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Research paper

The explanation of two semi-recursive multibody methods for educational purpose

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ABSTRACT

In this paper, two often-employed alternatives to the semi-recursive method are described and compared in terms of velocity transformation. The text offers a simplified way to understand the theory behind the two semi-recursive approaches. Focusing on planar cases, particular attention is given to reference and joint coordinates. Consequently, a multibody modeling analyst that is familiar with global formulations can easily extend his/her knowledge to the semi-recursive approach.

Using semi-recursive methods, the open loops are formulated with a reduced set of coordinates, and the constraint equations are avoided in the dynamic equations of motion. Accordingly, computationally efficient forms of matrices and vectors will be generated to represent the dynamic equations of motion, which leads to enhanced numerical efficiency. The difference between the two studied alternatives is the definition of the reference point (i.e., the origin of the body frame) used to define the reference coordinates of the body. The reference point could be either (1) rigidly attached to the moving body for semi-recursive *I* or (2) coincident with the origin of the global frame for semi-recursive *II*. The latter leads to simple expressions of velocity transformation.

1. Introduction

In computational dynamics, a number of formalisms have been introduced to analyze the dynamics of mechanical systems. Among these formalisms, multibody system dynamics (MSD) offers a reliable, and easy-to-use tool to analyze the dynamics of complex systems. It is a systematic approach that can be applied to a wide range of problems [1].

Why semi-recursive methods?

As the word multibody implies, the approach is developed, in particular, to describe systems consisting of multiple bodies [2]. The multibody system dynamics approach is effective at analyzing the dynamics of bodies that interact through mechanical joints. In fact, joints are a fundamental aspect of a multibody dynamics system. They control and limit relative movement between interconnected bodies. In general, a system of interconnected bodies can be described using global methods [3]. The set of coordinates that defines absolute translations and rotations of each body of the system is used in global methods. In the global formulations, joints are

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accounted for by employing constraint equations that couple dependent coordinates. This way, the movement of one body influences the movement of other bodies based on joint interconnection types [4].

Semi-recursive dynamic algorithms are widely used in multibody applications. Their popularity is due to their numerical efficiency, which makes them suitable for computationally critical applications such as real-time simulation [5]. The approach has been implemented in commercial multibody software, such as RecurDyn, and has been successfully used in many practical applications. Joint coordinates are used between the bodies in semi-recursive methods [6,7]. When using the semi-recursive approach, it is often possible to synchronize computational time with actual time. This real-time simulation produces a number of benefits as it allows the user, hardware, and/or software to be engaged with the multibody simulation [8]. Naturally, real-time simulation requires that the methods used to solve the equations of motion and the hardware used are capable of carrying out the computation within a pre-defined time period. Therefore, the description of the equations of motion must be optimal for the system under investigation. Because the approach is often case dependent, it can be challenging to determine the most efficient multibody approach. However, it has been demonstrated that semi-recursive methods perform more efficiently than global approach when a system consists of a large number of bodies [9–11]. In addition, the semi-recursive method has proven to be faster than the articulated inertia method for the five-bar pendulum example in [12].

A brief literature review of semi-recursive methods

To improve the computational efficiency of the multibody dynamics formulations, several authors from robotic fields developed the fully recursive algorithm [13–15] and composite inertia method [16–18] to solve open loop systems. Both approaches are based on using inverse dynamics to solve the recursive Newton–Euler formulation. Furthermore, using variational and vector calculus, Bae et al. [14,15,19] formulated the dynamic equations of motion with a minimal set of joint coordinates, which is well suited for the parallel computation of open and closed loop systems. Employing the concept of velocity transformations, Jerkovsky [20] first proposed the semi-recursive method for the dynamic formulation to transform Cartesian coordinates into joint coordinates. With the semi-recursive method, the joints of the closed loops must be cut open to form the spanning tree. The joint reaction forces in the open loops are eliminated due to velocity transformations. Therefore, the differential and algebraic equations (DAE) and constraint stabilization are not required in the open loop mechanism, which greatly reduces the complexity of the problem. Using the same idea, different semi-recursive methods were proposed by Avello et al. [21], Negrut et al. [22], Kim [23] and Jalón et al. [24]. The main difference among those works is the location of the reference points (origin of body frame) that define the Cartesian coordinates. The reference point is located at the body in [21–23] and at the origin of the global frame in [24]. In addition, a subsystem synthesis method in [23] has proven to be more efficient than the conventional recursive method for the dynamic analysis of vehicle applications consisting of more than two suspension subsystems.

The closed loops in the semi-recursive method need to be handled by imposing a set of closure of the open-loop constraint equations. Consequently, different constraint stabilization approaches can be used to fulfill the closure of the open-loop constraint equations in semi-recursive methods at the position, velocity and acceleration levels. Among these stabilization approaches, coordinate partitioning [12,25], the penalty method [21,26], and augmented Lagrangian index-3 formulations with projections [9,11] are actively used in the semi-recursive formulation.

Motivation of educational purpose for semi-recursive methods

In terms of theoretical explanation, the design of student projects, and the manual of multibody software, the educational literature of multibody dynamics is extensive. The methods of teaching and applying multibody system are presented in [27] for undergraduate, graduate and doctoral student levels. A variety of different issues that can be taught for multibody dynamics training of engineering students is explained in [28]. Later, [29] illustrates how to design specific courses on multibody systems and emphasizes the importance of software development. The multibody modeling of bicycle dynamics is introduced in [30]. In the bicycle framework, a detailed derivation of a bicycle model for educational purposes is offered in [31]. A course to teach the kinematic and dynamic analysis of three-dimensional multibody systems based on natural coordinates and MATLAB programming language is presented in [32,33]. [34] describes a multi-disciplinary project in the field of multibody system modeling and computer simulation to educate undergraduate-level students. The educational software for teaching the kinematic analysis of planar and spatial mechanisms is described in the work of [35]. Each of above mentioned studies offers a unique viewpoint for enhanced understanding of multibody system dynamics. This is important to ensure proper use of multibody system dynamics methods.

There are a large number of studies, textbooks, and book chapters related to semi-recursive approaches. However, the literature available on semi-recursive approaches mainly explains the approach for spatial cases. As is well known, the spatial implementation is often difficult as it involves rotations in three-dimensional space. Realizing a full understanding of the semi-recursive approach is challenging, particularly for analysts without a history in multibody system dynamics. The available literature concerning semi-recursive approaches typically targets multibody researchers and doctoral level students. A good entry-level explanation that relates semi-recursive approaches to the often-used global formulation does not exist. To address this missing aspect, this paper focuses only on simple planar cases and examines the nature of the physical interpretation of velocity transformation matrices for two semi-recursive methods. In addition, the principle of virtual work is introduced to formulate the dynamic equations of motion, which can treat all coordinate systems generally.

This paper introduces an "original" semi-recursive approach as explained in the work of [3,21]. As there have been significant developments since the publication of the book, the paper will also explain a recently introduced semi-recursive approach which is

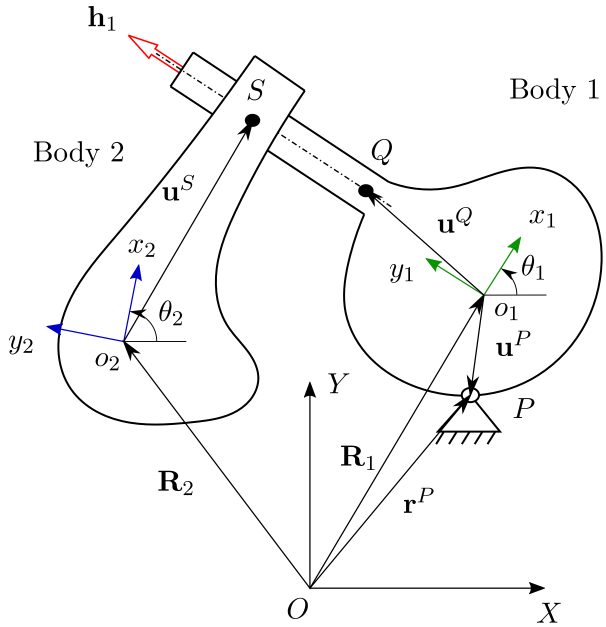


Fig. 1. A planar open loop system consists of two bodies that is constrained by one revolute joint and one prismatic joint. The kinematics are described in terms of reference point coordinates.

based on the intermediate body reference coordinates [24]. The rest of this paper is organized as follows: Section 2 introduces the review of two coordinates reduction methods. The recursive kinematics for semi-recursive *I* are presented in Section 3. Section 4 introduces a variant of semi-recursive *II*. Section 5 compares the kinematics of two semi-recursive methods through one open loop example. Section 6 studies the equations of motion for an open loop and the equations of motion for a closed loop with using both of the semi-recursive methods introduced in Section 7. Finally, Section 8 provides a summary and conclusion.

2. Review of coordinates reduction method

In the kinematics of a multibody system, the position, velocity and acceleration of each particle in the system are defined. Particles in a rigid body are assumed not to move relative to each other [4]. For this reason, describing their location with respect to the global frame by introducing a body-specific frame of reference is advised. This body frame is a mathematical representation without physical significance and can be located anywhere, even on non-material points of the body. Motion of the body frame can be described via reference point coordinates. These coordinates determine the position of the reference point of the body with Cartesian coordinates and a set of orientation coordinates of the body frame with respect to the global frame [3]. In planar cases, the number of reference point coordinates can be computed as $3n_b$ (two Cartesian coordinates and one rotation angle for one body). The designation “n_b” signifies the number of the bodies.

Fig. 1 shows a planar open loop system that consists of two bodies. The first body is constrained by the ground using revolute joint, and the second body is constrained by the first body using prismatic joint. The system is described via reference point coordinates. In this case, the vector of reference point coordinates $\mathbf{q}$ of the system consists of a total of six components:

$$\mathbf{q} = \begin{bmatrix}\mathbf{R}_1^T & \theta_1 & \mathbf{R}_2^T & \theta_2\end{bmatrix}^T, \tag{1}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the position vectors of the body frames with respect to the global frame, θ_1 and θ_2 are the rotational angles, which describe the orientation of the body frames with respect to the global frame.

2.1. Constrained kinematics

When using reference point coordinates, motion limitations due to joints in Fig. 1 are accounted for using a set of constraint equations. The body 1 and the ground are connected via a revolute joint at point P, and therefore the position vector of point P remains the same for both body 1 and the ground while constrained by the revolute joint. In addition, the prismatic joint allows only relative translation between the two bodies along the prismatic joint axis. One constraint equation will eliminate the relative rotation between the two bodies, and another will eliminate the relative translation between the two bodies which is perpendicular to the prismatic joint axis [4]. Therefore, the constraint equations of the open loop system in Fig. 1 are written as a function of reference point coordinates in a vector form as:

$$\mathbf{C}(\mathbf{q}) = \begin{bmatrix} \mathbf{R}_1 + \mathbf{u}^P - \mathbf{r}^P \\ \theta_2 - \theta_1 - \theta_c \\ (\tilde{\mathbf{I}}_2\mathbf{h}_1)^T(\mathbf{R}_1 + \mathbf{u}^Q - \mathbf{R}_2 - \mathbf{u}^S) \end{bmatrix} = \mathbf{0}, \tag{2}$$

where $\mathbf{r}^P$ is the position vector of revolute joint point P, $\mathbf{u}^P$ is the position vector of the joint point P with respect to the origin of body frame of body 1, θ_c is the constant rotation angle between two body frames, $\mathbf{u}^Q$ is the position vector of the joint point Q with respect to the origin of body frame of body 1, $\mathbf{u}^S$ is the position vector of the joint point S with respect to the origin of body frame of body 2, $\mathbf{h}_1$ is the unit vector that points in the direction of the prismatic joint axis as shown in Fig. 1, $\tilde{\mathbf{I}}_2\mathbf{h}_1$ represents one of the two possible unit vectors that are perpendicular to the vector $\mathbf{h}_1$, the local representation of the vector $\mathbf{h}_1$ and the skew symmetric matrix $\tilde{\mathbf{I}}_2$ are given by as:

$$\bar{\mathbf{h}}_1 = \frac{\bar{\mathbf{u}}_1^S - \bar{\mathbf{u}}_1^Q}{l_{Q,S}}, \quad \tilde{\mathbf{I}}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3}$$

where $\bar{\mathbf{u}}_1^S$ and $\bar{\mathbf{u}}_1^Q$ are the local representation of vectors $\mathbf{u}^S$ and $\mathbf{u}^Q$ with respect to body 1, and $l_{Q,S}$ is the distance between points Q and S.

The velocity constraint equations are obtained by differentiating Eq. (2) with respect to time as:

$$\dot{\mathbf{C}} = \mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}} = \mathbf{0}, \tag{4}$$

where $\mathbf{C}_{\mathbf{q}}$ is the constraint Jacobian matrix. It is written as follows:

$$\mathbf{C}_{\mathbf{q}} = \begin{bmatrix} \mathbf{I}_2 & \tilde{\mathbf{I}}_2\mathbf{u}^P & \mathbf{0} & \mathbf{0} \\ 0 & -1 & 0 & 1 \\ -\mathbf{h}_1^T\tilde{\mathbf{I}}_2 & -\mathbf{h}_1^T(\mathbf{R}_1 - \mathbf{R}_2 - \mathbf{u}^S) & \mathbf{h}_1^T\tilde{\mathbf{I}}_2 & -\mathbf{h}_1^T\mathbf{u}^S \end{bmatrix}, \tag{5}$$

where $\mathbf{I}_2$ is the identity matrix with 2×2 dimensions. A more detailed derivation of the Jacobian matrix is given in Appendix A.

Since the number of kinematic constraint equations is less than the number of reference point coordinates, the set of reference point coordinates can be reduced as a function of a subset of the independent coordinates [4] using constraint kinematics. In this section, two different coordinate reduction methods are reviewed: (a) the coordinate partitioning method and (b) the semi-recursive *I* method. For the simplification, only scleronomic constraints are considered for open loops.

2.2. Coordinate partitioning method

The Jacobian matrix $\mathbf{C}_{\mathbf{q}}$ from Eq. (5) has the full row rank in the example of Fig. 1, at least one nonsingular square submatrix of $\mathbf{C}_{\mathbf{q}}$ will exist with the rank of n_c [36]. To apply the coordinate partitioning method [36], the Jacobian matrix $\mathbf{C}_{\mathbf{q}}$ is partitioned into $\mathbf{C}_{\mathbf{q}^d} \in \mathbb{R}^{n_c \times n_c}$ and $\mathbf{C}_{\mathbf{q}^i} \in \mathbb{R}^{n_c \times (3n_b - n_c)}$, where $\mathbf{C}_{\mathbf{q}^d}$ is the dependent columns and $\mathbf{C}_{\mathbf{q}^i}$ is the independent columns of the Jacobian matrix. The n_c is the number of constraint equations. The velocity constraint equations from Eq. (4) can be rewritten as:

$$\mathbf{C}_{\mathbf{q}^d}\dot{\mathbf{q}}^d + \mathbf{C}_{\mathbf{q}^i}\dot{\mathbf{q}}^i = \mathbf{0}, \tag{6}$$

where $\dot{\mathbf{q}}^d$ is the dependent reference point velocities and $\dot{\mathbf{q}}^i$ is the independent reference point velocities.

Therefore, the reference point coordinates can be reduced to a minimum number of independent coordinates at the velocity level as follows.

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\mathbf{q}}^d \\ \dot{\mathbf{q}}^i \end{bmatrix} = \begin{bmatrix} -(\mathbf{C}_{\mathbf{q}^d})^{-1}\mathbf{C}_{\mathbf{q}^i} \\ \mathbf{I}_{3n_b - n_c} \end{bmatrix} \dot{\mathbf{q}}^i = \mathbf{H}\dot{\mathbf{q}}^i, \tag{7}$$

where $\mathbf{I}_{3n_b - n_c}$ is an identity matrix with $(3n_b - n_c) \times (3n_b - n_c)$ dimensions and velocity transformation matrix $\mathbf{H}$ relates the reference point velocities and independent reference point velocities. The velocity transformation matrix $\mathbf{H}$ is written as:

$$\mathbf{H} = \begin{bmatrix} -(\mathbf{C}_{\mathbf{q}^d})^{-1}\mathbf{C}_{\mathbf{q}^i} \\ \mathbf{I}_{3n_b - n_c} \end{bmatrix} \in \mathbb{R}^{3n_b \times n_c}. \tag{8}$$

2.3. Semi-recursive I

Semi-recursive *I* is another option that can reduce the size of the coordinates used to model the system. In semi-recursive *I*, the closed loop system must be partitioned into several open loops. Contrary to reference point coordinates, a number and type of joint coordinates describe the relative body motion at the joints under investigation. This is because joint coordinates only describe motions that are kinematically admissible. Accordingly, the number of joint coordinates is equal to the number of degrees of freedom of a multibody system in open loop multibody systems.

As shown in Fig. 2, the same open loop system as in Fig. 1 is expressed using joint coordinates. By accounting for kinematic limitations due to the joints, the system can be described using joint coordinates as:

$$\mathbf{z} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \tag{9}$$

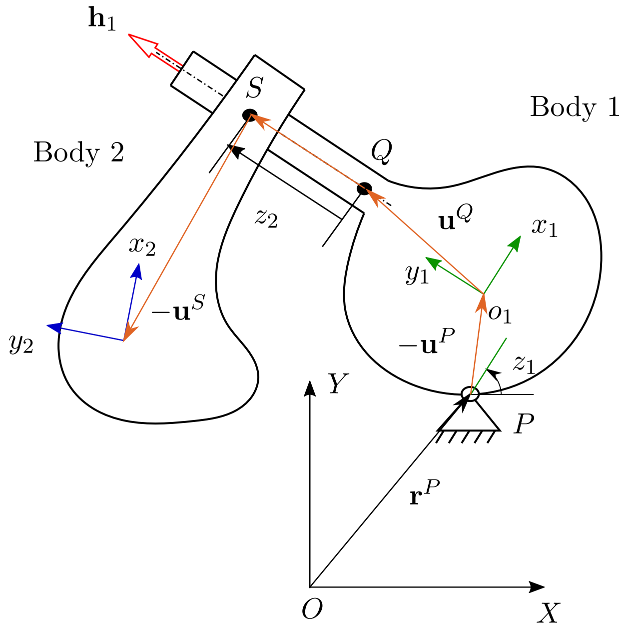


Fig. 2. Same open loop system as Fig. 1, but the kinematics are described in terms of joint coordinates.

where the joint coordinate z_1 is associated with the revolute joint P, defined using the angle measured from the X axis of the global frame to the X axis of the body frame 1. Joint coordinate z_2 is associated with the prismatic joint. It is a translation coordinate and measured from the joint point Q to point S.

As can be seen in Fig. 2, the reference point coordinates for the system are expressed recursively as a function of the joint coordinates as:

$$\mathbf{q}(\mathbf{z}) = \begin{bmatrix} \mathbf{r}^P - \mathbf{u}^P \\ z_1 \\ \mathbf{r}^P - \mathbf{u}^P + \mathbf{u}^Q + z_2\mathbf{h}_1 - \mathbf{u}^S \\ z_1 + \theta_c \end{bmatrix}. \tag{10}$$

Differentiating the reference point coordinates from Eq. (10) with respect to time yields:

$$\dot{\mathbf{q}}(\mathbf{z}, \dot{\mathbf{z}}) = \mathbf{V}\dot{\mathbf{z}}, \quad \text{with} \quad \mathbf{V} = \begin{bmatrix} \tilde{\mathbf{I}}_2(\mathbf{R}_1 - \mathbf{r}^P) & \mathbf{0} \\ 1 & 0 \\ \tilde{\mathbf{I}}_2(\mathbf{R}_2 - \mathbf{r}^P) & \mathbf{h}_1 \\ 1 & 0 \end{bmatrix}, \tag{11}$$

where $\dot{\mathbf{q}}$ is the reference point velocities, $\dot{\mathbf{z}}$ is the relative velocities, $\mathbf{V}$ is the velocity transformation matrix that relates the reference point velocities and relative velocities. A more detailed derivation of the velocity transformation matrix is provided in Appendix A.

According to Eqs. (5) and (11), the following relationship can be obtained.

$$\mathbf{C_q}\mathbf{V} = \mathbf{0}, \tag{12}$$

where the matrix $\mathbf{V}$ is the null space of the Jacobian matrix of the constraints in Eq. (5).

Both coordinate reduction methods will lead to a minimal set of independent coordinates to define total system kinematic response. However, some differences can be found when comparing the velocity transformation matrix for the coordinate partitioning method in Eq. (8) and semi-recursive I in Eq. (11).

- The shortcoming of the coordinate partitioning method is that an incorrect selection of independent coordinates can lead to an ill-conditioned or singular matrix [37] of the velocity transformation matrix $\mathbf{H}$. For the example of the open loop in Fig. 2, θ_1 and θ_2 cannot be selected together as independent coordinates, according to Eq. (5), due to the singular square matrix $\mathbf{C}_{\mathbf{q}^d}$. Usually, this selection can be carried out manually or using Gaussian elimination with column pivoting [38]. But Gaussian elimination might be computationally expensive.
- The velocity transformation matrix $\mathbf{V}$ in semi-recursive I has a triangular structure with all zeros above the diagonal. It is obtained directly without any factoring matrix operations, which greatly reduces the numerical effort. In addition, the sparse part of matrix $\mathbf{V}$ in semi-recursive I can lead to efficient computation, and it can be used for the subsequent matrix operation [3].

Since the coordinate partitioning method is not the focus of this paper, only the semi-recursive method is studied in the rest of the work.

3. Recursive kinematics for semi-recursive *I*

As explained earlier, the reference point coordinates for the open loop system can be related to joint coordinates using velocity transformations. In this way, the reference point velocities $\dot{\mathbf{q}}$ can be projected into a set of independent relative velocities $\dot{\mathbf{z}}$ [21].

$$\dot{\mathbf{q}} = \mathbf{V}\dot{\mathbf{z}}. \tag{13}$$

where the velocity transformation matrix $\mathbf{V}$ can be obtained directly by taking the partial derivative of the reference point coordinates with respect to the joint coordinates as follows.

$$\mathbf{V} = \begin{bmatrix} \frac{\partial \mathbf{q}_1}{\partial z_1} & \frac{\partial \mathbf{q}_1}{\partial z_2} & \cdots & \frac{\partial \mathbf{q}_1}{\partial z_{n_c}} \\ \frac{\partial \mathbf{q}_2}{\partial z_1} & \frac{\partial \mathbf{q}_2}{\partial z_2} & \cdots & \frac{\partial \mathbf{q}_2}{\partial z_{n_c}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{q}_{n_b}}{\partial z_1} & \frac{\partial \mathbf{q}_{n_b}}{\partial z_2} & \cdots & \frac{\partial \mathbf{q}_{n_b}}{\partial z_{n_c}} \end{bmatrix} \in \mathbb{R}^{3n_b \times n_c}, \tag{14}$$

where it is assumed that the open loop system comprises n_b rigid bodies and allows n_c degrees of freedom. Clearly, the row corresponds to the body and the column corresponds to the joint coordinates of the open loop system.

One advantage of semi-recursive *I* is that the matrix $\mathbf{V}$ can be obtained on an element-by-element basis, using a parallel computation technique. In the following sections, the velocity transformation matrix $\mathbf{V}$ is explained by means of recursive position and velocity problems.

3.1. Semi-recursive I: revolute joint kinematics

Consider an open loop multibody system that comprises a base element and a branch with an arbitrary number of rigid bodies. The position vector and rotation angle of the arbitrary body can be computed recursively starting from the base body to the branch. With body i in Fig. 3 for example, two contiguous bodies $i-1$ and i are connected by a revolute joint J_j that constrains the relative rotation of body $i-1$ with respect to body i in the direction of the revolute joint axis (perpendicular to the XY plane). The position vector and rotation angle of previous body $i-1$ are presumed known. The reference point coordinates for body i are described using the reference point coordinates of the previous body and the joint coordinates as follows.

$$\mathbf{q}_i = \begin{bmatrix} \mathbf{R}_{i-1} + \mathbf{a}_{i-1} + \mathbf{b}_i \\ \theta_{i-1} + z_j \end{bmatrix}, \quad \text{with} \quad \mathbf{a}_{i-1} = \mathbf{r}_j - \mathbf{R}_{i-1}, \quad \mathbf{b}_i = \mathbf{R}_i - \mathbf{r}_j, \tag{15}$$

where $\mathbf{r}_j$ is the position vector of the revolute joint point J_j, $\mathbf{a}_{i-1}$ is the translational vector that goes from the origin of body frame o_{i-1} to revolute joint point J_j, $\mathbf{b}_i$ is the translational vector that goes from the revolute joint point J_j to the origin of body frame o_i, z_j is the relative angle associated with revolute joint J_j, which is measured from the X axis of body frame $i-1$ to X axis of body frame i.

The time derivative of the reference point coordinates of Eq. (15) is given by:

$$\dot{\mathbf{q}}_i = \begin{bmatrix} \dot{\mathbf{R}}_{i-1} + \dot{\theta}_{i-1}\tilde{\mathbf{I}}_2(\mathbf{r}_j - \mathbf{R}_{i-1}) + (\dot{\theta}_{i-1} + \dot{z}_j)\tilde{\mathbf{I}}_2(\mathbf{R}_i - \mathbf{r}_j) \\ \dot{\theta}_{i-1} + \dot{z}_j \end{bmatrix}, \tag{16}$$

where $\dot{\mathbf{R}}_{i-1}$ is the time derivative of $\mathbf{R}_{i-1}$, $\dot{\theta}_{i-1}$ is the angular velocity of body $i-1$, and $\dot{z}_j$ is the relative velocity.

Eq. (16) can be reorganized in a matrix manner as follows.

$$\dot{\mathbf{q}}_i = \mathbf{B}_i^{i-1}\dot{\mathbf{q}}_{i-1} + \mathbf{V}_i^j \dot{z}_j, \tag{17}$$

where $\dot{\mathbf{q}}_{i-1}$ is the reference point velocities for body $i-1$, the matrix $\mathbf{B}_i^{i-1}$ is the transformation matrix associated with bodies $i-1$ and i, and the vector $\mathbf{V}_i^j$ is the component of the velocity transformation matrix $\mathbf{V}$ from independent to dependent velocities, which gives the unit translational and rotational direction to the relative velocity $\dot{z}_j$. Both are expressed as:

$$\mathbf{B}_i^{i-1} = \begin{bmatrix} \mathbf{I}_2 & \tilde{\mathbf{I}}_2(\mathbf{R}_i - \mathbf{R}_{i-1}) \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}, \quad \mathbf{V}_i^j = \begin{bmatrix} \tilde{\mathbf{I}}_2(\mathbf{R}_i - \mathbf{r}_j) \\ 1 \end{bmatrix} \in \mathbb{R}^{3\times 1}, \tag{18}$$

where $\mathbf{I}_2$ is the identity matrix of 2×2 size.

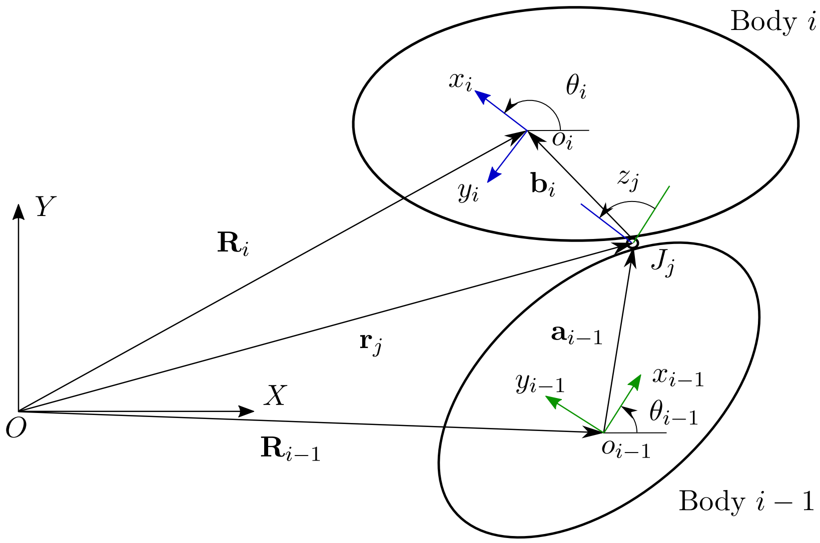


Fig. 3. Two bodies connected by revolute joints in the planar case. The kinematics are described in terms of reference point coordinates and joint coordinates.

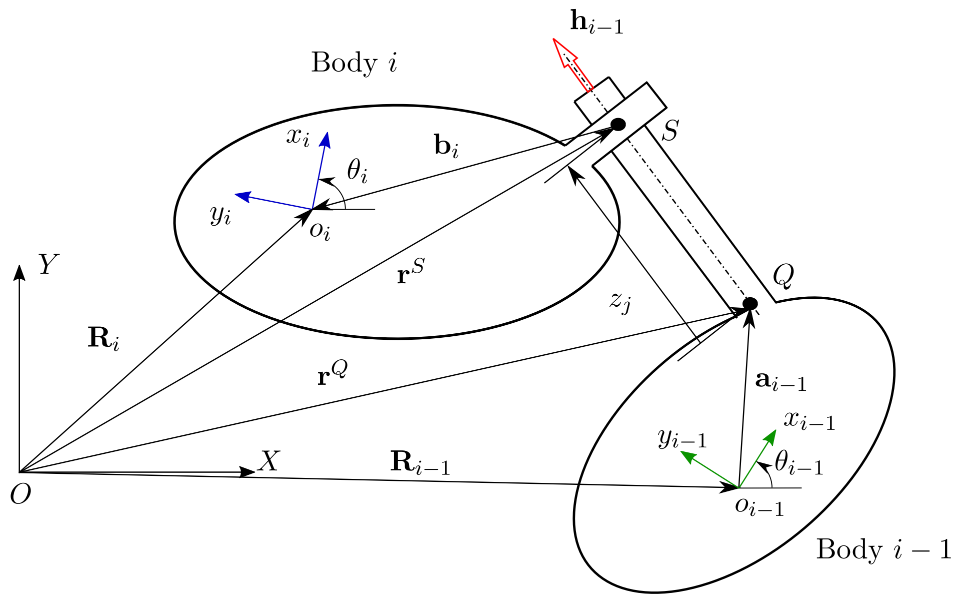


Fig. 4. Two bodies connected by prismatic joints in the planar case. The kinematics are described in terms of reference point coordinates and joint coordinates.

3.2. Semi-recursive I: prismatic joint kinematics

As shown in Fig. 4, two contiguous bodies $i-1$ and i are connected by a prismatic joint. The prismatic joint constrains the relative translation of body $i-1$ with respect to body i in the direction of the prismatic axis. Again, the reference point coordinates for body i are described as the function of joint coordinates as:

$$\mathbf{q}_i = \begin{bmatrix} \mathbf{R}_{i-1} + \mathbf{a}_{i-1} + z_j \mathbf{h}_{i-1} + \mathbf{b}_i \\ \theta_{i-1} + \theta_c \end{bmatrix}, \quad \text{with} \quad \mathbf{a}_{i-1} = \mathbf{r}^Q - \mathbf{R}_{i-1}, \quad \mathbf{b}_i = \mathbf{R}_i - \mathbf{r}^S, \tag{19}$$

where joint coordinate z_j is the translation coordinate associated with the prismatic joint. It is a signed distance and measured from joint point Q to point S along the direction of $\mathbf{h}_{i-1}$. The $\mathbf{h}_{i-1}$ is the unit vector that points in the direction of the prismatic axis, $\mathbf{r}^Q$ is the position vector of starting point Q, and $\mathbf{r}^S$ is the position vector of ending point S. All vectors are shown in Fig. 4.

The reference point velocities are computed as the time-derivative of the reference point coordinates of Eq. (19), which results in the same formulation as Eq. (17). Matrix $\mathbf{B}_i^{i-1}$ maintains the same form as Eq. (18), but the vector $\mathbf{V}_i^j$, which is the unit direction vector of body i given by prismatic joint J_j, is expressed as follows.

$$\mathbf{V}_i^j = \begin{bmatrix} \mathbf{h}_{i-1} \\ 0 \end{bmatrix} \in \mathbb{R}^{3\times 1}. \tag{20}$$

To understand the derivation of velocity transformation matrix $\mathbf{V}$, the revolute and prismatic joints in planar cases are considered in Sections 3.1 and 3.2. The introduced procedure can be applied to more complex mechanical constraints, such as point-to-line and cam-follower constraints. A more detailed derivation can be found in Appendix B.

As explained in Eq. (17), the reference point velocity of body i is expressed recursively in terms of the reference point velocity of the previous body $i-1$ and the relative joint velocity between $i-1$ and i. The process of eliminating the reference point velocity $\dot{\mathbf{q}}_{i-1}$ in Eq. (17) can continue in the same manner with the help of the recursive relations. Finally, the vector of the generalized global velocity for body i arrives at the linear combination of matrix $\mathbf{V}_i$ and the components of $\dot{\mathbf{z}}$, as:

$$\dot{\mathbf{q}}_i = \mathbf{V}_i \dot{\mathbf{z}}, \quad \text{with} \quad \mathbf{V}_i = \begin{bmatrix} \mathbf{V}_i^1 & \cdots & \mathbf{V}_i^{n_c} \end{bmatrix} \in \mathbb{R}^{3\times n_c}, \quad \dot{\mathbf{z}} = \begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_{n_c} \end{bmatrix} \in \mathbb{R}^{n_c\times 1}, \tag{21}$$

where $\mathbf{V}_i$ is the base of the allowable velocities $\dot{\mathbf{q}}_i$ and the components of $\dot{\mathbf{z}}$ are the coefficients of that base [3]. More detailed information related to dimension reduction is explained in Appendix C.

The reference point accelerations can be written as the time derivative of Eq. (21), as:

$$\ddot{\mathbf{q}}_i = \mathbf{V}_i \ddot{\mathbf{z}} + \dot{\mathbf{V}}_i \dot{\mathbf{z}}, \quad \text{with} \quad \dot{\mathbf{V}}_i = \begin{bmatrix} \dot{\mathbf{V}}_i^1 & \cdots & \dot{\mathbf{V}}_i^{n_c} \end{bmatrix} \in \mathbb{R}^{3\times n_c}, \quad \ddot{\mathbf{z}} = \begin{bmatrix} \ddot{z}_1 \\ \vdots \\ \ddot{z}_{n_c} \end{bmatrix} \in \mathbb{R}^{n_c\times 1}, \tag{22}$$

where $\dot{\mathbf{V}}_i$ is the time-derivative of $\mathbf{V}_i$, and $\ddot{\mathbf{z}}$ is the vector of relative accelerations. The time derivative of $\mathbf{V}_i^j$ is expressed according to the type of kinematic joints.

$$\begin{aligned} &\text{Revolute joint:} \quad \dot{\mathbf{V}}_i^j = \begin{bmatrix} \tilde{\mathbf{I}}_2(\dot{\mathbf{R}}_i - \dot{\mathbf{r}}_j) \\ 0 \end{bmatrix} \in \mathbb{R}^{3\times 1}, \\ &\text{Prismatic joint:} \quad \dot{\mathbf{V}}_i^j = \begin{bmatrix} \dot{\theta}_i \tilde{\mathbf{I}}_2 \mathbf{h}_{i-1} \\ 0 \end{bmatrix} \in \mathbb{R}^{3\times 1}. \end{aligned} \tag{23}$$

4. Semi-recursive method *II*

With using semi-recursive methods, the dynamic equations for open loops are first formulated in reference coordinates. Afterwards, those reference coordinates are projected onto a set of joint coordinates via velocity transformation. Normally, the definition of reference coordinates depends on the location of the reference point. With different reference point locations, the semi-recursive method can be classified into two groups.

- As explained in Section 2, the reference point of the body frame can be rigidly attached to the material point (including the center of gravity) or non-material point of the moving body. This reference coordinate is referred to as the reference point coordinate. In semi-recursive *I*, velocity transformation is used to project the reference point velocities onto relative velocities.
- In some cases, however, the reference point coincides with the origin of the global frame but is rigidly attached to the moving body [24,39]. This reference coordinate is referred to here as the intermediate body reference coordinate. Therefore, velocity transformation in semi-recursive *II* is used to project the intermediate body reference velocities onto relative velocities.

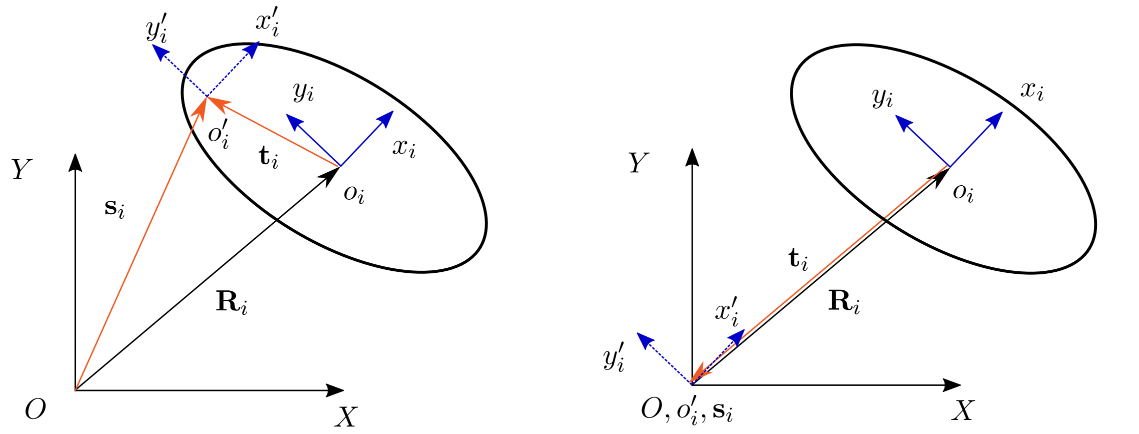
In the following paragraphs, semi-recursive *II* for planar cases is described in detail.

4.1. Reference point and intermediate body reference coordinates

In many practical cases, it is beneficial to locate the reference point of the body frame at the origin of the global frame. In this paper, this body frame is called the intermediate body reference frame. The origin of the intermediate body reference frame coincides with the origin of the global frame. The orientation of the intermediate body reference frame is rigidly attached to the moving body, which has the same orientation as the body frame [24]. For this reason, all bodies in the mechanical system share the same reference point over time. In such cases, the recursive kinematics for contiguous bodies is simpler than those obtained when the reference point is rigidly attached to the body.

To understand the relationship between reference point coordinates and intermediate body reference coordinates, the kinematics of the intermediate body reference frame at different locations is introduced below in Fig. 5. As depicted in Fig. 5(a), the origin of the intermediate body reference frame o_i' is located at the material point of the moving body i. According to Eq. (41), the position and velocity vectors of the point o_i' with respect to the origin of the global frame is expressed as:

$$\begin{aligned} \mathbf{s}_i &= \mathbf{R}_i + \mathbf{A}_i \bar{\mathbf{t}}_i, \\ \dot{\mathbf{s}}_i &= \dot{\mathbf{R}}_i + \dot{\theta}_i \tilde{\mathbf{I}}_2 \mathbf{A}_i \bar{\mathbf{t}}_i, \end{aligned} \tag{24}$$



(a) The origin of the intermediate body reference frame is the material point of moving body i

(b) The origin of the intermediate body reference frame is non-material point of the moving body i

Fig. 5. Kinematics of intermediate body reference frame at different locations.

where $\mathbf{s}_i$ is the position vector of reference point o_i' with respect to the origin of the global frame O, $\dot{\mathbf{s}}_i$ is the time derivative of vector $\mathbf{s}_i$, and $\bar{\mathbf{t}}_i$ is the position vector of reference point o_i' with respect to body frame in the body frame i.

However, this intermediate body reference frame can also be located at a non-material point of the moving body, such as the origin of the global frame, as shown in Fig. 5(b). In this case, the position vector of reference point $\mathbf{s}_i$ is a null vector, and the local position vector $\bar{\mathbf{t}}_i$ can be expressed as:

$$\bar{\mathbf{t}}_i = -\mathbf{A}_i^T \mathbf{R}_i. \tag{25}$$

Substituting $\bar{\mathbf{t}}_i$ from Eq. (25) into Eq. (24) results in the following.

$$\dot{\mathbf{R}}_i = \dot{\mathbf{s}}_i + \dot{\theta}_i \tilde{\mathbf{I}}_2 \mathbf{R}_i, \tag{26}$$

The intermediate body reference velocities are introduced here to describe the kinematics of the body i as:

$$\dot{\mathbf{Z}}_i = \begin{bmatrix} \dot{\mathbf{s}}_i \\ \dot{\theta}_i \end{bmatrix}. \tag{27}$$

According to Eq. (26), the reference point velocities and their time derivatives are expressed as a relationship between intermediate body reference velocities and accelerations, as:

$$\begin{aligned} \dot{\mathbf{q}}_i &= \mathbf{D}_i \dot{\mathbf{Z}}_i, \quad \text{with} \quad \mathbf{D}_i = \begin{bmatrix} \mathbf{I}_2 & \tilde{\mathbf{I}}_2 \mathbf{R}_i \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{R}^{3\times3}, \\ \ddot{\mathbf{q}}_i &= \mathbf{D}_i \ddot{\mathbf{Z}}_i + \mathbf{e}_i, \quad \text{with} \quad \mathbf{e}_i = \dot{\mathbf{D}}_i \dot{\mathbf{Z}}_i = \begin{bmatrix} -\dot{\theta}_i^2 \mathbf{R}_i \\ 0 \end{bmatrix} \in \mathbb{R}^{3\times1}, \end{aligned} \tag{28}$$

where $\ddot{\mathbf{Z}}_i$ is the vector of intermediate body reference acceleration.

4.2. Projection of intermediate body reference coordinates into joint coordinates based on a physical explanation

To understand the relationship between intermediate body reference coordinates and joint coordinates, revolute and prismatic joints in a planar case are considered in the following sections for semi-recursive *II*. Projection kinematics are introduced using a physical explanation. A mathematical explanation to obtain the projection matrix is presented in Appendix D.

Semi-recursive II: revolute joint kinematics

Analogous to Fig. 3, two contiguous bodies $i-1$ and i are connected by a revolute joint J_j in Fig. 6. However, the system is described with using intermediate body reference coordinates. The origin of both intermediate body reference frames coincide with the origin of the global frame. As depicted in Fig. 6, the position vector of revolute joint J_j and the rotation angle of body i can be expressed as the function of the intermediate body reference coordinates.

$$\begin{aligned} \mathbf{r}_j &= \mathbf{s}_{i-1} + \mathbf{A}_{i-1} \bar{\mathbf{r}}_{i-1}^{J_j} = \mathbf{s}_i + \mathbf{A}_i \bar{\mathbf{r}}_i^{J_j}, \\ \theta_i &= \theta_{i-1} + z_j, \end{aligned} \tag{29}$$

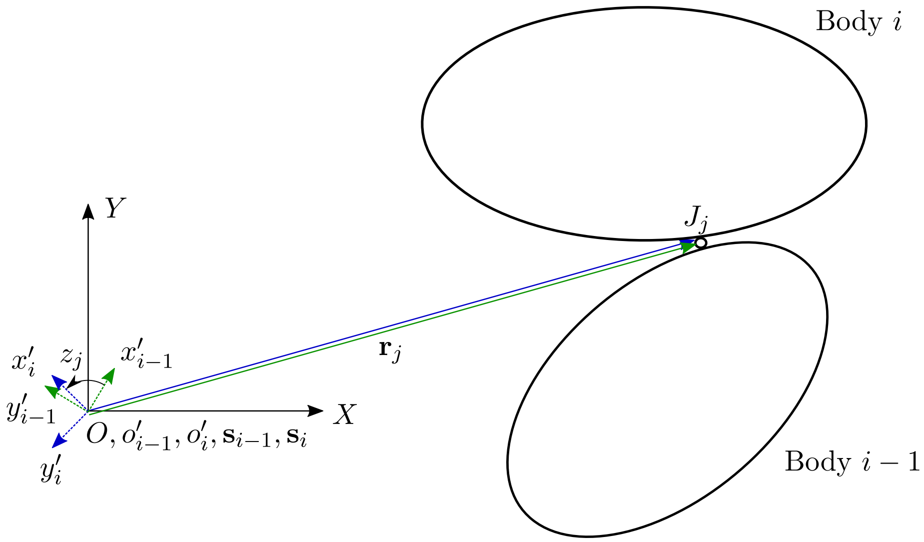


Fig. 6. Two bodies connected by revolute joints in planar case (as in Fig. 3). In this case, the kinematics are described in terms of intermediate body reference coordinates and joint coordinates.

where $\mathbf{s}_{i-1}$ and $\mathbf{s}_i$ are position vectors of the intermediate body reference frames with respect to the global frame, $\bar{\mathbf{r}}_{i-1}^{J_j}$ and $\bar{\mathbf{r}}_i^{J_j}$ are the position vectors of revolute joint J_j with respect to the intermediate body reference frame in intermediate body reference frames $i-1$ and i, and z_j is the relative angle associated with revolute joint J_j, which is measured from the X axis of the intermediate body reference frames $i-1$ to i.

Therefore, according to Eq. (29), the intermediate body reference coordinates for body i can be obtained from the intermediate body reference coordinates of the previous body $i-1$ and the joint coordinates.

$$\mathbf{Z}_i = \mathbf{Z}_{i-1} + \begin{bmatrix} \mathbf{A}_{i-1}\bar{\mathbf{r}}_{i-1}^{J_j} - \mathbf{A}_i\bar{\mathbf{r}}_i^{J_j} \\ z_j \end{bmatrix}. \tag{30}$$

The intermediate body reference velocities and accelerations for body i are computed as the time-derivatives of the intermediate body reference coordinates from Eq. (30):

$$\begin{aligned} \dot{\mathbf{Z}}_i &= \dot{\mathbf{Z}}_{i-1} + \mathbf{b}_j \dot{z}_j, \quad \text{with} \quad \mathbf{b}_j = \begin{bmatrix} -\tilde{\mathbf{I}}_2\mathbf{r}_j \\ 1 \end{bmatrix} \in \mathbb{R}^{3\times1}, \\ \ddot{\mathbf{Z}}_i &= \ddot{\mathbf{Z}}_{i-1} + \mathbf{b}_j \ddot{z}_j + \mathbf{d}_j, \quad \text{with} \quad \mathbf{d}_j = \dot{\mathbf{b}}_j \dot{z}_j = \begin{bmatrix} (\dot{\theta}_i^2 - \dot{\theta}_{i-1}^2)\mathbf{r}_j \\ 0 \end{bmatrix} \in \mathbb{R}^{3\times1}. \end{aligned} \tag{31}$$

Semi-recursive II: Prismatic joint kinematics

Analogous to Fig. 4, two contiguous bodies $i-1$ and i are connected by a prismatic joint J_j in Fig. 7. But the system is described with using intermediate body reference coordinates. The origin of both intermediate body reference frames coincide with the origin of the global frame. As depicted in the figure, the position vector of prismatic end point S and the rotation angle of body i can be expressed as the function of the intermediate body reference coordinates.

$$\begin{aligned} \mathbf{r}^S &= \mathbf{s}_{i-1} + \mathbf{A}_{i-1}\bar{\mathbf{r}}_{i-1}^Q + \mathbf{h}_{i-1} z_j = \mathbf{s}_i + \mathbf{A}_i \bar{\mathbf{r}}_i^S, \\ \theta_i &= \theta_{i-1} + \theta_c, \end{aligned} \tag{32}$$

where $\bar{\mathbf{r}}_{i-1}^Q$ is the position vector of starting point Q, $\bar{\mathbf{r}}_i^S$ is the position vector of ending point S, and z_j is the relative distance associated with the prismatic joint J_j, which is measured from the starting point Q to ending one S as shown in Fig. 7. Both are written with respect to the origin and within the intermediate body reference frame.

Therefore, according to Eq. (32), the vector of the intermediate body reference coordinates for body i can be obtained according to the intermediate body reference coordinates of previous body $i-1$ and the joint coordinates.

$$\mathbf{Z}_i = \mathbf{Z}_{i-1} + \begin{bmatrix} \mathbf{A}_{i-1}\bar{\mathbf{r}}_{i-1}^Q + \mathbf{h}_{i-1} z_j - \mathbf{A}_i\bar{\mathbf{r}}_i^S \\ \theta_c \end{bmatrix}. \tag{33}$$

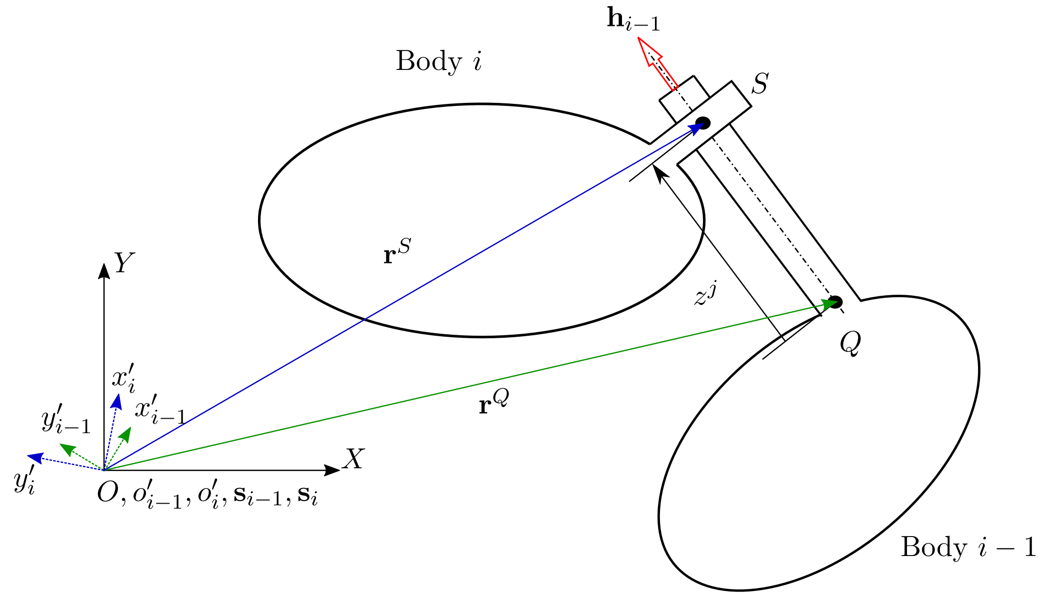


Fig. 7. Two bodies connected by prismatic joints in the planar case (as in Fig. 4). The kinematics are described in terms of intermediate body reference coordinates and joint coordinates.

The intermediate body reference velocities and accelerations for body i are computed as the time-derivatives of the intermediate body reference coordinates from Eq. (33).

$$\begin{aligned} \dot{\mathbf{Z}}_i &= \dot{\mathbf{Z}}_{i-1} + \mathbf{b}_j \dot{z}_j, \quad \text{with} \quad \mathbf{b}_j = \begin{bmatrix} \mathbf{h}_{i-1} \\ 0 \end{bmatrix} \in \mathbb{R}^{3\times 1}, \\ \ddot{\mathbf{Z}}_i &= \ddot{\mathbf{Z}}_{i-1} + \mathbf{b}_j \ddot{z}_j + \mathbf{d}_j, \quad \text{with} \quad \mathbf{d}_j = \dot{\mathbf{b}}_j \dot{z}_j = \begin{bmatrix} 2\dot{z}_j \dot{\theta}_{i-1} \tilde{\mathbf{I}}_2 \mathbf{h}_{i-1} \\ 0 \end{bmatrix} \in \mathbb{R}^{3\times 1}. \end{aligned} \tag{34}$$

Therefore, in view of Eqs. (31) and (34), a velocity transformation matrix $\mathbf{W}_i$ can be defined to relate the relative and the intermediate body reference coordinates.

$$\dot{\mathbf{Z}}_i = \mathbf{W}_i \dot{\mathbf{z}} = \mathbf{T}_i \mathbf{R}_d \dot{\mathbf{z}}, \quad \ddot{\mathbf{Z}}_i = \mathbf{W}_i \ddot{\mathbf{z}} + \dot{\mathbf{W}}_i \dot{\mathbf{z}} = \mathbf{T}_i (\mathbf{R}_d \ddot{\mathbf{z}} + \mathbf{d}), \tag{35}$$

where $\mathbf{W}_i \in \mathbb{R}^{3\times n_c}$ is the base of allowable velocities $\dot{\mathbf{Z}}_i$, $\mathbf{T}_i$ is the *path matrix* that defines the connectivity of the mechanism for body i in the open loop system, and the matrix $\mathbf{R}_d$, vectors $\mathbf{d}$ and $\mathbf{T}_i\mathbf{d}$ are written as:

$$\mathbf{R}_d = \begin{bmatrix} \mathbf{b}_1 & & \\ & \ddots & \\ & & \mathbf{b}_{n_c} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_{n_c} \end{bmatrix}, \quad \mathbf{T}_i \mathbf{d} = \frac{\mathrm{d}(\mathbf{T}_i \mathbf{R}_d)}{\mathrm{d}t} \dot{\mathbf{z}}. \tag{36}$$

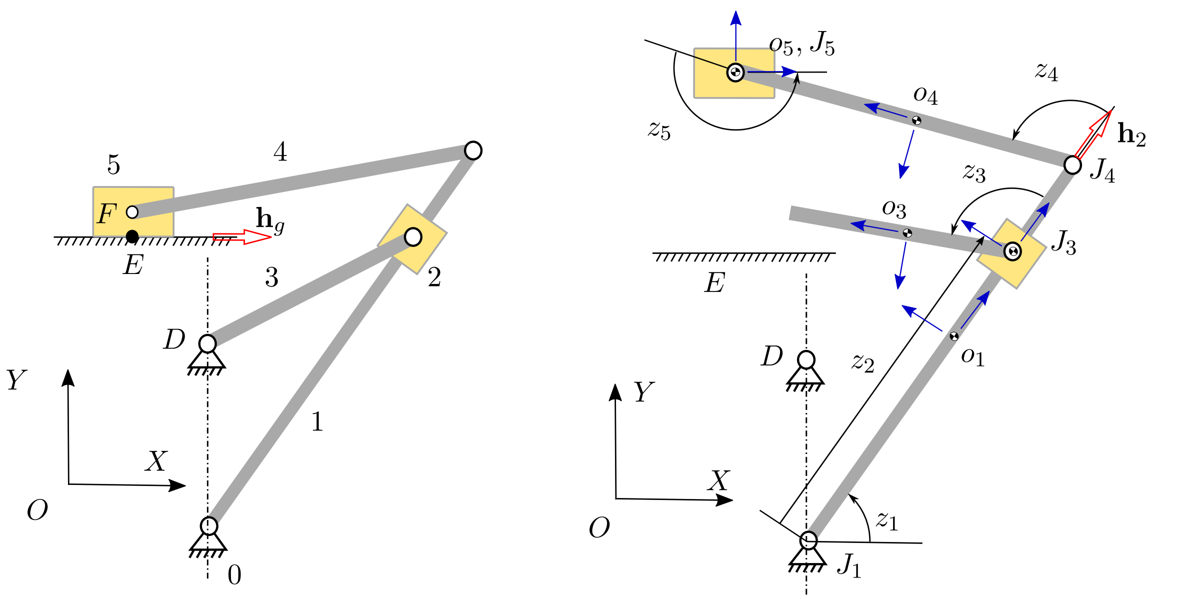
5. A discussion of kinematics of two semi-recursive methods through one example

To be able to apply both semi-recursive methods efficiently, it is important that the velocity transformation matrix for the open loop must be implemented correctly. In this section, an example is provided to introduce how the velocity transformation matrix is used for both semi-recursive methods.

An example of a closed loop system is depicted in Fig. 8(a). By cutting one revolute joint at point D and one prismatic joint between body 5 and ground, the closed-loop systems can be opened to make use of recursive kinematics and dynamics. Consequently, the open loop system has a tree structure with two branches. See Fig. 8(b). In this system, the ground is acting as the base body (body 0). The bodies and joints are labeled from the base to the branch. There is a total number of five degrees of freedom, which leads to a set of relative coordinates.

$$\mathbf{z} = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix}^T, \tag{37}$$

where z_1, z_3, z_4 and z_5 are rotational angles which correspond to revolute joints and z_2 is the translation coordinate that corresponds to prismatic joint in Fig. 8(b).



(a) Closed loop quick return mechanism

(b) Tree structure open loop by cutting the revolute joint at point D and prismatic joint between body 5 and ground

Fig. 8. Open and closed loop for quick return system.

The velocity transformation matrix $\mathbf{V}$ from Eq. (21) for open loop multibody system with using the semi-recursive I is written as:

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_1^1 & & & & \\ \mathbf{V}_2^1 & \mathbf{V}_2^2 & & & \\ \mathbf{V}_3^1 & \mathbf{V}_3^2 & \mathbf{V}_3^3 & & \\ \mathbf{V}_4^1 & & & \mathbf{V}_4^4 & \\ \mathbf{V}_5^1 & & & \mathbf{V}_5^4 & \mathbf{V}_5^5 \end{bmatrix} = \begin{bmatrix} \begin{matrix}\tilde{\mathbf{I}}_2(\mathbf{R}_1-\mathbf{r}_1)\\ 1\end{matrix} & & & & \\ \begin{matrix}\tilde{\mathbf{I}}_2(\mathbf{R}_2-\mathbf{r}_1)\\ 1\end{matrix} & \begin{matrix}\mathbf{h}_2\\ 0\end{matrix} & & & \\ \begin{matrix}\tilde{\mathbf{I}}_2(\mathbf{R}_3-\mathbf{r}_1)\\ 1\end{matrix} & \begin{matrix}\mathbf{h}_2\\ 0\end{matrix} & \begin{matrix}\tilde{\mathbf{I}}_2(\mathbf{R}_3-\mathbf{r}_3)\\ 1\end{matrix} & & \\ \begin{matrix}\tilde{\mathbf{I}}_2(\mathbf{R}_4-\mathbf{r}_1)\\ 1\end{matrix} & & & \begin{matrix}\tilde{\mathbf{I}}_2(\mathbf{R}_4-\mathbf{r}_4)\\ 1\end{matrix} & \\ \begin{matrix}\tilde{\mathbf{I}}_2(\mathbf{R}_5-\mathbf{r}_1)\\ 1\end{matrix} & & & \begin{matrix}\tilde{\mathbf{I}}_2(\mathbf{R}_5-\mathbf{r}_4)\\ 1\end{matrix} & \begin{matrix}\tilde{\mathbf{I}}_2(\mathbf{R}_5-\mathbf{r}_5)\\ 1\end{matrix} \end{bmatrix}, \tag{38}$$

where $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, $\mathbf{R}_4$, and $\mathbf{R}_5$ are the position vectors of the origin of the body frames. And $\mathbf{r}_1$, $\mathbf{r}_3$, $\mathbf{r}_4$, and $\mathbf{r}_5$ are the position vectors of the revolute joints. All of them are calculated recursively from the base body to the branch. See Eqs. (15) and (19). $\mathbf{h}_2$ is the unit vector of the translation direction.

The velocity transformation matrix $\mathbf{W}$ from Eq. (35) for open loop multibody system with using the semi-recursive II is written as:

$$\mathbf{W} = \mathbf{T}\mathbf{R}_d = \begin{bmatrix} \mathbf{I}_3 & & & & \\ \mathbf{I}_3 & \mathbf{I}_3 & & & \\ \mathbf{I}_3 & \mathbf{I}_3 & \mathbf{I}_3 & & \\ \mathbf{I}_3 & & & \mathbf{I}_3 & \\ \mathbf{I}_3 & & & \mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \begin{matrix}-\tilde{\mathbf{I}}_2\mathbf{r}_1\\ 1\end{matrix} & & & & \\ & \begin{matrix}\mathbf{h}_2\\ 0\end{matrix} & & & \\ & & \begin{matrix}-\tilde{\mathbf{I}}_2\mathbf{r}_3\\ 1\end{matrix} & & \\ & & & \begin{matrix}-\tilde{\mathbf{I}}_2\mathbf{r}_4\\ 1\end{matrix} & \\ & & & & \begin{matrix}-\tilde{\mathbf{I}}_2\mathbf{r}_5\\ 1\end{matrix} \end{bmatrix}. \tag{39}$$

As can be observed from the above Eqs. (38) and (39), the velocity transformation matrices for both approaches behave like the spatial cases [3,24]. Both can introduce the topology of the spanning tree, and the velocity transformation matrix in semi-recursive II can be straightforwardly obtained with the help of path matrix.

6. Equations of motion for the open loops

In this section, the principle of virtual work is introduced to formulate the dynamic equations of motion for planar cases making it possible for a multibody modeling analyst that is familiar with global formulations to easily extend that knowledge to a semi-recursive approach.

The virtual work done by the inertial forces of body i can be written as:

$$\delta W_{int,i} = \int_{V_i} \rho \ddot{\mathbf{r}}_i^T \mathrm{d}V_i \delta \mathbf{r}_i, = \int_{V_i} \rho \ddot{\mathbf{r}}_i^T \frac{\partial \mathbf{r}_i}{\partial \mathbf{z}} \mathrm{d}V_i \delta \mathbf{z} = \mathbf{Q}_{int,i}^T \delta \mathbf{z}, \tag{40}$$

where ρ is the body density, V_i is the volume of the body i, $\delta \mathbf{r}_i$ is the virtual displacement vector of the arbitrary point on a planar rigid body, and $\ddot{\mathbf{r}}_i$ is the second time-derivatives of $\mathbf{r}_i$. Both are written as [4]:

$$\begin{aligned} &\delta \mathbf{r}_i = \mathbf{L}_i \delta \mathbf{q}_i, \quad \text{with} \quad \mathbf{L}_i = \begin{bmatrix} \mathbf{I}_2 & \tilde{\mathbf{I}}_2 \mathbf{u}_i \end{bmatrix}, \\ &\ddot{\mathbf{r}}_i = \mathbf{L}_i \ddot{\mathbf{q}}_i + \dot{\mathbf{L}}_i \dot{\mathbf{q}}_i, \quad \text{with} \quad \dot{\mathbf{L}}_i = \begin{bmatrix} \mathbf{0}_2 & -\dot{\theta}_i \mathbf{u}_i \end{bmatrix}, \end{aligned} \tag{41}$$

where $\mathbf{u}_i$ is the position vector of an arbitrary point with respect to the body frame.

6.1. Semi-recursive I: Equations of motion for the open loops

According to Eqs. (21), (22), and (41), the components $\frac{\partial \mathbf{r}_i}{\partial \mathbf{z}}$ and $\ddot{\mathbf{r}}_i$ from Eq. (40) can be expressed with the kinematic relationship from semi-recursive I.

$$\begin{aligned} &\frac{\partial \mathbf{r}_i}{\partial \mathbf{z}} = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_i} \frac{\partial \mathbf{q}_i}{\partial \mathbf{z}} = \mathbf{L}_i \mathbf{V}_i \\ &\ddot{\mathbf{r}}_i = \mathbf{L}_i (\mathbf{V}_i \ddot{\mathbf{z}} + \dot{\mathbf{V}}_i \dot{\mathbf{z}}) + \dot{\mathbf{L}}_i \mathbf{V}_i \dot{\mathbf{z}}. \end{aligned} \tag{42}$$

Finally, substituting $\ddot{\mathbf{r}}_i$ from Eq. (42) into Eq. (40), the generalized inertial forces $\mathbf{Q}_{int,i}$ (associated with joint coordinates for semi-recursive I) can be expressed as:

$$\mathbf{Q}_{int,i} = \int_{V_i} \rho \mathbf{V}_i^T \mathbf{L}_i^T \left(\mathbf{L}_i (\mathbf{V}_i \ddot{\mathbf{z}} + \dot{\mathbf{V}}_i \dot{\mathbf{z}}) + \dot{\mathbf{L}}_i \mathbf{V}_i \dot{\mathbf{z}} \right) \mathrm{d}V_i = \overline{\mathbf{M}}_i \ddot{\mathbf{z}} - \overline{\mathbf{Q}}_{v,i}, \tag{43}$$

where the mass matrix $\overline{\mathbf{M}}_i$ and quadratic velocity vector $\overline{\mathbf{Q}}_{v,i}$ associated with joint coordinates for semi-recursive I are expressed as:

$$\overline{\mathbf{M}}_i = \mathbf{V}_i^T \mathbf{M}_i \mathbf{V}_i, \quad \overline{\mathbf{Q}}_{v,i} = \mathbf{V}_i^T (\mathbf{Q}_{v,i} - \mathbf{M}_i \dot{\mathbf{V}}_i \dot{\mathbf{z}}), \tag{44}$$

where the mass matrix is $\mathbf{M}_i = \int_{V_i} \rho \mathbf{L}_i^T \mathbf{L}_i \mathrm{d}V_i$ and the quadratic velocity vector is $\mathbf{Q}_{v,i} = \int_{V_i} \rho \mathbf{L}_i^T \dot{\mathbf{L}}_i \dot{\mathbf{q}}_i \mathrm{d}V_i$.

Similarly, the external force vector $\overline{\mathbf{Q}}_{e,i}$ associated with joint coordinates for semi-recursive I can be expressed as follows.

$$\overline{\mathbf{Q}}_{e,i} = \mathbf{V}_i^T \mathbf{Q}_{e,i}, \tag{45}$$

where $\mathbf{Q}_{e,i}$ includes all generalized external forces and is associated with reference point coordinates.

The virtual work of all forces and torques in a multibody system, including the applied and inertia forces from Eqs. (43) and (45), equals to zero.

$$\sum_{i=1}^{n_b} \delta \dot{\mathbf{z}}^T (\overline{\mathbf{M}}_i \ddot{\mathbf{z}} + \mathbf{C}_{\mathbf{z}}^T \lambda - \overline{\mathbf{Q}}_{v,i} - \overline{\mathbf{Q}}_{e,i}) = \mathbf{0}, \tag{46}$$

where $-\mathbf{C}_{\mathbf{z}}^T \lambda$ is the vector of the system generalized constraint forces of the open loop associated with joint coordinates. It can be written in terms of open loop constraint Jacobian matrix $\mathbf{C}_{\mathbf{z}}$ with respect to joint coordinates and the vector of Lagrange multipliers λ. The Jacobian matrix $\mathbf{C}_{\mathbf{z}}$ is expressed with using the chain rule:

$$\mathbf{C}_{\mathbf{z}} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}_i} \frac{\partial \mathbf{q}_i}{\partial \mathbf{z}} = \mathbf{C}_{\mathbf{q}_i} \mathbf{V}_i, \tag{47}$$

where $\mathbf{C}$ is the open loop constraint equations. The velocity transformation matrix $\mathbf{V}_i$ is the null space of Jacobian matrix $\mathbf{C}_{\mathbf{q}_i}$, and therefore the generalized constraint force $\mathbf{C}_{\mathbf{z}}^T \lambda$ from Eq. (46) is eliminated.

Considering that the virtual relative velocities are independent, the resulting differential equation for this planar open loop multibody system from Eq. (46) is:

$$\overline{\mathbf{M}} \ddot{\mathbf{z}} - \overline{\mathbf{Q}}_v - \overline{\mathbf{Q}}_e = \mathbf{0}, \tag{48}$$

where $\overline{\mathbf{M}}$ is the system mass matrix, $\overline{\mathbf{Q}}_v$ and $\overline{\mathbf{Q}}_e$ are system quadratic velocity and external force vectors that contain the matrices and vectors of individual $\overline{\mathbf{M}}_i$, $\overline{\mathbf{Q}}_{v,i}$ and $\overline{\mathbf{Q}}_{e,i}$.

6.2. Semi-recursive II: equations of motion for open loop

According to Eqs. (28), (35), and (41), the components $\frac{\partial \mathbf{r}_i}{\partial \mathbf{z}}$ and $\ddot{\mathbf{r}}_i$ from Eq. (40) can be expressed with the kinematic relationship from semi-recursive *II*.

$$\begin{aligned} \frac{\partial \mathbf{r}_i}{\partial \mathbf{z}} &= \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_i}\frac{\partial \mathbf{q}_i}{\partial \mathbf{Z}_i}\frac{\partial \mathbf{Z}_i}{\partial \mathbf{z}} = \mathbf{L}_i\mathbf{D}_i\mathbf{T}_i\mathbf{R}_d \\ \ddot{\mathbf{r}}_i &= \mathbf{L}_i\mathbf{D}_i\mathbf{T}_i(\mathbf{R}_d\ddot{\mathbf{z}} + \mathbf{d}) + \mathbf{L}_i\mathbf{e}_i + \dot{\mathbf{L}}_i\mathbf{D}_i\mathbf{T}_i\mathbf{R}_d\dot{\mathbf{z}} \end{aligned} \tag{49}$$

Similarly, substituting $\ddot{\mathbf{r}}_i$ from Eq. (49) into Eq. (40), the generalized inertial forces $\mathbf{Q}_{int,i}$ associated with joint coordinates for semi-recursive *II* can be expressed as follows.

$$\mathbf{Q}_{int,i} = \int_{V_i} \rho\mathbf{R}_d^T\mathbf{T}_i^T\mathbf{D}_i^T\mathbf{L}_i^T\left(\mathbf{L}_i\mathbf{D}_i\mathbf{T}_i(\mathbf{R}_d\ddot{\mathbf{z}} + \mathbf{d}) + \mathbf{L}_i\mathbf{e}_i + \dot{\mathbf{L}}_i\mathbf{D}_i\mathbf{T}_i\mathbf{R}_d\dot{\mathbf{z}}\right)\mathrm{d}V_i = \overline{\overline{\mathbf{M}}}_i\ddot{\mathbf{z}} - \overline{\overline{\mathbf{Q}}}_{v,i}, \tag{50}$$

where the mass matrix $\overline{\overline{\mathbf{M}}}_i$ and quadratic velocity vector $\overline{\overline{\mathbf{Q}}}_{v,i}$ associated with joint coordinates for semi-recursive *II* are expressed as:

$$\overline{\overline{\mathbf{M}}}_i = \mathbf{R}_d^T\mathbf{T}_i^T\mathbf{D}_i^T\mathbf{M}_i\mathbf{D}_i\mathbf{T}_i\mathbf{R}_d, \quad \overline{\overline{\mathbf{Q}}}_{v,i} = \mathbf{R}_d^T\mathbf{T}_i^T\mathbf{D}_i^T(\mathbf{Q}_{v,i} - \mathbf{M}_i\mathbf{e}_i - \mathbf{M}_i\mathbf{D}_i\mathbf{T}_i\mathbf{d}). \tag{51}$$

Similarly, the external force vector $\overline{\overline{\mathbf{Q}}}_{e,i}$ associated with joint coordinates for semi-recursive *II* can be expressed as:

$$\overline{\overline{\mathbf{Q}}}_{e,i} = \mathbf{R}_d^T\mathbf{T}_i^T\mathbf{D}_i^T\mathbf{Q}_{e,i}. \tag{52}$$

The virtual work of all forces and torques in a multibody system including the applied and inertia forces from Eqs. (50) and (52), equals zero.

$$\sum_{i=1}^{n_b}\delta\dot{\mathbf{z}}^T(\overline{\overline{\mathbf{M}}}_i\ddot{\mathbf{z}} + \mathbf{C}_{\mathbf{z}}^T\lambda - \overline{\overline{\mathbf{Q}}}_{v,i} - \overline{\overline{\mathbf{Q}}}_{e,i}) = \mathbf{0}, \tag{53}$$

The Jacobian matrix $\mathbf{C}_{\mathbf{z}}$ is expressed with using the chain rule.

$$\mathbf{C}_{\mathbf{z}} = \frac{\partial \mathbf{C}}{\partial \mathbf{q}_i}\frac{\partial \mathbf{q}_i}{\partial \mathbf{Z}_i}\frac{\partial \mathbf{Z}_i}{\partial \mathbf{z}} = \mathbf{C}_{\mathbf{q}_i}\mathbf{D}_i\mathbf{T}_i\mathbf{R}_d, \tag{54}$$

where it is noted that the term $\mathbf{D}_i\mathbf{T}_i\mathbf{R}_d$ is the null space of Jacobian matrix $\mathbf{C}_{\mathbf{q}_i}$, and therefore the generalized constraint force $\mathbf{C}_{\mathbf{z}}^T\lambda$ from Eq. (53) is eliminated.

Again, assuming that the relative virtual velocities $\delta\dot{\mathbf{z}}$ are independent, the resulting differential equation for this planar open loop multibody system from Eq. (53) is:

$$\overline{\overline{\mathbf{M}}}\ddot{\mathbf{z}} - \overline{\overline{\mathbf{Q}}}_v - \overline{\overline{\mathbf{Q}}}_e = \mathbf{0}, \tag{55}$$

where $\overline{\overline{\mathbf{M}}}$ is the system mass matrix and $\overline{\overline{\mathbf{Q}}}_v$ and $\overline{\overline{\mathbf{Q}}}_e$ are system quadratic velocity and external force vectors that collect the matrices and vectors of individual $\overline{\overline{\mathbf{M}}}_i$, $\overline{\overline{\mathbf{Q}}}_{v,i}$ and $\overline{\overline{\mathbf{Q}}}_{e,i}$.

Both approaches are derived based on the virtual work from Eq. (40). They theoretically lead to exactly the same equations with the same system matrix. Compare Eqs. (48) with (55).

7. Equations of motion for closed loop with multiple bodies

The dynamics of a closed loop system can be analyzed by adding constraint equations to enforce the closure of the open loops. Take the example in Fig. 8. Fig. 8(b) depicts a tree structure open loop that consists of five rigid bodies constrained by different mechanical joints. To form the closed loop in Fig. 8(a), the open loop from Fig. 8(b) can be closed by enforcing one revolute constraint between body 3 and ground and one prismatic constraint between body 5 and ground. Those constraints are called closure of the open-loop constraints.

The position vector of point D from Fig. 8(a) remains the same for both body 3 and the ground while constrained by the revolute joint. The constraint equation associated with point D can be written as follows.

$$\overline{\mathbf{C}}^r(\mathbf{z}) = \mathbf{r}_3^D - \mathbf{r}^D = \mathbf{0}, \tag{56}$$

where $\mathbf{r}_3^D$ is the position vector of point D from body 3 and $\mathbf{r}^D$ is the position vector of point D from the ground.

In Fig. 8(a), one constraint equation will eliminate the relative rotation, and another will eliminate the relative translation between body 5 and ground E. The kinematic constraint conditions of the prismatic joint can be written as:

$$\overline{\mathbf{C}}^p(\mathbf{z}) = \begin{bmatrix} \theta_5 - \theta_g - \theta_c \\ (\tilde{\mathbf{I}}_2\mathbf{h}_g)^T(\mathbf{r}^E - \mathbf{r}_5^F) \end{bmatrix} = \mathbf{0}, \tag{57}$$

where θ_5 is the rotation angle of body 5, θ_g is the rotation angle of the ground, $\mathbf{h}_g$ is the unit vector of the translational direction along the ground, $\mathbf{r}^E$ is the position vector of point E and $\mathbf{r}_5^F$ is the position vector of point F on body 5.

The position, velocity and acceleration vectors of the closure of the open-loop constraint are subsequently written in the following general form.

$$\begin{aligned}&\overline{\mathbf{C}}(\mathbf{z},t)=\mathbf{0},\\&\dot{\overline{\mathbf{C}}}=\overline{\mathbf{C}}_{\mathbf{Z}}\dot{\mathbf{z}}+\overline{\mathbf{C}}_t=\mathbf{0},\\&\ddot{\overline{\mathbf{C}}}=\overline{\mathbf{C}}_{\mathbf{Z}}\ddot{\mathbf{z}}+\dot{\overline{\mathbf{C}}}_{\mathbf{Z}}\dot{\mathbf{z}}+\dot{\overline{\mathbf{C}}}_t=\mathbf{0},\end{aligned} \tag{58}$$

where $\overline{\mathbf{C}}_{\mathbf{Z}}$ is the closure of the open-loop constraint Jacobian matrix, $\overline{\mathbf{C}}_t$ is the partial derivative of the closure of the open-loop constraint equations with respect to time, $\dot{\overline{\mathbf{C}}}_{\mathbf{Z}}$ is the time-derivative of $\overline{\mathbf{C}}_{\mathbf{Z}}$, and $\dot{\overline{\mathbf{C}}}_t$ is the time-derivative of $\overline{\mathbf{C}}_t$.

7.1. Semi-recursive I: Equation of motion for closed loops

In the closed loop system, the components of the vector of joint coordinates $\mathbf{z}$ cannot be independent due to the closure of the open-loop constraints. As for the closed loop system, reaction forces that correspond to closure of the open-loop constraints can be accounted for by incorporating Lagrange multipliers λ into the equations of motion of the open loops from Eq. (48). The constraint equations at the position level from (58) can be included into the equations of motion to ensure virtual displacements do not violate the constraints. The set of equations of motion from Eq. (48) for an open loop is a system of ordinary differential equations (ODE), and the constraint equations from Eq. (58) are a set of algebraic equations. Combining both sets, a system of differential algebraic equations (DAE) can be generated as follows.

$$\begin{aligned}&\overline{\mathbf{M}}\ddot{\mathbf{z}}+\overline{\mathbf{C}}_{\mathbf{Z}}^T\lambda=\overline{\mathbf{Q}}_v+\overline{\mathbf{Q}}_e,\\&\overline{\mathbf{C}}(\mathbf{z},t)=\mathbf{0}.\end{aligned} \tag{59}$$

The second-order equations from Eq. (59) might be unstable during numerical time integration due to the amplified numerical errors introduced by the integration procedure [38]. Therefore, Baumgarte constraint stabilization [40] is used here to ensure that closure of the open-loop constraint equations fulfilled. When using Baumgarte constraint stabilization method, the right hand of Eq. (58) can be rewritten as:

$$\overline{\mathbf{C}}_{\mathbf{Z}}\ddot{\mathbf{z}}-\mathbf{Q}_c+2a\dot{\overline{\mathbf{C}}}+b^2\overline{\mathbf{C}}=\mathbf{0}, \tag{60}$$

where $\mathbf{Q}_c=-\dot{\overline{\mathbf{C}}}_{\mathbf{Z}}\dot{\mathbf{z}}-\dot{\overline{\mathbf{C}}}_t$, parameter b is the spring constant and a is the damping coefficient [3] that weight the velocity and position constraint violations.

By combining Eq. (60) with Eq. (59), the equations of motion incorporating Baumgarte constraint stabilization for semi-recursive I can be written as follows.

$$\begin{bmatrix}\overline{\mathbf{M}} & \overline{\mathbf{C}}_{\mathbf{Z}}^T\\ \overline{\mathbf{C}}_{\mathbf{Z}} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{z}}\\ \lambda\end{bmatrix}=\begin{bmatrix}\overline{\mathbf{Q}}_v+\overline{\mathbf{Q}}_e\\ \mathbf{Q}_c-2a\dot{\overline{\mathbf{C}}}-b^2\overline{\mathbf{C}}\end{bmatrix}, \tag{61}$$

7.2. Semi-recursive II: equation of motion for closed loops

Similarly, reaction forces that correspond to closure of the open-loop constraints from Eq. (58) can be accounted for by employing Lagrange multipliers λ into the equations of motion of the open loops from Eq. (55) for a closed loop system as follows.

$$\begin{aligned}&\overline{\overline{\mathbf{M}}}+\overline{\mathbf{C}}_{\mathbf{Z}}^T\lambda=\overline{\overline{\mathbf{Q}}}_e+\overline{\overline{\mathbf{Q}}}_v,\\&\overline{\mathbf{C}}(\mathbf{z},t)=\mathbf{0}.\end{aligned} \tag{62}$$

Similarly, by combining Eq. (60) with Eq. (62), the equations of motion incorporating Baumgarte constraint stabilization for semi-recursive II can be written:

$$\begin{bmatrix}\overline{\overline{\mathbf{M}}} & \overline{\mathbf{C}}_{\mathbf{Z}}^T\\ \overline{\mathbf{C}}_{\mathbf{Z}} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{z}}\\ \lambda\end{bmatrix}=\begin{bmatrix}\overline{\overline{\mathbf{Q}}}_v+\overline{\overline{\mathbf{Q}}}_e\\ \mathbf{Q}_c-2a\dot{\overline{\mathbf{C}}}-b^2\overline{\mathbf{C}}\end{bmatrix}. \tag{63}$$

8. Conclusion

With the help of the constraint kinematics of the open loop, the multibody system can be described with a reduced number of dynamic equations by using semi-recursive approaches. Therefore, this method is also referred to as the coordinate reduction method. The semi-recursive formulations described in this work are not new, but entry level explanations that not currently available in the literature are provided here. Therefore, this theoretical derivation, which relates the semi-recursive approach to the often-used global formulation using simple planar cases, is novel.

The pros and cons of two coordinate reduction methods, *i.e.,* coordinate partitioning and semi-recursive *I*, are introduced and compared. The semi recursive method is superior in terms of the selection of independent coordinates and more sparse matrices. Two semi-recursive methods are introduced using a planar multibody mechanism. The difference between both methods is the location of the reference point that is usually used to compute the kinematics of the body. One is rigidly attached to the moving body, and the other coincides with the origin of the global frame. As all bodies share the same reference point with using the semi-recursive *II*, the topology of the spanning tree are simpler than those obtained with using semi-recursive *I*.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Derivation of Jacobian matrix and velocity transformation matrix for open loop system in Fig. 1

The time derivative of constraint equations in Eq. (2) is given by:

$$\begin{aligned}
&\dot{\mathbf{C}}(1,1) = \dot{\mathbf{R}}_1 + \dot{\theta}_1 \tilde{\mathbf{I}}_2 \mathbf{u}^P = \mathbf{0},\\
&\dot{\mathbf{C}}(2,1) = -\dot{\theta}_1 + \dot{\theta}_2 = 0,\\
&\dot{\mathbf{C}}(3,1) = \dot{\theta}_1 \mathbf{h}_1^T \tilde{\mathbf{I}}_2 \tilde{\mathbf{I}}_2 (\mathbf{R}_1 + \mathbf{u}^Q - \mathbf{R}_2 - \mathbf{u}^S) - \mathbf{h}_1^T \tilde{\mathbf{I}}_2 (\dot{\mathbf{R}}_1 + \dot{\theta}_1 \tilde{\mathbf{I}}_2 \mathbf{u}^P - \dot{\mathbf{R}}_2 - \dot{\theta}_2 \tilde{\mathbf{I}}_2 \mathbf{u}^S),\\
&\qquad\quad = -\mathbf{h}_1^T \tilde{\mathbf{I}}_2 \dot{\mathbf{R}}_1 - \dot{\theta}_1 \mathbf{h}_1^T (\mathbf{R}_1 - \mathbf{R}_2 - \mathbf{u}^S) + \mathbf{h}_1^T \tilde{\mathbf{I}}_2 \dot{\mathbf{R}}_2 - \dot{\theta}_2 \mathbf{h}_1^T \mathbf{u}^S = 0,
\end{aligned} \tag{A.1}$$

where $\mathbf{u}^P = \mathbf{A}_1 \overline{\mathbf{u}}_1^P$, $\mathbf{A}_1$ is the rotation matrix of body 1, $\overline{\mathbf{u}}_1^P$ is the local representation of $\mathbf{u}^P$, $\dot{\mathbf{u}}^P = \dot{\mathbf{A}}_1 \overline{\mathbf{u}}_1^P = \dot{\theta}_1 \tilde{\mathbf{I}}_2 \mathbf{u}^P$, $\tilde{\mathbf{I}}_2^T = -\tilde{\mathbf{I}}_2$, and $\tilde{\mathbf{I}}_2 \tilde{\mathbf{I}}_2 = -\mathbf{I}_2$. Thus the Jacobian matrix can be organized as Eq. (5).

The time derivative of Eq. (10) is given by:

$$\dot{\mathbf{q}} = \begin{bmatrix} -\dot{z}_1 \tilde{\mathbf{I}}_2 \mathbf{u}^P \\ \dot{z}_1 \\ -\dot{z}_1 \tilde{\mathbf{I}}_2 \mathbf{u}^P + \dot{z}_1 \tilde{\mathbf{I}}_2 \mathbf{u}^Q + \dot{z}_2 \mathbf{h}_1 + z_2 \dot{z}_1 \tilde{\mathbf{I}}_2 \mathbf{h}_1 - \dot{z}_1 \tilde{\mathbf{I}}_2 \mathbf{u}^S \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} \dot{z}_1 \tilde{\mathbf{I}}_2 (\mathbf{R}_1 - \mathbf{r}^P) \\ \dot{z}_1 \\ \dot{z}_1 \tilde{\mathbf{I}}_2 (\mathbf{R}_2 - \mathbf{r}^P) + \dot{z}_2 \mathbf{h}_1 \\ \dot{z}_1 \end{bmatrix}. \tag{A.2}$$

Appendix B. Velocity transformation matrix for more complex mechanical joints

B.1. Point to line constraint

Fig. B.9 depicts a system where a rigid body is constrained such that one end of beam-like body can slide along a line that makes an angle of θ_c with respect to the global frame. Using the relative coordinates approach, this constraint can be expressed using a combination of a prismatic joint and a revolute joint. This system has a total of two degrees of freedom, which leads to a set of relative coordinates:

$$\mathbf{z} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^T, \tag{B.1}$$

where z_1 is the relative coordinate associated with the prismatic joint and z_2 is the relative coordinate associated with the revolute joint.

In the case of the system shown in Fig. B.9, the reference point coordinates of body 1 are described with respect to the relative coordinates as follows:

$$\mathbf{q}_1 = \begin{bmatrix} \mathbf{r}_1 + z_1 \mathbf{h}_c + \mathbf{a}_1 \\ \theta_c + z_2 \end{bmatrix}, \quad \text{with} \quad \mathbf{a}_1 = \mathbf{R}_1 - \mathbf{r}^P, \tag{B.2}$$

where $\mathbf{r}_1$ is the position vector of the starting point of prismatic joint, $\mathbf{a}_1$ is the translational vector that goes from the revolute joint point P to the origin of body frame o_1, $\mathbf{r}^P$ is the position vector of point P, and $\mathbf{R}_1$ is the position vector of the origin of body frame o_1.

The velocity transformation for the body 1 can be obtained by taking the partial derivative of $\mathbf{q}_1$ from Eq. (B.2) with respect to **z**.

$$\mathbf{V}_1 = \begin{bmatrix} \mathbf{h}_c & \tilde{\mathbf{I}}_2 (\mathbf{R}_1 - \mathbf{r}^P) \\ 0 & 1 \end{bmatrix}. \tag{B.3}$$

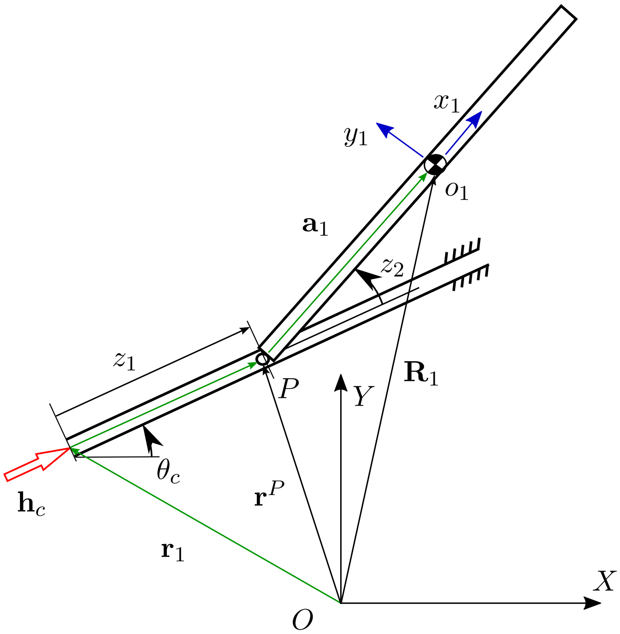

Fig. B.9. Combination of revolute joint and prismatic joint.

B.2. Cam-follower constraint

Fig. B.10(a) shows a cam follower system with the cam constrained by one revolute joint to the ground. The cam and follower are always assumed to be in contact at point P. There are three degrees of freedom, which leads to a set of relative coordinates.

$$\mathbf{z} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T, \tag{B.4}$$

where z_1 is the polar coordinate for the cam in Fig. B.10(b), z_2 is the relative angle that is associated with the revolute joint and z_3 is associated with the rotational angle of body 2. See Fig. B.10(a).

As shown in Fig. B.10(b), the outline of the cam can be described in polar coordinates. The rotational angle z_1 increases counterclockwise from 0 to 2π and $s_1 = s_1(z_1)$ is the corresponding radii [41]. Accordingly, the position vector of contact point P with respect to the revolute joint J_1 in the local frame is not a constant vector, which is the function of the angle z_1.

$$\bar{\mathbf{a}}_1 = \bar{\mathbf{a}}_1\left(z_1, s_1(z_1)\right). \tag{B.5}$$

In the case of the system shown in Fig. B.10(a), the reference point coordinates of body 2 are described with respect to the relative coordinates as follows:

$$\mathbf{q}_2 = \begin{bmatrix} \mathbf{r}_1 + \mathbf{a}_1 + \mathbf{b}_2 \\ z_3 \end{bmatrix}, \quad \text{with} \quad \mathbf{a}_1 = \mathbf{r}^P - \mathbf{r}_1, \quad \mathbf{b}_2 = \mathbf{R}_2 - \mathbf{r}^P, \tag{B.6}$$

where $\mathbf{r}_1$ is the position vector of the revolute joint J_1, $\mathbf{a}_1$ is the translational vector that goes from the revolute joint point J_1 to contact point P, $\mathbf{b}_2$ is the translational vector that goes from the contact point P to the origin of body frame o_2, $\mathbf{r}^P$ is the position vector of point P, and $\mathbf{R}_2$ is the position vector of the origin of body frame o_2.

The velocity transformation for the body 2 can be obtained by taking the partial derivative of $\mathbf{q}_2$ from Eq. (B.6) with respect to $\mathbf{z}$.

$$\mathbf{V}_2 = \begin{bmatrix} \dfrac{\partial \mathbf{a}_1}{\partial z_1} & \tilde{\mathbf{I}}_2(\mathbf{r}^P - \mathbf{r}_1) & \tilde{\mathbf{I}}_2(\mathbf{R}_2 - \mathbf{r}^P) \\ 0 & 0 & 1 \end{bmatrix}. \tag{B.7}$$

Appendix C. Dimensional reduction through velocity transformations

According to Eq. (17), the reference point velocities for body $i-1$ can be expressed explicitly as:

$$\dot{\mathbf{q}}_{i-1} = \mathbf{B}_{i-1}^{i-2}\dot{\mathbf{q}}_{i-2} + \mathbf{V}_{i-1}^{j-1}\dot{z}_{j-1}, \tag{C.1}$$

where $\dot{\mathbf{q}}_{i-2}$ is the reference point velocities for body $i-2$, the matrix $\mathbf{B}_{i-1}^{i-2}$ is the transformation matrix associated with bodies $i-2$ and $i-1$, and the vector $\mathbf{V}_{i-1}^{j-1}$ is the component of velocity transformation matrix $\mathbf{V}$ associated with joint J_{j-1} for body $i-1$.

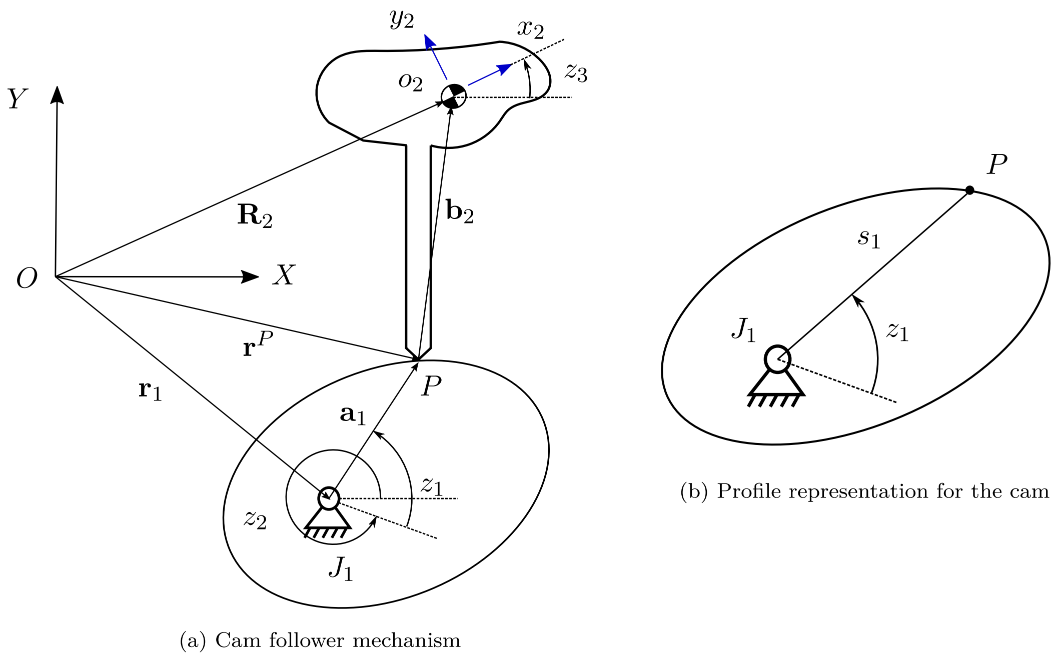

Fig. B.10. Kinematics of cam follower mechanism.

Substituting Eq. (C.1) into Eq. (17), the reference point velocity $\dot{\mathbf{q}}_{i-1}$ will be eliminated from the equation as:

$$\dot{\mathbf{q}}_i = \mathbf{B}_i^{i-2}\dot{\mathbf{q}}_{i-2} + \mathbf{V}_i^{j-1}\dot{z}_{j-1} + \mathbf{V}_i^{j}\dot{z}_j, \tag{C.2}$$

where the matrix $\mathbf{B}_i^{i-2}$ is the transformation matrix associated with bodies $i-2$ and i. It is given by:

$$\mathbf{B}_i^{i-2} = \mathbf{B}_i^{i-1}\mathbf{B}_{i-1}^{i-2} = \begin{bmatrix} \mathbf{I}_2 & \tilde{\mathbf{I}}_2(\mathbf{R}_i - \mathbf{R}_{i-2}) \\ \mathbf{0} & 1 \end{bmatrix}, \tag{C.3}$$

where $\mathbf{R}_{i-2}$ is the position vector of the origin of the body frame $i-2$, and the vector $\mathbf{V}_i^{j-1}$ is the component of velocity transformation matrix $\mathbf{V}$ that represents the unit direction of body i given by revolute or prismatic joints J^{j-1}.

Vector $\mathbf{V}_i^{j-1}$ for both joint take the form:

$$\begin{aligned} &\text{Revolute joint:} \quad \mathbf{V}_i^{j-1} = \mathbf{B}_i^{i-1}\mathbf{V}_{i-1}^{j-1} \overset{(18)}{=} \begin{bmatrix} \tilde{\mathbf{I}}_2(\mathbf{R}_i - \mathbf{r}_{j-1}) \\ 1 \end{bmatrix}, \\ &\text{Prismatic joint:} \quad \mathbf{V}_i^{j-1} = \mathbf{B}_i^{i-1}\mathbf{V}_{i-1}^{j-1} \overset{(20)}{=} \begin{bmatrix} \mathbf{h}_{i-2} \\ 0 \end{bmatrix}, \end{aligned} \tag{C.4}$$

where $\mathbf{r}_{j-1}$ is the position vector of revolute joint J_{j-1}, and $\mathbf{h}_{i-2}$ is the unit vector of the prismatic joint J_{j-1}.

Appendix D. Semi-recursive II: a mathematical method to obtain the projection matrix

To project the intermediate body reference velocities $\dot{\mathbf{Z}}_i$ onto the relative velocities $\dot{\mathbf{z}}$, one can easily substitute $\dot{\mathbf{q}}_i$ and $\dot{\mathbf{q}}_{i-1}$ from Eq. (28) into the recursive relation of Eq. (17) as:

$$\mathbf{D}_i\dot{\mathbf{Z}}_i = \mathbf{B}_i^{i-1}(\mathbf{D}_{i-1}\dot{\mathbf{Z}}_{i-1}) + \mathbf{V}_i^{j}\dot{z}_j. \tag{D.1}$$

Since matrices $\mathbf{D}_i$ and $\mathbf{B}_i$ have following expressions.

$$\mathbf{D}_i^{-1} = \begin{bmatrix} \mathbf{I}_2 & -\tilde{\mathbf{I}}_2\mathbf{R}_i \\ \mathbf{0} & 1 \end{bmatrix}, \quad \mathbf{D}_i^{-1}\mathbf{B}_i^{i-1}\mathbf{D}_{i-1} = \mathbf{I}_3. \tag{D.2}$$

Multiplying $\mathbf{D}_i^{-1}$ with Eq. (D.1) leads to:

$$\dot{\mathbf{Z}}_i = \dot{\mathbf{Z}}_{i-1} + \mathbf{b}_j\dot{z}_j, \tag{D.3}$$

where the vector of $\mathbf{b}_j$ appearing in Eqs. (D.3) is associated with different types of mechanical joints J_j, as:

$$\begin{aligned} &\text{Revolute joint:} \quad \mathbf{b}_j = \mathbf{D}_i^{-1}\mathbf{V}_i^j \overset{(18)}{=} \begin{bmatrix} -\tilde{\mathbf{I}}_2\mathbf{r}_j \\ 1 \end{bmatrix} \in \mathbb{R}^{3\times1}, \\ &\text{Prismatic joint:} \quad \mathbf{b}_j = \mathbf{D}_i^{-1}\mathbf{V}_i^j \overset{(20)}{=} \begin{bmatrix} \mathbf{h}_{i-1} \\ 0 \end{bmatrix} \in \mathbb{R}^{3\times1}, \end{aligned} \tag{D.4}$$

where above expression is same as Eqs. (31) and (34).

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