Assignment 1

Last date: January 29, 2025

- In all of the questions, you can assume discrete sample space and discrete random variables.
- In the class, I used the following Hoffding's inequality:

  Let $X_1, \ldots, X_n$ be independent r.v's such that $0 \leq X_i \leq 1$ for all $i \in [n]$. Let $S_n = X_1 + \cdots + X_n$. Then for all $t > 0$, we have

  $$\Pr(|S_n - E[S_n]| \geq t) \leq 2exp(-2t^2/n)$$

  We can apply Hoffding's inequality as long as $X_1, \ldots, X_n$ are bounded. In particular, if $a_i \leq X_i \leq b_i$ for all $i \in [n]$ (a_i and b_i can be negative), we have

  $$\Pr(|S_n - E[S_n]| \geq t) \leq 2exp(-2t^2/\sum_i (b_i - a_i)^2)$$

  Note that if we substitute $a_i = 0$ and $b_i = 1$, we recover the first inequality.

Prove the following statements (1 marks each):

1. For any two events A and B, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.
2. For any two random variables X and Y, $E[X+Y] = E[X] + E[Y]$.
3. $Var(cX) = c^2 Var(X)$ where c is a constant and X is a random variable.
4. Prove that with probability at least $1 - 1/k$, a uniformly random permutation $\sigma : [n] \to [n]$ has at most k fixed points.
5. Consider a particle that does an unbiased random walk on real line. It starts at position 0. For any i, if the particle is at i, it moves to position $i+1$ with probability $1/2$ and to position $i-1$ with probability $1/2$. Prove that after n steps, with at least $1 - 10/\sqrt{n}$ probability, the distance of the particle from start, i.e., 0 is at most $\sqrt{n \ln n}$.

Solve/Prove the following statements (2 marks each):

1. If X and Y are independent random variables then $E[XY] = E[X]E[Y]$.

2. $E[X] = \Pr(A)E[X|A] + \Pr(A^c)E[X|A^c]$ where X is a random variable and A is an event.

3. If $X_1, \ldots, X_n$ are independent random variables then $Var(\sum_{i=1}^n X_i) = \sum_{i=1}^n Var(X_i)$.

4. Prove Markov inequality, i.e., if X is any random variable that takes only non negative values then $\Pr(X \geq cE[X]) \leq \frac{1}{c}$.

5. Prove Chebyshev's inequality, i.e., $\Pr(|X - E[X]| \geq t) \leq \frac{Var(X)}{t^2}$ (Hint: use Markov inequality).

6. Show that, for more than 2 events, pairwise independence does not imply independence.

7. We have a standard 6-sided dice. Let X be the number of times 6 appears over n throes of the dice. Let p be the probability of the event $X \geq n/4$. Compare the upper bounds on p that you can obtain using Markov's inequality, Chebyshev's inequality, and Hoffding's bounds.

(3-marks):

1. It is promised that a given coin is either fair ($\Pr(Head) = 1/2$) or biased with $\Pr(Head) = 1/2 + \epsilon$ where $0 < \epsilon < 1/2$. Show that $100/\epsilon^2$ coin tosses are sufficient to correctly determine the type of coin (fair or biased) with at least $4/5$ probability, i.e., give an algorithm that will need at most $100/\epsilon^2$ coin tosses, and should have the following guarantee: if the coin is fair the algorithm will return 'fair' with probability at least $4/5$, and if the coin is biased then algorithm will return 'biased' with probability at least $4/5$.

2. Let μ be some fixed quantity and $\epsilon > 0$ be any fixed small number. Consider a randomized algorithm R which has the following guarantee: the value returned by the algorithm say $\hat{\mu}$ satisfies $\mu - \epsilon < \hat{\mu} < \mu + \epsilon$ with probability at least $2/3$. We wish an algorithm that achieves the same guarantee (that is value returned should be in $[\mu - \epsilon, \mu + \epsilon]$) but with very high probability say at least $1 - \delta$ for a small $\delta > 0$. Show that the following algorithm achieves the same.

   run the algorithm R independently $t = 100000 \log 1/\delta$ times[1]. Return the median of these t outputs.

   Note: This is called median trick. If we can estimate a quantity with probability c where $c > 1/2$ is any constant using at most s samples then we can also estimate that quantity with confidence $1 - \delta$ for any $\delta > 0$, with only logarithmic blow up in the number of samples (i.e., using at most $O(s \log 1/\delta)$ samples).

[1] as we only care about Big-O dependence, I have put big constant 100000 here which should make the things easy

3. We show that the expected reward $E[R(T)]$ of the algorithm ETC satisfies $E[R(T)] = O(T^{2/3}K^{1/3}(\log T)^{1/3})$. Note that we have assumed that the support of all reward distributions is in $[0, 1]$, i.e, all rewards are in $[0, 1]$. Suppose all rewards are in $[0, x]$ instead of $[0, 1]$. Now determine $E[R(T)]$ (now x should also come in the final expression of expected regret).

   (Hint: Carefully go through the proof of $E[R(T)] = O(T^{2/3}K^{1/3}(\log T)^{1/3})$. Find exact places where it was used that rewards are in $[0, 1]$ and modify accordingly.)

4. Fix some large T (number of rounds). Give the best upper bound (that you can) on the expected regret of the ETC algorithm after T rounds [2] on the following bandit instances consisting of only two arms:

   (a) both arm have mean of $1/2$.

   (b) one arm has mean of $1/2$ and another arm has mean of $1/2+1000\frac{(\log T)^{1/3}}{T^{1/3}}$.

   (c) one arm has mean of $1/2$ and another arm has mean of $1/2 + \frac{1}{\sqrt{T}}$

[2] You need to provide $f(T)$ (as small as possible) so that $E[R(T)] \leq f(T)$