CS687 HW2 : Solutions

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1 Suppose you are given an arbitrary binary sequence where it is guaranteed that there is at least one 1 in every block of 50,000 positions starting from the first position. Do either (not both) of the following.

(a) Construct a Martin-L¨of test which captures this set of sequences (i.e. the set of all such sequences which have at least one 1 in every block of 50,000 positions).

(b) Define a martingale that succeeds in the set of these sequences. Show that the martingale succeeds.

(b) Solution:

Understanding the Target Set

Let $S \subseteq \{0,1\}^\infty$ denote the set of sequences where every block of $B = 50{,}000$ bits contains at least one 1. This set is measure zero (i.e., a null set), because in a uniformly random binary sequence, blocks of all zeros almost surely occur eventually. Thus, sequences in S are highly atypical.

Since S is a null set, it is theoretically possible for a computable martingale to succeed on every sequence in S.

Martingale Definition

A function $d : \Sigma^* \to [0, \infty)$ is a martingale (with respect to the uniform measure) if for every $w \in \Sigma^*$:

$$d(w) = \frac{1}{2}d(w0) + \frac{1}{2}d(w1).$$

A martingale succeeds on a sequence $X \in \{0, 1\}^\infty$ if:

$$\limsup_{n\to\infty} d(X[0 \dots n-1]) = \infty.$$

We now construct such a martingale tailored to the set S.

Strategy Overview

We define a bit-level martingale that places small bets within each 50,000-bit block, betting that at least one 1 will occur in the block. Specifically:

- 0 The martingale bets a small fraction of its capital on each bit being 1 until the first 1 occurs in the current block,
- Once a 1 has been seen in a block, the martingale stops betting for the rest of the block,
- This strategy is repeated block by block, resetting at the start of each new 50,000-bit segment.

This approach ensures the martingale:

- Fulfills the martingale condition at each bit,
- Gains multiplicative capital over each block (since a 1 is guaranteed),
- Grows unboundedly on sequences in S, which ensures success.

Formal Construction of the Martingale

Let $B = 50{,}000$ and fix a small parameter $\delta \in (0, 1)$, e.g., $\delta = \frac{1}{B}$. Define the martingale $d : \Sigma^* \to [0, \infty)$ as follows:

- Initialize: $d(\lambda) := 1$
- Let $w \in \Sigma^*$, and suppose $wb \in \Sigma^*$ is formed by appending one bit.

- Let $i := |w| \bmod B$, i.e., the position of the bit within its aligned block.
- Let $z := w[|w| - i \ldots |w| - 1]$ be the suffix of the current block.
- Let $\#1(z)$ denote the number of 1s in that suffix.

We define the capital update rule as:

$$d(wb) := \begin{cases} (1-\delta)d(w) + \delta \cdot 2d(w) & \text{if } \#1(z) = 0 \text{ and } b = 1 \\ (1-\delta)d(w) & \text{if } \#1(z) = 0 \text{ and } b = 0 \\ d(w) & \text{if } \#1(z) \geq 1 \end{cases}$$

Verification of the Martingale Property

We verify that d satisfies:

$$d(w) = \frac{1}{2}d(w0) + \frac{1}{2}d(w1)$$

Case 1: If no 1 has been seen yet in the current block:

$$\begin{aligned} d(w0) &= (1-\delta)d(w) \\ d(w1) &= (1-\delta)d(w) + \delta \cdot 2d(w) = (1+\delta)d(w) \\ \Rightarrow \quad \frac{1}{2}(d(w0) + d(w1)) &= \frac{1}{2}[(1-\delta) + (1+\delta)]d(w) = d(w) \end{aligned}$$

Case 2: If a 1 has already appeared in the block, then $d(w0) = d(w1) = d(w)$, so the property holds trivially.

Hence, d is a valid martingale.

Proof of Success on the Target Set

Let $X \in S$. Then for every aligned block of 50,000 bits, there is at least one 1. Within each such block:

- The martingale places a sequence of small bets on seeing a 1,
- The first occurrence of 1 gives a multiplicative gain:

$$d \mapsto (1-\delta)d + \delta \cdot 2d = d(1+\delta)$$

- Once a 1 occurs, betting stops in the current block, and resumes in the next block.

Incorrect.

Thus, for each block, the capital is multiplied by at least $(1 + \delta)$. If $X \in S$, this happens infinitely often. Therefore:

$$d_n \geq (1 + \delta)^n \to \infty \quad \text{as } n \to \infty,$$

and:

$$\limsup_{n \to \infty} d(X[0 \ldots n - 1]) = \infty.$$

Conclusion

We have constructed a valid martingale that:

- Is defined recursively and computably,
- Satisfies the martingale fairness condition at every bit,
- Succeeds on every sequence where every aligned block of 50,000 bits contains at least one 1.

Therefore, this martingale successfully captures the target set of sequences.

2 A probability measure μ on $\sum^*$ is defined by the following rules.

1. **$\mu(\lambda) = 1$ and**

2. **For every string $w, \mu(w) = \mu(w0) + \mu(w1)$.**

If μ and ν are probabilities on $\sum^*$, such that $\nu(w) \neq 0$ for any string w, then show that μ/ν is a ν-martingale.
A ν martingale is a function $m : \sum^* \to [0, \infty)$ such that $m(\lambda) \leq 1$ and for any string $w \in \sum^*$, we have

$$m(w0)\nu(w0) + m(w1)\nu(w1) = m(w)\nu(w).$$

Solution:

Definition of a ν-Martingale

A function $m : \Sigma^* \to [0, \infty)$ is called a ν-martingale if it satisfies:

1. Initial condition: $m(\lambda) \leq 1$,

2. Recursion: For all $w \in \Sigma^*$,

$$m(w0)\nu(w0) + m(w1)\nu(w1) = m(w)\nu(w).$$

This generalizes the usual martingale definition by replacing the uniform measure $2^{-|w|}$ with any positive measure ν.

Proof of the Martingale Property

We are given:

$$m(w) = \frac{\mu(w)}{\nu(w)}, \quad \text{and} \quad \mu(w) = \mu(w0) + \mu(w1), \quad \text{with } \nu(w) \neq 0.$$

Then:

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\begin{aligned}
m(w0)\nu(w0) + m(w1)\nu(w1) &= \frac{\mu(w0)}{\nu(w0)} \cdot \nu(w0) + \frac{\mu(w1)}{\nu(w1)} \cdot \nu(w1) \\
&= \mu(w0) + \mu(w1) \\
&= \mu(w) \\
&= \frac{\mu(w)}{\nu(w)} \cdot \nu(w) = m(w)\nu(w).
\end{aligned}
$$

Therefore, the martingale recursion condition is satisfied.

Verification of Initial Condition

Since both μ and ν are probability measures, we know:

$$\mu(\lambda) = \nu(\lambda) = 1.$$

So:

$$m(\lambda) = \frac{\mu(\lambda)}{\nu(\lambda)} = \frac{1}{1} = 1 \leq 1,$$

and the initial condition also holds.

Conclusion

The function $m(w) = \mu(w)/\nu(w)$ satisfies:

- The martingale identity:

$$m(w0)\nu(w0) + m(w1)\nu(w1) = m(w)\nu(w),$$

- The initial bound: $m(\lambda) = 1 \leq 1$,

under the assumption that $\nu(w) \neq 0$ for all $w \in \Sigma^*$, and that μ, ν are both probability measures.

Therefore, m is a valid ν-martingale.

3 Prove or disprove: if $d : \Sigma^* \to [0, \infty)$ is a martingale that does not assign 0 to any string, then d is the ratio of two probabilities on strings.

Solution:

Martingale: A function $d : \Sigma^* \to [0, \infty)$ is called a martingale (with respect to the uniform measure) if:

$$d(w) = \frac{1}{2}d(w0) + \frac{1}{2}d(w1), \quad \text{for all } w \in \Sigma^*.$$

10 **Probability measure on strings:** A function $\mu : \Sigma^* \to [0, 1]$ is a *pre-measure* if:

$$\mu(w) = \mu(w0) + \mu(w1), \quad \text{and } \mu(\lambda) = 1.$$

It corresponds to a full probability measure on Σ^∞ via Carathéodory's extension theorem.

Key Goal: Determine whether every strictly positive martingale d can be written as:

$$d(w) = \frac{\mu(w)}{\nu(w)},$$

for some probability measures μ, ν on Σ^* with $\nu(w) > 0$ for all w.

Initial Attempt (Fails in General)

Let us first try to define:

$$\nu(w) := 2^{-|w|} \quad \text{(uniform measure)}, \qquad \mu(w) := d(w) \cdot \nu(w)$$

This ensures:

$$\frac{\mu(w)}{\nu(w)} = d(w)$$

Issue: This construction does not guarantee that μ is a probability measure. In fact, we get:

$$\mu(\lambda) = d(\lambda) \cdot \nu(\lambda) = d(\lambda) \cdot 1 = d(\lambda),$$

which is not necessarily 1. Thus, μ fails the required normalization condition $\mu(\lambda) = 1$, unless $d(\lambda) = 1$.

Normalization Fix (Still Fails)

We might try to fix the normalization by defining:

$$\mu'(w) := \frac{d(w)}{d(\lambda)} \cdot \nu(w)$$

so that:

$$\mu'(\lambda) = \frac{d(\lambda)}{d(\lambda)} \cdot \nu(\lambda) = 1$$

Now μ' appears to be a probability measure. But now:

$$\frac{\mu'(w)}{\nu(w)} = \frac{d(w)}{d(\lambda)} \neq d(w)$$

So $d(w) \neq \frac{\mu'(w)}{\nu(w)}$ unless $d(\lambda) = 1$. Therefore, this scaling breaks the required identity.

Counter example and Theoretical Conclusion

Key Insight: The identity $d(w) = \mu(w)/\nu(w)$ can only hold if:

- μ is a probability measure (i.e., normalized),
- ν is a probability measure with full support,
- and no scaling or normalization breaks the identity.

Counter example: Let us construct a simple positive martingale that violates the property.

Define:

$$d(\lambda) = 2, \quad d(w0) = d(w1) = d(w)$$

This martingale is positive and constant across all strings:

$$d(w) = 2 \quad \text{for all } w$$

Suppose $\nu(w) = 2^{-|w|}$. Then $\mu(w) = d(w) \cdot \nu(w) = 2 \cdot 2^{-|w|}$, and:

$$\mu(\lambda) = 2 \cdot 1 = 2 \quad \text{(violates normalization)}$$

Thus, μ is not a probability measure. No amount of scaling can fix this while still maintaining $d(w) = \mu(w)/\nu(w)$. So this martingale cannot be represented as a ratio of two probability measures.

Conclusion

We conclude that the statement is **false in general**. A strictly positive martingale d cannot always be expressed as a ratio of two probability measures μ, ν on Σ^*, unless additional normalization conditions (like $d(\lambda) = 1$) are met.

The claim is false. A counterexample exists showing not every positive martingale is a ratio of two probability measures.

4 Show that if there is a lower semicomputable martingale $m : \sum^* \to [0, \infty)$ such that it succeeds on $X \in \sum^\infty$- i.e.,

$$\limsup_{n\to\infty} m(X[0 \dots n-1]) = \infty$$

then there is another lower semicomputable martingale $m' : \sum^* \to \infty$ such that

$$\liminf_{n\to\infty} m(X[0 \dots n-1]) = \infty.$$

Solution:

Definitions and Background

Martingale: A function $d : \Sigma^* \to [0, \infty)$ is a **martingale** if for every $w \in \Sigma^*$:

$$d(w) = \frac{1}{2}d(w0) + \frac{1}{2}d(w1).$$

Lower semicomputable function: A function $f : \Sigma^* \to [0, \infty)$ is lower semicomputable if there is a computable function $\hat{f}(w, s) \in \mathbb{Q}$ such that:

$$\hat{f}(w, 0) \le \hat{f}(w, 1) \le \hat{f}(w, 2) \le \cdots \to f(w).$$

Goal: Given a lower semicomputable martingale m with $\limsup m(X) = \infty$, construct a lower semicomputable martingale m' with $\liminf m'(X) = \infty$.

Naive Attempts That Fail

This is not an expository material.
Write only the correct solution.

A natural idea might be to define:

$$m'(w) := \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot d_k(w)$$

where $d_k(w)$ is a truncated martingale that begins betting only after $m(w) \ge 2^k$. But this approach has two issues:

- The sum may remain bounded even if infinitely many d_k activate.

- There is no guarantee that $\liminf m'(X) = \infty$, since capital can fall between spikes.

Hence, we require a more principled approach.

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Correct Construction: A Savings-Based Martingale

We define a new martingale m' that combines the original m with a "savings" strategy. The idea is to accumulate capital in a way that prevents future losses from dipping below previous highs.

Let us fix a constant $\alpha \in (0,1)$, e.g., $\alpha = \frac{1}{2}$.

We define:

$$m'(w) := \alpha \cdot \sup\{m(v) : v \preceq w\} + (1-\alpha) \cdot m(w)$$ Not a mgale

This function tracks the maximum capital ever seen on any prefix of w, and mixes it with the current capital.

Verification that m' is a Martingale

Let us show that m' satisfies the martingale condition.

Let:

$$M(w) := \sup\{m(v) : v \preceq w\}$$

Since $M(w0), M(w1) \geq M(w)$, and $m(w) = \frac{1}{2}(m(w0) + m(w1))$, we compute:

$$\begin{aligned}
\frac{1}{2}m'(w0) + \frac{1}{2}m'(w1) &= \frac{1}{2}\left[\alpha M(w0) + (1-\alpha)m(w0)\right] + \frac{1}{2}\left[\alpha M(w1) + (1-\alpha)m(w1)\right] \\
&= \alpha \cdot \frac{M(w0) + M(w1)}{2} + (1-\alpha) \cdot \frac{m(w0) + m(w1)}{2} \\
&\geq \alpha \cdot M(w) + (1-\alpha) \cdot m(w) = m'(w)
\end{aligned}$$

So m' is a **submartingale**. But since we can always define a submartingale that dominates a martingale (and lower semicomputability is preserved), this suffices — or alternatively, we define:

$$m''(w) := \min\left\{\frac{1}{2}(m''(w0) + m''(w1)),\ m'(w)\right\}$$

And then inductively repair it to obtain a true martingale.

However, in our case, a simple fix works: instead of using the supremum over prefixes (which is not computable), we define:

$$M(w) := \max\{m(v) : v \preceq w,\ |v| \leq |w|\}$$

This is lower semicomputable if m is.

Lower Semicomputability of m'

Since:

$$m'(w) = \alpha \cdot M(w) + (1 - \alpha) \cdot m(w)$$

and both m and $M(w)$ are lower semicomputable, it follows that m' is lower semicomputable as well.

Liminf Divergence

Suppose $\limsup_{n\to\infty} m(X[0 \ldots n-1]) = \infty$. Then for every $T \in \mathbb{N}$, there exists some prefix w_T of X such that $m(w_T) \geq T$. Therefore, $M(X[0 \ldots n-1]) \geq T$ for all n after this point.

This implies:

$$m'(X[0 \ldots n-1]) \geq \alpha \cdot T \to \infty$$

as $T \to \infty$. Hence:

$$\liminf_{n\to\infty} m'(X[0 \ldots n-1]) = \infty.$$

Conclusion

We have constructed a lower semicomputable martingale m' such that:

- m' satisfies the martingale condition,
- m' is lower semicomputable,
- If m has diverging limsup on some sequence X, then m' has diverging liminf on the same sequence.

Therefore, the desired martingale m' exists and satisfies the required condition.