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MATH415B - Klaus Lux
2/5/18
Homework 4

11 If $\phi : R \to S$ is a ring homomorphism, define $\bar{\phi} : R[x] \to S[x]$ by $\phi(a_n x^n + ... + a_0) \to \phi(a_n)x^n + ...\phi(a_0)$. Show that $\bar{\phi}$ is a ring homomorphism. (This exercise is referred to in Chapter 33.)

For two elements $f, g \in R[X]$.

$$\begin{aligned}
\bar{\phi}(f(x)) + \bar{\phi}(g(x)) &:= \bar{\phi}(a_n x^n + ... + a_0) + \bar{\phi}(b_n x^n + ... + b_0) \\
&= \phi(a_n)x^n + ... + \phi(a_0) + \phi(b_n)x^n + ... + \phi(b_0) \\
&= (\phi(a_n) + \phi(b_n))x^n + ... + \phi(a_0) + \phi(b_0) \\
&= \phi(a_n + b_n)x^n + ... + \phi(a_0 + b_0) \\
&= \bar{\phi}(f(x) + g)
\end{aligned}$$

and

$$\begin{aligned}
\bar{\phi}(f(x))\bar{\phi}(g(x)) &= \bar{\phi}(a_n x^n + ... + a_0)\bar{\phi}(b_m x^m + ... + b_0) \\
&= (\phi(a_{n+m})\phi(b_0) + ... + \phi(a_0)\phi(b_{m+n}))x^{n+m} \\
&\quad + (\phi(a_{n+m-1})\phi(b_0) + ... + \phi(a_0)\phi(b_{m+n-1}))x^{n+m-1} \\
&\quad + ... \\
&\quad + \phi(a_0)\phi(b_0)x^0 \\
&= (\phi(a_{n+m}b_0) + ... + \phi(a_0 b_{m+n}))x^{n+m} \\
&\quad + (\phi(a_{n+m-1}b_0) + ... + \phi(a_0 b_{m+n-1}))x^{n+m-1} \\
&\quad + ... \\
&\quad + \phi(a_0 b_0)x^0 \\
&= \phi(a_{n+m}b_0 + ... + a_0 b_{m+n})x^{n+m} \\
&\quad + \phi(a_{n+m-1}b_0 + ... + a_0 b_{m+n-1})x^{n+m-1} \\
&\quad + ... \\
&\quad + \phi(a_0 b_0)x^0 \\
&= \bar{\phi}(f(x)g(x))
\end{aligned}$$

19 1. Let D be an integral domain and $f, g \in D[x]$. Prove that $deg(f \circ g) > deg(f) + deg(g)$.

Proof. Let f be a polynomial of degree n and g be a polynomial of degree m.

$$\begin{aligned}
(f \circ g)(x) &:= (a_n x^n + ... + a_0) \circ (b_m x^m + ... + b_0) \\
&= (a_n(b_m x^m + ... + b_0))^n + (a_{n-1}(b_m x^m + ... + b_0))^{n-1} + ... + a_0
\end{aligned}$$

So we can see that the highest term in the polynomial will be

$$a_n b_m x^{nm}$$

and because n and m are the orders of f, g we know that $a_n, b_m \neq 0$. Because there are no zero divisors this implies that $a_n b_m \neq 0$. □

2. Show, by example that for a commutative ring R it is possible that $deg(fg) < deg(f) + deg(g)$

   For $f(x), g(x) \in Z_4[x]$ let $f(x) = 2x, g(x) = 2$ we can see $f(x)g(x) = 0$.

21 Let $f(x)$ belong to $F[x]$ where F is a field let a be a zero of $f(x)$ of multiplicity n and write $f(x) = (x-a)^n q(x)$. If $b \neq a$ is a zero of $q(x)$, show that b has the same multiplicity as a zero of $q(x)$ that it does for $f(x)$.

Let n be the multiplicity of $f(x)$'s zero at b. By the factor theorem if $f(b) = 0$ then $(x-b)|f(x)$ because $x-b$ does not divide $x-a$, it is a factor of $q(x)$ so the multiplicity of $q(x)$ at b is not less then that of $f(x)$ at b, and because $q(x)$ is a factor of $f(x)$ it also must be no greater.

33 Consider the homomorphism $\bar{\phi} : Z \to Z_M$ given by the mapping $\bar{\phi}(x) \to x \bmod m$. We know this is a valid homomorphis because modulo addition and multiplication preserve the properties of a homomorphism.

By question 11 this implies that $\phi : Z[x] \to Z_m[x]$ is also a homomorphism.

46 Prove that $Q[x]/ < x^2 - 2 >$ is a ring isomorphism to $Q[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in Q\}$.

Consider the homomorphism $\phi : Q[x] \to Q[\sqrt{2}]$ given by the mapping $\phi(q(x)) = q(\sqrt{2})$. ϕ is well defined

$$\phi(a(x) + b(x)) = a(\sqrt{2}) + b(\sqrt{2}) = \phi(a(x)) + \phi(b(x))$$

$$\phi(a(x)b(x)) = a(\sqrt{2})b(\sqrt{2}) = \phi(a(x))\phi(b(x))$$

In addition if $\phi(a(x)) = 0$ then $a(\sqrt{2}) = 0$, this means that either $a(x) = 0$ or $a(x)|(x-\sqrt{2})$. Implying that any element of $Q[x]$ divisible by $(x - \sqrt{2})$ gets mapped to 0 in $Q(\sqrt{2})$ so

$$\text{Ker } \phi =< x^2 - 2 >$$

Also note that ϕ is onto as for all $q \in Q[\sqrt{2}]$

$$q = a + b\sqrt{2}$$

because $a, b \in Q$

$$ax + b = \phi^{-1}(q) \in Q[\sqrt{2}]$$

So $\phi(Q[x]) \approx Q[\sqrt{2}]$. By the first isomorphism theorem

$$Q[x]/\text{Ker}\phi \approx Q[x]/ < x^2 + 2 >\approx Q[\sqrt{2}]$$

50 Let R be a ring and x be an indeterminate. Prove that the rings $R[x]$ and $R[x^2]$ are ring-isomorphic.

Proof. Consider the mapping $\phi : R[x] \to R[x^2]$ given by function composition with x^2 that is to say $\phi(f(x)) \to f(x^2)$ this is a homomorphism by question 21. We can see that for any g in $R[x^2]$,

$$g(x) = a_n(x^2)^n + a_{n-1}(x^2)^{n-1} + ... + a_0$$

there exists a unique

$$\phi^{-1}(g(x)) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0 \in R[x]$$

So ϕ is one to one and onto. □

GAP 16.1 Use GAP to factor x^{p-1} in $Z_p[x]$ for $p = 3, 5, 7, 11$.

$p = 3$

$$x + 1, x + 2$$

$p = 5$

$$x + 1, x - 1, x^2 + 1$$

$p = 7$

$$x + 1, x + 2, x + 3, x + 4, x + 5, x + 6$$

$p = 11$

$$x + 1, x + 2...x + 11$$

GAP 16.2 Make a conjecture: $x^{p-1} - 1$ is has every nonzero element of $Z_p[x]$ as a factor.

GAP 16.3

GAP 16.4

GAP 16.5

GAP 16.6

D2L Question Determine all the automorphisms of $Z[x]$, the ring of polynomials with integer coefficients.