CS687 2023 HW1

Due: February 17, 2024

Notation. $M(i)\downarrow$ means that the Turing machine M halts on input i.

1. Recall that in the notes, we defined a language A to be decidable if A and A^c are Turing acceptable. Show that this is equivalent to saying that language A is decidable if and only if there is a Turing machine that accepts every string in A and halts, and rejects every string not in A and halts. [10]

2. Prove that every infinite computably enumerable language contains an infinite decidable subset. [10]

3. (Data Processing Inequality) Show that if x is any finite string, and $f : \Sigma^* \to \Sigma^*$ is any total computable function, then $C(f(x)) \leq C(x) + O(1)$. This says that you can never *increase* information through computation, you can either preserve it or decrease it. [10]

4. (Upsetting the apple cart) Imagine that you have a hard-drive filled with useful data. You place it on a stick of dynamite and light the fuse. Making the stick of dynamite, lighting the fuse etc. are all computable processes. Yet, in a few seconds, your "disorder" in the data will increase. How does this relate to the data processing inequality? [5]

5. Let x be an arbitrary finite binary string. Do all permutations of x have the same plain Kolmogorov complexity of x? If so, prove your claim. Otherwise, construct a counterexample, and prove that your string has some permutation which has a significantly different Kolmogorov complexity. [10]

6. (Characteristic sequence of the halting problem) Let χ_H be the infinite binary sequence defined as follows. For any number i, the i^{th} bit of χ_H is 1 if the i^{th} Turing machine halts, and 0 otherwise. Since the Halting problem is uncomputable, χ_H is uncomputable. However, show that prefixes of χ_H are compressible: for all sufficiently large n, show that $C(\chi_H[0 \dots n-1]) \leq \log n + O(1)$, where $\chi[0 \dots n-1]$ denotes the n-length prefix of χ_H. (This shows that even when a string is uncomputable, it may be highly compressible.) [10]

7. (Chaitin's Omega, modified version) Let P be a prefix-free set of strings, which form the domain of the universal prefix-free machine M. Let ω be an infinite binary sequence defined by
$$\omega = \sum_{\substack{p \in \mathbb{P}, \\ M(p)\downarrow}} \frac{1}{2^{|p|}}.$$
Show that prefixes of ω are incompressible: for all sufficiently large n, $C(\omega[0 \dots n-1]) \geq n - O(1)$. (The purpose of this question is to show that there are uncomputable strings which are also incompressible, in contrast to the previous question.) [10]

8. Show, using an argument using plain Kolmogorov complexity, that for all sufficiently large n, most strings of length n will not have a contiguous stretch of more than $\log^2 n$ zeroes anywhere in the string. (This shows how you can prove that some event happens with very high probability, using the theory of plain Kolmogorov complexity.) [10]