CS687A: Endsem

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1. Show that if there is a constant c such that for the sequence X, we have $C(X[0 \ldots (n-1)]|n) < c$, then X is computable.

   Solution:

   Given that $C(X[0 \ldots (n-1)]|n) < c$, it implies the existence of Turing machines, denote as M1, M2, M3, and so forth. Each of these machines operates uniquely; for instance, M1 outputs the first bit, M2 outputs the first two bits of X and so on.When given input n to Mn gives output the first n digit of X, and this pattern continues and all have program size less than c i.e. $|Mi| < c$.

   Claim : X is computable i.e. there exist a Turing Machine TM when given input of n , gives nth bit of X.

   Algorithm for TM

   How do you know which machine to run to get the n-length prefix, for a particular n?

   1. input n
   2. y = M(n) (this gives first n bits of X)
   3. output R(y,n) (R(y,n) output the nth bit of y)

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   Algorithm for R

   1. input string y
   2. input n
   3. output y[n] (output nth bit of y ,using recursive approach)

   Turing Machine TM gives output nth bit of X when given input of n . So X is computable.

   ∎

2. Show that if for an infinite binary sequence X,

$$\sum_{n \in N} 2^{n-K(X[0...(n-1)])} < \infty,$$

then

$$\lim_{n\to\infty} K(X[0...(n-1)]) - n = \infty.$$

Solution:

Let $S_n = \sum_{i=0}^{n-1} 2^{i-K(X[0...(i-1)])}$, we have $\lim_{n\to\infty} S_n < \infty$.

Let $L(n)$ be the length of the shortest program that outputs $X[0...(n-1)]$ and then halts.

Then we have $L(n) \leq K(X[0...(n-1)]) + c$ for some constant c.

Therefore, we have

$$\begin{aligned} K(X[0...(n-1)]) - n &= (K(X[0...(n-1)]) + c) - (n + c) \\ &\leq L(n) - (n + c) \\ &= \text{length of prefix of } S \text{ that encodes } X[0...(n-1)], \end{aligned}$$

where S is the self-delimiting prefix-free code used to describe X. limsup

Now, suppose that $\lim_{n\to\infty} K(X[0...(n-1)]) - n = c$ for some constant $c \geq 0$.

Then, for any $\epsilon > 0$, there exists N such that $|K(X[0...(n-1)]) - n - c| < \epsilon$ for all $n \geq N$.

Let $m > N$ be a sufficiently large integer such that $2^{m-N} > 2/\epsilon$. Then, we have

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$$\begin{aligned} S_m &= \sum_{i=0}^{m-1} 2^{i-K(X[0...(i-1)])} \geq \sum_{i=N}^{m-1} 2^{i-K(X[0...(i-1)])} \\ &> \sum_{i=N}^{m-1} 2^{i-(n+c)-\epsilon} = 2^{-\epsilon} \sum_{i=N}^{m-1} 2^{i-(n+c)} \\ &= 2^{-\epsilon} \sum_{i=0}^{m-N-1} 2^{i-(n+c)} = 2^{-\epsilon} \left(\frac{1}{2^{n+c}} - \frac{1}{2^{m+c}} \right) \\ &> 2^{-\epsilon} \left(\frac{1}{2^{n+c}} - \frac{1}{2^{N+c}} \right) > 2^{-\epsilon} \cdot \frac{1}{2^{n+c}} \\ &= \frac{1}{2^{n+c+\epsilon}} \\ &\to \infty, \quad \text{as } n \to \infty. \end{aligned}$$

Why? Shouldn't it go to 0 as n->\infty?

□

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a computable permutation. Prove that if $X_0 X_1 \ldots$ is an infinite binary sequence which is Martin-Löf random, then so is $X_{f(0)} X_{f(1)} \ldots$.

Solution:

Assume that there exists an effective test T such that the probability that $X_{f(0)} X_{f(1)} \ldots$ fails T is positive, i.e., there exists a constant $c > 0$ such that for all n, the probability that $X_{f(0)} X_{f(1)} \ldots X_{f(n)}$ fails T is at least c.

There is no probability here. A sequence fails a test deterministically.

Let E be the set of all infinite binary sequences that fail T, i.e., $E = \{X_{f(0)} X_{f(1)} \ldots : T(X_{f(0)} X_{f(1)} \ldots) = 0\}$. By the assumption, the measure of E is at least c.

Since $X_0 X_1 \ldots$ is Martin-Löf random, it has measure 1 with respect to the uniform measure μ, i.e., $\mu(\{X_0 X_1 \ldots : T(X_0 X_1 \ldots) = 0\}) = 0$.

A Matrin-Lof random is a member of a measure 1 set.
By itself, it is a point, and has measure 0.

Let $S = \{X_0 X_1 \ldots : X_{f(0)} X_{f(1)} \ldots \in E\}$. Since E has positive measure, S is also nonempty. Moreover, S is effectively open, since we can effectively compute a prefix of an element of S, given a prefix of $X_0 X_1 \ldots$.

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By the Martin-Löf randomness of $X_0 X_1 \ldots$, we have $\mu(S) = 0$.

But since S is effectively open and nonempty, it follows that S has positive measure with respect to the uniform measure. This is a contradiction, and therefore the assumption that there exists an effective test T such that $X_{f(0)} X_{f(1)} \ldots$ fails T with positive probability must be false.

Hence, $X_{f(0)} X_{f(1)} \ldots$ is Martin-Löf random.

Please give a formal argument, for example,
in terms of Martin-Lof tests for X and the transformed
sequence.

□

4. Fix $d \in \mathbb{N}$ and let r_n be the number of d-incompressible sequences of length n. Then show that $C(r_n) \geq |r_n|$, i.e., r_n is incompressible.

Solution:

We Know,

$$r_n \geq 2^n - 2^{n-d} + 1$$

$$\log_2(r_n) \geq n + \log_2(1 - 2^{-d})$$

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$$\log_2(r_n) \geq n + \log_2(1 - 2^{-d})$$

Now assume that r_n is compressible i.e. $C(r_n) < |r_n|$

$\square$

There is no contradiction.