Linear Algebra: A Gentle Introduction

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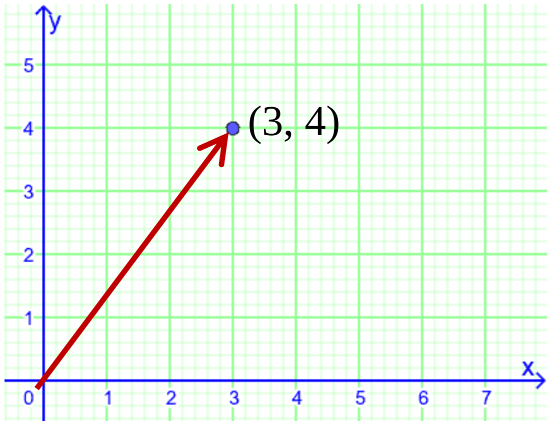
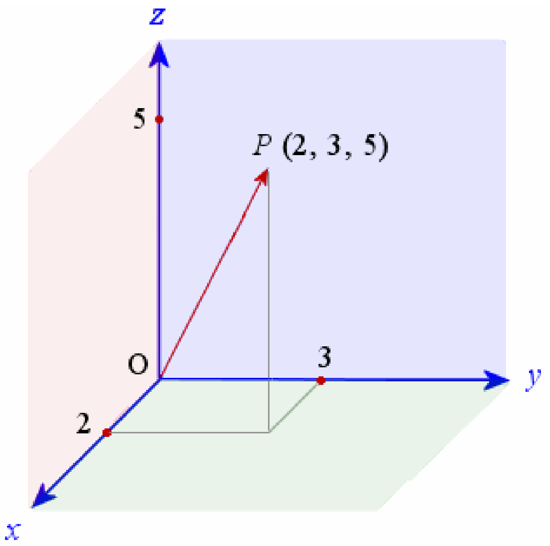
Vectors and Spaces

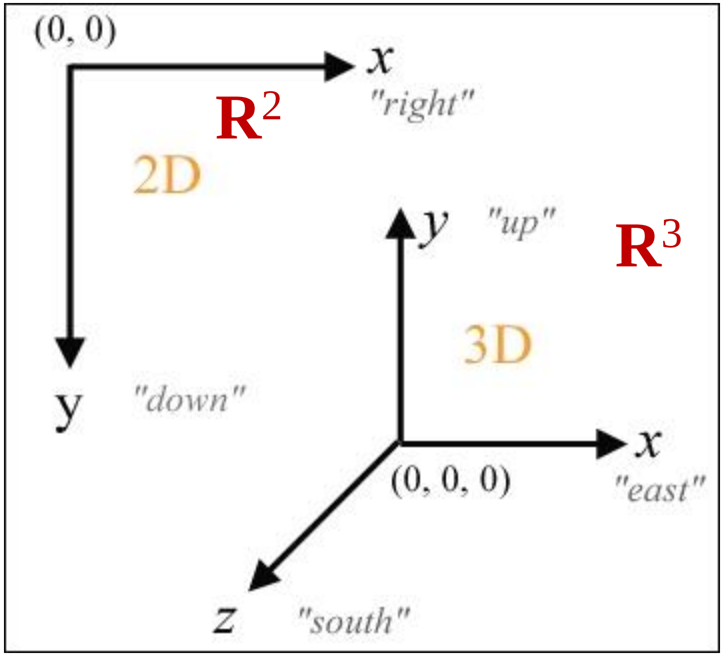


Column vector $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$



$$p = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Vectors and Spaces



A space $\mathbf{R}^n$ consists of all vectors v with n components

It is possible to reach any point in $\mathbf{R}^n$ by combining n non-colinear points with appropriate weights

Column Space

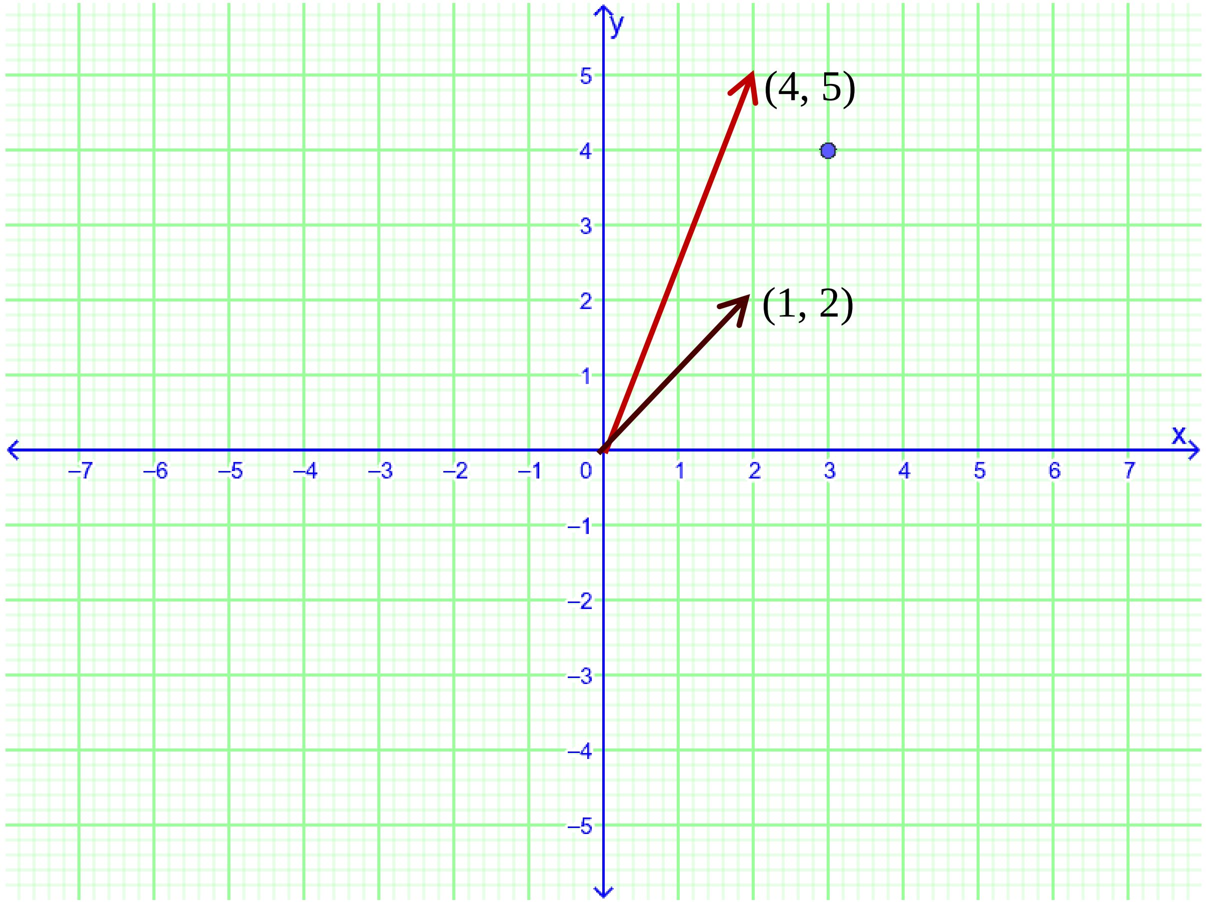
$$A = \begin{bmatrix} | & | & | & \cdots & | \\ c_1 & c_2 & c_3 & \cdots & c_k \\ | & | & | & \cdots & | \end{bmatrix}$$

What space can be formed by the weighted combinations of these columns? → This is called the column space of A

Column Space

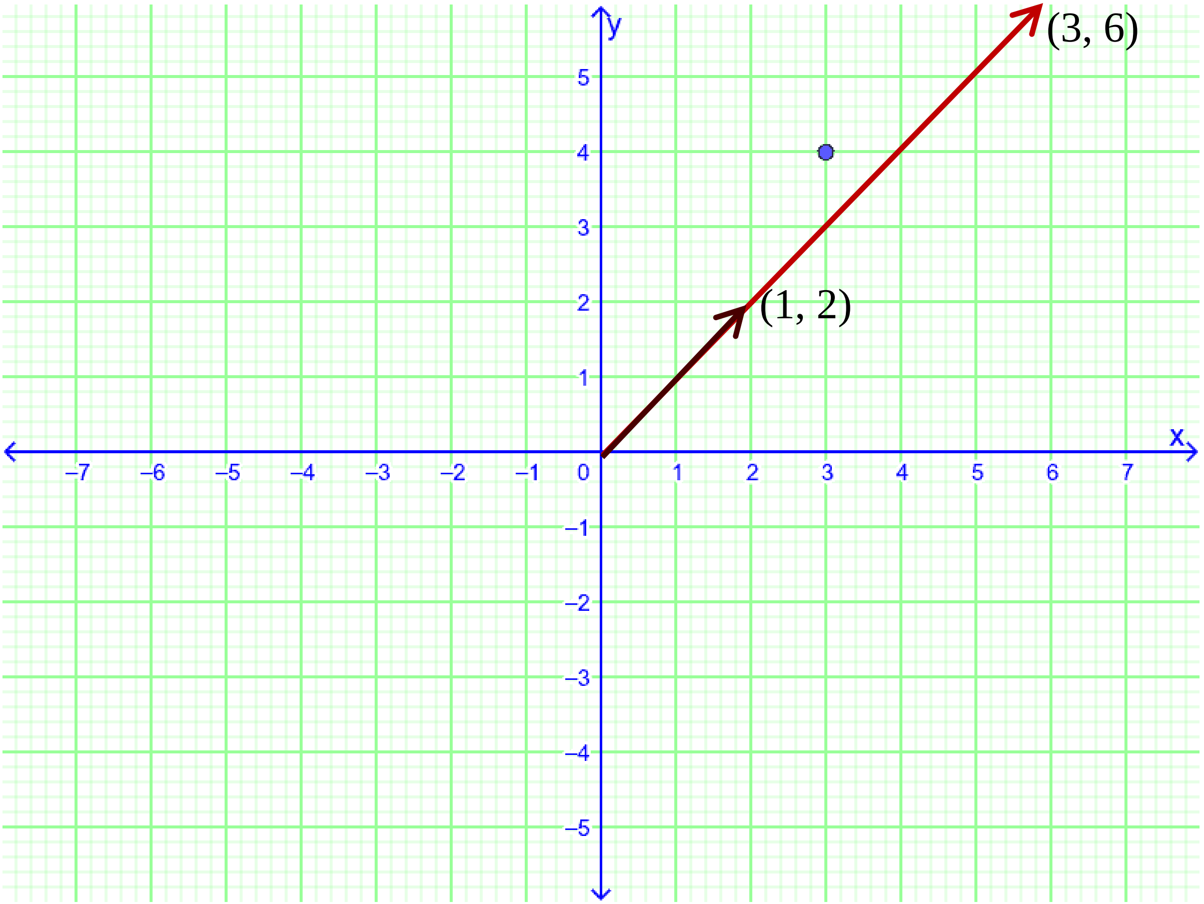
$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ What is $C(A)$?

$\boldsymbol{C(A)} = \mathbf{R}^2$



Column Space

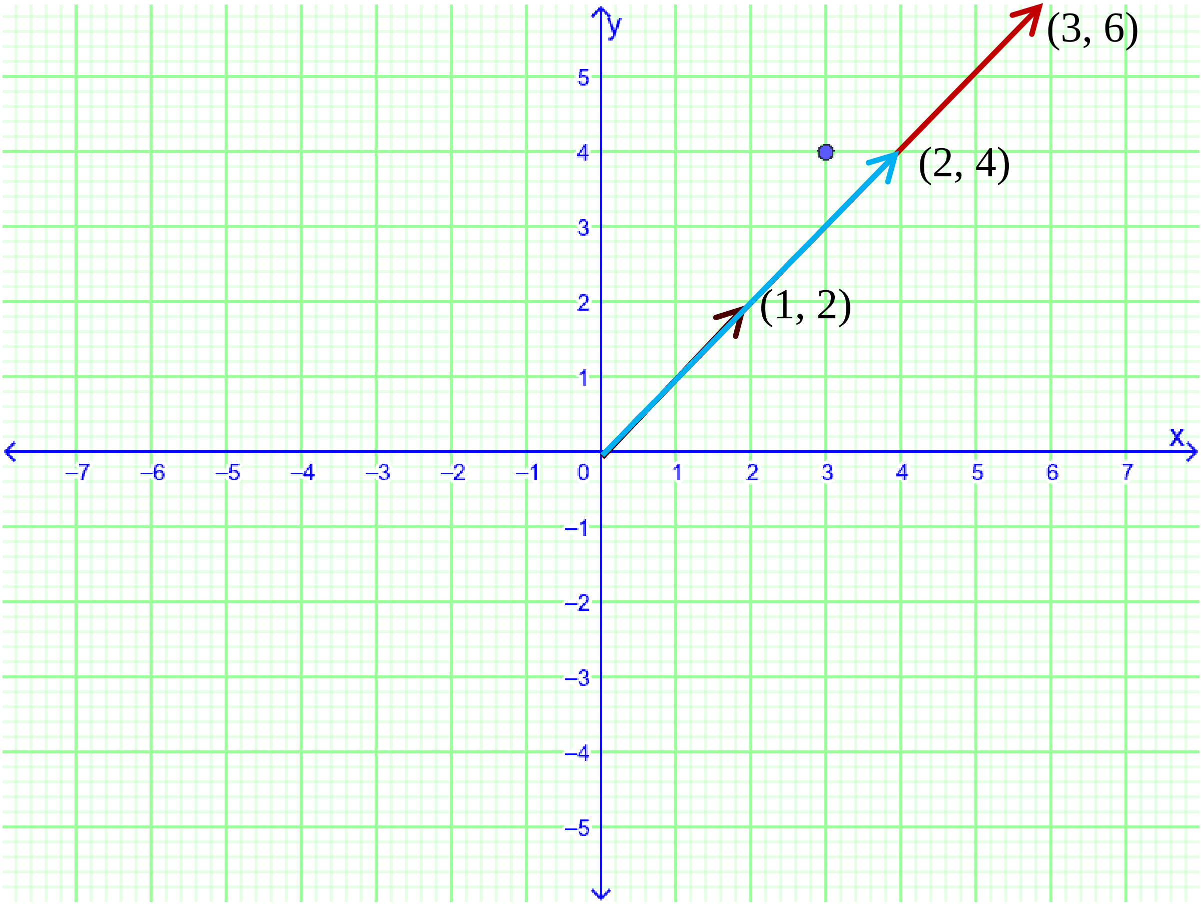
$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ What is $C(A)$?



$\boldsymbol{C(A)} = \mathbf{R}^{\mathbf{1}}$

Column Space

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$$ What is $C(A)$?

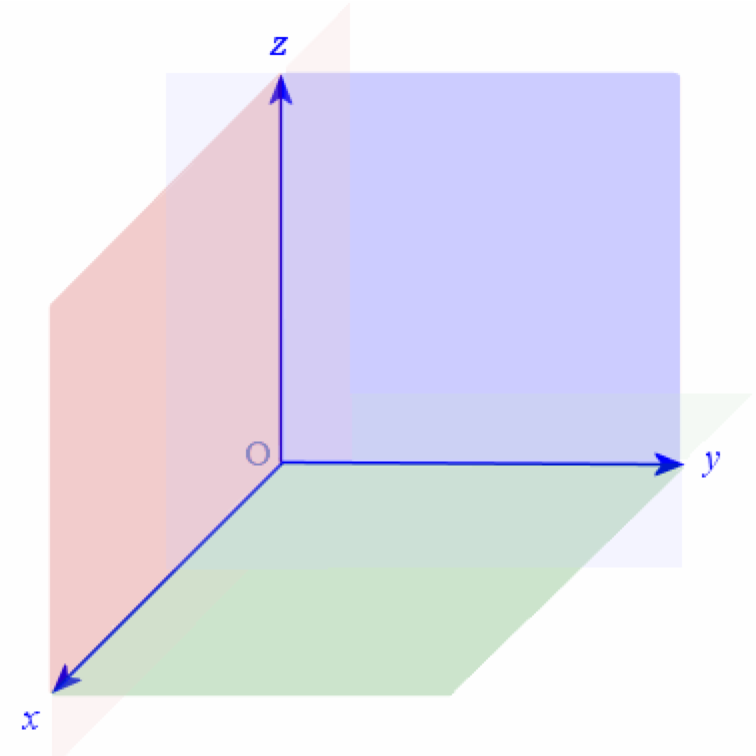


$\boldsymbol{C(A) = \mathbf{R}^1}$

Column Space

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ What is $C(A)$?

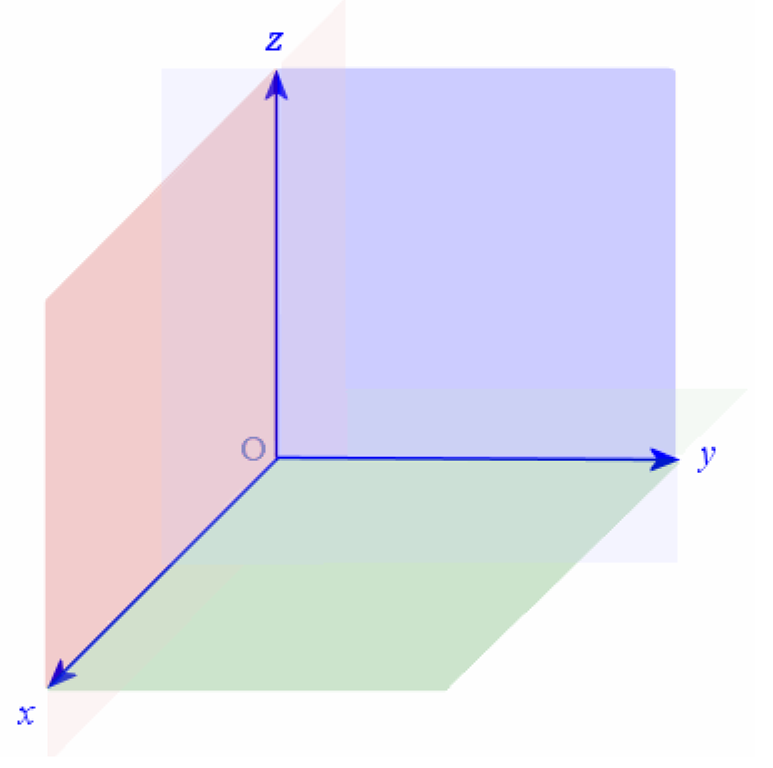
$\boldsymbol{C(A)} = \mathbf{R}^3$



Column Space

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix}$$ What is $C(A)$?

$C(A) = \mathbf{R}^2$



Column Space

$$Ax = \begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} | \\ c_1 \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ c_2 \\ | \end{bmatrix} + x_3 \begin{bmatrix} | \\ c_3 \\ | \end{bmatrix} = b$$

The b lies within the column space of A

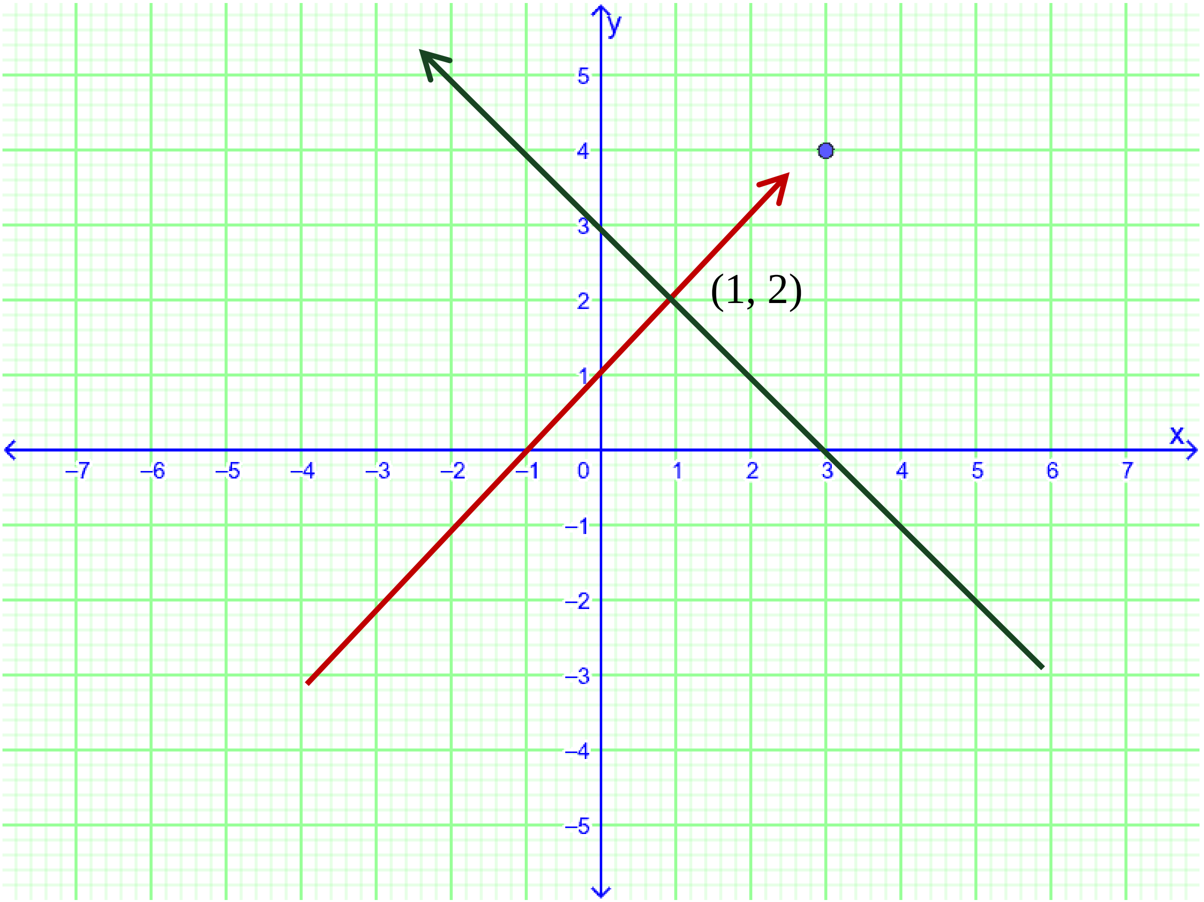
Definition: The column space consist of all linear combinations of the columns, i.e. the combinations of all possible vectors Ax.

This is denoted as $C(A)$

Set of all linear combinations is also called the span

Column Space

$$-x + y = 1$$
$$x + y = 3$$



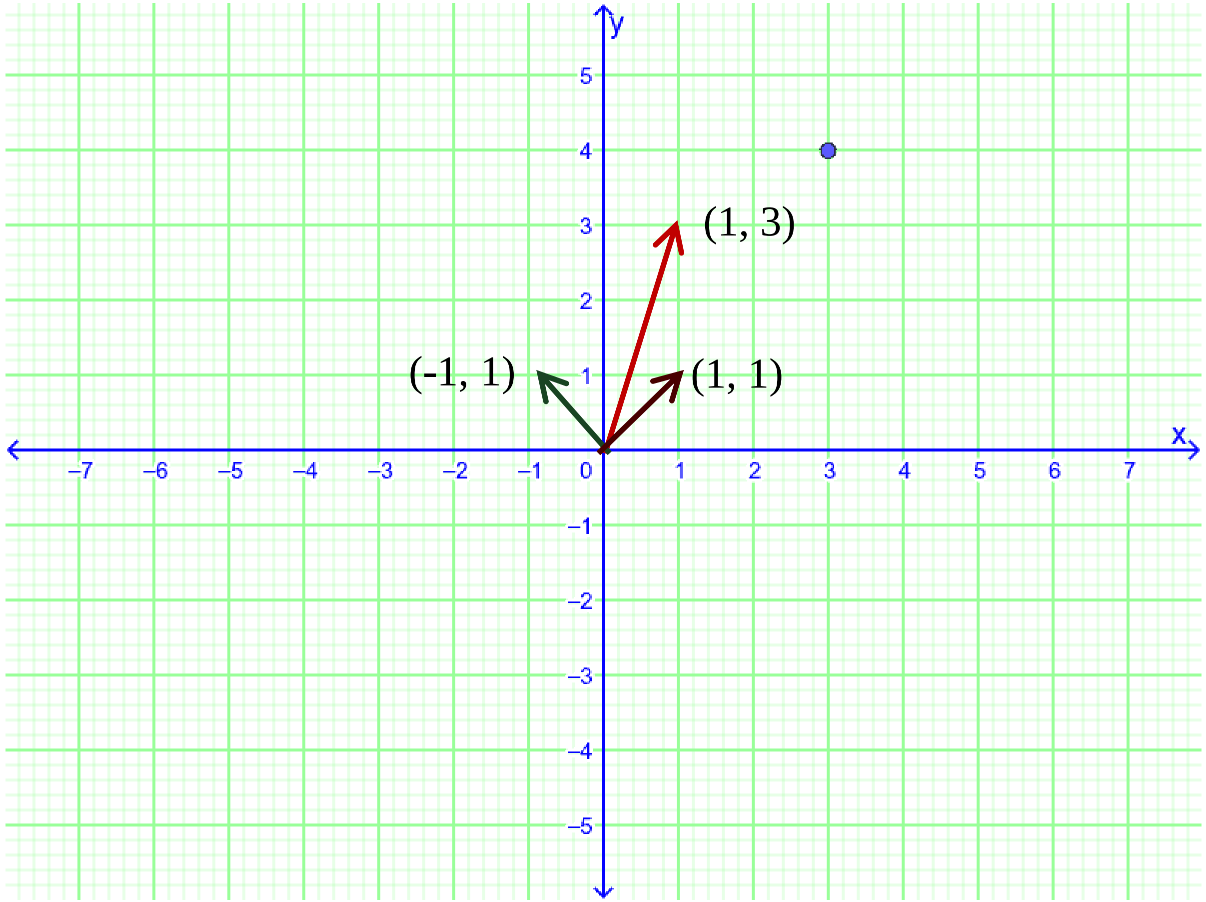
Column Space

$$-x + y = 1$$
$$x + y = 3$$

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$x \begin{bmatrix} -1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$x = 1,\ y = 2$



Row Space

$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$

The **column** space of a matrix is the vector space generated by linear combinations of the **column** vectors

The **row** space of a matrix is the vector space generated by linear combinations of the **row** vectors

How to solve $Ax = b$?

$$Ax = \begin{bmatrix} | & | & | & \cdots & | \\ c_1 & c_2 & c_3 & \cdots & c_k \\ | & | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ | \\ x_k \end{bmatrix} = \begin{bmatrix} b_1 \\ | \\ b_k \end{bmatrix} = b$$

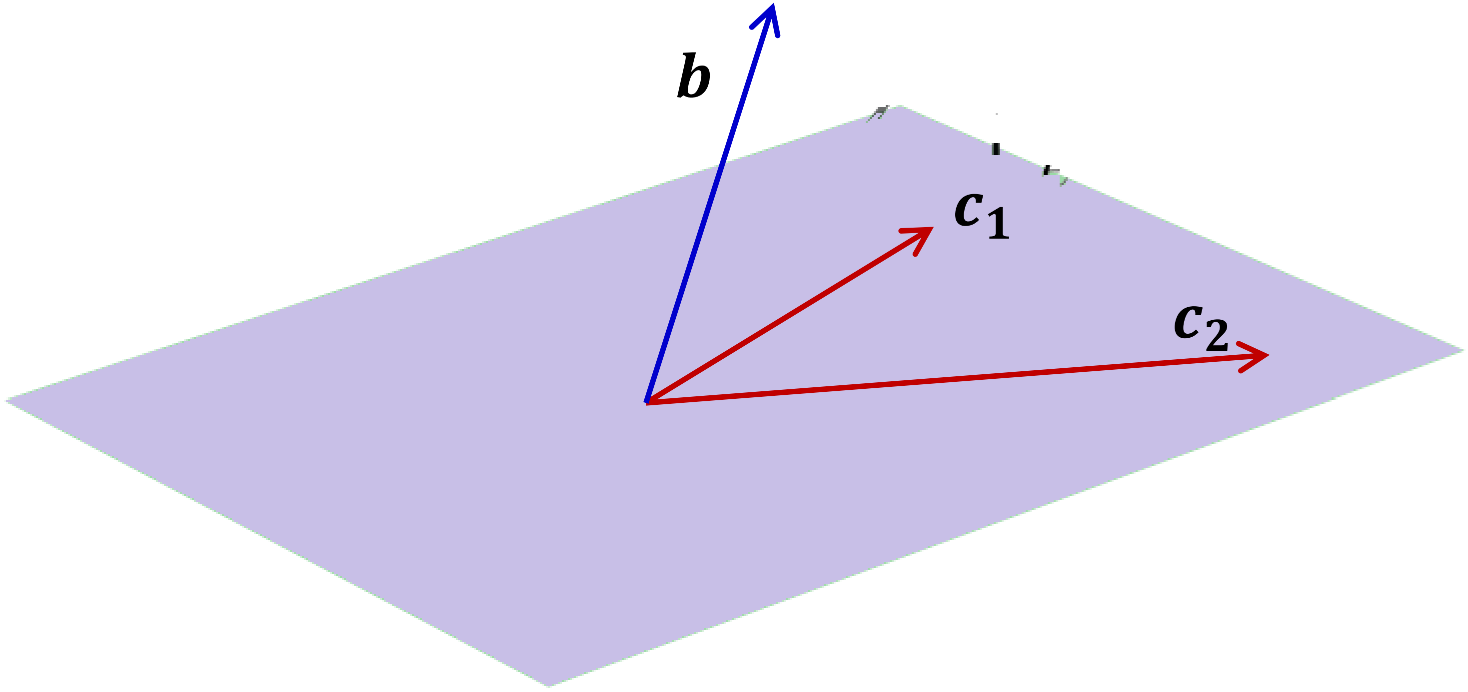
How to solve this?

$$\boldsymbol{x = A^{-1} b}$$

When is $Ax = b$ is solvable?

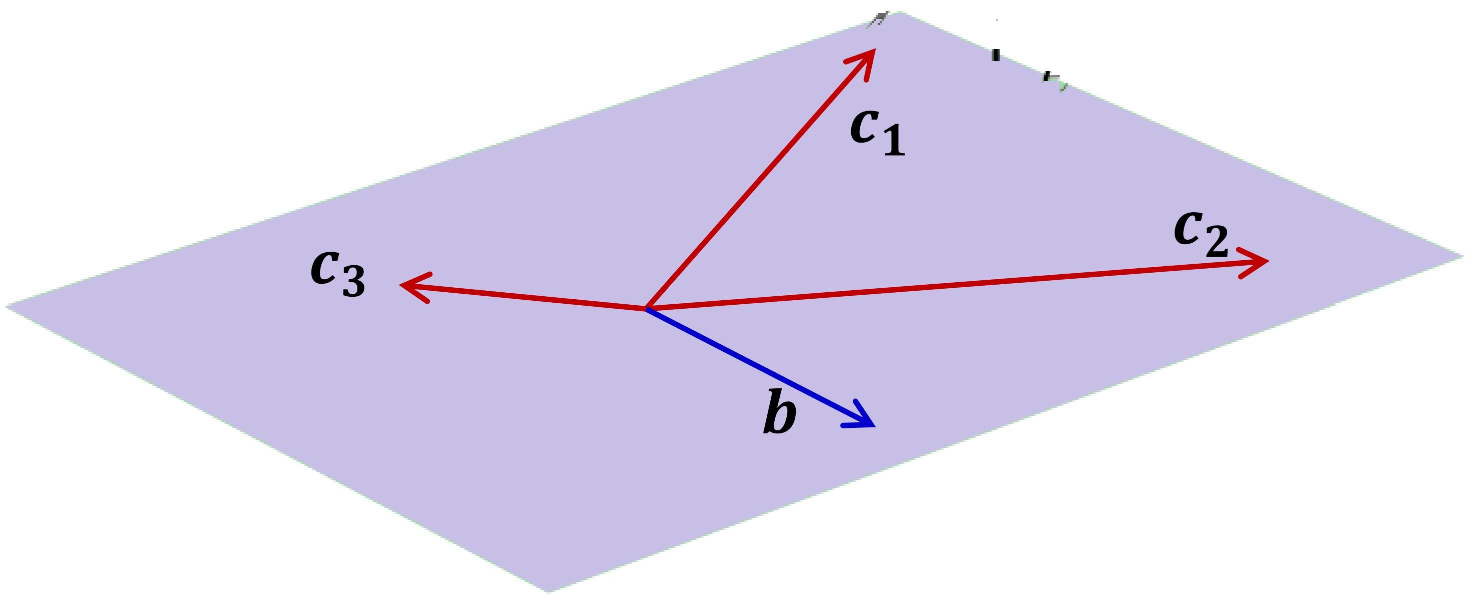
How to solve $Ax = b$?

Condition 1: b must lie in $C(A)$



How to solve $Ax = b$?

<u>Condition 2:</u> A has to be invertible → There should not be more than one way of getting to b from the columns of A



Linearly Independence

$$\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} \text{ are linearly dependent if}$$

$$w_1 \begin{bmatrix} | \\ c_1 \\ | \end{bmatrix} + w_2 \begin{bmatrix} | \\ c_2 \\ | \end{bmatrix} + w_3 \begin{bmatrix} | \\ c_3 \\ | \end{bmatrix} = \begin{bmatrix} | \\ 0 \\ | \end{bmatrix} \text{ for some } w_1, w_2, w_3 \neq 0$$

Why? Because in that case $w_1\boldsymbol{C_1} + w_2\boldsymbol{C_2}$ = - $w_3\boldsymbol{C_3}$
$\therefore$ $\boldsymbol{C_3}$ can be expressed as a combination of $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$

Rank (*C*): The number of linearly independent columns of ***C***.

Linearly Independence

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$$ Are these linearly independent?

$$5\begin{bmatrix}0\\0\\0\end{bmatrix} + 0\begin{bmatrix}1\\2\\3\end{bmatrix} + 0\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}|\\0\\|\end{bmatrix}$$

$\therefore$ They are not linearly independent

A zero vector cannot be in a linearly independent set

Linearly Independence

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}5\\7\\9\end{bmatrix}$$ Are these linearly independent?

$$1\begin{bmatrix}1\\2\\3\end{bmatrix} + 1\begin{bmatrix}4\\5\\6\end{bmatrix} + -1\begin{bmatrix}5\\7\\9\end{bmatrix} = \begin{bmatrix}|\\0\\|\end{bmatrix}$$

$\therefore$ They are not linearly independent

Basis

Definition: Basis of a vector space V is a set of vectors that:

- Are linearly independent
- Span V

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \quad \text{is a basis of } \mathbf{R}^3$$

Definition: Basis of a vector space V is not unique

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \text{ and } \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}5\\1\end{bmatrix} \quad \text{are two basis of } \mathbf{R}^2$$

Basis

Definition: Basis of a vector space V is a set of vectors that:

- Are linearly independent
- Span V

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}9\\8\\7\end{bmatrix}, \begin{bmatrix}5\\7\\9\end{bmatrix}$$ spans $\mathbf{R}^3$, but not linearly independent

→ Not a basis of $\mathbf{R}^3$