Outdoor Localization using GPS

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GPS Localization

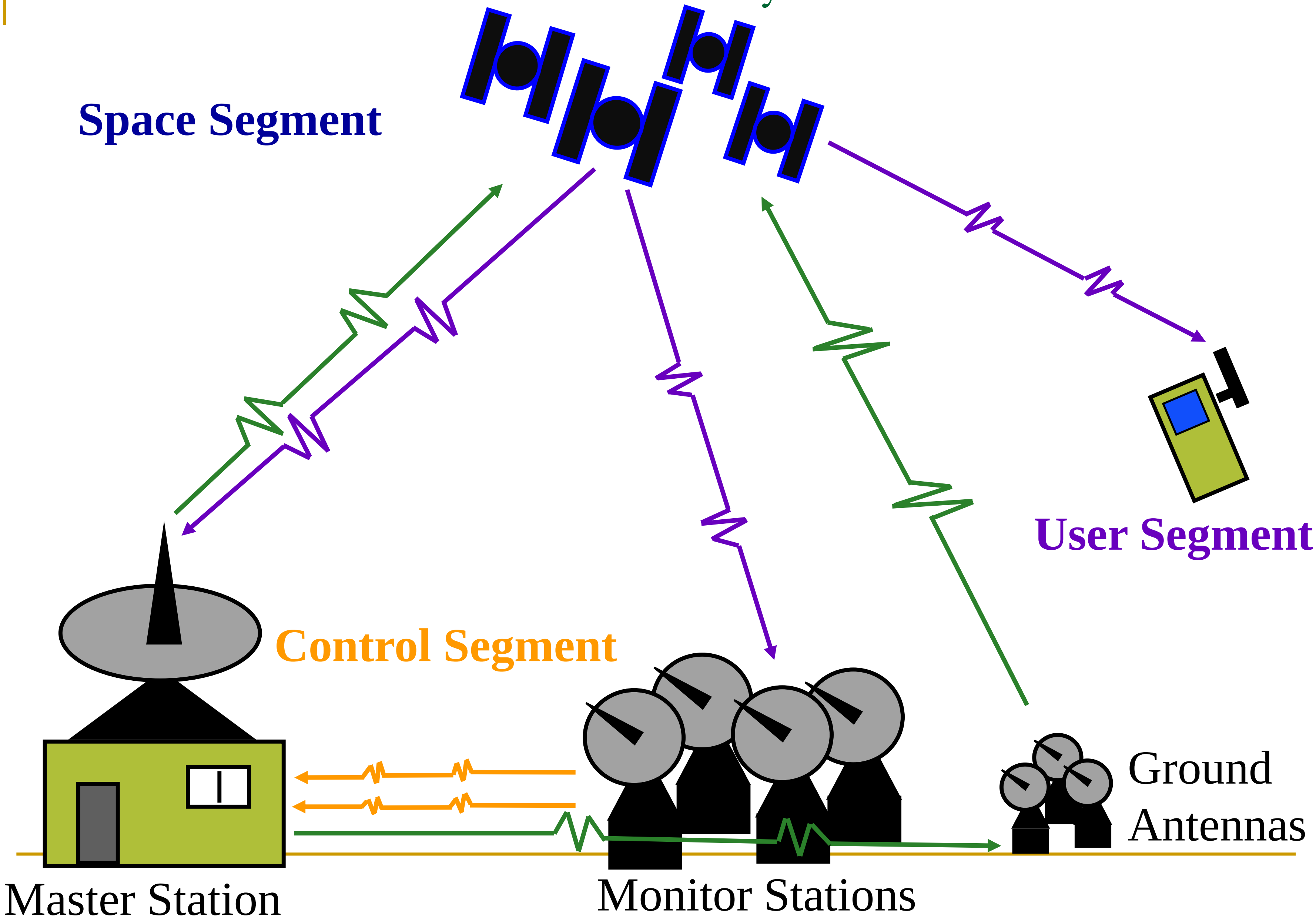
https://www.shutterstock.com/nb/video/clip-13833572-localization-gps-navigation-path-finding-city-routing
https://commons.wikimedia.org/wiki/File:Earth_Orbits.svg

Various Application Areas

Sector	Specific Analytical Focus
Agriculture	Precision agriculture technologies and practices
Electricity	Electrical system reliability and efficiency
Finance	Time stamps for high-frequency trading
Location-based services	Smartphone apps and consumer devices that use location services to deliver services and experiences
Mining	Efficiency gains, cost reductions, and increased accuracy
Maritime	Navigation, port operations, fishing, and recreational boating
Oil and gas	Positioning for offshore drilling and exploration
Surveying	Productivity gains, cost reductions, and increased accuracy in professional surveying
Telecommunications	Improved reliability and bandwidth utilization for wireless networks
Telematics	Efficiency gains, cost reductions, and environmental benefits through improved vehicle dispatch and navigation

https://www.gps.gov/governance/advisory/meetings/2019-11/gallaher.pdf

Entire GPS System

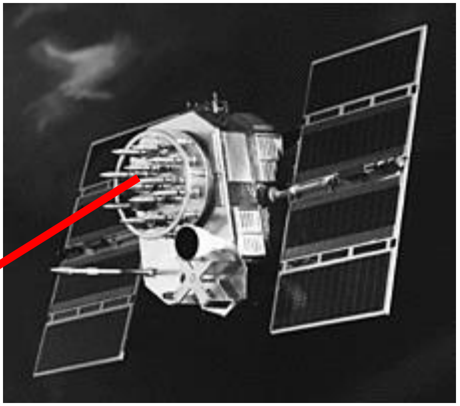


https://cupdf.com/document/introduction-to-the-global-positioning-system-56bc80026faa1.html

GPS Localization

Sends navigation messages at ~50 bps

- ❑ **Transmits satellites exact location**



$t_s = T$

Radio waves travel at the speed of light. If GPS signal leaves satellite at time "T"…

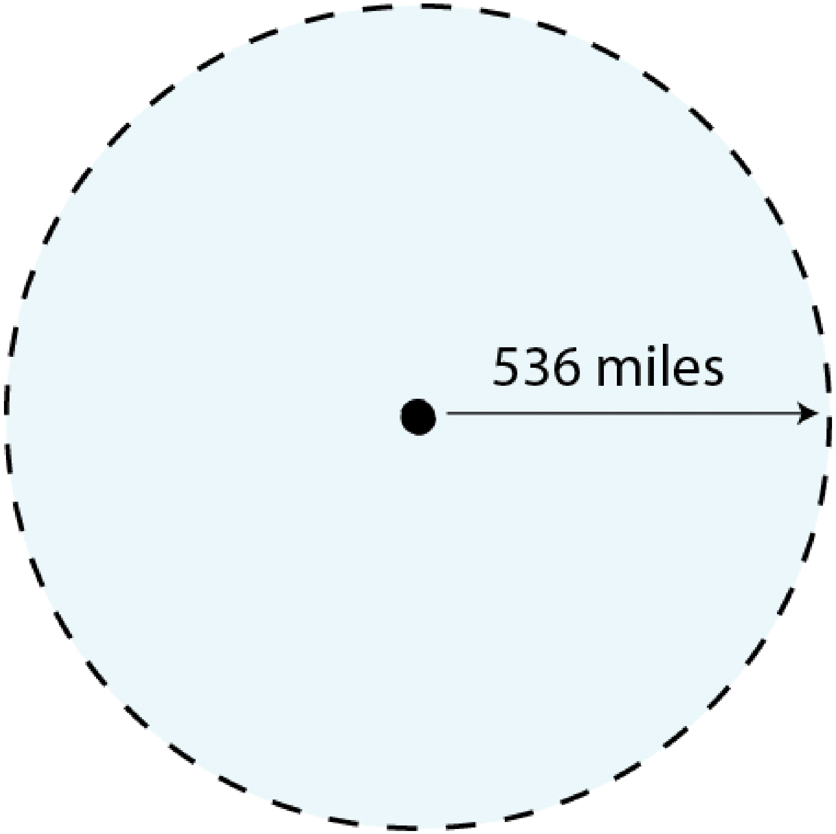
$r = (t_m - t_s)c$

$t_m = T{+}3$

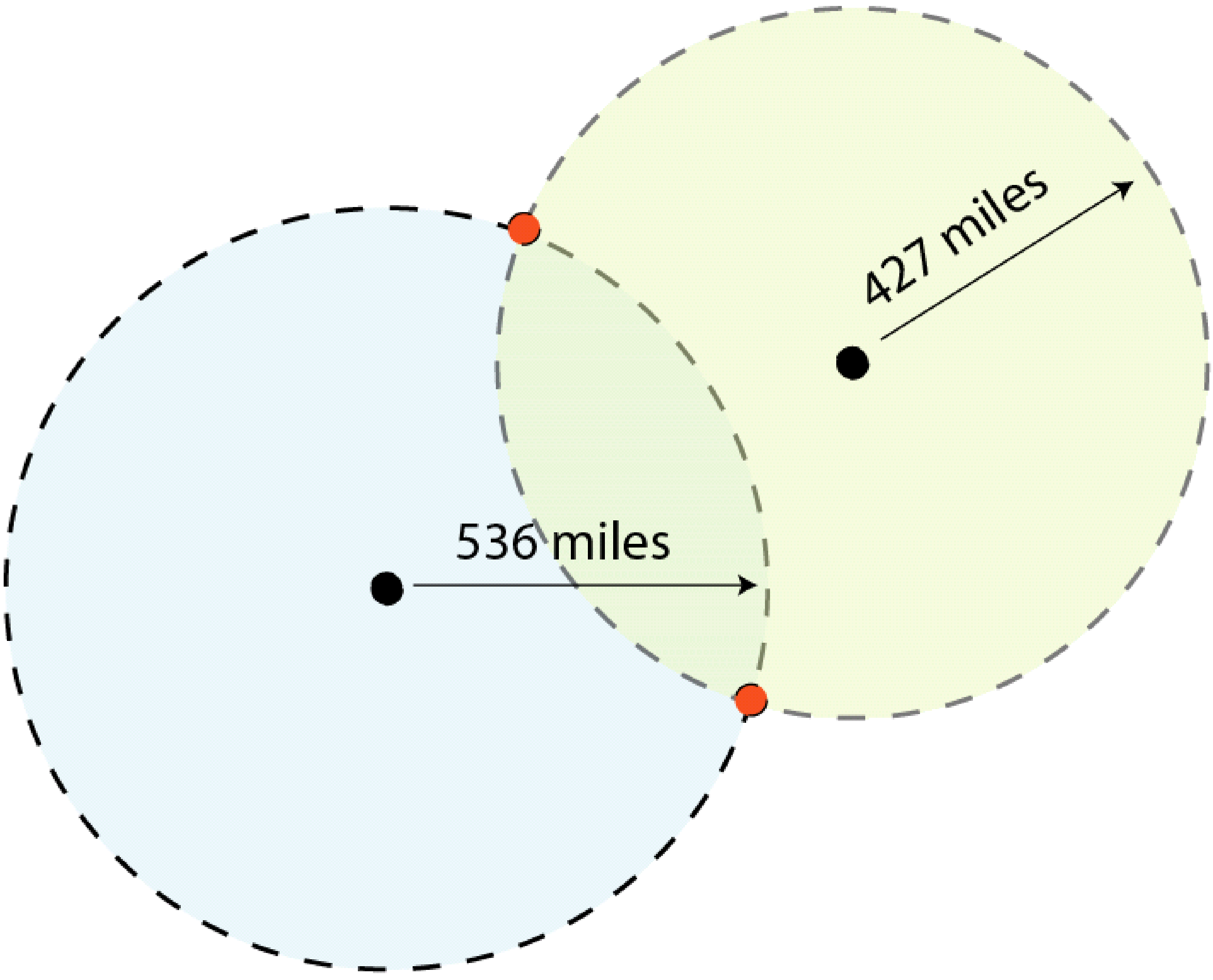
…and is picked up by the receiver at time "T + 3."

Then distance between satellite and receiver = "3 times the speed of light"

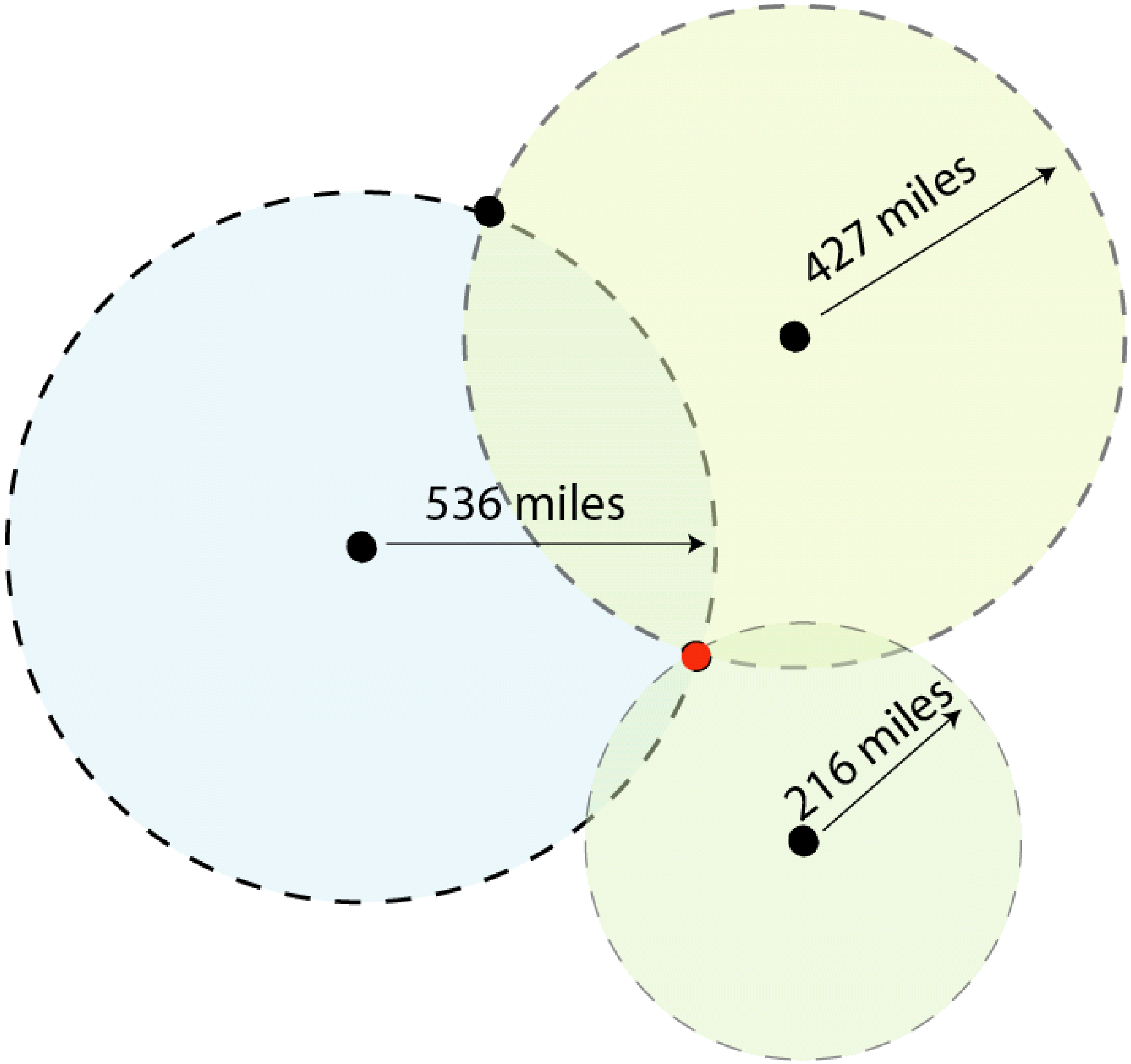
GPS Localization



GPS Localization

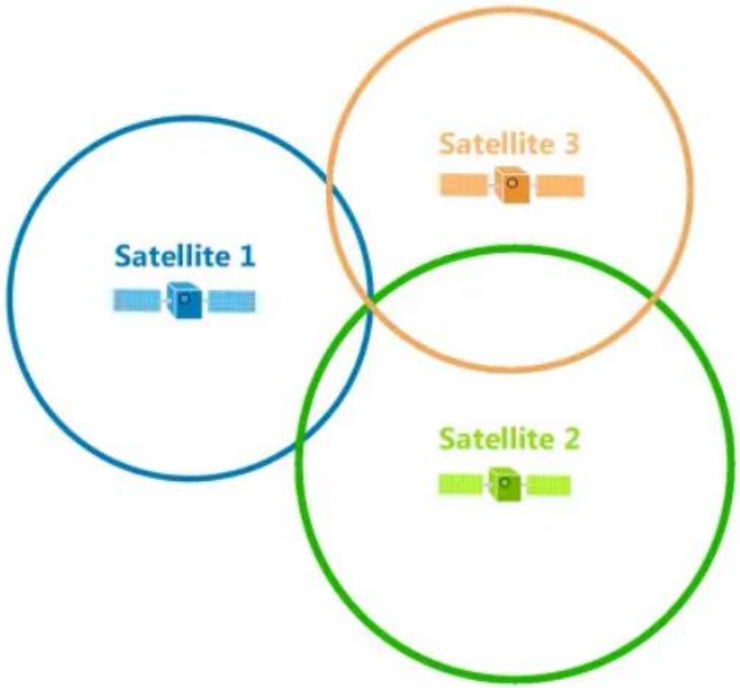


GPS Localization (Trilateration)



https://www.slideshare.net/gutzy6/what-is-gps

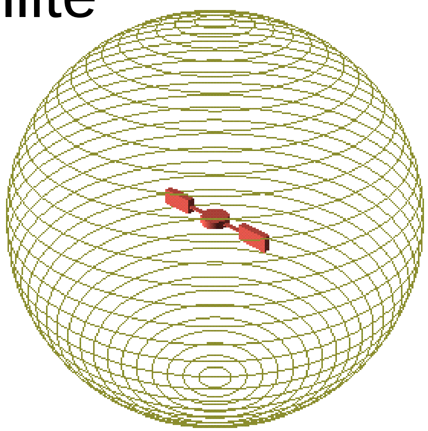
GPS Localization



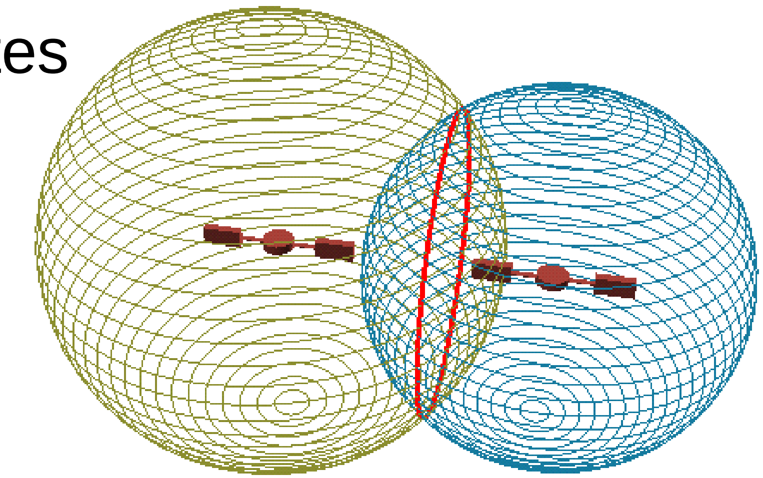
https://gisgeography.com/trilateration-triangulation-gps/

GPS Localization

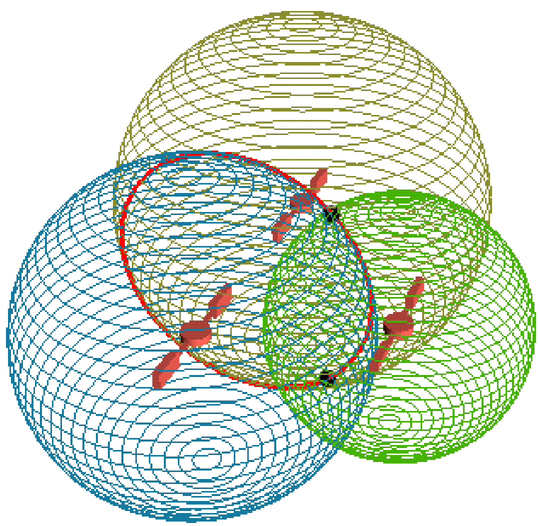
1 Satellite

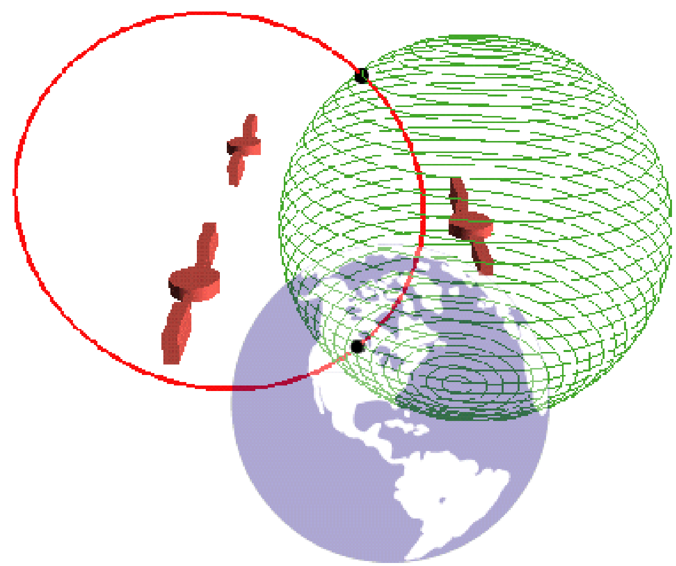


2 Satellites



3 Satellites





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GPS Localization

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r_1^2$$
$$(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 = r_2^2$$
$$(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2 = r_3^2$$
$$(x - x_4)^2 + (y - y_4)^2 + (z - z_4)^2 = r_4^2$$

$$2(x_2 - x_1)x + 2(y_2 - y_1)y + 2(z_2 - z_1)z = (r_1^2 - r_2^2) - (x_1^2 - x_2^2) - (y_1^2 - y_2^2) - (z_1^2 - z_2^2)$$
$$2(x_3 - x_2)x + 2(y_3 - y_2)y + 2(z_3 - z_2)z = (r_2^2 - r_3^2) - (x_2^2 - x_3^2) - (y_2^2 - y_3^2) - (z_2^2 - z_3^2)$$
$$2(x_4 - x_3)x + 2(y_4 - y_3)y + 2(z_4 - z_3)z = (r_3^2 - r_4^2) - (x_3^2 - x_4^2) - (y_3^2 - y_4^2) - (z_3^2 - z_4^2)$$

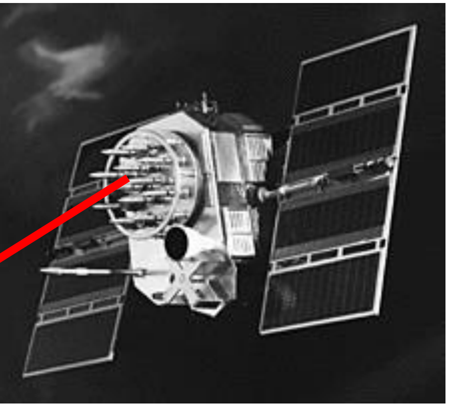
$$AX = \begin{bmatrix} 2(x_2 - x_1) & 2(y_2 - y_1) & 2(z_2 - z_1) \\ 2(x_3 - x_2) & 2(y_3 - y_2) & 2(z_3 - z_2) \\ 2(x_4 - x_3) & 2(y_4 - y_3) & 2(z_4 - z_3) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (r_1^2 - r_2^2) - (x_1^2 - x_2^2) - (y_1^2 - y_2^2) - (z_1^2 - z_2^2) \\ (r_2^2 - r_3^2) - (x_2^2 - x_3^2) - (y_2^2 - y_3^2) - (z_2^2 - z_3^2) \\ (r_3^2 - r_4^2) - (x_3^2 - x_4^2) - (y_3^2 - y_4^2) - (z_3^2 - z_4^2) \end{bmatrix} = B$$

Hundreds of kms of errors due to unsynchronized clocks

GPS Localization

Sends navigation messages at ~50 bps

- **Transmits satellites exact location**



$t_s = T$

Radio waves travel at the speed of light. If GPS signal leaves satellite at time "T"…

$$r = (t_m - t_s)c + \delta c$$

$t_m = T+3$

…and is picked up by the receiver at time "T + 3."

Then distance between satellite and receiver = "3 times the speed of light"

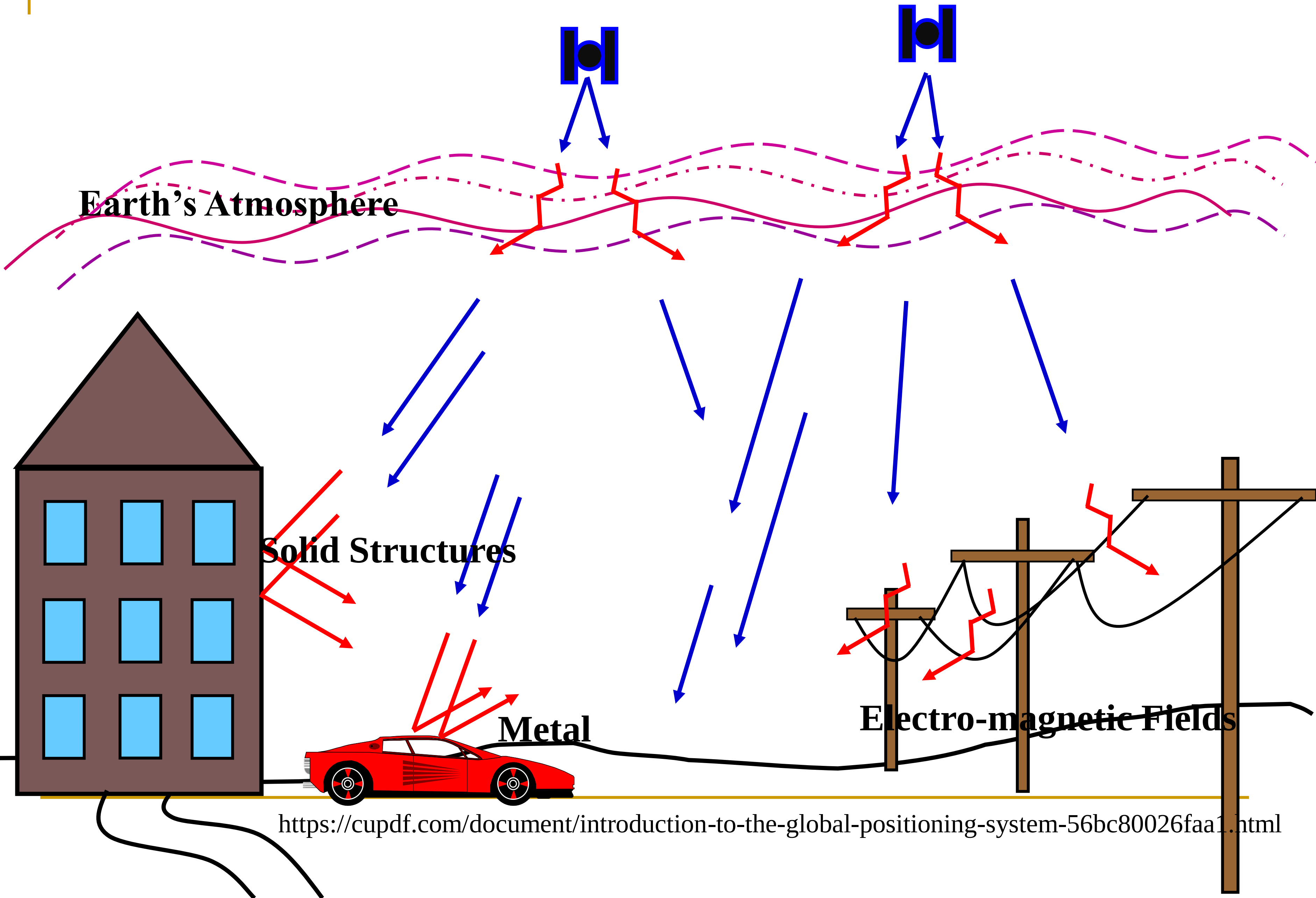
GPS Localization

$$
\begin{aligned}
(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 &= (r_1 - \Delta)^2 \\
(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 &= (r_2 - \Delta)^2 \\
(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2 &= (r_3 - \Delta)^2 \\
(x - x_4)^2 + (y - y_4)^2 + (z - z_4)^2 &= (r_4 - \Delta)^2 \\
(x - x_5)^2 + (y - y_5)^2 + (z - z_5)^2 &= (r_5 - \Delta)^2
\end{aligned}
$$

$$
\begin{aligned}
2(x_2 - x_1)x + 2(y_2 - y_1)y + 2(z_2 - z_1)z &= (r_1^2 - r_2^2) - 2(r_1 - r_2)\Delta - (x_1^2 - x_2^2) - (y_1^2 - y_2^2) - (z_1^2 - z_2^2) \\
2(x_3 - x_2)x + 2(y_3 - y_2)y + 2(z_3 - z_2)z &= (r_2^2 - r_3^2) - 2(r_2 - r_3)\Delta - (x_2^2 - x_3^2) - (y_2^2 - y_3^2) - (z_2^2 - z_3^2) \\
2(x_4 - x_3)x + 2(y_4 - y_3)y + 2(z_4 - z_3)z &= (r_3^2 - r_4^2) - 2(r_3 - r_4)\Delta - (x_3^2 - x_4^2) - (y_3^2 - y_4^2) - (z_3^2 - z_4^2) \\
2(x_5 - x_4)x + 2(y_5 - y_4)y + 2(z_5 - z_4)z &= (r_4^2 - r_5^2) - 2(r_4 - r_5)\Delta - (x_4^2 - x_5^2) - (y_4^2 - y_5^2) - (z_4^2 - z_5^2)
\end{aligned}
$$

$$
AX = \begin{bmatrix} 2(x_2 - x_1) & 2(y_2 - y_1) & 2(z_2 - z_1) & 2(r_1 - r_2) \\ 2(x_3 - x_2) & 2(y_3 - y_2) & 2(z_3 - z_2) & 2(r_2 - r_3) \\ 2(x_4 - x_3) & 2(y_4 - y_3) & 2(z_4 - z_3) & 2(r_3 - r_4) \\ 2(x_5 - x_4) & 2(y_5 - y_4) & 2(z_5 - z_4) & 2(r_4 - r_5) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \Delta \end{bmatrix} = \begin{bmatrix} (r_1^2 - r_2^2) - (x_1^2 - x_2^2) - (y_1^2 - y_2^2) - (z_1^2 - z_2^2) \\ (r_2^2 - r_3^2) - (x_2^2 - x_3^2) - (y_2^2 - y_3^2) - (z_2^2 - z_3^2) \\ (r_3^2 - r_4^2) - (x_3^2 - x_4^2) - (y_3^2 - y_4^2) - (z_3^2 - z_4^2) \\ (r_4^2 - r_5^2) - (x_4^2 - x_5^2) - (y_4^2 - y_5^2) - (z_4^2 - z_5^2) \end{bmatrix} = B
$$

Sources of Inaccuracies



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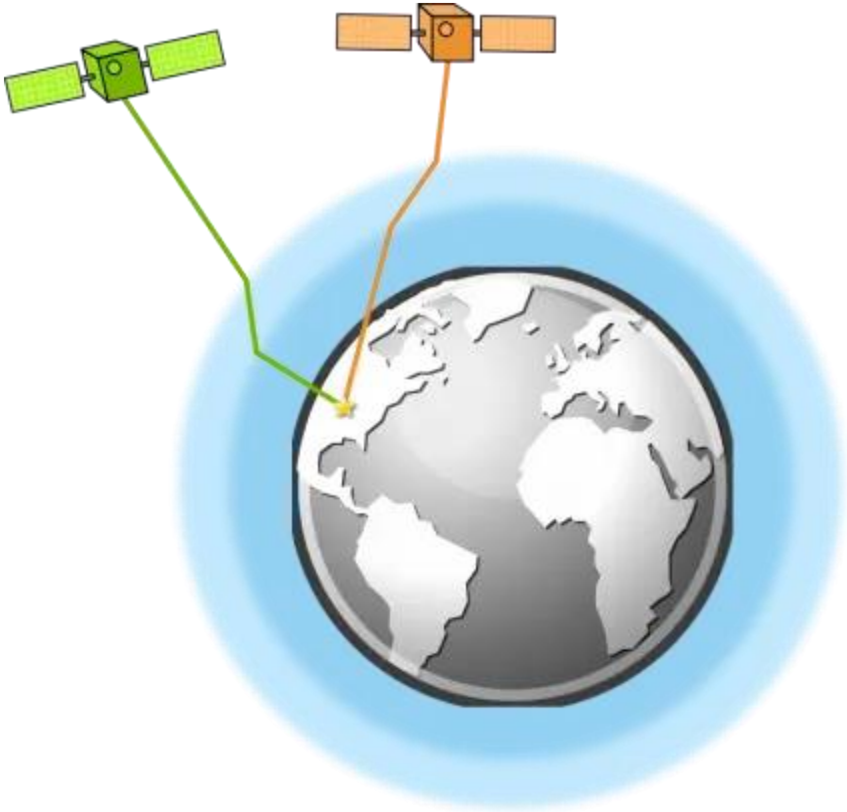
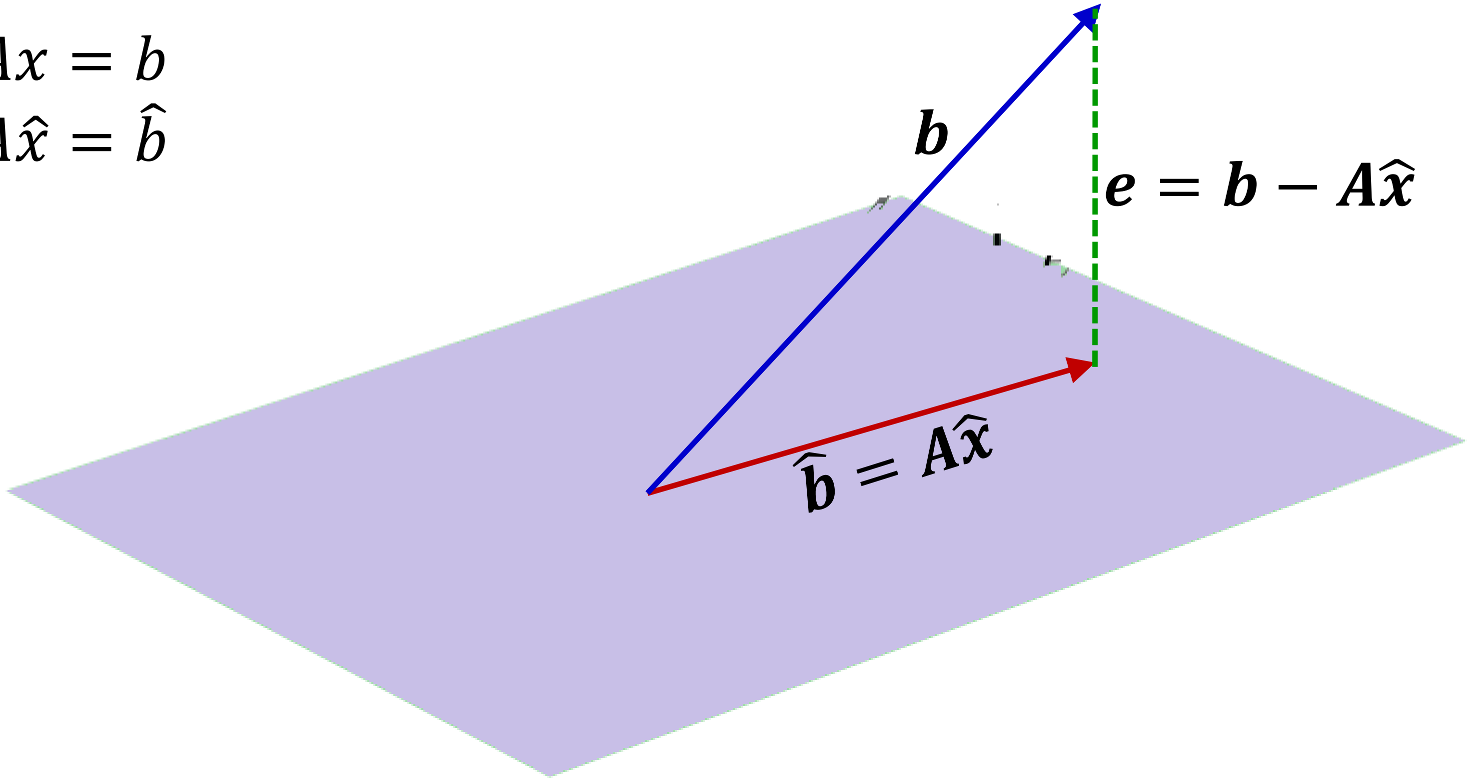


https://gisgeography.com/gps-accuracy-hdop-pdop-gdop-multipath/

GPS Localization

$$Ax = b$$
$$A\hat{x} = \hat{b}$$

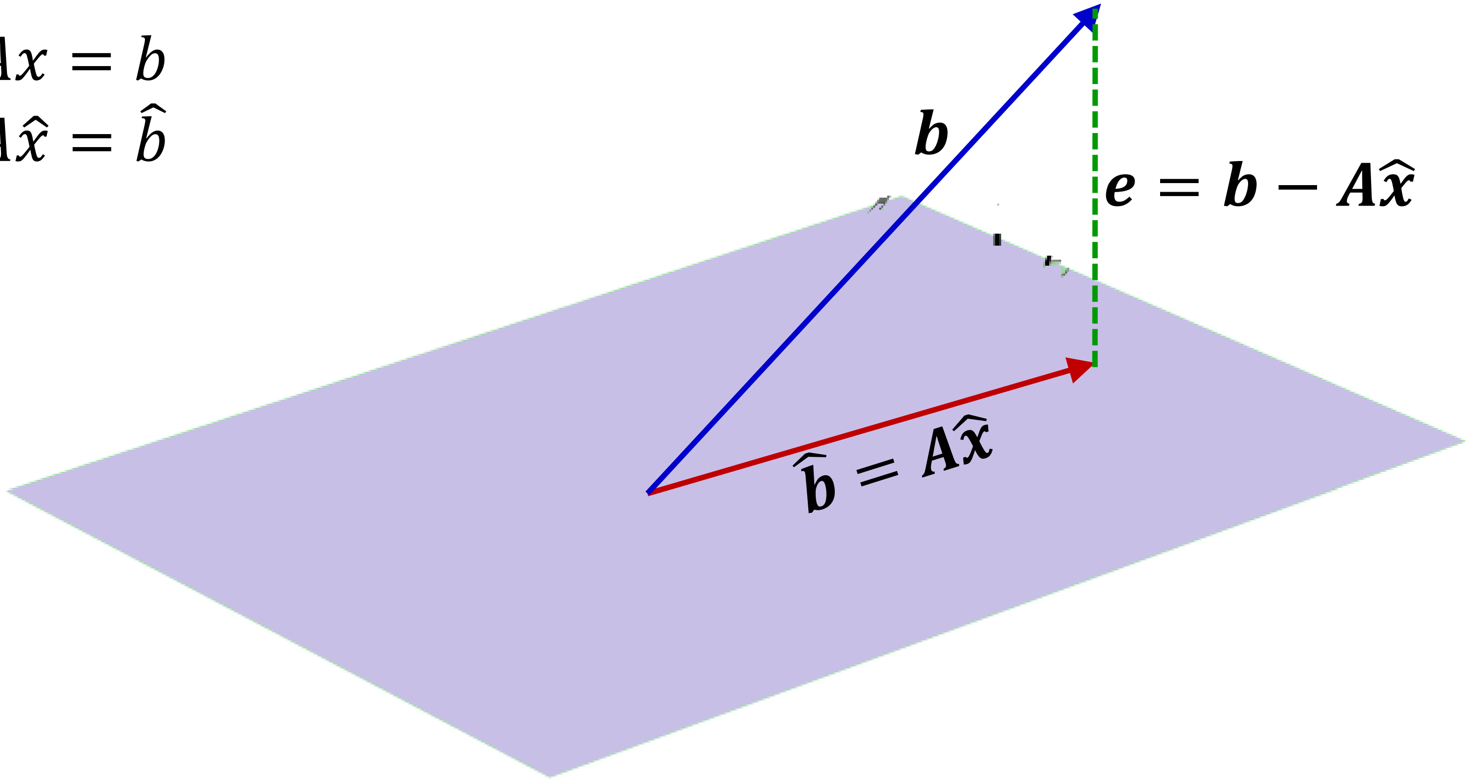


$$A^T e = 0 \quad \rightarrow A^T(b - A\hat{x}) = 0 \rightarrow \; A^T A\hat{x} = A^T b$$

$$\boxed{\hat{x} = (A^T A)^{-1} A^T b}$$

GPS Localization (Minimize Mean Square Error)

$$Ax = b$$
$$A\hat{x} = \hat{b}$$



$$||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2 = (\mathbf{A}\mathbf{x} - \mathbf{b})^{\mathrm{T}}(\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} - 2\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{b}$$

$$2\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} - 2\mathbf{A}^{\mathrm{T}}\mathbf{b} = 0 \Leftrightarrow \mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

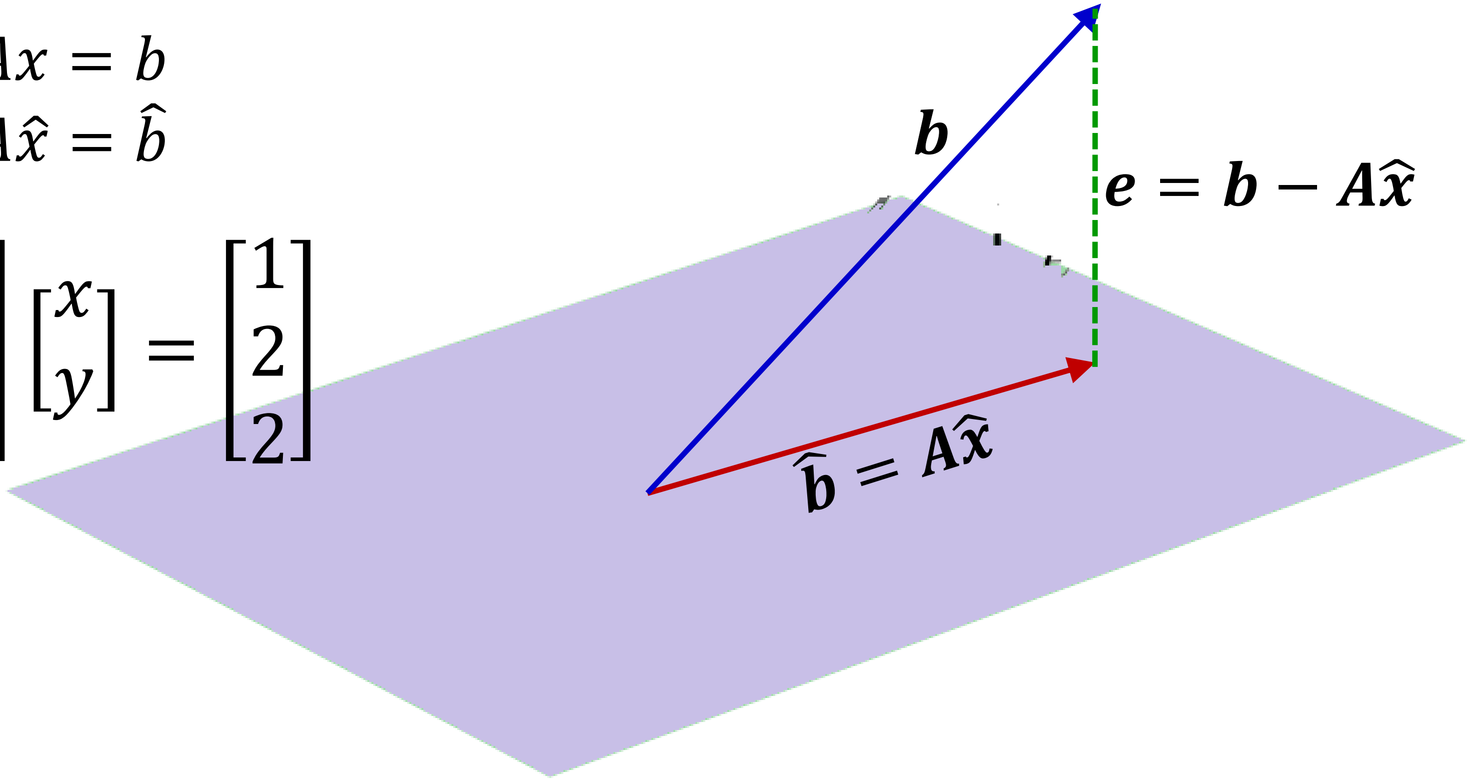
$$\boxed{x = (A^T A)^{-1} A^T b}$$

1-3 meters of errors

GPS Localization (Minimize Mean Square Error)

$$Ax = b$$
$$A\hat{x} = \hat{b}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$



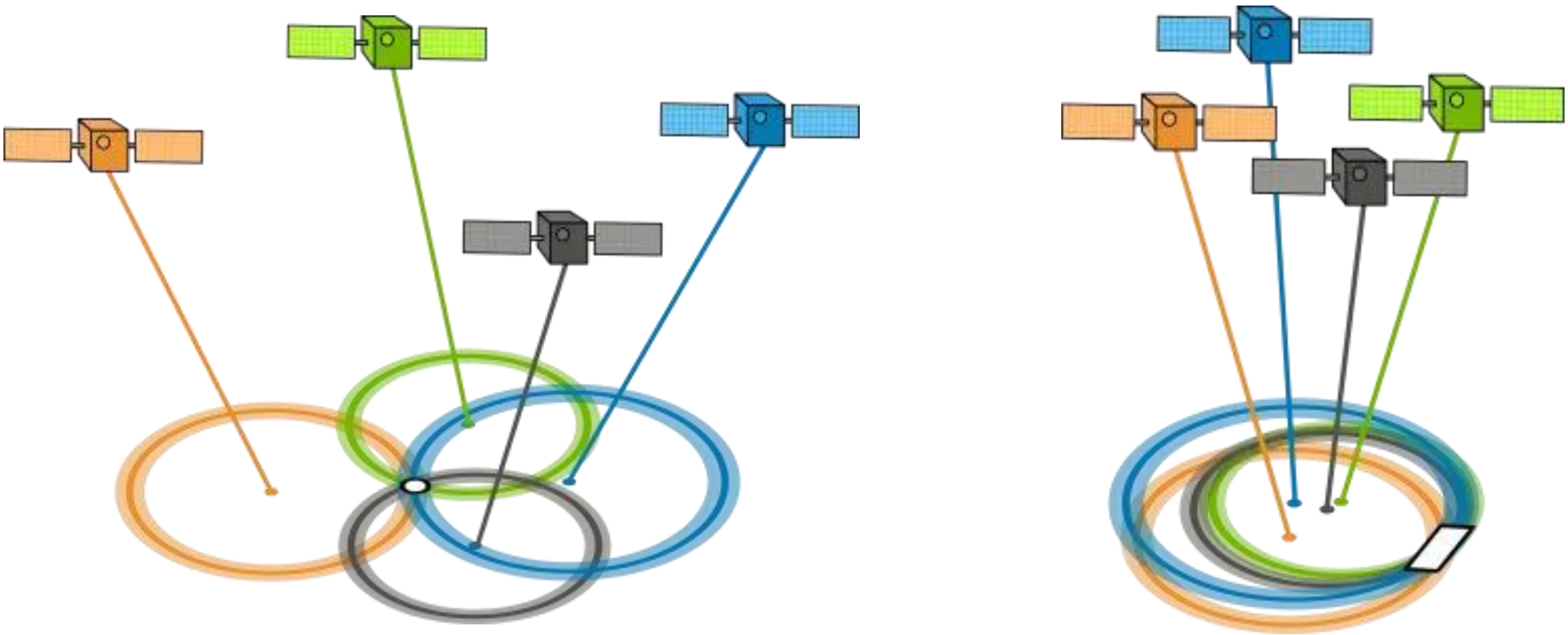
$$||\mathbf{Ax} - \mathbf{b}||_2^2 = (\mathbf{Ax} - \mathbf{b})^{\mathrm{T}}(\mathbf{Ax} - \mathbf{b}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{b}$$

$$2\mathbf{A}^{\mathrm{T}}\mathbf{Ax} - 2\mathbf{A}^{\mathrm{T}}\mathbf{b} = 0 \Leftrightarrow \mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

$$\boxed{x = (A^T A)^{-1} A^T b}$$

1-3 meters of errors

Dilution of Precision



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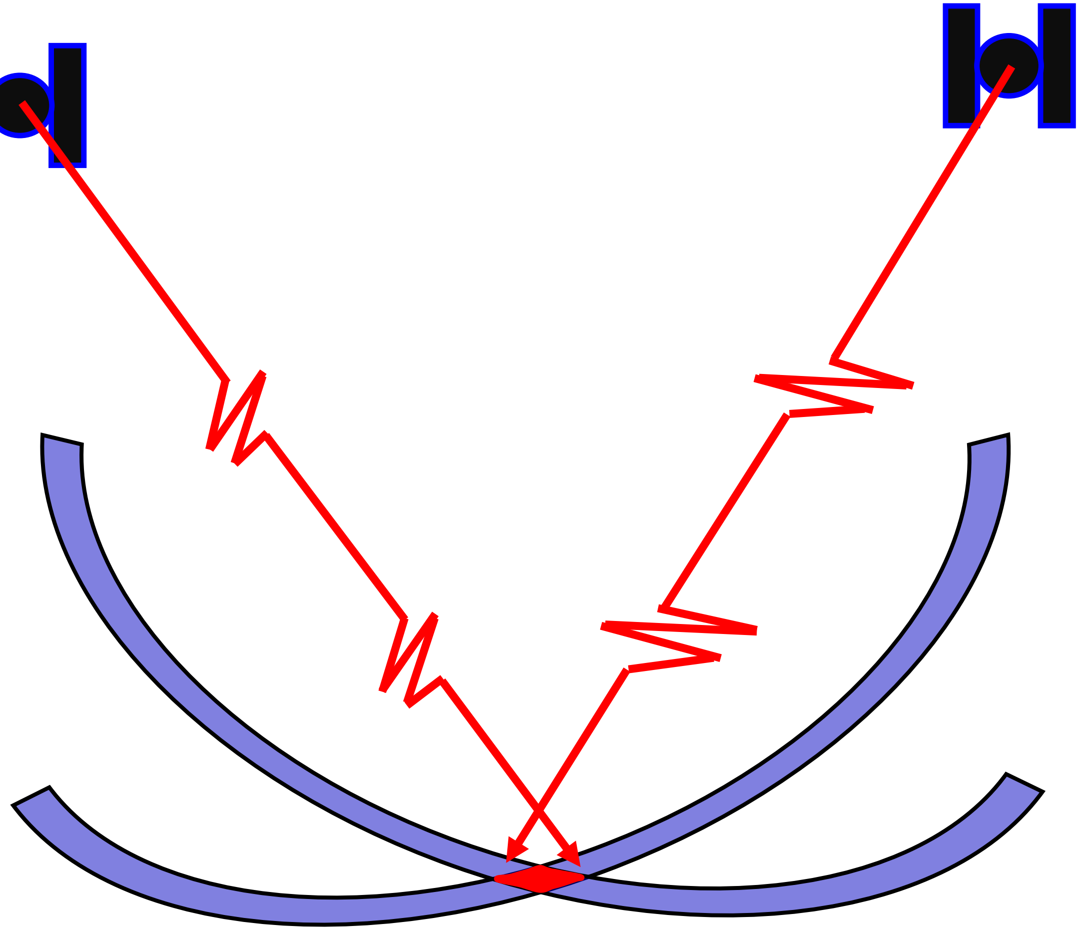

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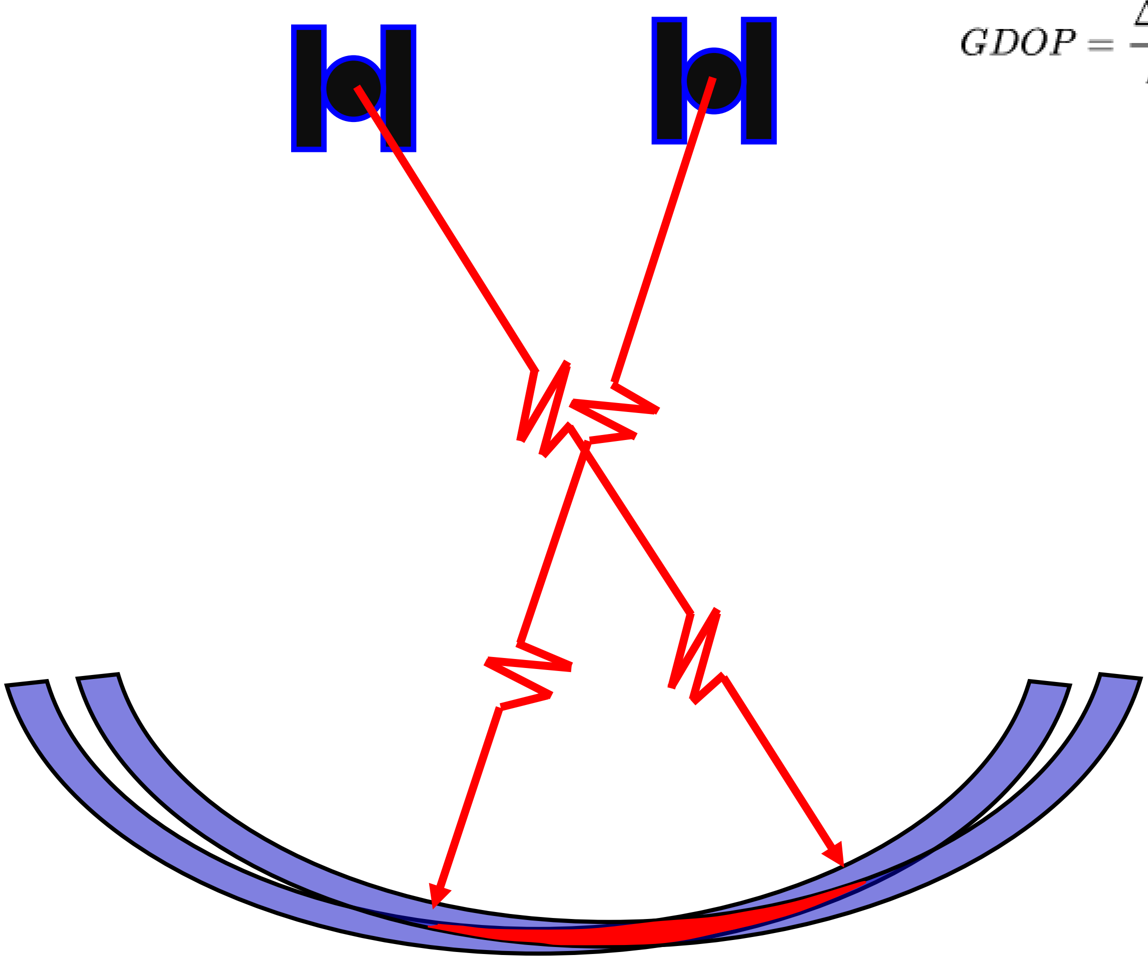

$$GDOP = \frac{\Delta(\text{Output Location})}{\Delta(\text{Measured Data})}$$

Dilution of Precision

$$GDOP = \frac{\Delta(\text{Output Location})}{\Delta(\text{Measured Data})}$$