G}$ whose origin is the center of mass of the multi body system, and whose orientation is that of the inertial frame A. Note that the use of $\overline{G}$ is an abuse of notation, as we *do not* define an orientation for frame G (see Figure 2).

Then, the total momentum and the composite rigid body inertia (CRBA) of the system expressed w.r.t. the $\overline{G}$ frame are given by:

$$
{}_{\overline{G}}\mathrm{h} = {}_{\overline{G}}X^{\overline{B}}{}_{\overline{B}}\mathrm{h}, \qquad {}_{\overline{G}}I^{C} = {}_{\overline{G}}X^{\overline{B}}{}_{\overline{B}}I^{C\,\overline{B}}X_{\overline{G}}. \tag{29}
$$

In the robotics literature, these quantities are known as *Centroidal Momentum* and as *Centroidal Composite Rigid Body Inertia (CCRBI)* [27], respectively. The structure of these *centroidal* quantities is the following [27] :

$$
{}_{\overline{G}}\mathrm{h} = \begin{bmatrix} m{}^{A}\dot{c} \\ {}_{\overline{G}}\mathrm{h}^{a} \end{bmatrix}, \qquad {}_{\overline{G}}I^{C} = \begin{bmatrix} m1_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & L^{C} \end{bmatrix}, \tag{30}
$$

with $m \in \mathbb{R}$ the total mass of the robot, ${}^{A}c \in \mathbb{R}^{3}$ the center of the mass of the robot expressed in the inertial frame, ${}_{\overline{G}}\mathrm{h}^{a} \in \mathbb{R}^{3}$ the total angular momentum and $L^{C} \in \mathbb{R}^{3\times3}$ the locked 3D inertia of the robot , both expressed in $\overline{G}$.

Given a base frame B, note that the matrix $A_{G,B} \in \mathbb{R}^{6\times(n+6)}$ that multiplied by the generalized robot velocities vector ν^{B} gives the *centroidal momentum* can be easily obtained from the floating base mass matrix. In fact, by combining (15) and (29) one has:

$$
{}_{\overline{G}}\mathrm{h} = A_{G,B}\nu^{B}, \quad A_{G,B} = {}_{\overline{G}}X^{\overline{B}}\left[{}_{\overline{B}}I^{C} \; F_{B}\right]. \tag{31}
$$

The $A_{G,B}$ matrix is known as the *Centroidal Momentum Matrix* [26,27].

5.2 Average 6D Velocity

In the humanoids whole-body control literature, it is common to define the *average 6D velocity* v_{G} of the robot as [27]:

$$
\mathrm{v}_{G} = ({}_{\overline{G}}I^{C})^{-1}{}_{\overline{G}}\mathrm{h} = \begin{bmatrix} \frac{1}{m}(m{}^{A}\dot{o}_{G}) \\ (L^{C})^{-1}{}_{\overline{G}}\mathrm{h}^{a} \end{bmatrix} = \begin{bmatrix} {}^{A}\dot{o}_{G} \\ \omega_{G} \end{bmatrix}. \tag{32}
$$

By definition, the linear part of the *average 6D velocity* (i.e. the first three components of v_{G}) is simply the velocity of the center of mass of the system. Its angular part (i.e. the last three components) is called the *average angular velocity* ω_{G} [17,6,22], even if this name is an abuse of the term *angular velocity* because ω_{G} is not defined as the angular velocity of an orientation frame. In fact, a rotation matrix $R(s) \in SO(3)$ such that $R(s)R^{\top}(s) = \omega_{G}^{\wedge}$ exists only for a limited class of multibody systems [30]. The *average angular velocity* has a precise physical meaning if a multibody system is evolving without external forces acting on it. If a generalized impulse blocks all its joint motions instantaneously, the resulting rigid body would evolve with an angular velocity ω_G. For this reason, ω_G is also known as the *locked angular velocity* in geometrical mechanics literature [21].

The relationship between the generalized robot velocities vector ν^B and the *centroidal velocity* v_G can be easily obtained by combining (31) and (32):

$$\begin{aligned}\mathrm{v}^G &= \left({}_{\overline{G}}I^C\right)^{-1} A_{G,B}\nu^B = \left({}_{\overline{G}}I^C\right)^{-1} {}_{\overline{G}}X^{\overline{B}} \left[{}_{\overline{B}}I^C \; F_B\right] \nu^B = \\ &= \left[{}^{\overline{G}}X_{\overline{B}} \; \left({}_{\overline{G}}I^C\right)^{-1} {}_{\overline{G}}X^{\overline{B}} F_B\right] \nu^B.\end{aligned}$$

Noting that this matrix has the same structure of the floating base jacobian (10), we borrow the notation we use for the Jacobians of links, even if v_G is not defined as the mixed velocity of a well defined frame. So, we define $J_{G,B}$ and $S_{G,B}$ such that:

$$\nu^G = J_{G,B}\nu^B, \tag{33}$$

$$J_{G,B} := \begin{bmatrix} {}^{\overline{G}}X_{\overline{B}} & S_{G,B} \end{bmatrix} := \begin{bmatrix} {}^{\overline{G}}X_{\overline{B}} & ({}_{\overline{G}}I^C)^{-1} {}_{\overline{G}}X^{\overline{B}} F_B \end{bmatrix}. \tag{34}$$

5.3 Centroidal change of variables

5.3.1 Including the center of mass in the robot state

For the sake of including the center of mass in the multibody dynamics, we can define a new robot position as follows:

$$q^{G[B]} = ({}^A c, {}^A R_B, s). \tag{35}$$

This implies that:

$$\nu^{G[B]} = \begin{bmatrix} \mathrm{v}_{G[B]} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} {}^A\dot{c} \\ ({}^A\dot{R}_B {}^A R_B^\top)^\vee \\ \dot{s} \end{bmatrix}. \tag{36}$$

The corresponding transformation matrix is given by, as explained in subsection 4.1:

$${}^{G[B]}T_B = \begin{bmatrix} J^l_{G[B],B} \\ 0_{3\times(3+n)} \; 1_{(3+n)\times(3+n)} \end{bmatrix}, \tag{37}$$

where $J^l_{G[B],B}$ is the center of mass jacobian, i.e. the matrix such that

$${}^A\dot{c} = J^l_{G[B],B}\nu_B.$$

Note that this is also equal to the first three rows of (34), i.e. $J^l_{G[B],B} = J^l_{G,B}$. The change of base introduced by ${}^{G[B]}T_B$ is the one used in [28].

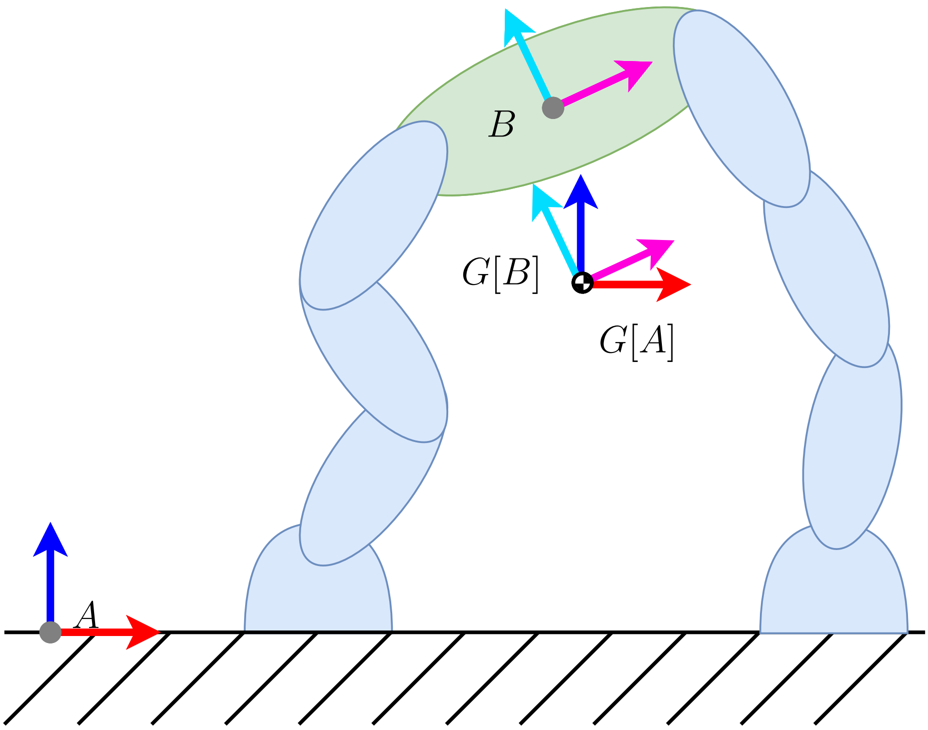

Fig. 2 Graphical depiction of frames A, B, $G[A]$ and $G[B]$ for an example two legged robot.

5.3.2 Block diagonalization of the mass matrix

Using the definition of *average 6D velocity* of the multibody system, we can define a *centroidal* generalized joint velocities vector in which we combine the average 6D velocity and the joint velocities :

$$\nu^G = \begin{bmatrix} \mathrm{v}_G \\ \dot{s} \end{bmatrix}. \tag{38}$$

Let us remark (again) that there is no such thing as a G frame: the notation ν^G is an abuse of notation. In particular, it does not make sense to write q^G : the change of variables induced by the use of the average 6D velocity is only a change of variables in the velocity space, while for the position space it is usually convenient to continue to use the $q^{G[B]}$ position introduced before.

The mapping ${}^G T_B$ from the *floating base* generalized velocities ν^B to ν^G has the same structure of the change of variables introduced in (18):

$$\nu^G = {}^G T_B \nu^B, \quad {}^G T_B = \begin{bmatrix} {}^{\overline{G}}X_{\overline{B}} & S_{G,B} \\ 0_{n\times 6} & 1_{n\times n} \end{bmatrix}. \tag{39}$$

This induces a change of variables also on the generalized robot forces, as detailed in (19), i.e. :

$$ {}_G T^B = {}^G T_B^{-\top} = \begin{bmatrix} {}_{\overline{G}}X^{\overline{B}} & 0_{6\times n} \\ -S_{G,B}^{\top}\, {}_{\overline{G}}X^{\overline{B}} & 1_n \end{bmatrix}. \tag{40}$$

Using the transformation ${}^G T_B$ in (23), one can obtain the equations of motion with $(q^{G[B]}, \nu^G)$ as state, i.e. :

$$\dot{q}^{G[B]} = \left({}^A\dot{c}, {}^A\omega^{\wedge}_{A,B}(\nu^G)^A R_B, \dot{s}\right), \tag{41a}$$

$$M_G(q^{G[B]})\dot{\nu}^G + C_G(q^{G[B]}, \nu^G)\nu^G + G_G = \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix} + \sum_{L\in\mathfrak{L}} J^\top_{L,G} \mathrm{f}^x_L, \tag{41b}$$

where ${}^A\omega_{A,B}(\nu^G)$ is the angular velocity of the link B as a function of the *centroidal* generalized joint velocity ν_G.

The equations of motion (41b) incorporate in the first six rows what is usually called the *centroidal dynamics*, together with the joint dynamics in a single set of equations of motions. The specific features of such a representation of the equations of motion are highlighted in the following theorem, and they have been already exploited in [24].

Lemma 4 *For the equations of motion given by* (41b)*, the following results hold.*

1. *The (centroidal) mass matrix M_G has a block diagonal structure, i.e. :*

$$M_G(q^{G[B]}) = \begin{bmatrix} m1_3 & 0_{3\times 3} & 0_{3\times n} \\ 0_{3\times 3} & L^C(q^{G[B]}) & 0_{3\times n} \\ 0_{3\times 3} & 0_{3\times 3} & H_G(s) \end{bmatrix} \tag{42}$$

where $m \in \mathbb{R}$ is the total mass of the system, $L^C \in \mathbb{R}^{3\times 3}$ is the locked *3D inertia matrix of the system expressed in the center of mass and with the orientation of the absolute frame A and $H_G(s) \in \mathbb{R}^{n\times n}$ is the centroidal joint mass matrix.*

2. *The (centroidal) gravity term G_G is independent from the robot state and has the following structure:*

$$G_G = \begin{bmatrix} -m^A g \\ 0_{3\times 1} \\ 0_{n\times 1} \end{bmatrix} \tag{43}$$

where ${}^A g \in \mathbb{R}^3$ is the gravitational acceleration expressed in the frame A.

3. *Assuming that Property 2 holds for M_B and C_B, the (centroidal) coriolis matrix C_G has a block diagonal structure, i.e. :*

$$C_G(q^{G[B]}, \nu^G) = \begin{bmatrix} 0_{3\times 3} & 0_{3\times(n+3)} \\ 0_{3\times(n+3)} & C^{aj}_G(q^{G[B]}, \nu^G) \end{bmatrix} \tag{44}$$

where $C^{aj}_G(q^{G[B]}, \nu^G) \in \mathbb{R}^{(3+n)\times(3+n)}$ is the centroidal coriolis matrix for the angular and joint part of the dynamics.

The proof is given in the Appendix.

6 Applications to humanoid robotics: switching fixed-base dynamics

When the multi-body system represents a humanoid robot, the base frame B is usually placed at the robot waist [35]. The main reason why for this choice is that it induces branch-induced sparsity of the system mass matrix and Jacobians [8].

There are some cases, however, where it would be more intuitive to place the base frame B at some other robot links than its waist. For instance, assume that the humanoid robot is balancing on a single foot. Then, the robot behaves very similarly to a fixed-base industrial manipulator whose base frame is usually attached to the link connecting the robot and the inertial frame. For this reason, in the case of a humanoid robot balancing on one foot, it is common to choose the base frame as the frame associated with the foot frame [25]. Furthermore, during bipedal locomotion, it is also convenient to have the base frame that switches from one foot to the other depending on the robot link in contact with the environment [12, Section 3.2]. The choice of placing the base frame at a robot feet has also been used for inertial parameters estimation purposes [14].

In light of the above, for the humanoid robotics community it is paramount to have tools allowing one to deduce the equations of motion independently of the link that is in contact with the environment. The next Lemma presents the equations of motion of the joint space assuming that an arbitrary fixed support link F cannot move w.r.t. the inertial frame A, i.e. holonomic constraints hold on the free-floating system.

Lemma 5 *Assume that the floating base system* (10b) *is subject to holonomic constraints of the form*

$$^{A}H_F(q) = H_K, \tag{45}$$

where H_K is a constant element of $SE(3)$. Then, the evolution of the joint angles s is characterised by the following equations of motion:

$$M_F^{fixed}\ddot{s} + C_F^{fixed}(\dot{s}, s)\dot{s} + G_F^{fixed}(s) = \tau + \sum_{L \in \mathfrak{L} - F} {J_{L,F}^{\top}}^{fixed} \mathrm{f}_L^x, \tag{46}$$

where

$$\begin{aligned}
P &= \begin{bmatrix} 0_{n\times 6} & 1_{n\times n}\end{bmatrix}, && (47a)\\
M_F^{fixed} &= PM_F P^\top = P\,{}_F T^B M_B\, {}^B T_F P^\top, && (47b)\\
C_F^{fixed} &= PC_F P^\top = P\left({}_F T^B\left(M_B\, {}^B\dot{T}_F + C_B\, {}^B T_F\right)\right) P^\top, && (47c)\\
G_F^{fixed} &= PG_F = P\,{}_F T^B G_B, && (47d)\\
J_{L,F}^{fixed} &= J_{L,F} P^\top = J_{L,F}\, {}^F T_B P^\top, && (47e)
\end{aligned}$$

and the expression of M_F, C_F, G_F are given by (23) *with $D = F$.*

Furthermore, the external force/torques f_F^x such that (45) *holds are given by:*

$$\mathrm{f}_F^x = (J_{F,F} M_F^{-1} J_{F,F}^\top)^{-1} J_{F,F} M_F^{-1}\left(C_F \nu^F + G_F - \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix} - \sum_{L\in\mathfrak{L}-F} J_{L,F}^\top \mathrm{f}_L^x\right) \tag{48}$$

where in view of (9)*, one has:*

$$J_{F,F} = \begin{bmatrix} 1_6 & 0_{6\times n}\end{bmatrix}. \tag{49}$$

The proof is given in the appendix. The Lemma above states the equations of motion of the joint space s when the holonomic constraint (45) holds. The Lagrange multipliers ensuring that this constraint is satisfied are given by Eq. (48). Note that the quantity ${}_{F[A]}f$ is a six dimensional vector composed of a *contact* force-torque both expressed in the frame having the orientation of the inertial frame A and the origin of the constrained frame F. Hence, the wrench ${}_{F[A]}\mathrm{f}$ is of particular interests for control purposes because imposing inequality constraints on it can allow to break the contact only at specific configurations [29]. Let us remark, however, that to evaluate the contact wrench f^x_F one has to invert the matrix $J_{F,F} M_F J^\top_{F,F}$. Then, one has the following result.

Corollary 1 *Assume that the constraint* (45) *holds. Then, the matrix* $J_{F,F} M_F J^\top_{F,F}$ *corresponds to the* Composite Rigid Body Inertia *of the robot:*

$$J_{F,F} M_F J^\top_{F,F} = {}_{\overline{F}}I^C.$$

Consequently, this matrix is always invertible.

The above result is a straightforward combination of (49) and (13) . It conveys the possibility of always computing the *contact* force-torque for any system configuration as long as the holonomic constraints acting on the system represents a single robot frame subject to constraints.

7 CONCLUSIONS

In this paper, we have investigated the role of the base frame choice in free-floating mechanical systems and shown how it can be exploited to unify the equations of motion used for the control design of humanoid robots. In particular, we have first introduced a formalism to model the arbitrary choice of the base frame: this formalism is based on defining proper variable changes for the state of the system and leaves invariant the fundamental properties of the underlying mechanical system. Then, we have applied the formalism to the domain of humanoid robotics. More precisely, we have first shown how to obtain the robot *joint dynamics* when any robot link is in contact with the environment. Then, we have applied the proposed formalism to unify two sets of equations usually used in humanoid robot control: the robot equations of motions expressed w.r.t. a base frame and the so-called *centroidal dynamics*, thus providing the user with a single set of $n+6$ equations of motion. Future work will be directed towards investigating how the specific structure of the base frame change transformation can be used in free-floating robot control and estimation, as already partially done in [24] and [32].

Proof of Lemma 3

1. The robot position q^B can be transformed in q^D simply by using the *forward kinematics* (8) of D considering B to be the base frame:

$$q^D = ({}^A H_D(q^B), s) = ({}^A H_B {}^B H_D(s), s)$$

where for simplicity we write $({}^A o_B, {}^A R_B)$ as ${}^A H_B$.

In view of (9) the transformation between ν^B and ν^D can be easily obtained using the jacobian of the frame D considering B as the floating base, i.e. :

$$\nu^D = \begin{bmatrix} \mathrm{v}_D \\ \dot{s} \end{bmatrix} = \begin{bmatrix} J_{D,B}\nu^B \\ \dot{s} \end{bmatrix} = \begin{bmatrix} J_{D,B} \\ 0_{n\times 6} \; 1_n \end{bmatrix} \nu^B = \begin{bmatrix} {}^{\overline{D}}X_{\overline{B}} & S_{D,B} \\ 0_{n\times 6} & 1_n \end{bmatrix} \nu^B.$$

From which we get:

$$ {}^{D}T_B = \begin{bmatrix} {}^{\overline{D}}X_{\overline{B}} & S_{D,B} \\ 0_{n\times 6} & 1_n \end{bmatrix}.$$

2. The power P transmitted to the robot by a generalized force ϕ_B on a robot moving with velocity ν^B is independent of the arbitrary choice of the floating base, i.e. :

$$P = \phi_B^\top \nu^B = \phi_D^\top \nu^D.$$

By substituting $\nu^B = {}^{B}T_D \nu^D$ in the second term and transposing, we get:

$$(\nu^D)^\top {}^{B}T_D^\top \phi_B = (\nu^D)^\top \phi_D, \quad {}^{B}T_D^\top \phi_B = \phi_D$$

From which we get:

$$ {}_{D}T^B = {}^{B}T_D^\top = {}^{D}T_B^{-\top} = \begin{bmatrix} {}_{\overline{D}}X^{\overline{B}} & 0_{6\times n} \\ -(S_{D,B})^\top {}_{\overline{D}}X^{\overline{B}} & 1_{n\times n} \end{bmatrix}.$$

3. The *mixed* velocity v_L of a link L is invariant w.r.t. the frame considered as the floating base, which implies that:

$$\mathrm{v}_L = J_{L,B}\nu^B = J_{L,D}\nu^D.$$

By substituting $\nu^B = {}^{B}T_D\nu^D$ on the right hand side of the above equation, we get:

$$J_{L,B}{}^{B}T_D\nu^D = J_{L,D}\nu^D,$$

which in turn implies that:

$$J_{L,D} = J_{L,B}{}^{B}T_D.$$

■

Proof of Theorem 2

1. First, notice that the time derivative of (17b) yields:

$$\dot{\nu}^D = {}^{D}T_B\dot{\nu}^B + {}^{D}\dot{T}_B\nu^B. \tag{50}$$

Then, the equations of motion (23)-(23) can be obtained by substituting $\dot{\nu}_B$ obtained from (50) into (10b), and by multiplyng the obtained equation times ${}_{D}T^B$.

2. In view of (11), i.e.:

$$K = \frac{1}{2} {\nu^B}^\top M_B \nu^B,$$

and of $\nu^B = {}^B T_D \nu^D$ one has :

$$K = \frac{1}{2} (\nu^D)^\top {}^B T_D^\top M_B {}^B T_D \nu^B.$$

From which we obtain that the kinematic energy is also given by:

$$K = \frac{1}{2} (\nu^D)^\top M_D \nu^D, \quad M_D{=}{}_D T^B M_B {}^B T_D.$$

3. The condition that $\dot{M}_B - 2C_B$ is skew-symmetric is equivalent to the condition that $\dot{M}_B = C_b + C_B^\top$. So, we will demonstrate that $\dot{M}_B - C_B - C_B^\top = 0_{(n+6)\times(n+6)}$ implies $\dot{M}_D - C_D - C_D^\top = 0_{(n+6)\times(n+6)}$. Let us write $\dot{M}_D - C_D - C_D^\top$ using (23a) and (24), i.e. :

$$\begin{aligned} \dot{M}_D - C_D - C_D^\top = \\ = \underbrace{{}_D \dot{T}^B M_B {}^B T_D + {}_D T^B \dot{M}_B {}^B T_D + {}_D T^B M_B {}^B \dot{T}_D}_{\dot{\mathbf{M}}_D} \\ \underbrace{- {}_D T^B M_B {}^B \dot{T}_D - {}_D T^B C_B {}^B T_D}_{-C_D} \\ \underbrace{- {}^B \dot{T}_D^\top M_B {}_D {T^B}^\top - {}^B T_D^\top {C_B}^\top {}_D {T^B}^\top}_{-C_D^\top} \end{aligned}$$

Noting that from (19) we have that ${}_D {T^B}^\top = {}^B T_D$, we can write:

$$\begin{aligned} \dot{M}_D - C_D - C_D^\top = \\ {}_D T^B \left(\dot{M}_B - C_B - C_B^\top \right) {}^B T_D + \\ + {}_D \dot{T}^B M_B {}^B T_D - {}_D \dot{T}^B M_B {}^B T_D + \\ + {}_D T^B M_B {}^B \dot{T}_D - {}_D T^B M_B {}^B \dot{T}_D \end{aligned}$$

Using the hypothesis that $\dot{M}_B - C_B - C_B^\top = 0_{(n+6)\times(n+6)}$ we can then conclude that $\dot{M}_D - C_D - C_D^\top = 0_{(n+6)\times(n+6)}$.

■

Proof of Lemma 4

1. The *centroidal* mass matrix can be obtained by applying (39) to (23), i.e. :

$$M_G = {}_G T^B M_B {}^B T_G. \tag{51}$$

By exploiting the structure of M_B and ${}_G T^B$ in (13) and (40), and recalling that ${}_{\overline{G}}X^{\overline{B}} = {}^{\overline{B}}X_{\overline{G}}^{\top}$ one has:

$$M_G = \begin{bmatrix} {}_{\overline{G}}X^{\overline{B}} & 0_{6\times n} \\ S_{B,G}^{\top} & 1_{n\times n} \end{bmatrix} \begin{bmatrix} {}_{\overline{B}}I^C & F_B \\ F_B^{\top} & H_B \end{bmatrix} \begin{bmatrix} {}^{\overline{B}}X_{\overline{G}} & S_{B,G} \\ 0_{6\times n} & 1_{n\times n} \end{bmatrix} = \tag{52a}$$

$$= \begin{bmatrix} {}_{\overline{G}}X^{\overline{B}}{}_{\overline{B}}I^{C\,\overline{B}}X_{\overline{G}} & {}_{\overline{G}}X^{\overline{B}}\left({}_{\overline{B}}I^C S_{B,G} + F_B\right) \\ \left(S_{B,G\,\overline{B}}^{\top}I^C + F_B^{\top}\right){}^{\overline{B}}X_{\overline{G}} & H_G \end{bmatrix} \tag{52b}$$

with $H_G = S_{B,G\,\overline{B}}^{\top}I^C S_{B,G} + S_{B,G}^{\top}F_B + F_B^{\top}S_{B,G} + H_B$.
In view of (29) and (30), then ${}_{\overline{G}}X^{\overline{B}}{}_{\overline{B}}I^{C\,\overline{B}}X_{\overline{G}}$ can be written as the *centroidal composite rigid body inertia*, i.e. :

$${}_{\overline{G}}X^{\overline{B}}{}_{\overline{B}}I^{C\,\overline{B}}X_{\overline{G}} = {}_{\overline{G}}I^C = \begin{bmatrix} m1_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & L^C \end{bmatrix}. \tag{53}$$

Using (34), it is possible to write $S_{B,G}$ as:

$$S_{B,G} = -{}^{\overline{B}}X_{\overline{G}}S_{G,B} = -\left({}_{\overline{B}}I^C\right)^{-1} F_B. \tag{54}$$

Substituting (54) in the off-diagonal terms of (52b) it is possible to show that the off-diagonal terms are equal to $0_{6\times n}$.

2. Exploiting the gravity generalized forces structure given by (12) and the structure of M_G given in (42), we can write G_G as:

$$G_G = -M_G \begin{bmatrix} {}^A g \\ 0_{3\times 1} \\ 0_{n\times 1} \end{bmatrix} = \begin{bmatrix} -m{}^A g \\ 0_{3\times 1} \\ 0_{n\times 1} \end{bmatrix}.$$

3. From the structure of the centroidal dynamics (41b), and from the Newton equation $m{}^A\ddot{c} - m{}^A g = \sum_{L\in\mathfrak{L}} {}^A f_L^x$ we have that the C_G matrix has the following structure:

$$\begin{bmatrix} 0_{3\times 3} & 0_{3\times(n+3)} \\ C_G^{\text{offDiag}} & C_G^{\text{aj}} \end{bmatrix}. \tag{55}$$

From the assumption given by Property 2 that $(\dot{M}_B - 2C_B)^{\top} = (2C_B - M_B)$ by applying Theorem 2.3 we obtain that:

$$C_G + C_G^{\top} = \frac{\dot{M}_G}{2}. \tag{56}$$

Plugging the the sparsity patterns (55) and (42) in the (56), extracting the top left $3\times n$ subblock one obtains that:

$$0_{3\times(n+3)} + \left(C_G^{\text{offDiag}}\right)^{\top} = 0_{3\times(n+3)}$$

from which:

$$C_G^{\text{offDiag}} = 0_{(n+3)\times 3}.$$

■

Proof of Lemma 5

By deriving w.r.t. time the holonomic constraint (45) and by applying the definition of the mixed velocity (2), one gets that the holonomic constraint implies that:

$$\mathrm{v}_F = 0. \tag{57}$$

This velocity constraint can be transformed in a linear constraint on the system velocity using the floating base jacobian (9) :

$$J_{F,B}(q^B)\nu^B = 0. \tag{58}$$

Furthermore, the constraint (58) can be simplified by expressing the floating base velocity using F as the floating base frame, in this case the Jacobian simplifies to a simply selection matrix:

$$J_{F,F}\nu^F = \begin{bmatrix} 1_6 \; 0_{6\times n} \end{bmatrix} \nu^F = 0. \tag{59}$$

Given that $J_{F,F}$ is constant w.r.t. to time and configuration, the same constraints holds at the acceleration level:

$$\dot{\mathrm{v}}_F = J_{F,F}\dot{\nu}^F = \begin{bmatrix} 1_6 \; 0_{6\times n} \end{bmatrix} \dot{\nu}^F = 0. \tag{60}$$

The equations (46) can then be obtained by using Theorem 2 to write the dynamics (10b) using F as the floating base frame, by substituting the constraints (59) and (60) in the obtained dynamics and considering only the last n rows results in the equation .

The expression of the contact force/torque (48) can be obtained by substituting the expression of the free-floating acceleration $\dot{\nu}^F$ obtained solving (10b) for $\dot{\nu}^F$:

$$\dot{\nu}^F = M_F^{-1}\left(C_F \nu^F + G_F - \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix} - \sum_{L \in \mathfrak{L} - F} J_{L,F}^\top \mathrm{f}_L^x - J_{F,F}^\top \mathrm{f}_F^x . \right) \tag{61}$$

Equation (48) is obtained substituting (61) in the acceleration constraint $J_{F,F}\dot{\nu}^F = 0$ and solving for f_F^x.

■

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