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CSC445 - Alon Efrat
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Homework 4

1. Pondered deeply.

2. **Algorithm** The only change we need to make here is to terminate the algorithm once we relax every edge on the graph, without changing a single distance value. Implementing this would be as simple as adding a terminate flag set to True before the loop relaxing every edge, and setting it to False whenever a distance changes, then breaking out of the main loop. If this flag is still True once we finished checking every edge we terminate the algorithm.
   Correctness Assume we've relaxed every edge and updated no distance values for any vertex of the graph. Let v_1 be our start vertex and v_n an arbitrary vertex. Assume $v_1, v_2, ..., v_n$ is the shortest path between v_1 and v_n. Note that $n \leq k$; this will be important for our run time. We can see that for $1 < i < j \leq n$

   $$\delta(1, j) = \delta(v_i, v_j) + \delta(v_j, v_i)$$

   as $v_1 \to v_j, v_1 \to v_j\ v_i \to v_j$ are all sub-paths of the shortest path $v_1 \to v_n$. Let $w_{i,i+1}$ be the weight on the edge between v_i and v_{i+1} along the shortest path. Note that $w_{i,i+1} = \delta(v_i, v_{i+1})$ and that $d[v_1] = 0$ so because $d[v_2]$ and because the shortest path didn't change on the last step when we check the edge between i and $i + 1$,

   $$d[v_2] = d[v_1] + w_{1,2} = w_{1,2} = \delta(w_1, w_2)$$

   And because $d[v_2]$ didn't change either

   $$d[v_3] = d[v_2] + w_{1,2} = \delta(v_1, v_2) + w_{1,2} = \delta(w_1, w_2) + \delta(w_2, w_3) = \delta(w_1, w_3)$$

   Proceeding in this way, we can see that

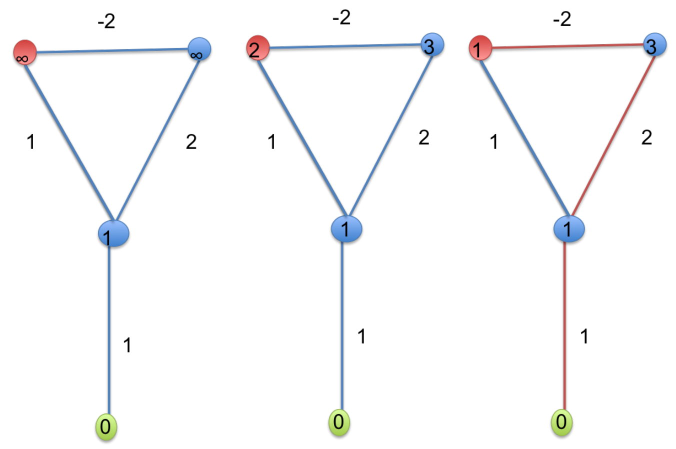
   $$d[v_n] = \delta(v_1, v_n)$$

   So we've found the shortest path to any arbitrary node.
   Runtime Note that after relaxing the edge every edge once

   $$d[v_2] = d[v_1] + w_{1,2} = \delta(v_1, v_2)$$

   by the argument above after we've relaxed every edge n times (so after we've relaxed every edge k times) we've found the shortest path to every node. This gives us a final runtime on the order of $O(k|E|)$.

3. This is Bellman Ford, finding the shortest path between the green node and the red node. The graph is drawn with every node's respective weight value at the end of every relaxation of every edge. Edges are relaxed from highest to lowest.



Because there are 4 vertices, every edge will be relaxed 3 times. We can report the shortest path from green to red (or from green to any node) after the algorithm has run by repeatedly taking the predecessor of our end node.

4. Let $V = \{v_1, v_2, ...v_n\}$ and $E = \{(v_1, v_2), (v_2, v_3), ..., (v_{n-1}, v_n)\}$ if $s = v_1$ and $v = v_n$. After $\Omega(n^2)$ operations $d[v] = \infty$ but it will be correct by the time the algorithm terminates.

5. (a) For this algorithm to run in $\Omega(n^2)$ time, for every node I would only store $d[v_i]$, its list of edges, and whether or not it has been expanded. I would NOT use a priority queue to slow down the algorithm from $\Omega(m + n \log n)$ to $\Omega(n^2)$. Instead I would simply store the fringe as a list, and every time search it in linear time for the lowest d value on the fringe. Because there are a maximum of n nodes on the fringe and in the worst case scenario I would have to pick every node out of the fringe to expand it, the runtime here would be on the order of $\Omega(n^2)$

   (b) I would store the same attributes as above only this time I would use a priority queue for the fringe to bring Dijkstra's algorithm back down to the expected runtime.

   (c) For every node v_i, I would store $d[v_i]$, a list of its edges and its predecessor (that is the last node that overrode its distance value). The worst case time is $O(nm)$, so our runtime goals are satisfied by the original algorithm. [1]

6. **Algorithm**
Searching Create an array d to store $|V|$ numbers, one for each vertex. Create an array p to store $|V|$ pointers to vertices, one for each vertex. This is the array we will use to keep track of the predecessor for any given vertex. Set $d[s] = 0$ and all other elements of d to ∞. Let $v_1, v_2, ..., v_n$ be all elements with orders between t and s read in there topological order.

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For i = 1 to i = n:
  For all neighbors 'r' of v_i:
    if d[v_i] + w < d[r]: (w = edge weight between v_i and r)
      d[r] = w + d[v_i]
      p[r] = v_i
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[1]That is: for i in range $1, n-1$ check every edge $e_{r,s}$ of weight w from nodes v_r to v_s and if $d[v_r] + w < d[v_s]$ set $d[v_s] = d[v_r] + w$, and set v_r to be the predecessor of v_s. Pretty clearly we can see the expected and worst case runtime of this algorithm is the number of nodes times the number of edges.

Reporting Recursively record the predecessor of s until you get back to t, reverse this list and report it.
Correctness Let $T(v)$ be the topological order for any vertex v. No vertex can have an edge to a vertex with a lower topological order (either directly or by induction indirectly). So updating $d[a]$ for a vertex a will not change $d[b]$ if $T(a) < T(b)$. This means that $d[s]$ will never change from us expanding vertices of lower topological order. When we expand the node after s in topological order call it r. We know $d[r]$ will never change, that is to say we know there can't be a path from s to anything after r back r so $d[r]$ must be correct. By induction on this idea once we get to t, $d[t]$ must be correct and every predecessor must be set correctly.
Runtime First note that there are $O(|V|)$ elements with topological orders between s, t. Reading elements in order by one off the ordering we have to check every edges of every vertices in this list. If you assume the number of edges per vertex is constant our runtime is

$$O(|V|c) = O(|V|)$$

However notice that we are checking every edge for $O(|V|)$ vertices so this may be $O(|E|)$ in itself giving us a final runtime of if you don't assume a constant number of edges

$$O(|V| + |E|)$$

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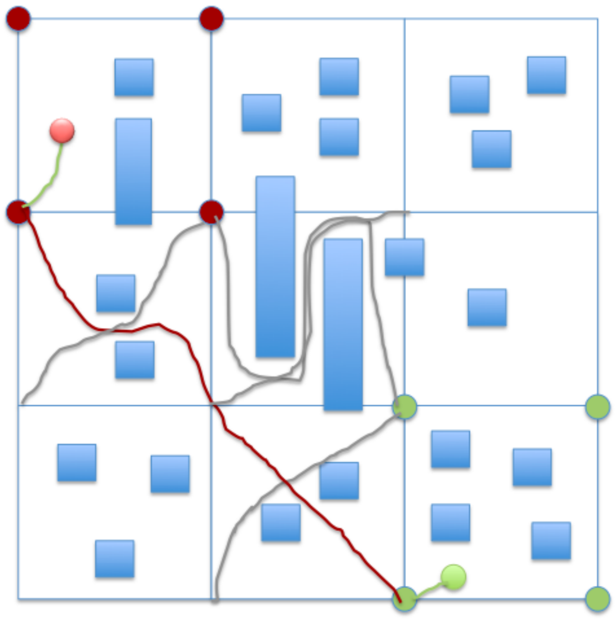
7. **Algorithm** The algorithm here is the same as in the last question the only thing we need to change is to set all values of d except for $d[s]$ to -1 and change the if statement from

8. **Algorithm** Add two vertices to the graph, one with zero weight edges to all blue points, and another with zero weight edges from every orange point to itself. We will consider these two nodes our start and end points.
For every vertex of the graph, eliminate every edge to vertex with an x coordinate lower then or equal to its own. This will make the graph only flow one way, and there will be no cycles. That is to say it is now a DAG. So we can apply a topological sort to order the vertices of our DAG. For every remaining edge of the graph, set its edge weight to be the euclidean distance between the points that the start and end vertices represent. Now run the algorithm above for shortest path on a DAG described in question 6. **Correctness** Suppose there exists a shorter path between a orange node and a blue node then the one we found in this algorithm. This implies that there is a shorter path between our start and end vertices, as the distance between our start and end vertices is the same as the minimum distance between any blue node and any orange node.
We can also see intuitively that removing the edges that decrease our x coordinate does not impact the shortest path through the graph, as no shortest path would ever include steps that traveled in the negative x direction. In addition, because our edge weights correspond to euclidean distance, we know that this is the minimum distance path by correctness of the shortest path of a DAG.
Runtime The runtime to remove every backwards edge and compute the distance between any two points with an edge are both on the order $O(|E|)$. A topological sort runs in $O(|E| + |V|)$ time and the shortest path algorithm for a sorted DAG also runs in $O(|E| + |V|)$ time.

9. I would use divide an concer to break up the shortest path problem into sub-problems at different scales then takel those sub problems with Dijkstra algorithm. to start divide the

map of the USA into one square mile ”blocks.” For each of these blocks I would create graphs with edges representing the roads and vertices representing houses in that block. Finding the shortest path within a block could be done with Dijkstra’s algorithm, as the number of edges and vertices within one block should be much more manageable then the whole map. For every block I would also compute the shortest path between each of its corners, e.g. the shortest path from the south west corner to the north west corner. To answer shortest path questions between blocks, I would create a graph where the north west corner of each block was a vertex with edges to other blocks that corresponded to the distance to travel to any adjacent block. First we could use Dijkstra’s algorithm on the blocks graph to find the shortest path between any one of the four corners, the block you start in to any one of the four corners of the block you finish in. Then we could use Dijkstra’s algorithm to get from your start to the corner of the block your path starts from and from the place your path stops to your destination. Below is an illustration of the algorithm



Gray paths are the pre-computed paths between blocks (drawn in only 3 squares), red paths are the paths the algorithm has chosen between blocks, and green paths are the paths between endpoints and the main path. Note that this algorithm does not necessarily find the minimum path, as any path between blocks must pass through some corner of a block but in a problem like finding the shortest path from the east coast to the west coast this distance should be relatively small. If the search space on the block graph is still too large you could combine 10 by 10 chunks of blocks in the same way that we constructed a graph of blocks (calculating the shortest path between chunks from the shortest paths across individual blocks) and do a Dijkstra search on that graph before we go down to the level of blocks. We see how we can add higher level graphs to get good ideas of how to get from one general area to another, and lower level graphs to get from the startpoint or to the endpoint from any given area.

10. If there is a negative weight cycle there would be a negative entry along the main diagonal of your output. To see why this first note that any negative weight cycle must trivially

contain $|V|$ or fewer nodes. Let $a_1 \rightarrow a_2 \rightarrow ... \rightarrow a_n \rightarrow a_1$ then the shortest path between a_1 and itself with $|V|$ or fewer nodes is one or more loops through the negative cycle. Because the algorithm always finds the shortest path (with $|V|$ or fewer nodes) it will find this one so the distance of the shortest path from a_1 to itself will negative in the output.

11. The output matrix from the Warshall-Floud algorithm gives us $\delta(a, b)$ (that is the *length of* the shortest path for a to b. For any two vertices of our graph a, b to find a shortest path from a to b note that if a node c is along this path

$$\delta(a, b) = \delta(a, c) + \delta(c, b)$$

And we know that at least one neighbor of a must be along a shortest path so check every neighbor of a to see if they meet this criterion. Once we've found one that does we can report a, consider the node we found our startpoint and repeat the algorithm above until we arrive at b.
To prove that this algorithm runs in optimal time note that we are reporting k elements where k is the number of vertices along the shortest path. Because k is on the order of $|V|$ we know that the number of operations it takes to report our solution alone means this algorithm must run in $O(|V|)$ time. So a solution is optional if it runs in linear time. Because every node has an assumed constant number of neighbors we only need to check a fixed number of edges per node till we find the next that vertex that satisfies the equation above, this will be the next vertex along the shortest path. Meaning our solution runs in constant time per vertex, and linear time altogether. Hence our solution is optimal.