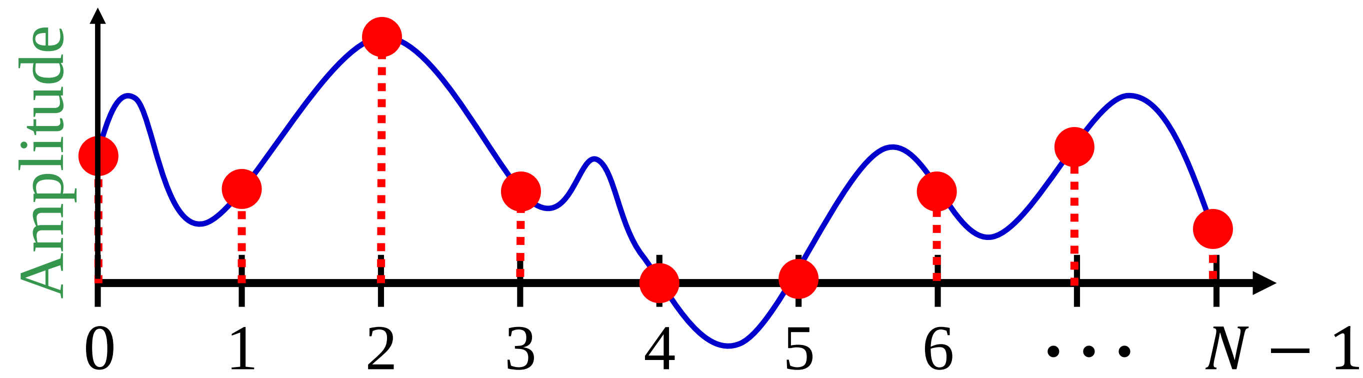
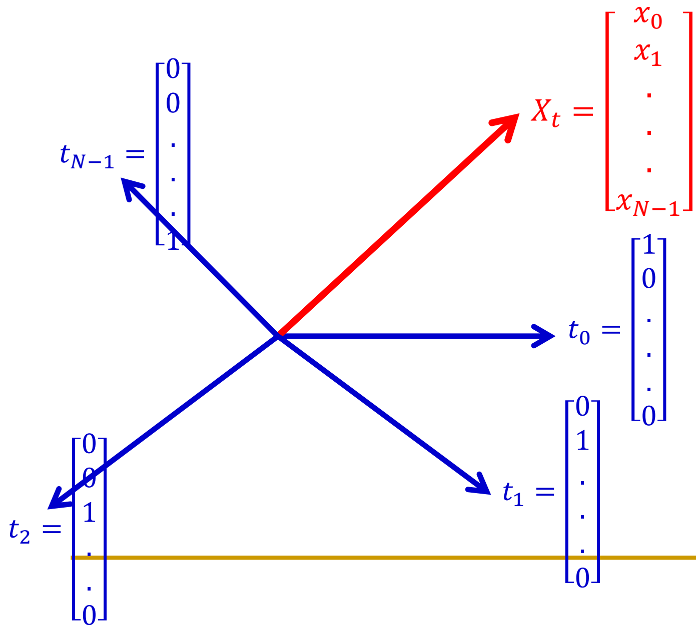
Discrete Fourier Transform: A Gentle Introduction

Amitangshu Pal

Time Domain Signals and Time Basis



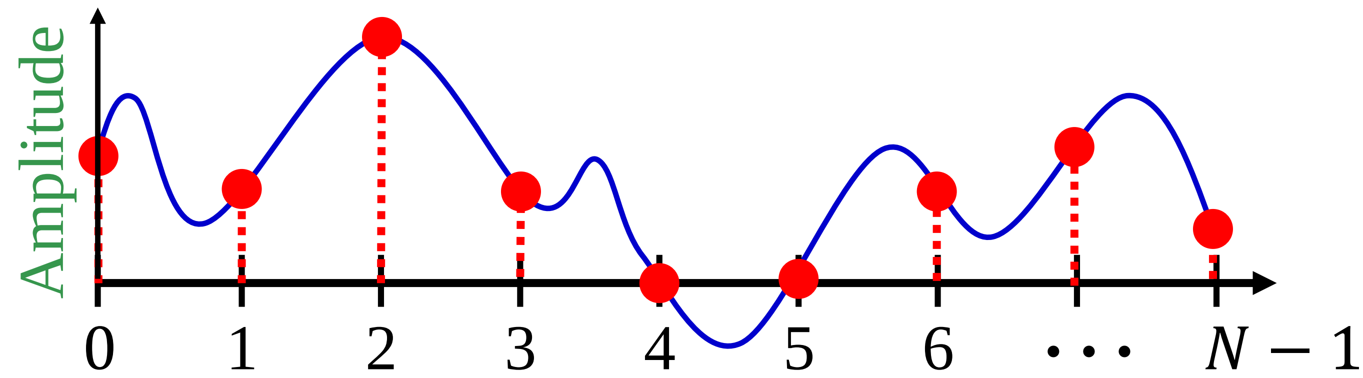
$$X_t = [x_0, x_1, x_2, \dots, x_{N-1}]^T$$



$$\boldsymbol{X_t} = x_0 \begin{bmatrix}1\\0\\.\\.\\.\\0\end{bmatrix} + x_1 \begin{bmatrix}0\\1\\.\\.\\.\\0\end{bmatrix} + \dots + x_{N-1}\begin{bmatrix}0\\0\\.\\.\\.\\1\end{bmatrix} = \begin{bmatrix}1&0&.&.&0\\0&1&.&.&0\\.&.&.&.&0\\.&.&.&.&0\\.&.&.&.&0\\0&0&.&.&1\end{bmatrix}\begin{bmatrix}x_0\\x_1\\.\\.\\.\\x_{N-1}\end{bmatrix}$$

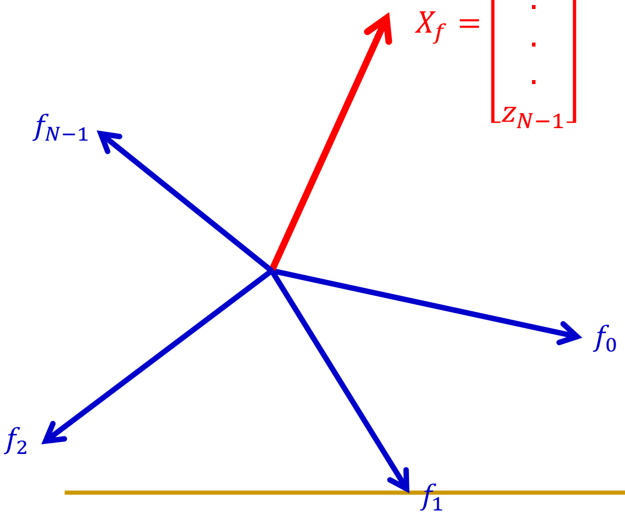
Basis vectors (Time) Signal

Time Domain Time To Frequency Domain



$$X_t = [x_0, x_1, x_2, \ldots, x_{N-1}]^T$$

$$X_f = \begin{bmatrix} z_0 \\ z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_{N-1} \end{bmatrix}$$



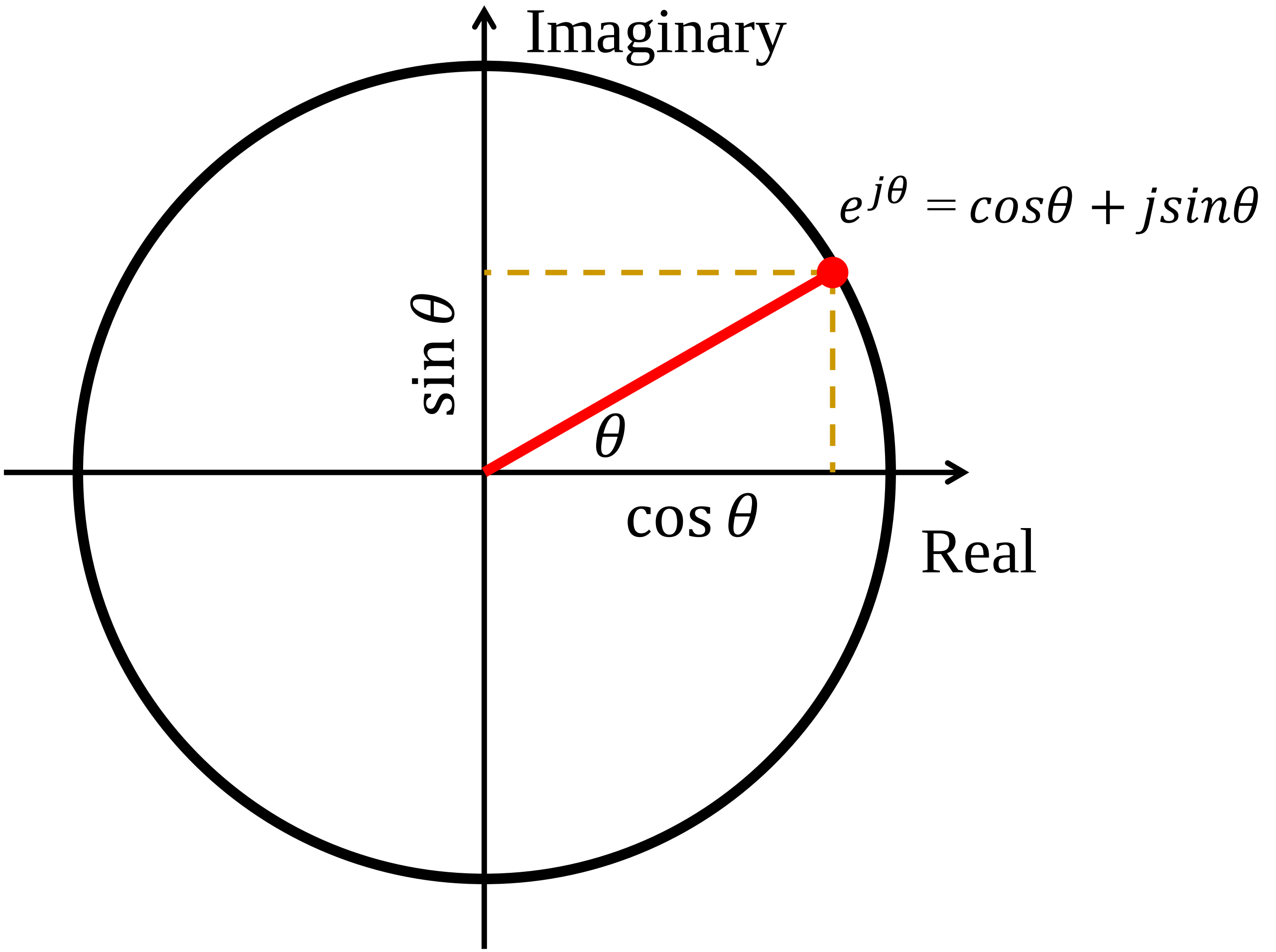
$$\boldsymbol{X_f} = \begin{bmatrix} | & | & | & \cdots & | \\ f_0 & f_1 & f_2 & \cdots & f_{N-1} \\ | & | & | & \cdots & | \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_{N-1} \end{bmatrix}$$

A different basis (F)

Same signal in other basis

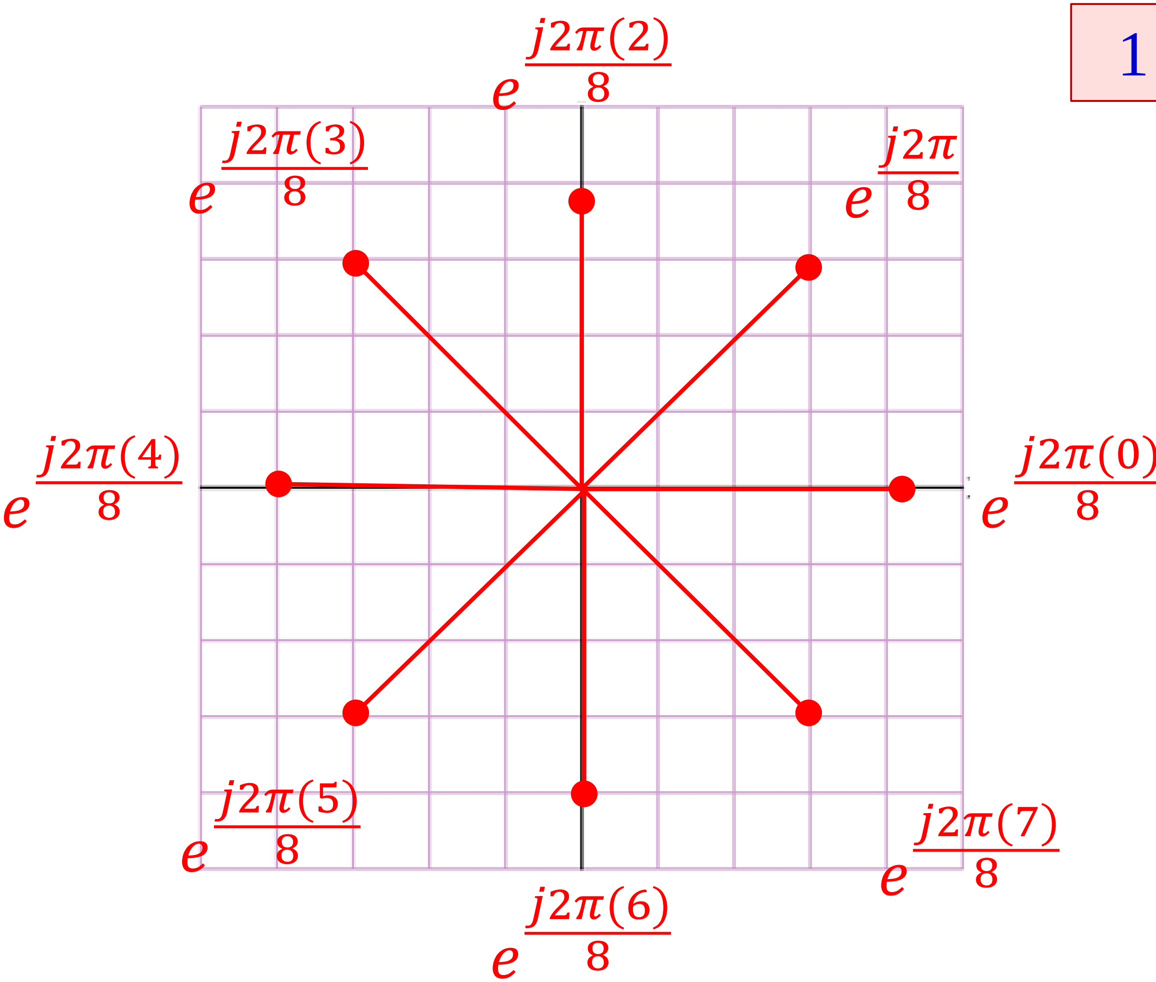
$$\boxed{IX_t = FX_f}$$

Euler's Theorem



Imagine Rotation as a Vector

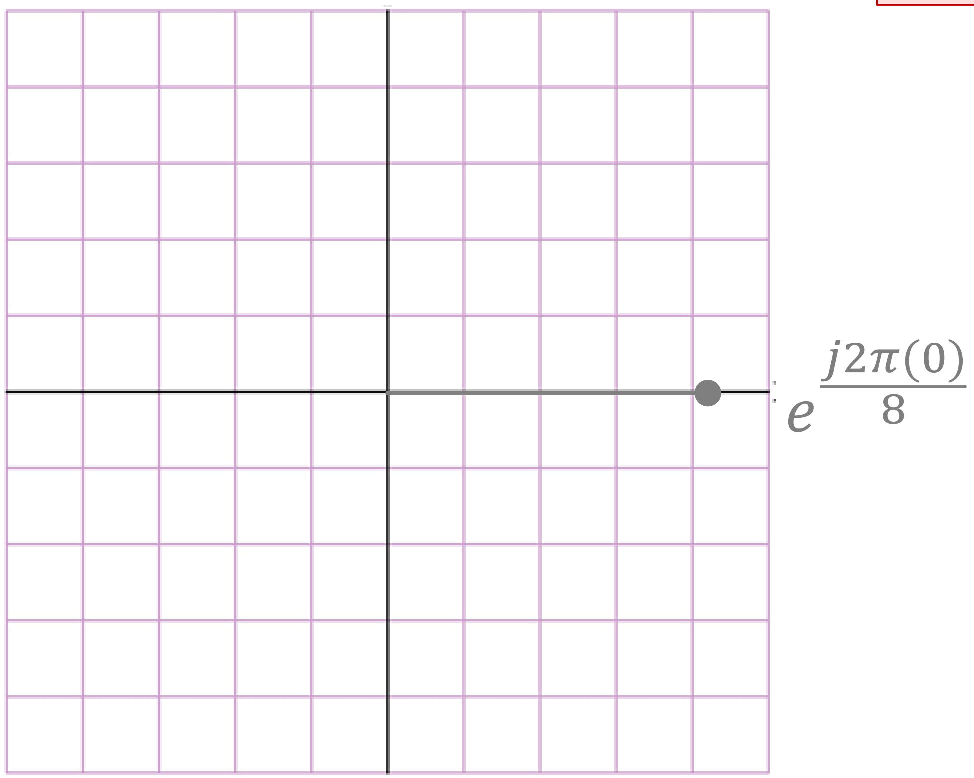
1 cycle in 8 steps



$$\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(1)}{8}} \\ e^{\frac{j2\pi(2)}{8}} \\ \cdots \\ e^{\frac{j2\pi(8)}{8}} \end{bmatrix}$$

Imagine Rotation as a Vector

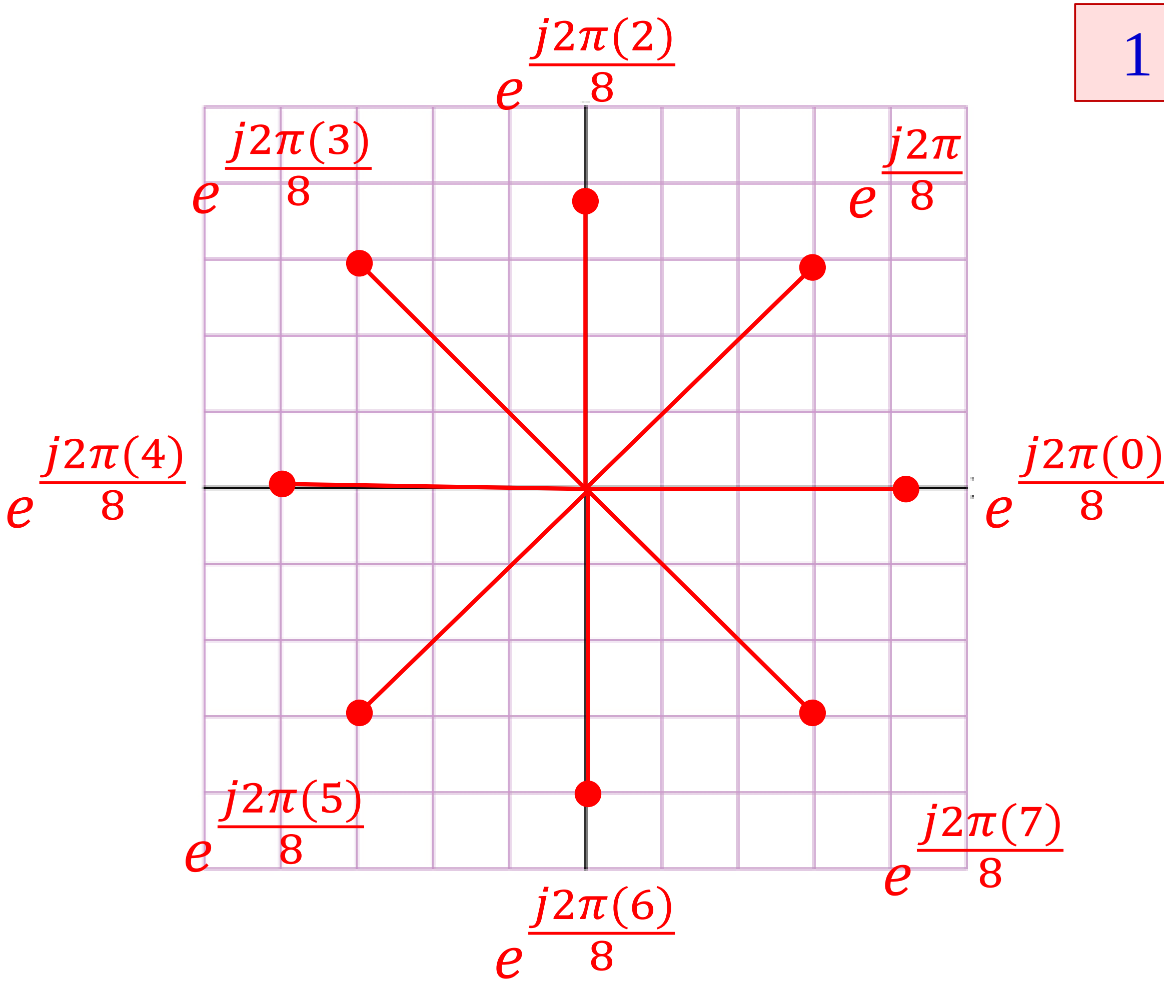
0 cycle in 8 steps



$$\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ \cdots \\ e^{\frac{j2\pi(0)}{8}} \end{bmatrix}$$

f_0

Imagine Rotation as a Vector



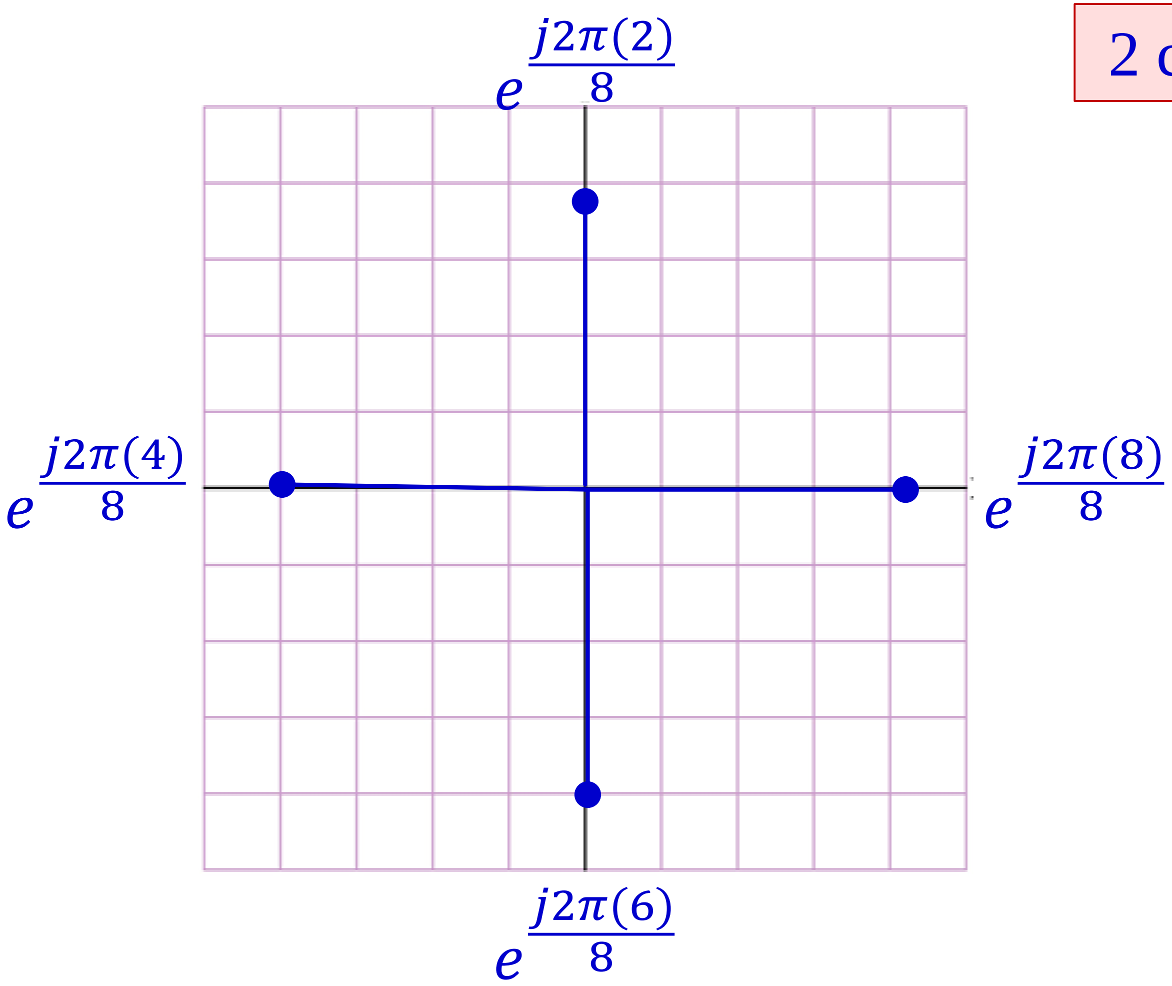
1 cycle in 8 steps

$$\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(1)}{8}} \\ e^{\frac{j2\pi(2)}{8}} \\ \dots \\ e^{\frac{j2\pi(7)}{8}} \end{bmatrix}$$

f_1

Imagine Rotation as a Vector

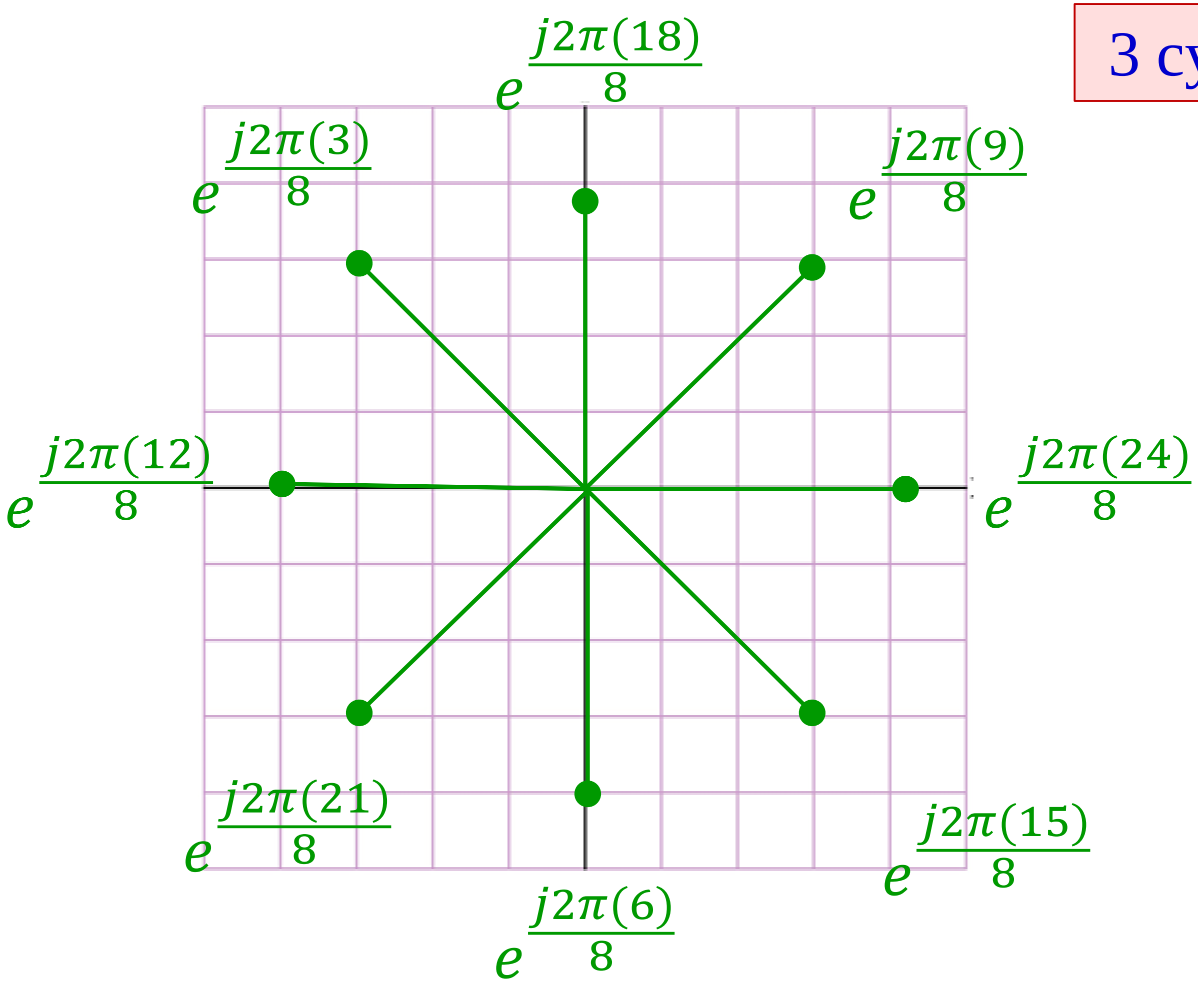
2 cycles in 8 steps



$$\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(2)}{8}} \\ e^{\frac{j2\pi(4)}{8}} \\ \cdots \\ e^{\frac{j2\pi(14)}{8}} \end{bmatrix}$$

f_2

Imagine Rotation as a Vector



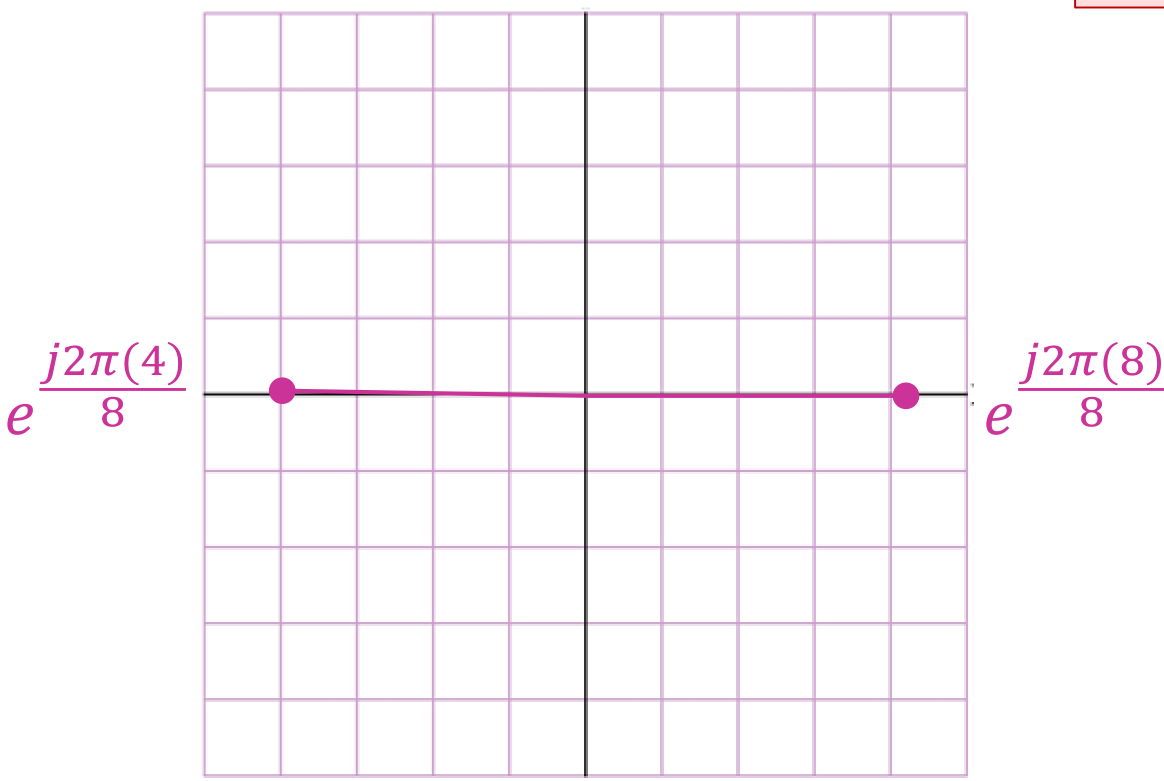
3 cycles in 8 steps

$$\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(3)}{8}} \\ e^{\frac{j2\pi(6)}{8}} \\ \dots \\ e^{\frac{j2\pi(21)}{8}} \end{bmatrix}$$

f_3

Imagine Rotation as a Vector

4 cycles in 8 steps



$$\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(4)}{8}} \\ e^{\frac{j2\pi(8)}{8}} \\ \cdots \\ e^{\frac{j2\pi(28)}{8}} \end{bmatrix}$$

f_4

Imagine Rotation as a Vector

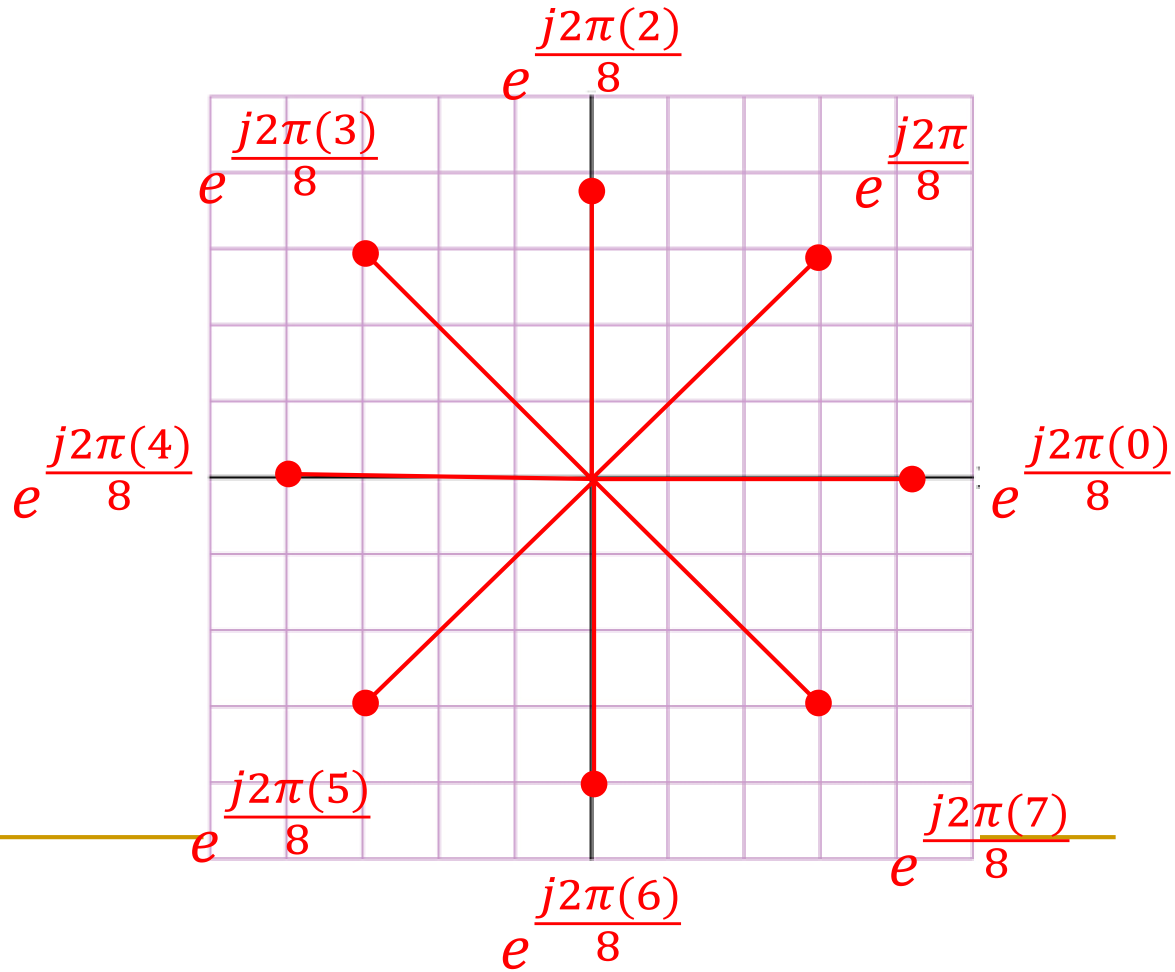
$$
\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \end{bmatrix}
\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(1)}{8}} \\ e^{\frac{j2\pi(2)}{8}} \\ e^{\frac{j2\pi(3)}{8}} \\ e^{\frac{j2\pi(4)}{8}} \\ e^{\frac{j2\pi(5)}{8}} \\ e^{\frac{j2\pi(6)}{8}} \\ e^{\frac{j2\pi(7)}{8}} \end{bmatrix}
\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(2)}{8}} \\ e^{\frac{j2\pi(4)}{8}} \\ e^{\frac{j2\pi(6)}{8}} \\ e^{\frac{j2\pi(8)}{8}} \\ e^{\frac{j2\pi(10)}{8}} \\ e^{\frac{j2\pi(12)}{8}} \\ e^{\frac{j2\pi(14)}{8}} \end{bmatrix}
\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(3)}{8}} \\ e^{\frac{j2\pi(6)}{8}} \\ e^{\frac{j2\pi(9)}{8}} \\ e^{\frac{j2\pi(12)}{8}} \\ e^{\frac{j2\pi(15)}{8}} \\ e^{\frac{j2\pi(18)}{8}} \\ e^{\frac{j2\pi(21)}{8}} \end{bmatrix}
\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(4)}{8}} \\ e^{\frac{j2\pi(8)}{8}} \\ e^{\frac{j2\pi(12)}{8}} \\ e^{\frac{j2\pi(16)}{8}} \\ e^{\frac{j2\pi(20)}{8}} \\ e^{\frac{j2\pi(24)}{8}} \\ e^{\frac{j2\pi(28)}{8}} \end{bmatrix}
\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(5)}{8}} \\ e^{\frac{j2\pi(10)}{8}} \\ e^{\frac{j2\pi(15)}{8}} \\ e^{\frac{j2\pi(20)}{8}} \\ e^{\frac{j2\pi(25)}{8}} \\ e^{\frac{j2\pi(30)}{8}} \\ e^{\frac{j2\pi(35)}{8}} \end{bmatrix}
\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(6)}{8}} \\ e^{\frac{j2\pi(12)}{8}} \\ e^{\frac{j2\pi(18)}{8}} \\ e^{\frac{j2\pi(24)}{8}} \\ e^{\frac{j2\pi(30)}{8}} \\ e^{\frac{j2\pi(36)}{8}} \\ e^{\frac{j2\pi(42)}{8}} \end{bmatrix}
\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(7)}{8}} \\ e^{\frac{j2\pi(14)}{8}} \\ e^{\frac{j2\pi(21)}{8}} \\ e^{\frac{j2\pi(28)}{8}} \\ e^{\frac{j2\pi(35)}{8}} \\ e^{\frac{j2\pi(42)}{8}} \\ e^{\frac{j2\pi(49)}{8}} \end{bmatrix}
$$

$\boxed{f_0}$ $\boxed{f_1}$ $\boxed{f_2}$ $\boxed{f_3}$ $\boxed{f_4}$ $\boxed{f_5}$ $\boxed{f_6}$ $\boxed{f_7}$

The vectors $f_0, f_1, \dots, f_7$ are ORTHOGONAL

Imagine Rotation as a Vector

$$[1 \quad 1 \quad 1 \quad \dots \quad 1].\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(1)}{8}} \\ e^{\frac{j2\pi(2)}{8}} \\ \dots \\ e^{\frac{j2\pi(7)}{8}} \end{bmatrix} = e^{\frac{j2\pi(0)}{8}} + e^{\frac{j2\pi(1)}{8}} + \dots + e^{\frac{j2\pi(7)}{8}}$$



Imagine Rotation as a Vector

$$\begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(0)}{8}} \end{bmatrix} \begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(1)}{8}} \\ e^{\frac{j2\pi(2)}{8}} \\ e^{\frac{j2\pi(3)}{8}} \\ e^{\frac{j2\pi(4)}{8}} \\ e^{\frac{j2\pi(5)}{8}} \\ e^{\frac{j2\pi(6)}{8}} \\ e^{\frac{j2\pi(7)}{8}} \end{bmatrix} \begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(2)}{8}} \\ e^{\frac{j2\pi(4)}{8}} \\ e^{\frac{j2\pi(6)}{8}} \\ e^{\frac{j2\pi(8)}{8}} \\ e^{\frac{j2\pi(10)}{8}} \\ e^{\frac{j2\pi(12)}{8}} \\ e^{\frac{j2\pi(14)}{8}} \end{bmatrix} \begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(3)}{8}} \\ e^{\frac{j2\pi(6)}{8}} \\ e^{\frac{j2\pi(9)}{8}} \\ e^{\frac{j2\pi(12)}{8}} \\ e^{\frac{j2\pi(15)}{8}} \\ e^{\frac{j2\pi(18)}{8}} \\ e^{\frac{j2\pi(21)}{8}} \end{bmatrix} \begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(4)}{8}} \\ e^{\frac{j2\pi(8)}{8}} \\ e^{\frac{j2\pi(12)}{8}} \\ e^{\frac{j2\pi(16)}{8}} \\ e^{\frac{j2\pi(20)}{8}} \\ e^{\frac{j2\pi(24)}{8}} \\ e^{\frac{j2\pi(28)}{8}} \end{bmatrix} \begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(5)}{8}} \\ e^{\frac{j2\pi(10)}{8}} \\ e^{\frac{j2\pi(15)}{8}} \\ e^{\frac{j2\pi(20)}{8}} \\ e^{\frac{j2\pi(25)}{8}} \\ e^{\frac{j2\pi(30)}{8}} \\ e^{\frac{j2\pi(35)}{8}} \end{bmatrix} \begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(6)}{8}} \\ e^{\frac{j2\pi(12)}{8}} \\ e^{\frac{j2\pi(18)}{8}} \\ e^{\frac{j2\pi(24)}{8}} \\ e^{\frac{j2\pi(30)}{8}} \\ e^{\frac{j2\pi(36)}{8}} \\ e^{\frac{j2\pi(42)}{8}} \end{bmatrix} \begin{bmatrix} e^{\frac{j2\pi(0)}{8}} \\ e^{\frac{j2\pi(7)}{8}} \\ e^{\frac{j2\pi(14)}{8}} \\ e^{\frac{j2\pi(21)}{8}} \\ e^{\frac{j2\pi(28)}{8}} \\ e^{\frac{j2\pi(35)}{8}} \\ e^{\frac{j2\pi(42)}{8}} \\ e^{\frac{j2\pi(49)}{8}} \end{bmatrix}$$

The vectors $f_0, f_1, \dots, f_{N-1}$ are ORTHOGONAL → they are linearly independent

Vectors $f_0, f_1, \dots, f_{N-1}$ form a BASIS in N-dimensional space

Time Domain Time To Frequency Domain

$$IX_t = FX_f$$

$$\begin{bmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & . & . & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} | & | & | & \cdots & | \\ f_0 & f_1 & f_2 & \cdots & f_{N-1} \\ | & | & | & \cdots & | \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ . \\ . \\ . \\ z_{N-1} \end{bmatrix}$$

Basis vectors (Time) — Time domain signal — Basis vectors (Frequency) — Frequency domain signal

DFT is the representation of a signal in a different (frequency) basis

Discrete Fourier Transform

$$IX_t = FX_f \quad \rightarrow \quad X_f = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \dots \\ z_{N-1} \end{bmatrix} = F^{-1}X_t = (F^*)^T X_t = \begin{bmatrix} ----- & f_0^* & ----- \\ ----- & f_1^* & ----- \\ ----- & f_2^* & ----- \\ \dots & \dots & \dots \\ ----- & f_{N-1}^* & ----- \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \dots \\ x_{N-1} \end{bmatrix}$$

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \dots \\ z_7 \end{bmatrix} = \begin{bmatrix} e^{\frac{-j2\pi(0)}{8}} & e^{\frac{-j2\pi(0)}{8}} & e^{\frac{-j2\pi(0)}{8}} & e^{\frac{-j2\pi(0)}{8}} & e^{\frac{-j2\pi(0)}{8}} & e^{\frac{-j2\pi(0)}{8}} & e^{\frac{-j2\pi(0)}{8}} & e^{\frac{-j2\pi(0)}{8}} \\ e^{\frac{-j2\pi(0)}{8}} & e^{\frac{-j2\pi(1)}{8}} & e^{\frac{-j2\pi(2)}{8}} & e^{\frac{-j2\pi(3)}{8}} & e^{\frac{-j2\pi(4)}{8}} & e^{\frac{-j2\pi(5)}{8}} & e^{\frac{-j2\pi(6)}{8}} & e^{\frac{-j2\pi(7)}{8}} \\ e^{\frac{-j2\pi(0)}{8}} & e^{\frac{-j2\pi(2)}{8}} & e^{\frac{-j2\pi(4)}{8}} & e^{\frac{-j2\pi(6)}{8}} & e^{\frac{-j2\pi(8)}{8}} & e^{\frac{-j2\pi(10)}{8}} & e^{\frac{-j2\pi(12)}{8}} & e^{\frac{-j2\pi(14)}{8}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ e^{\frac{-j2\pi(0)}{8}} & e^{\frac{-j2\pi(7)}{8}} & e^{\frac{-j2\pi(14)}{8}} & e^{\frac{-j2\pi(21)}{8}} & e^{\frac{-j2\pi(28)}{8}} & e^{\frac{-j2\pi(35)}{8}} & e^{\frac{-j2\pi(42)}{8}} & e^{\frac{-j2\pi(49)}{8}} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \dots \\ x_7 \end{bmatrix}$$

$$z_2 = e^{\frac{-j2\pi(0)}{8}} x_0 + e^{\frac{-j2\pi(2)}{8}} x_1 + e^{\frac{-j2\pi(4)}{8}} x_2 + e^{\frac{-j2\pi(6)}{8}} x_3 + e^{\frac{-j2\pi(8)}{8}} x_4 + e^{\frac{-j2\pi(10)}{8}} x_5 + e^{\frac{-j2\pi(12)}{8}} x_6 + e^{\frac{-j2\pi(14)}{8}} x_7 = \sum_{n=0}^{7} x_n e^{\frac{-j2\pi.2.n}{8}}$$

$$z_m = \sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi.m.n}{N}}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt$$

Inverse Discrete Fourier Transform

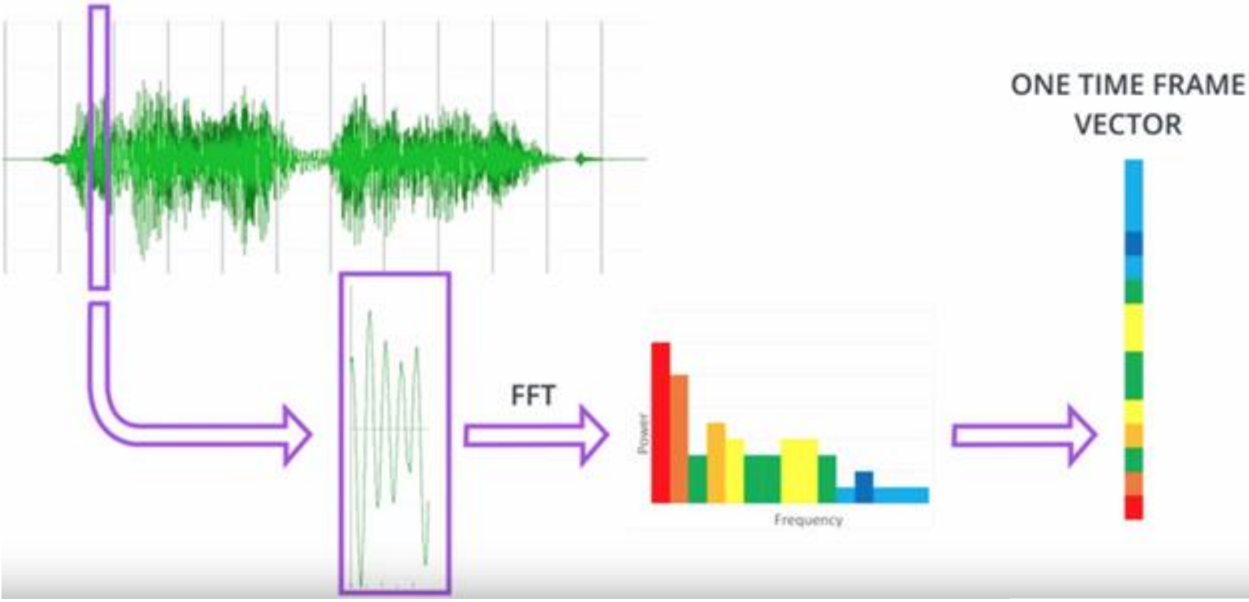
$$IX_t = FX_f \quad \rightarrow \quad X_t = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \cdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} | & | & | & \cdots & | \\ f_0 & f_1 & f_2 & \cdots & f_{N-1} \\ | & | & | & \cdots & | \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \cdots \\ z_{N-1} \end{bmatrix}$$

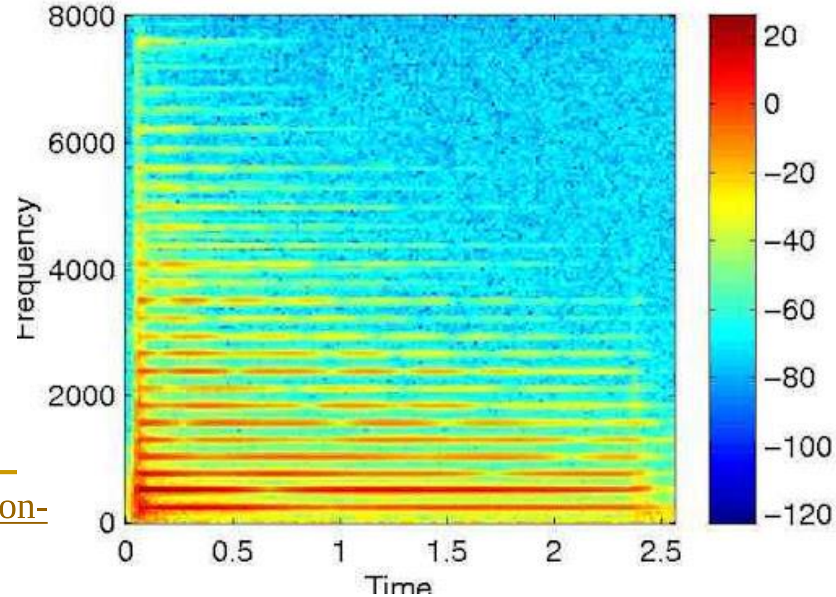
$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \cdots \\ x_7 \end{bmatrix} = \begin{bmatrix}
e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(0)}{8}} \\
e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(1)}{8}} & e^{\frac{j2\pi(2)}{8}} & e^{\frac{j2\pi(3)}{8}} & e^{\frac{j2\pi(4)}{8}} & e^{\frac{j2\pi(5)}{8}} & e^{\frac{j2\pi(6)}{8}} & e^{\frac{j2\pi(7)}{8}} \\
e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(2)}{8}} & e^{\frac{j2\pi(4)}{8}} & e^{\frac{j2\pi(6)}{8}} & e^{\frac{j2\pi(8)}{8}} & e^{\frac{j2\pi(10)}{8}} & e^{\frac{j2\pi(12)}{8}} & e^{\frac{j2\pi(14)}{8}} \\
e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(3)}{8}} & e^{\frac{j2\pi(6)}{8}} & e^{\frac{j2\pi(9)}{8}} & e^{\frac{j2\pi(12)}{8}} & e^{\frac{j2\pi(15)}{8}} & e^{\frac{j2\pi(18)}{8}} & e^{\frac{j2\pi(21)}{8}} \\
e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(4)}{8}} & e^{\frac{j2\pi(8)}{8}} & e^{\frac{j2\pi(12)}{8}} & e^{\frac{j2\pi(16)}{8}} & e^{\frac{j2\pi(20)}{8}} & e^{\frac{j2\pi(24)}{8}} & e^{\frac{j2\pi(28)}{8}} \\
e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(5)}{8}} & e^{\frac{j2\pi(10)}{8}} & e^{\frac{j2\pi(15)}{8}} & e^{\frac{j2\pi(20)}{8}} & e^{\frac{j2\pi(25)}{8}} & e^{\frac{j2\pi(30)}{8}} & e^{\frac{j2\pi(35)}{8}} \\
e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(6)}{8}} & e^{\frac{j2\pi(12)}{8}} & e^{\frac{j2\pi(18)}{8}} & e^{\frac{j2\pi(24)}{8}} & e^{\frac{j2\pi(30)}{8}} & e^{\frac{j2\pi(36)}{8}} & e^{\frac{j2\pi(42)}{8}} \\
e^{\frac{j2\pi(0)}{8}} & e^{\frac{j2\pi(7)}{8}} & e^{\frac{j2\pi(14)}{8}} & e^{\frac{j2\pi(21)}{8}} & e^{\frac{j2\pi(28)}{8}} & e^{\frac{j2\pi(35)}{8}} & e^{\frac{j2\pi(42)}{8}} & e^{\frac{j2\pi(49)}{8}}
\end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \cdots \\ z_7 \end{bmatrix}$$

$$x_2 = e^{\frac{j2\pi(0)}{8}} z_0 + e^{\frac{j2\pi(2)}{8}} z_1 + e^{\frac{j2\pi(4)}{8}} z_2 + e^{\frac{j2\pi(6)}{8}} z_3 + e^{\frac{j2\pi(8)}{8}} z_4 + e^{\frac{j2\pi(10)}{8}} z_5 + e^{\frac{j2\pi(12)}{8}} z_6 + e^{\frac{j2\pi(14)}{8}} z_7 = \sum_{n=0}^{7} z_n e^{\frac{j2\pi.2.n}{8}}$$

$$x_n = \frac{1}{N} \sum_{m=0}^{N-1} z_m e^{\frac{j2\pi.m.n}{N}} \qquad x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t}\, df$$

Spectrogram





https://medium.com/nerd-for-tech/indian-accent-speech-recognition-2d433eb7edac
http://coding-geek.com/how-shazam-works/

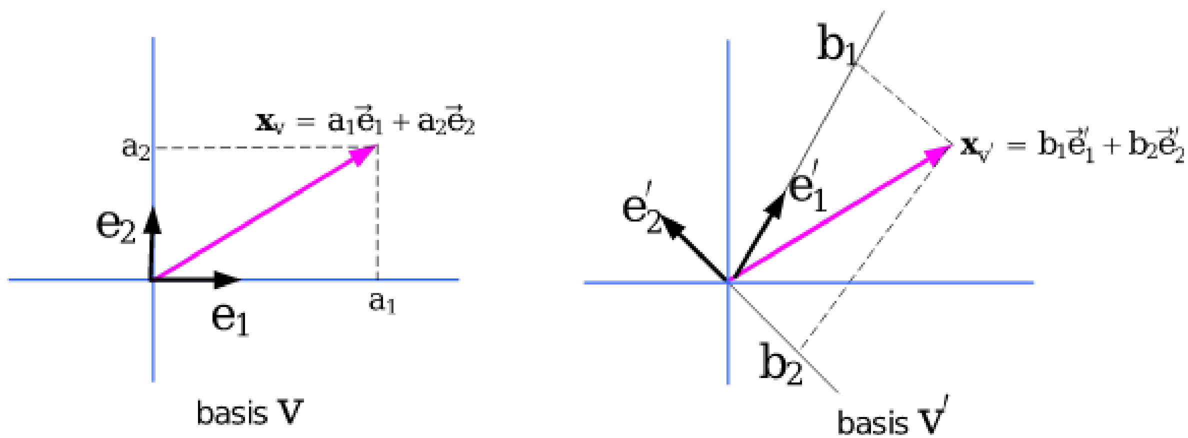
Extra

Given a vector $\mathbf{x}$, it is represented w.r.t. basis $v = \{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ as

$$\mathbf{x}_v = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n$$

and w.r.t. basis $v' = \{\vec{e}'_1, \vec{e}'_2, \cdots, \vec{e}'_n\}$ as

$$\mathbf{x}_{v'} = b_1\vec{e}'_1 + b_2\vec{e}'_2 + \cdots + b_n\vec{e}'_n$$



The same vector having different representation depending on basis used

But the vector itself is invariant under any change of basis, hence $\mathbf{x}_v = \mathbf{x}_{v'}$

$$a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n = b_1\vec{e}'_1 + b_2\vec{e}'_2 + \cdots + b_n\vec{e}'_n$$

https://12000.org/my_notes/similarity_transformation_and_SVD/index.htm

Extra

$$v = \begin{bmatrix} e^{\frac{j2\pi(0)}{N}} & e^{\frac{j2\pi(1)}{N}} & e^{\frac{j2\pi(2)}{N}} & \ldots & e^{\frac{j2\pi(N-1)}{N}} \end{bmatrix}$$

$$\sqrt{v^* v^T} = \sqrt{\begin{bmatrix} e^{\frac{-j2\pi(0)}{N}} & e^{\frac{-j2\pi(1)}{N}} & e^{\frac{-j2\pi(2)}{N}} & \ldots & e^{\frac{-j2\pi(N-1)}{N}} \end{bmatrix} . \begin{bmatrix} e^{\frac{j2\pi(0)}{N}} \\ e^{\frac{j2\pi(1)}{N}} \\ e^{\frac{j2\pi(2)}{N}} \\ \ldots \\ e^{\frac{j2\pi(N-1)}{N}} \end{bmatrix}} = \sqrt{N}$$