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MATH415B - Klaus Lux
3/26/1
Homework 8

1 Gallian

1.1 Chapter 20

27 Prove or disprove that $Q(\sqrt{3})$ and $Q(\sqrt{-3})$ are ring isomorphic.
$Q(\sqrt{3})$ is not isomorphic to $Q(\sqrt{-3})$

Proof. Let $\phi : Q(\sqrt{-3}) \to Q(\sqrt{3})$ be an isomorphic mapping.

$$\begin{aligned}
\phi(3+\sqrt{-3}^2) &= \phi(0) \\
&= 0 \\
\phi(3+\sqrt{-3}^2) &= \phi(3) + \phi(\sqrt{-3}^2) \\
&= \phi(1)+\phi(1)+\phi(1)+\phi(\sqrt{-3})\phi(\sqrt{-3}) \\
&= 1+1+1+\phi(\sqrt{-3})^2 \\
&= 3+\phi(\sqrt{-3})^2 = 0
\end{aligned}$$

Because ϕ is onto there exists $a \in Q(\sqrt{3})$ such that $a^2 = -3$ and $Q(\sqrt{3}) \subseteq \mathbb{R}$ so $a \in \mathbb{R}$. □

28 For any prime p, find a field of characteristic p that is not perfect.
Consider the Field $Z_p(\sqrt[p]{2})$, this field clearly has characteristic p and

$$Z_p(\sqrt[p]{2}) = \{a + b\sqrt[p]{2} | a, b \in Z_p\}$$

Also

$$(a+b\sqrt[p]{2})^p = \sum_{k=0}^{p} \binom{p}{k} a^{n-k} b \sqrt[p]{2}^k$$

Because p is prime, p divides $\binom{p}{k}$ for all values of k except $k = 0$ and $k = p$. All but the first and last terms of the expansion will be 0 so for any element of $Z_p(\sqrt[p]{2})$.

$$(a+b\sqrt[p]{2})^p = a^p + (b\sqrt[p]{2})^p = (a^p + 2b^p) \in Z_p$$

Implying $[Z_p(\sqrt[p]{2})]^p \neq Z_p(\sqrt[p]{2})$.

30 Show that $x^4 + x + 1$ does not have multiple zero's in any extension field of Z_2.

$$\begin{aligned}
f(x) &= x^4 + x + 1 \\
f'(x) &= 4x^3 + 1 = 1
\end{aligned}$$

$$\gcd(f(x), f'(x)) = 1$$

So $f(x), f'(x)$ do not have a common factor of positive degree. The result follows by Cirterion for Multiple Zeros.

33 Let F be a field of characteristic $p \neq 0$. Show that the polynomial ring $f(x) = x^{p^n} - x$ over F has distinct zeros.

$$f(x) = x^{p^n} - x$$
$$f'(x) = p^n x^{p^n - 1} - 1 = 0^n x^{p^n - 1} = -1$$
$$\gcd(f(x), f'(x)) = 1$$

so $f(x)$ cannot have multiple zeros in any extension field of F since F was an arbitrary field we know that it cannot have multiple zeros in any field. And clearly 1 is a zero in any field.

34 Find the splitting field for $f(x) = (x^2 + x + 2)(x^2 + 2x + 2)$ over $F = Z_{3p}$ and write $f(x)$ as a product of linear factors. $F(\sqrt{i})$ is a splitting field of $f(x)$

$$f(x) = (x^2 + x + 2)(x^2 + 2x + 2) = x^4 + 3x^3 + 6x^2 + 6x + 4 = x^4 + 1$$

And in $F(\sqrt{i})$

$$x^4 + 1 = (x + \sqrt{i})(x - \sqrt{i})(x + i\sqrt{i})(x - i\sqrt{i})$$

1.2 Chapter 21

2 Let E be the algebraic closure of F. Show that every polynomial in $F[x]$ splits in E. Let $f(x) \in F[x]$ and let $a_0, a_1, ..., a_n$ be roots of $f(x)$ then $a_0, a_1, ..., a_n \in E$ so in E,

$$f(x) = (x - a_0)(x - a_1)....(x - a_n)$$

8 Find the degree of a basis for $Q(\sqrt{3} + \sqrt{5})$ over $Q(\sqrt{15})$. Find the degree and a basis for $Q(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2})$ over Q.
The set $\{1, \sqrt{3}\}$ is a basis for $Q(\sqrt{3} + \sqrt{5})$ over $Q(\sqrt{15})$. $\sqrt{5} = \frac{1,\sqrt{15}}{\sqrt{3}}$ so $\sqrt{3}, \sqrt{5} \in Q(\sqrt{15})(\sqrt{3})$ hence any linear combination of the two is also in $Q(\sqrt{15})(\sqrt{3})$. They are also linearly independant as $a + b\sqrt{3} = 0$ has no solutions for rational $a, b \in Q(\sqrt{15})$ $\{1, \sqrt[4]{2}, \sqrt[3]{2}\}$ is a basis for $Q(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2})$, as

$$\sqrt{2} = \sqrt[4]{2}^2$$

If the basis is linearly depended hen there exists some a,b, and c such that

$$a + b\sqrt[4]{2} + c\sqrt[3]{2} = 0$$

10 Let a be a complex number that is algebraic over Q show that $\sqrt{a}$ is algebraic over Q. Why does this prove that $\sqrt[2]{a}$ is algebraic over Q?

14 Find the minimal polynomial for $\sqrt{-3} + \sqrt{2}$ over Q.

2 GAP

22.1

22.2

22.3