CS687 2023 HW2

Due: April 20, 2025

1. Suppose you are given an arbitrary binary sequence where it is guaranteed that there is at least one 1 in every block of 50,000 positions starting from the first position. Do either (not both) of the following. **[10]**

   (a) Construct a Martin-Löf test which captures this set of sequences (*i.e.* the set of all such sequences which have at least one 1 in every block of 50,000 positions).

   (b) Define a martingale that succeeds on the set of these sequences. Show that the martingale succeeds.

2. A probability measure μ on Σ^* is defined by the following rules.

   1. $\mu(\lambda) = 1$ and
   2. For every string w, $\mu(w) = \mu(w0) + \mu(w1)$.

   If μ and ν are probabilities on Σ^*, such that $\nu(w) \neq 0$ for any string w, then show that μ/ν is a ν-martingale.

   A ν martingale is a function $m : \Sigma^* \to [0, \infty)$ such that $m(\lambda) \leq 1$ and for any string $w \in \Sigma^*$, we have

   $$m(w0)\nu(w0) + m(w1)\nu(w1) = m(w)\nu(w).$$

   [10]

3. Prove or disprove: if $d : \Sigma^* \to [0, \infty)$ is a martingale which does not assign 0 to any string, then d is the ratio of two probabilities on strings. **[10]**

4. Show that if there is a lower semicomputable martingale $m : \Sigma^* \to [0, \infty)$ such that it succeeds on $X \in \Sigma^\infty$- *i.e.*,

   $$\limsup_{n\to\infty} m(X[0 \dots n-1]) = \infty,$$

   then there is another lower semicomputable martingale $m' : \Sigma^* \to \infty$ such that

   $$\liminf_{n\to\infty} m(X[0 \dots (n-1)] = \infty.$$

   [10]