Uncertainty Analysis

$V = 100V \pm 1V$ — convention: 95% confident that error is within 1 volt

1. Systematic error – fixed, bias error – consistent, repeatable
2. Random error – lack of repeatability – background noise, uncontrolled variables, etc.

Sources
① calibration error – non-linear behaviour, hysteresis, etc.
② loading errors – intrusive measurement device affects measurement
③ spatial error – sensor affected by environment

$P = V \cdot I \qquad V = 100\,V \pm 2\,V \qquad I = 10\,A \pm 0.2\,A$

$\delta P = \delta V \cdot \frac{\partial P}{\partial V} + \delta I \cdot \frac{\partial P}{\partial I}$ ← terms could be +ve, −ve

replace δV with w_V ← uncertainty

Use root mean square formula

$$w_P = \left(\left(w_V \frac{\partial P}{\partial V}\right)^2 + \left(w_I \frac{\partial P}{\partial I}\right)^2 \right)^{1/2}$$

$$= \left((2 \cdot 10)^2 + (0.2 \cdot 100)^2 \right)^{1/2} = 28.3\,W$$

$\therefore P = 1000\,W \pm 28.3\,W$ — more reasonable value, consistent with 95% confidence interval

Experiment → Result $R = f(x_1, x_2 \dots x_n)$

measurands:
$x_1 \pm w_1$
$x_2 \pm w_2$ ← uncertainty in measurand
$\vdots$
$x_n \pm w_n$ $\quad w_R$ = uncertainty in result

$$w_R = \left(\left(w_1 \frac{\partial R}{\partial x_1}\right)^2 + \left(w_2 \frac{\partial R}{\partial x_2}\right)^2 + \dots \left(w_n \frac{\partial R}{\partial x_n}\right)^2 \right)^{1/2}$$

If $R = C x_1^a x_2^b \dots x_n^N$ $\quad R = f(x_1, x_2, x_3 \dots x_n)$

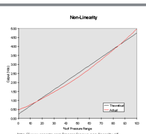
$$w_R = \left(\left(w_1 \frac{\partial R}{\partial x_1}\right)^2 + \left(w_2 \frac{\partial R}{\partial x_2}\right)^2 + \dots \left(w_n \frac{\partial R}{\partial x_n}\right)^2 \right)^{1/2}$$

$\frac{\partial R}{\partial x_1} = a C x_1^{a-1} x_2^b \dots x_n^N = a \frac{R}{x_1}$

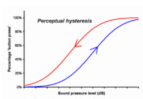
$\frac{\partial R}{\partial x_2} = b C x_1^a x_2^{b-1} \dots x_n^N = b \frac{R}{x_2}$

$$\frac{w_R}{R} = \left(\left(w_1 \cdot \frac{a}{x_1}\right)^2 + \left(w_2 \cdot \frac{b}{x_2}\right)^2 + \dots \left(w_n \cdot \frac{N}{x_n}\right)^2 \right)^{1/2}$$

1. Non-linearity – deviation from linear behaviour — systematic



2. Hysteresis – different measured values for increasing/decreasing load — systematic



3. Non-repeatability — random error

Random Errors and Calibration

Systematic error B_x – fixed, bias error, independent of sample size

Random error – noise, uncontrolled variables, depends on sample # through the t-distribution

$P_{\bar{x}} = \pm t \cdot \frac{S_x}{\sqrt{n}}$ — t value for 95% confidence interval, $\alpha = 0.025$ (mean) – for large sample, $n > 30$, $t = 2$

$P_x = \pm t \cdot S_x$ — single measurement

Hypothesis Testing

Hypothesis test – choice between two hypotheses

Null hypothesis – claim about the population parameter that is assumed to be true

$H_0: \mu = 355\,ml$

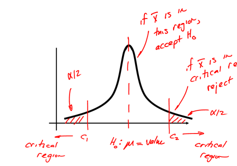
Alternative hypothesis claim about population parameter that is true if H_0 is false

$H_1: \mu \neq 355\,ml$

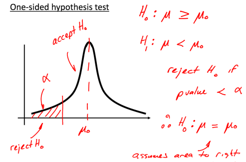
P-value approach – 7 step process
1) State parameter of interest
2) Null hypothesis
3) Alternative hypothesis
4) Select the test statistic (model) for the calculation
5) Determine the rejection region(s) "Reject H0 if …"
6) Calculate value of test statistic, compare with α
7) Make decision

Critical value approach – fixed α value
1. State null and alternative hypotheses
2. Give the test statistic - selects the model for the calculation
3. Determine the rejection and non-rejection regions
4. Calculate value of test statistic, make a decision

Significance level of test α = probability value



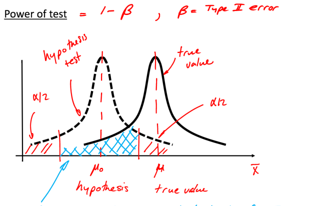
One-sided hypothesis test $H_0: \mu \geq \mu_0$, $H_1: \mu < \mu_0$



reject H_0 if p-value < α

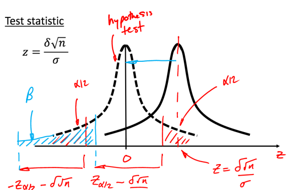
$\therefore H_0: \mu = \mu_0$ assumes area to right of center is included

Power of test $= 1 - \beta$, β = Type II error



β = area under curve = probability of Type II error occurring

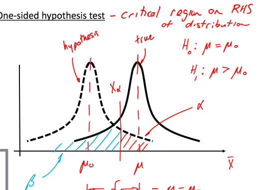
Test statistic $z = \frac{\delta \sqrt{n}}{\sigma}$



$$\beta = P\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma} < z < z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$$

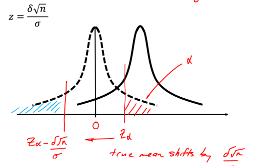
One-sided hypothesis test – critical region on RHS of distribution

$H_0: \mu = \mu_0$, $H_1: \mu > \mu_0$



$\delta = \mu - \mu_0$

Test statistic $z = \frac{\delta \sqrt{n}}{\sigma}$ $\qquad \beta = P\left(z < z_\alpha - \frac{\delta\sqrt{n}}{\sigma}\right)$



Most effective method to decrease α and β is to increase n

$\beta = P(z < -z_\beta) = P\left(z < z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$ (rearranged)

$\therefore -z_\beta = z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}$, $n = \frac{(z_\beta + z_{\alpha/2})^2 \sigma^2}{\delta^2}$

Two-sided Test

$$n \approx \frac{(z_\beta + z_{\alpha/2})^2 \sigma^2}{\delta^2}$$

- assumes $P\left(z < -z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) \cong 0$

One-sided Test

$$n \approx \frac{(z_\beta + z_{\alpha/2})^2 \sigma^2}{\delta^2}$$

remember all z values $(z_\alpha, z_{\alpha/2}, z_\beta)$ must be positive

$P(L < \mu < U) = 1 - \alpha$ — confidence interval bounds, limits

$$P\left(\bar{X} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

point estimate; margin of error; upper bound; significance level

Confidence interval, bounds – variance known, unknown

$P(L < \mu_1 - \mu_2 < U) = 1 - \alpha$

$U, L = (\bar{X}_1 - \bar{X}_2) \pm z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ ← σ_1, σ_2 known

$(\bar{X}_1 - \bar{X}_2) \pm t_{\alpha/2,(n_1+n_2-2)} \cdot s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ ← σ_1, σ_2 unknown, presumed equal

$(\bar{X}_1 - \bar{X}_2) \pm t_{\alpha/2,k} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$ ← σ_1, σ_2 unknown, presumed $\sigma_1 \neq \sigma_2$ — k from formula

One-sided confidence bounds, variance known and unknown

$P(L < \mu_1 - \mu_2) = 1 - \alpha$ $\qquad P(\mu_1 - \mu_2 < U) = 1 - \alpha$

$L = \bar{X}_1 - \bar{X}_2 - z_\alpha \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ $\qquad U = \bar{X}_1 - \bar{X}_2 + z_\alpha \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

$L = \bar{X}_1 - \bar{X}_2 - t_{\alpha,(n_1+n_2-2)} \cdot s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ $\qquad U = \bar{X}_1 - \bar{X}_2 + t_{\alpha,(n_1+n_2-2)} \cdot s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

$L = \bar{X}_1 - \bar{X}_2 - t_{\alpha,k} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$ $\qquad U = \bar{X}_1 - \bar{X}_2 + t_{\alpha,k} \cdot \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

Case 1: $\sigma_1 = \sigma_2 = \sigma$ ← unknown but assumed equal case of $s_1 \cong s_2$

$$T = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

→ use t dist table with $k = n_1 + n_2 - 2$ dof

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

← pooled estimate of s_1 and s_2, weighted average

Case 2: $\sigma_1 \neq \sigma_2$ ← unknown, unequal

$$T^* = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$$ approx.

$$k = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$$

← dof, round down to nearest integer to find t values

Models and Linear Regression

$$b = \frac{S_{xy}}{S_{xx}} = \frac{\text{sum of cross products, } xy}{\text{sum of squares, } xx} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$ (slope)

$$b = \frac{\sum_{i=1}^n x_i y_i - \frac{\sum x_i \sum y_i}{n}}{\sum_{i=1}^n x_i^2 - \frac{\sum x_i \sum x_i}{n}} \qquad a = \bar{y} - b\bar{x}$$ (intercept)

$\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$

Least squares method – finds values for a and b that minimizes error at all points

SSE = error sum of squares $= \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2$ where n = # data values

minimize wrt a and b (calculus)

How "good" is fit of data by estimated regression line?

1. Standard error of estimators approach
$H_0: \beta = 0$ model as t distribution
$H_1: \beta \neq 0$

$T = \frac{(b - \beta)}{\left(\frac{Se}{\sqrt{S_{xx}}}\right)}$ ← β_0

assumes normal distribution, σ unknown, similar for all x values
$\therefore$ t dist, $k = n - 2$ d.o.f.

Why? a and b are estimates based on sample data of the true regression line for population

$Y = \alpha + \beta x$

Confidence interval

$\mu(Y/x_0) = \alpha + \beta x_0$ ← true regression line — population mean value of Y at x_0

$\hat{y}_0 = a + b x_0$ ← estimate based on sample

C.I. on $\mu(Y/x_0)$ is $P(L < \mu(Y/x_0) < U) = 1 - \alpha$

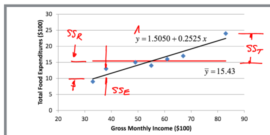
$$L, U = \underbrace{a + b x_0}_{\text{point estimate}} \pm \underbrace{t_{\frac{\alpha}{2}, n-2} \cdot Se \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}}}_{\text{margin}}$$

Prediction interval – y_0 observation, independent of data used to find a, b

Then $\hat{y}_0 = a + b x_0$ = estimate of new, future values of response Y at x_0

$$\text{P.I. on } Y_0 \text{ at } x_0 = \underbrace{a + b x_0}_{\text{point estimate}} \pm \underbrace{t_{\alpha/2, n-2} \cdot s_e \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}}}_{\text{margin}}$$

error from new measurement



$$SS_T = \sum_{i=1}^n (y_i - \bar{y})^2 = SS_R + SS_E = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^n (y_i - \hat{y})^2$$

$125.7 = 113.0 + 12.72$

Since SS_E small relative to SS_T, model fits data well

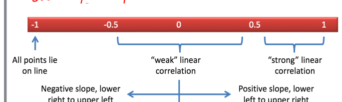
Coefficient of determination – proportion of SS_T "explained by" use of regression model

$$R^2 = \frac{SS_R}{SS_T} = \frac{SS_T - SS_E}{SS_T} = 1 - \frac{SS_E}{SS_T}$$

Sample correlation coefficient

$$r = \frac{1}{n-1} \sum \left(\frac{x_i - \bar{x}}{S_x}\right) \cdot \left(\frac{y_i - \bar{y}}{S_y}\right) = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$$

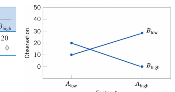
"standardized" observations



Factorial Experiments

Experiments
1. Observational study – based on observations during a time of normal operation
2. Designed experiment – observations made of results due to deliberate changes to control variables

Factor A	Factor B	
	B_{low}	B_{high}
A_{low}	10	20
A_{high}	30	0

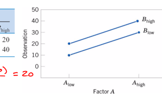


$A = 0$

$B = -10$

$AB = \frac{(10+0)}{2} - \frac{(30+20)}{2} = -20$ ← strong interaction effect, hides or masks main effect of A

Factor A	Factor B	
	B_{low}	B_{high}
A_{low}	10	20
A_{high}	30	40



$A = \frac{(30+40)}{2} - \frac{(10+20)}{2} = 20$

$B = \frac{(20+40)}{2} - \frac{(10+30)}{2} = 10$

$AB = \frac{(10+40)}{2} - \frac{(30+20)}{2} = 0$ ← no interaction

Table 7-4 The 2^2 Design for the Epitaxial Process Experiment

Treatment Combination	Factorial Effect A	B	AB	Thickness (μm)				Thickness (mm) Total	Average	Variance
(1)	−	−	+	14.037	14.165	13.972	13.907	56.081	14.020	0.0121
a	+	−	−	14.821	14.757	14.843	14.878	59.299	14.825	0.0026
b	−	+	−	13.880	13.860	14.032	13.914	55.686	13.922	0.0059
ab	+	+	+	14.888	14.921	14.415	14.932	59.156	14.789	0.0625

$\hat{\sigma}_{(1)} = 0.0121$, $\hat{\sigma}_a = 0.0026$, etc.

$\hat{\sigma}^2 = \frac{(0.0121 + 0.0026 + \dots)}{4} = 0.0208$

$se(\text{Effects}) = \sqrt{\frac{0.0208}{4 \cdot 2^{2-2}}} = 0.072$

Normalize effects using standard error

$se(\text{Effects}) = \sqrt{\frac{\hat{\sigma}^2}{n\, 2^{k-2}}}$ $\quad k$ = # factors → $k = 2$, n = # replicates

$2^k(n-1)$ d.o.f.

Main effect A = average response as A changes from low to high

A = [Average response at high A values] − [Average response at low A values]

Main effect B

B = [Average response at high B values] − [Average response at low B values]

Interaction effect = relative strength of interaction

AB = [Average response when A and B are at same level] − [Average response when A and B are at different levels]

Main, interaction effects

$A = \frac{1}{2n}[ab + a - b - (1)]$ $\quad B = \frac{1}{2n}[ab + b - a - (1)]$ $\quad AB = \frac{1}{2n}[ab + (1) - a - b]$

$A = -0.167$, $B = 2.83$, $AB = -19.5$, $se(Effects) = 7.907$

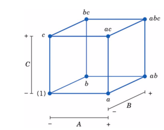
Since $\alpha = 0.05$, 95% Confidence Interval

$A = -0.167 \pm 2(7.907)$
$B = 2.83 \pm 2(7.907)$ — contain 0 in confidence interval, ∴ not significant

$AB = -19.5 \pm 2(7.907)$ ← significant

2^3 Factorial Design

$2^3 = 8$ treatment combinations



Main effects

$$A = \frac{1}{4n}\left(\underbrace{a + ab + ac + abc - b - c - bc - (1)}\right)$$

A	B	C	Label

Interaction Effects
- More difficult to interpret, indicates dependence of factors on each other
- Strong interaction can mask main effects

Table 7-8 Algebraic Signs for Calculating Effects in the 2^3 Design

Treatment Combination	I	A	B	AB	C	AC	BC	ABC
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

Response surface – surface plot in 3D space of response variable Y as function of factors (regressor variables) X_1 and X_2 (assumes 2 factors)

$$Y = \bar{y} + \left(\frac{A}{2}\right)X_1 + \left(\frac{B}{2}\right)X_2 + \left(\frac{AB}{2}\right)X_1 X_2$$

Method of steepest ascent – first order method to move quickly to general vicinity of optimum