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MATH415B - Klaus Lux
4/8/18
Homework 9

1 Gallian

1.1 Chapter 21

19 Suppose that $p(x) \in F[x]$ and E is a finite extension of F. If $p(x)$ is irreducible over F, and $\deg p(x)$ and $[E : F]$ are relatively prime, show that $p(x)$ is irreducible over E.

Let a be a zero of $p(x)$ and let $\deg p(x) = n$. Because $\{1, a, ..., a^n\}$ is a basis for $F(a)$ over F by Theorem 20.3 $[F(a) : F]$ is relatively prime to $[E : F]$. Because E is an extension of F, $[E(a) : E] \leq [F(a) : F]$. Also note that

$$[E(a) : F(a)][F(a) : F] = [E(a) : F] = [E(a) : E][E : F]$$

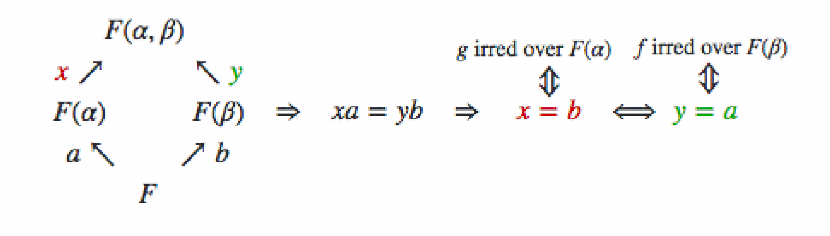
So $[F(a) : F]$ divides $[E(a) : E]$ implying $[F(a) : F] = [E(a) : E]$, so $a \notin E$ and p is irreducible in E.

24 Find a splitting field for $x^4 - x^2 - 2$ over Z_3.

$$x^4 - x^2 - 2 = (x^2 + 1)(x^2 - 2) = (x + i)(x - i)(x + \sqrt{2})(x - \sqrt{2})$$

So $Z_3(i, \sqrt{2})$ is a splitting field.

32 Let $f(x)$ and $g(x)$ be irreducible polynomials over a field F and let a and b belong to some extension E of F. If a is a zero of $f(x)$ and b is a zero of $g(x)$, show that $f(x)$ is irreducible over $F(b)$ if and only if $g(x)$ is irreducible over $F(a)$.



36 Suppose that a is algebraic over a field F. Show that a and $1 + a^{-1}$ have the same degree over F.

We can see that $1 + a^{-1} \in F(a)$ and that $a \in F(1 + a^{-1})$, meaning that $F(a) = F(1 + a^{-1}) = F(a, 1 + a^{-1})$ so

$$[F(a, 1 + a^{-1}) : F] = [F(a) : F] = [F(1 + a^{-1}) : F]$$

42 Suppose K is an extension of F of degree n. Prove that K can be written in the form $F(x_1, x_2, ..., x_n)$ from some $x_1, x_2, ..., x_n$ in K.

Proof. Because $[K : N]$ is finite K must be an algebraic extension of F. Let $[K : N] = n$, for all $k \in K$

$$k = f_0k_0 + f_1k_1 + ... + f_nk_n$$

Where $f_1, ..., f_n \in F$ and $k_1, ..., k_n \in K$ we can see that $f_0k_0 \in F(k_0)$, so

$$k = f_0k_0 + f_1k_1 + ... + f_nk_n \in F(k_0)(k_1)...(k_n) = F(k_0, k_1, ..., k_n)$$

□

1.2 Chapter 22

18 Suppose that $[E : Q] = 2$. Show that there is an integer d such that $E = Q(\sqrt{d})$ where d is not divisible by the square of any primes. Let $d = -1$. We can see that $\{1, i\}$ is a basis for $Q(i)$ over Q.

24 Show that any finite subgroup of the multiplicative group of a field is cyclic.

37 Let E be the splitting field of $f(x) = x^{p^n} - x$ over Z_p. Show that the set of zeros of $f(x)$ in E is closed under addition, subtraction, multiplication, and division (by nonzero elements). (This exercise is referred to in the proof of Therome 22.1)

2 GAP

22.10