Probability Basics: A Gentle Introduction

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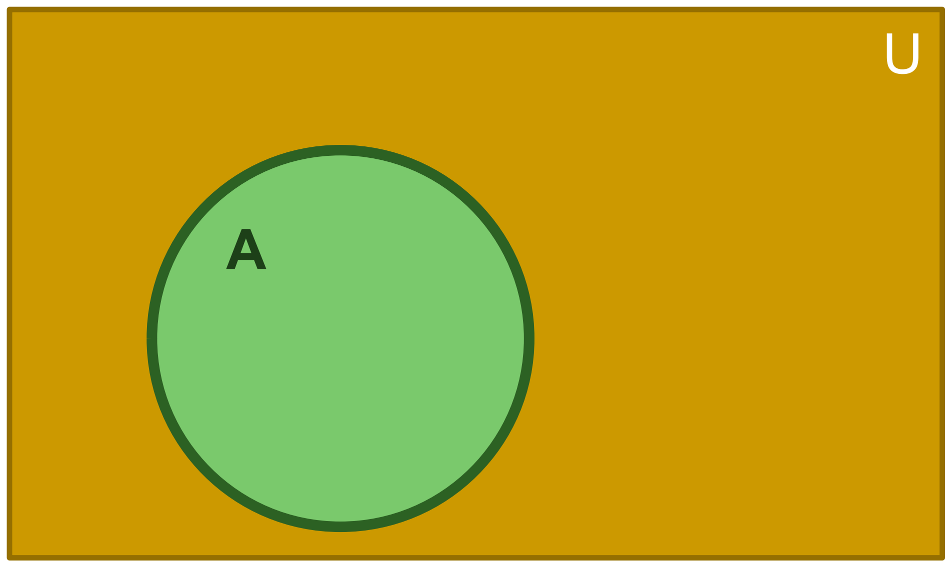
Probability – Sample Space & Events

- Sample space is a collection of all possible outcomes in an experiment

- An event is a subset of the sample space



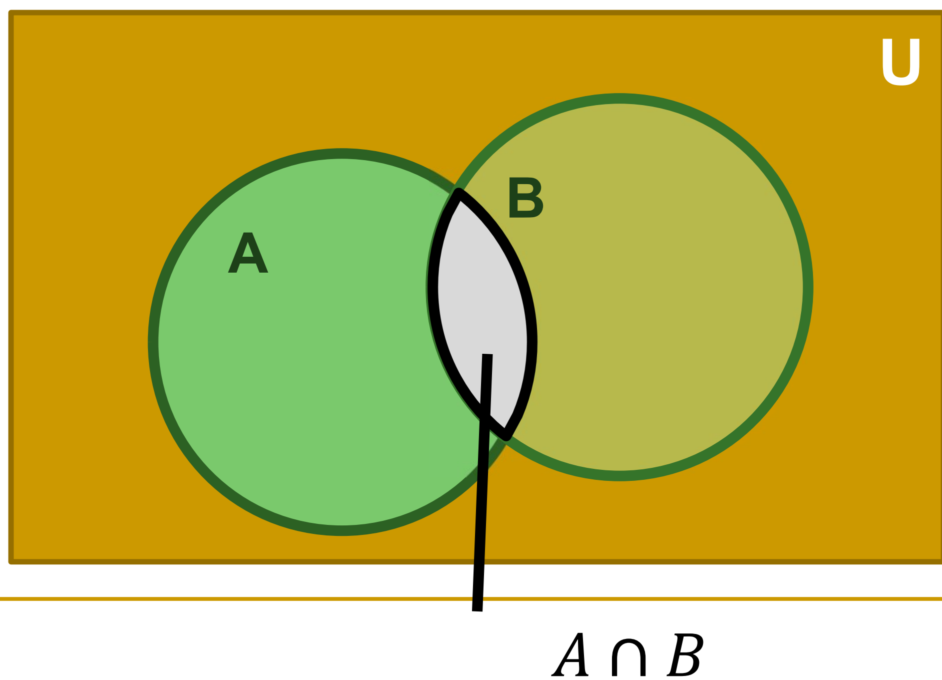
$$P[A] = \frac{|A|}{|U|}$$

Probability - Example

- Throw a fair coin thrice. What is the probability of getting exactly one tail?
  - All possible outcomes: S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
  - Event of our interest is exactly one tail, i.e. E = {HHT, HTH, THH}
  - $P(E) = \frac{|E|}{|S|} = \frac{3}{8}$

Probability - Basics

- Consider the set of integers from 1 to 10
- Let A be the event of getting an even number → 2, 4, 6, 8, 10
- Let B be the event of getting an integer greater than 5 → 6, 7, 8, 9, 10
- What is the probability that a randomly drawn integer will be both even and greater than 5



$$P[E] = \frac{|A \cap B|}{|U|} = \frac{3}{10}$$

Probability - Basics

- Consider the set of integers from 1 to 10
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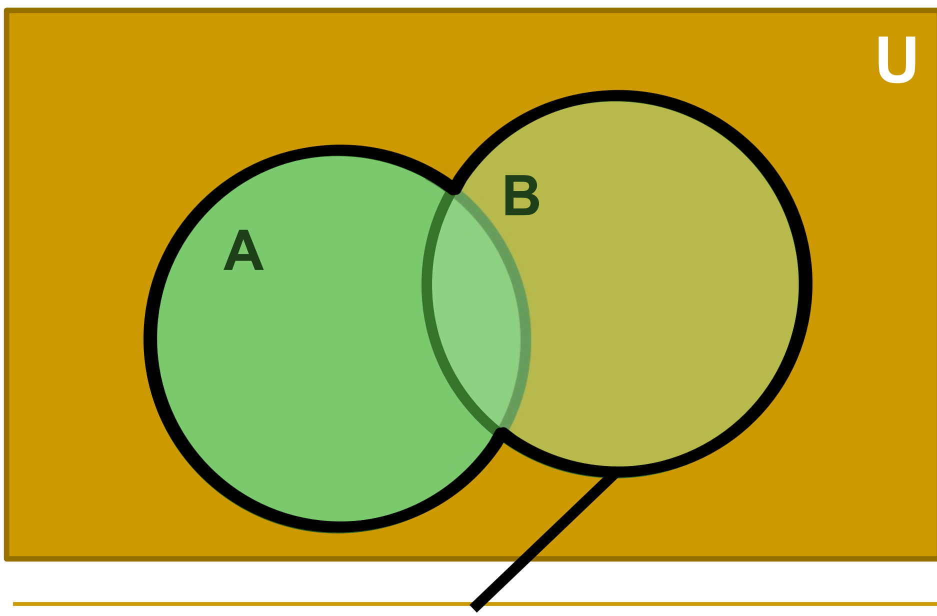


$A \cup B$

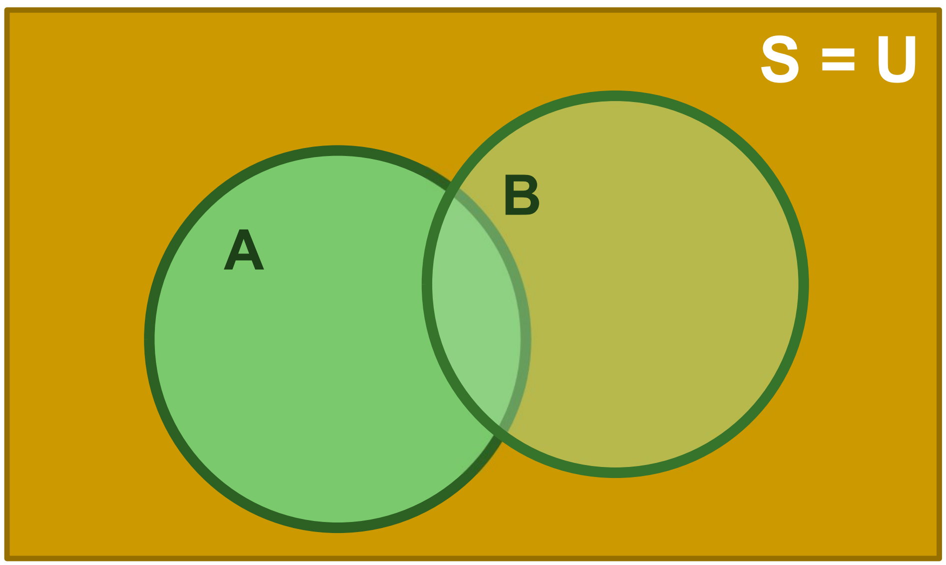
$$P[E] = \frac{|A \cup B|}{|U|}$$

$$= P[A] + P[B] - P[A \cap B]$$

Conditional Probability

- **Suppose A has already occurred**

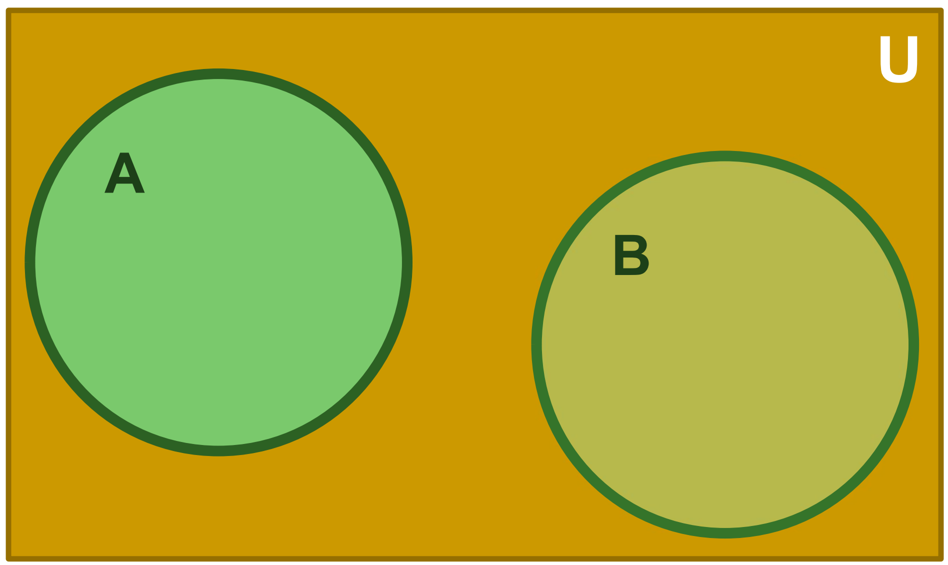
$$P[B|A] = \frac{size\ of\ A \cap B}{size\ of\ A} = \frac{\frac{size\ of\ A \cap B}{size\ of\ S}}{\frac{size\ of\ A}{size\ of\ S}} = \frac{P[A \cap B]}{P[A]}$$



$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

Independent Events

- Set A and B are independent if randomly picking from the set A tells us nothing about its membership in B



Are A and B independent?

Independent Events

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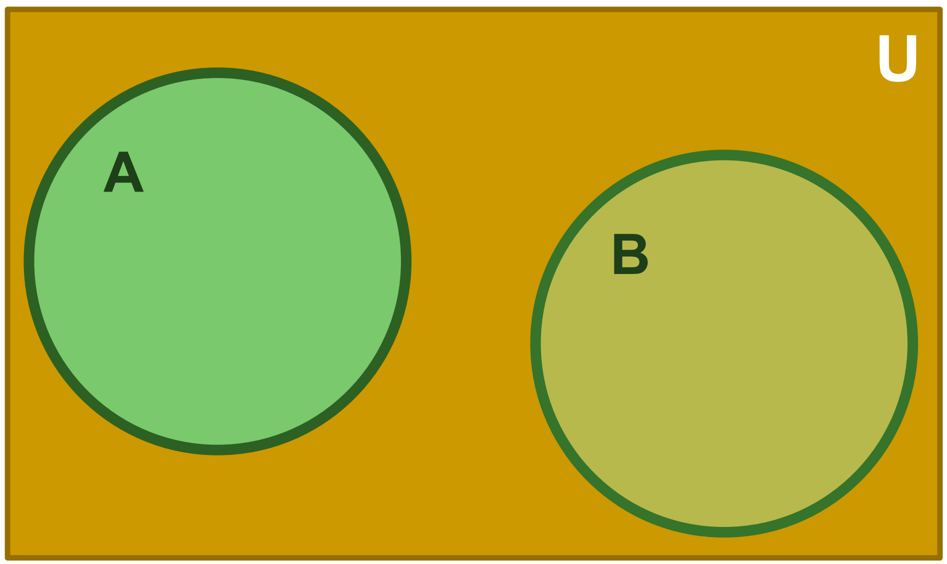

A and B are mutually exclusive

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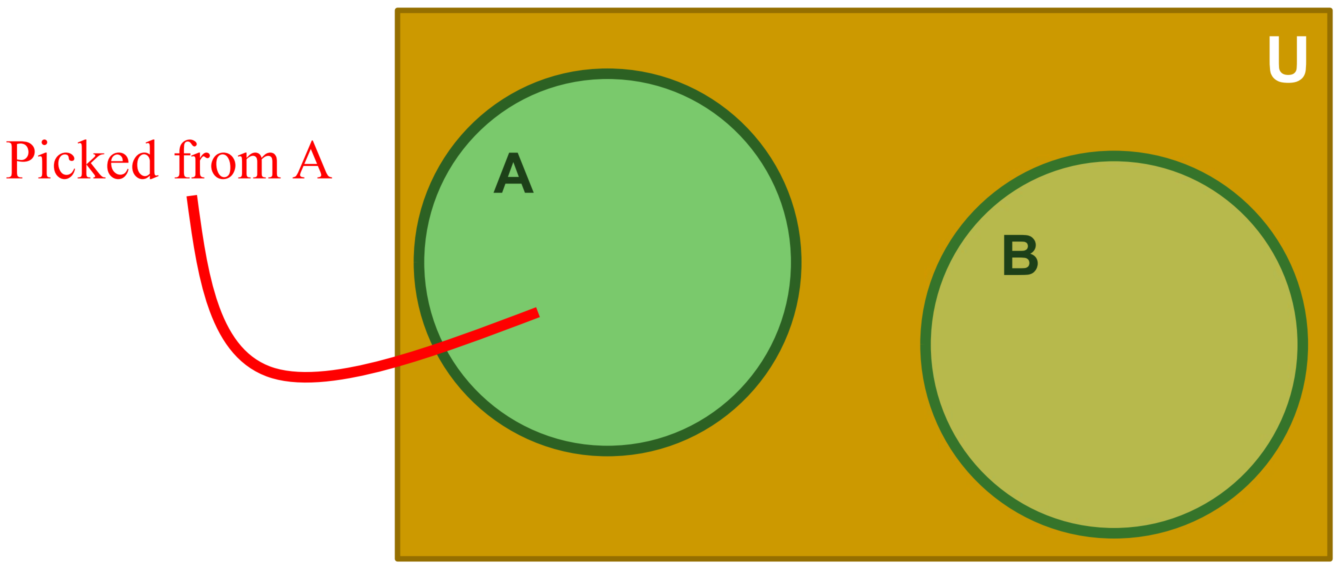

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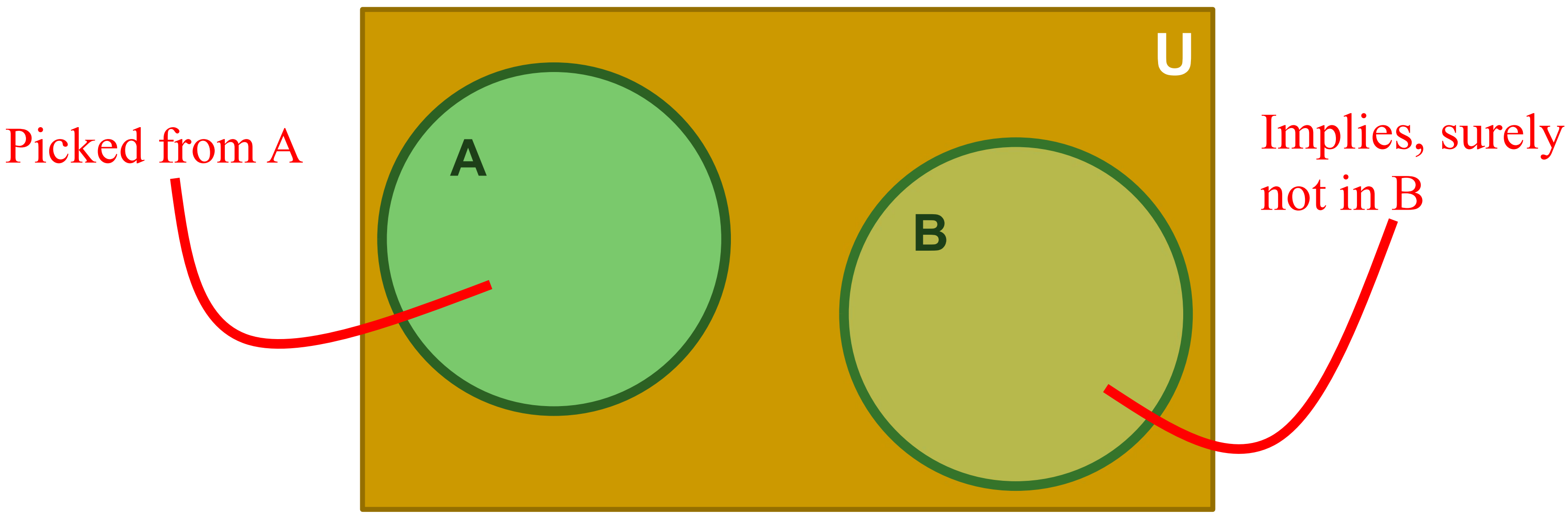


A and B are mutually exclusive

Independent Events

- Set A and B are independent if randomly picking from the set A tells us nothing about its membership in B

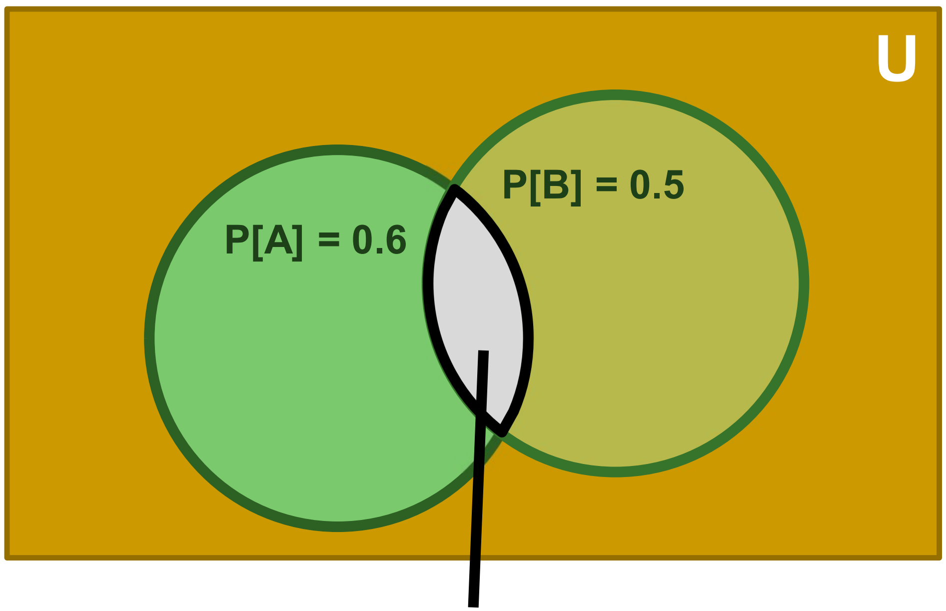
$$P[A|B] = P[A], \qquad P[B|A] = P[B]$$

$$P[A|B] = \frac{P[A \cap B]}{P[B]}, \qquad P[B|A] = \frac{P[A \cap B]}{P[A]}$$

$$P[A \cap B] = P[A|B].P[B]$$

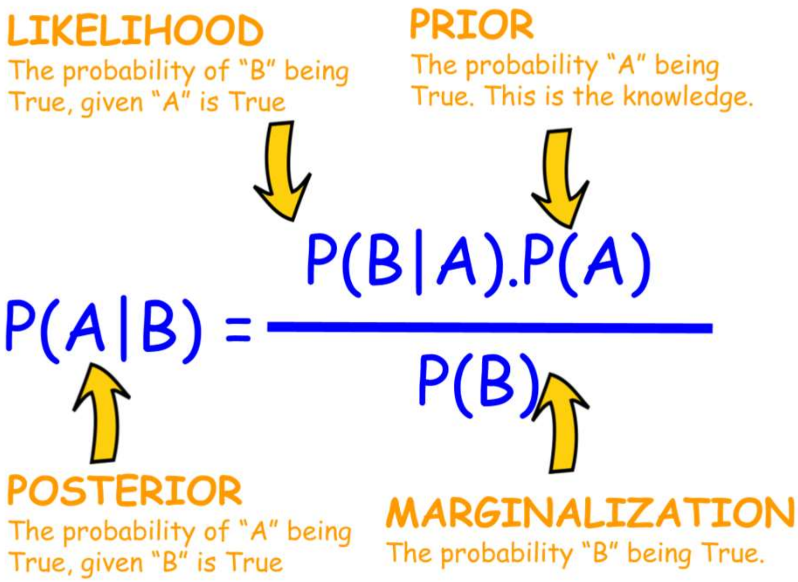
$$P[A \cap B] = P[A].P[B]$$

Independent Events



$$P[A \cap B] = 0.3$$

Bayes’ theorem



1 $$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)}} = \frac{P(B|A) \times P(A)}{P(B)}$$

2 $$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

Bayes' theorem

10% of patients in a clinic have liver disease. Five percent of the clinic's patients are alcoholics. Amongst those patients diagnosed with liver disease, 7% are alcoholics. You are interested in knowing the probability of a patient having liver disease, given that he is an alcoholic.

P(A) = probability of liver disease = 0.10
P(B) = probability of alcoholism = 0.05
P(B|A) = 0.07
P(A|B) = ?

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)} = \frac{0.07 \times 0.10}{0.05} = 0.14$$

In other words, if the patient is an alcoholic, their chances of having liver disease is 0.14 (14%)

Random Variable

Definition: Random Variable

A random variable is a rule that assigns a numerical value to an outcome of interest.

Example: Suppose we toss a coin 3 times, we define a random variable X as the number of outcomes where a tail appears.

This random variable can take values of $0, 1, 2$ *and* 3.

- The probability that the random variable takes a given value can be computed using the rules governing probability.
  - For example, the probability that $X = 1$ means there will be 1 tail. Symbolically, it is denoted as **P(X=1) = 3/8**

Probability Distribution

Definition: Probability distribution

A probability distribution is a definition of probabilities of the values of random variable.

Example: Then the probability distribution is given by

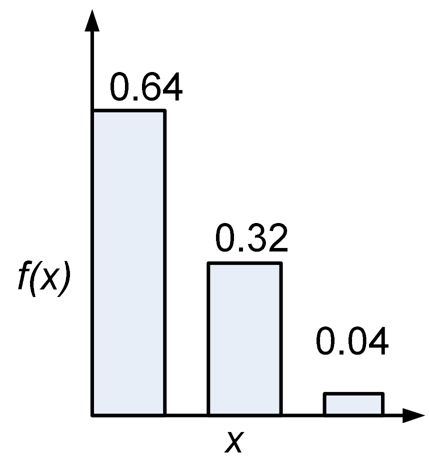
X	Probability
0	1/8
1	3/8
2	3/8
3	1/8

Probability Distribution

- In general, a probability distribution function takes the following form

x	x_1	$x_2 \ldots \ldots \ldots \ldots . . x_n$
$f(x) = P(X = x)$	$f(x_1)$	$f(x_2) \ldots \ldots . . f(x_n)$

x	0	1	2
$f(x)$	0.64	0.32	0.04



Descriptive measures

Given a random variable *X* in an experiment, we have denoted $f(x) = P(X = x)$, the probability that $X = x$. For discrete events $f(x) = 0$ for all values of x except $x = 0, 1, 2, \dots..$

Properties of discrete probability distribution

$0 \le f(x) \le 1$

$\sum f(x) = 1$

$\mu = \sum x.f(x)$ [is the mean]

$\sigma^2 = \sum (x-\mu)^2 . f(x)$ [is the variance]

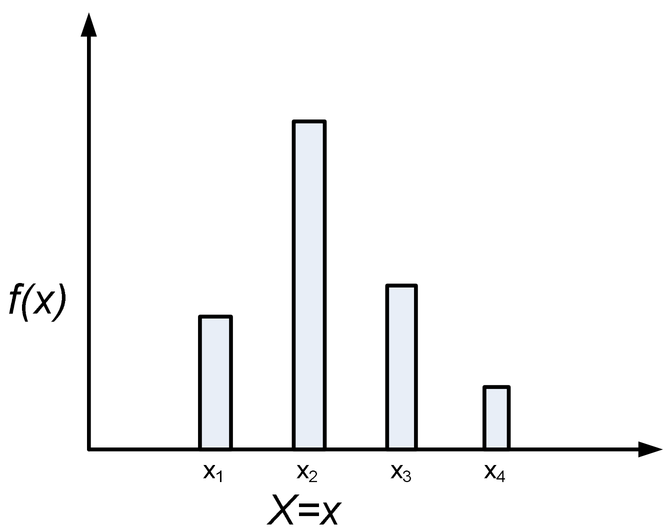
Note: For discrete uniform distribution, $f(x) = \frac{1}{n}$ with $x = 1, 2, \dots \dots, n$

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$

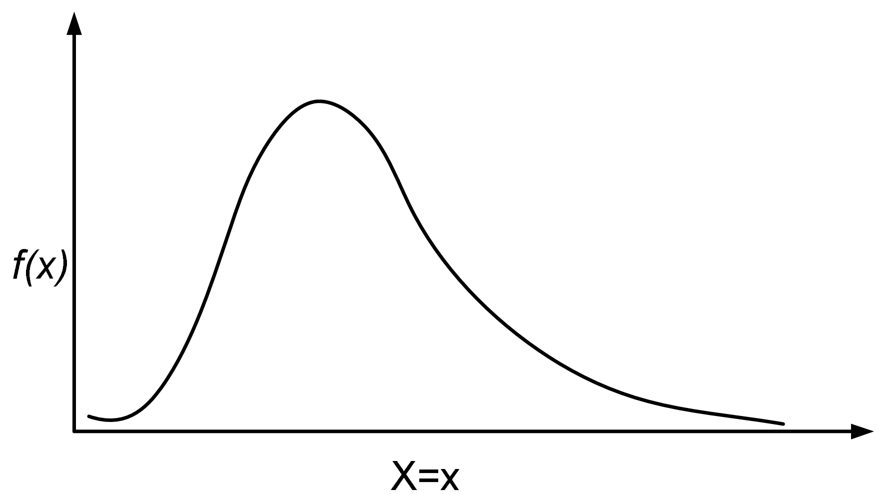
and $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$

Continuous Probability Distributions

- When the random variable of interest can take any value in an interval, it is called continuous random variable.
  - Every continuous random variable has an infinite, uncountable number of possible values (i.e., any value in an interval)



Discrete Probability distribution



Continuous Probability Distribution

Properties of Probability Density Function

The function $f(x)$ is a probability density function for the continuous random variable X, defined over the set of real numbers R, if

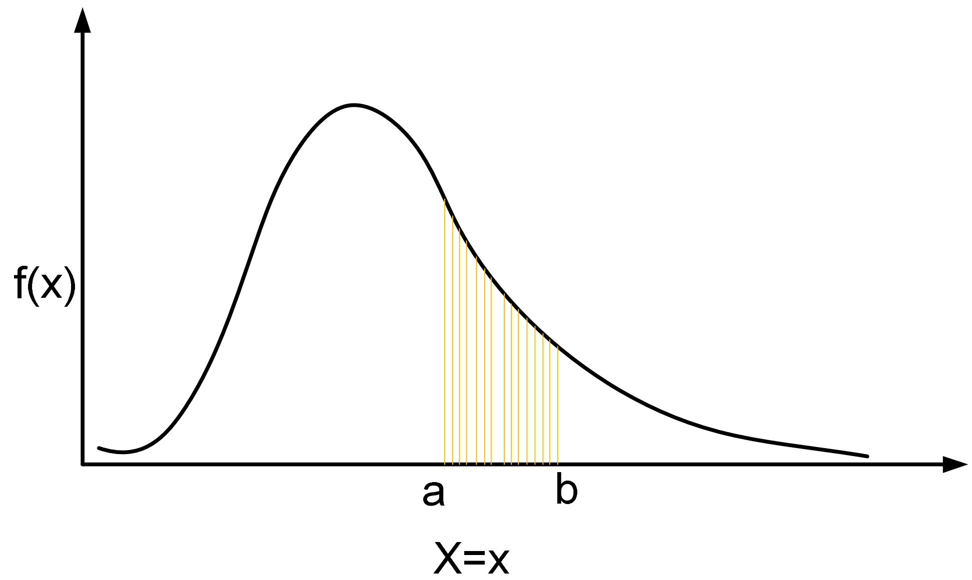
$f(x) \geq 0$, for all $x \in R$

$\int_{-\infty}^{\infty} f(x)dx = 1$

$P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx$

$\mu = \int_{-\infty}^{\infty} xf(x)\, dx$

$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$



Continuous Uniform Distribution

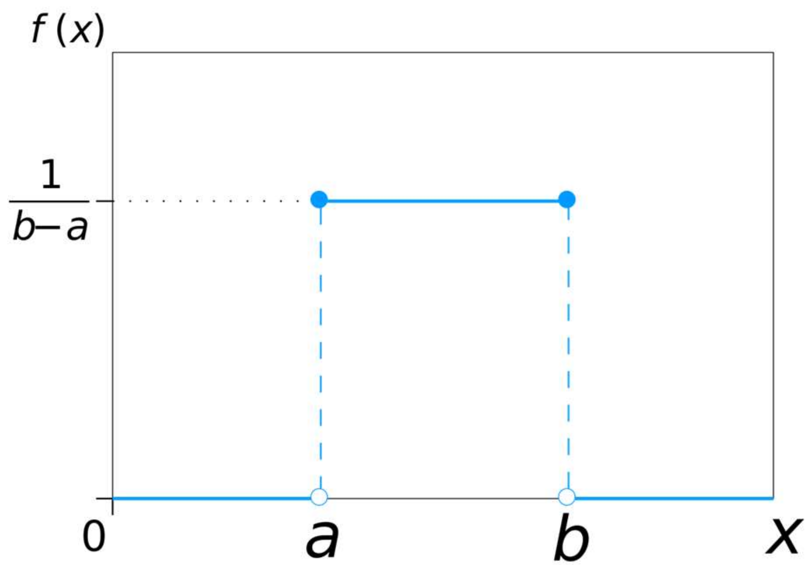
- One of the simplest continuous distribution in all of statistics is the continuous uniform distribution.

Definition: **Continuous Uniform Distribution**

The density function of the continuous uniform random variable X on the interval $[a, b]$ is:

$$f(x: a, b) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & Otherwise \end{cases}$$

Continuous Uniform Distribution



Note:

a) $\int_{\infty}^{-\infty} f(x)dx = \frac{1}{b-a} \times (b-a) = 1$

b) $P(c < x < d)$= $\frac{d-c}{b-a}$ where both c and d are in the interval (a,b)

c) $\mu = \frac{a+b}{2}$

d) $\sigma^2 = \frac{(b-a)^2}{12}$

Normal Distribution

- The most often used continuous probability distribution is the normal distribution; it is also known as **Gaussian** distribution.
- Its graph called the normal curve is the bell-shaped curve.
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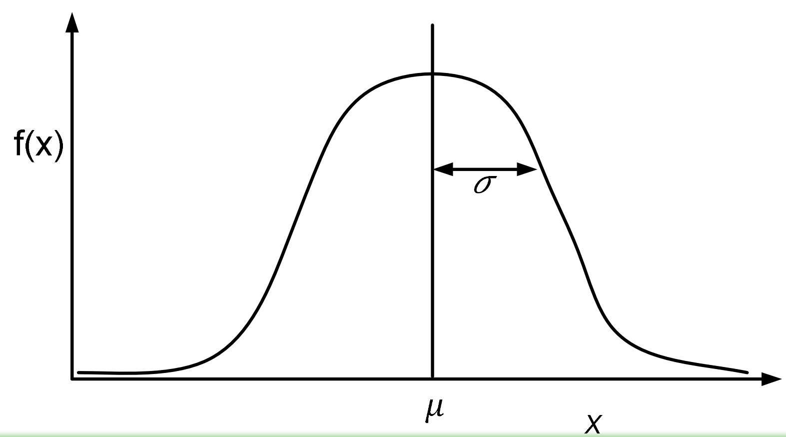
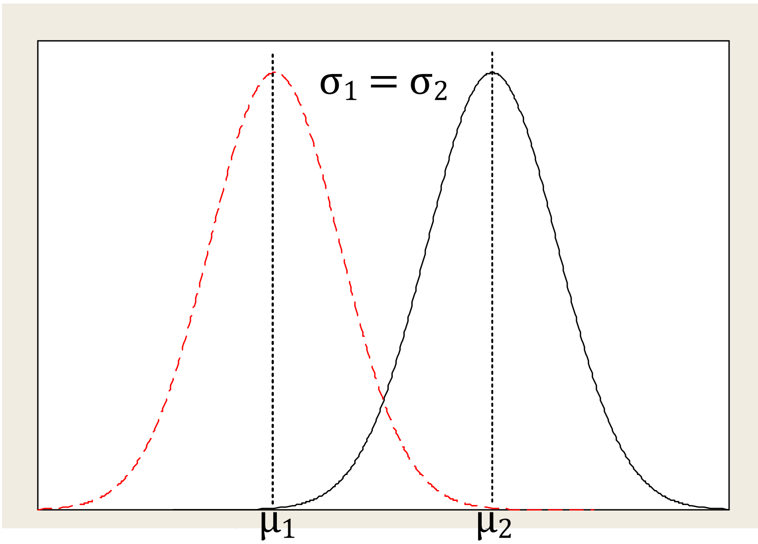
- 

Definition: Normal distribution

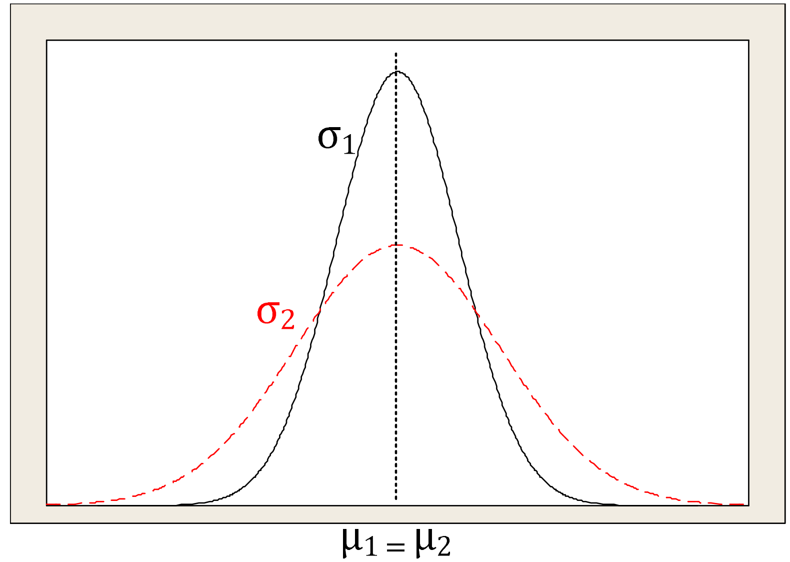
The density of the normal variable x with mean μ and variance σ^2 is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-{(x-\mu)^2}/{2\sigma^2}} \qquad -\infty < x < \infty$$

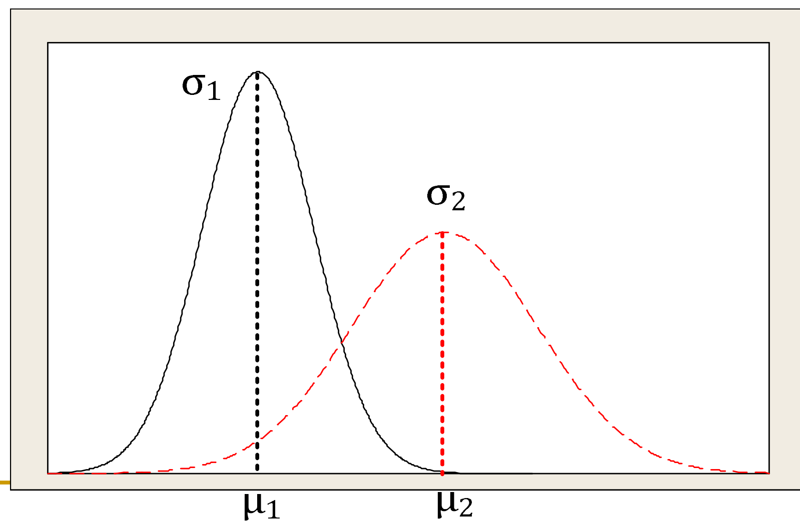
Normal Distribution



Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$



Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$



Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$

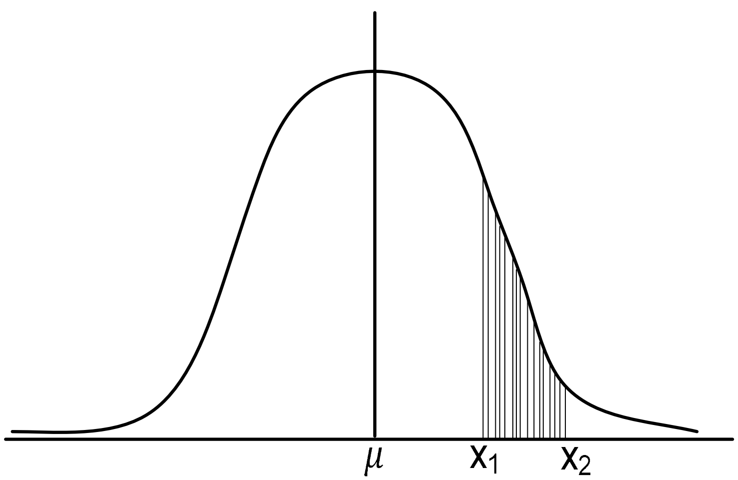
Properties of Normal Distribution

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx = 1$$

$$\mu = \int_{-\infty}^{\infty} x.f(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x.e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$$

$$\sigma^2 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2.e^{-\frac{1}{2}[{(x-\mu)}/{\sigma^2}]}dx$$

$$P(x_1 < x < x_2) = \frac{1}{\sigma\sqrt{2\pi}}\int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$$



Examples of Random Variables

Continuous-valued random variable:

Uniform:

$$p(x) = 1 \quad 0 \le x \le 1$$

Gaussian:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty \le x \le \infty$$

Discrete-valued random variable:

$$p(x_i) = \frac{1}{M} \quad i = 0,\ldots,M-1$$

N/A

Check: http://mathlets.org/mathlets/probability-distributions/