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MATH3066
Homework 1
26/4/17

1. (a)

P	Q	R	$(P \wedge Q)$	$(P \wedge Q) \Rightarrow R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

(b)

P	Q	R	$\sim P$	$(\sim P \wedge R)$	$Q \Rightarrow (\sim P \wedge R)$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	F	T

(c) $Q \Rightarrow (\sim P \wedge R) \vDash (P \wedge Q) \Rightarrow R$:
Because in order for $V((P \wedge Q) \Rightarrow R) = F$, $V(P) = V(Q) = T$ and $V(R) = F$ meaning $V(Q \Rightarrow (\sim P \wedge R)) = F$.

(d) $(P \wedge Q) \Rightarrow R \nvDash Q \Rightarrow (\sim P \wedge R)$:
as if $V(P) = V(Q) = V(R) = T$ then $V((P \wedge Q) \Rightarrow R) = T$ and $V(Q \Rightarrow (\sim P \wedge R)) = F$

2. (a)

1	(1)	$((P \wedge Q) \Rightarrow R) \wedge Q$	RA
1	(2)	Q	$1, \wedge - E$
3	(3)	P	RA
1,3	(4)	$(P \wedge Q)$	$2, 3 \wedge -I$
1	(5)	$(P \wedge Q) \Rightarrow R$	$1, \wedge - E$
1,3	(6)	R	$4, 5MP$
1	(7)	$P \Rightarrow R$	$6 \Rightarrow -I$

(b)

1	(1)	$(P \Rightarrow Q) \vee (R \Rightarrow Q)$	RA
2	(2)	$(P \Rightarrow Q)$	RA
3	(3)	$(R \Rightarrow Q)$	RA
4	(4)	$(P \wedge R)$	RA
4	(5)	P	$2 \wedge -E$
4	(6)	R	$2 \wedge -E$
2,4	(7)	Q	$2, 5MP$
3,4	(8)	Q	$3, 6MP$
1,4	(9)	Q	$1, 2, 3, 7, 8 \vee -E$
1	(10)	$(P \wedge R) \Rightarrow Q$	$1, 4CP$

(c)

1	(1)	$(P \vee Q)$	RA
2	(2)	$(P \vee R)$	RA
2	(3)	P	$2 \wedge -E$
2	(4)	R	$2 \wedge -E$
1,2	(5)	Q	$1, 3, 4 \vee -E$
1	(6)	$(P \wedge R) \Rightarrow Q$	$2, 5CP$

(d)

1	(1)	$(P \Rightarrow Q) \wedge (R \Rightarrow \sim Q)$	RA
2	(2)	R	RA
1	(3)	$(P \Rightarrow Q)$	$1 \wedge -E$
1	(4)	$(R \Rightarrow \sim Q)$	$1 \wedge -E$
1,2	(5)	$\sim Q$	$2, 4MP$
1,2	(6)	$\sim P$	$3, 5MT$
1,	(7)	$R \Rightarrow \sim P$	$2, 6CP$

3. (a)

P	Q	R	$(P \Rightarrow (\sim Q \wedge R))$	$(Q \Rightarrow (\sim P \wedge R))$	$(P \Rightarrow (\sim Q \wedge R)) \Rightarrow (Q \Rightarrow (\sim P \wedge R))$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

We can see from the truth table above that

$$(P \Rightarrow (\sim Q \wedge R)) \Rightarrow (Q \Rightarrow (\sim P \wedge R)) = F$$

when:

$$P = F, Q = T, R = F$$

We know this is the only counter-model as the truth table is exhaustive.

(b)

1	(1)	$P \Rightarrow (\sim Q \vee R)$	RA
2	(2)	Q	RA
3	(3)	$\sim (\sim P \vee R)$	RA (For RAA)
2	(4)	$(P \wedge \sim R)$	$3SI(S)$ (De Morgan's Law)
3	(5)	P	$5 \wedge -E$
3	(6)	$\sim R$	$5 \wedge -E$
1,3	(7)	$(\sim Q \vee R)$	$1, 7MP$
2,3	(8)	$(Q \wedge \sim R)$	$2, 8 \wedge -I$
2,3	(9)	$\sim (\sim Q \vee R)$	$8\ SI(S)$ (De Morgan's Law)
1,2,3	(10)	$(\sim Q \vee R) \wedge \sim (\sim Q \vee R)$	$\wedge - I$
1,2,3	(11)	$\sim Q \vee R$	$\wedge - E$
1,2,3	(11)	$\sim (Q \sim Q \vee R)$	$\wedge - E$
1,2	(12)	$\sim\sim (\sim P \vee R)$	$3, 11, 12RRA$
1,2	(13)	$(\sim P \vee R)$	$12DN$
1	(14)	$Q \Rightarrow (\sim P \vee R)$	$2, 13CP$
	(15)	$(P \Rightarrow (\sim Q \vee R)) \Rightarrow (Q \Rightarrow (\sim P \vee R))$	$1, 14CP$

4. *Conjecture:* for any WFF W:

$$\#W = 3c(W) + v(W)$$

Proof. Base Case:
Assume: A is a WWF consisting of a single propositional variable then $\#A = 1$, $v(A) = 1$ and $c(A) = 0$ so

$$\#A = 3c(A) + v(A)$$

.

Inductive Step:
Assume: N,M are WFF's such that $\#N = 3c(N) + v(N)$ and $\#M = 3c(M) + v(M)$:

$$3c((\sim N)) + v((\sim N)) = \#N + 3 = \#(\sim N)$$
$$3c((M \vee N)) + v((M \vee N)) = \#N + \#M + 3 = \#(M \vee N)$$
$$3c((M \wedge N)) + v((M \wedge N)) = \#N + \#M + 3 = \#(M \wedge N)$$
$$3c((M \Rightarrow N)) + v((M \Rightarrow N)) = \#N + \#M + 3 = \#(M \Rightarrow N)$$
$$3c((M \Leftrightarrow N)) + v((M \Leftrightarrow N)) = \#N + \#M + 3 = \#(M \Leftrightarrow N)$$

Because every WFF is either a single propositional variable, a negated WFF, or two WFF's conjoined by a logical connective mathematical induction proves our conjecture

□

5. (a)

P	Q	$(P \Rightarrow Q)$
T	T	T
T	F	F
F	T	T
F	F	T

We can see from the truth table that if $V(Q) = F$ then $V(P) = F$ implies $V(P \Rightarrow Q) = T$ and $V(P) = T$ implies $V(P \Rightarrow Q) = F$ so $V(P \Rightarrow Q) = V(\sim P)$
similarly if $V(P) = T$ then $V(Q) = T$ implies $V(P \Rightarrow Q) = T$ and $V(Q) = F$ implies $V(P \Rightarrow Q) = F$ so $V(P \Rightarrow Q) = V(Q)$

(b) if $V(X) = T$ then:

$V((X \Rightarrow \sim Y) \Rightarrow \sim (Z \Rightarrow X)) =$	
$V((X \Rightarrow \sim Y) \Rightarrow \sim T) =$	By the truth table for $\Rightarrow$
$V((X \Rightarrow \sim Y) \Rightarrow F) =$	
$V(V(\sim Y) \Rightarrow F) =$	by part a
$V(\sim\sim Y) =$	by part a
$V(Y)$	

if $V(X) = F$ then:

$V((X \Rightarrow \sim Y) \Rightarrow \sim (Z \Rightarrow X)) =$	
$V((X \Rightarrow \sim Y) \Rightarrow \sim V(\sim Z)) =$	by part a
$V((X \Rightarrow \sim Y) \Rightarrow V(Z)) =$	
$V(T \Rightarrow V(Z)) =$	By the truth table for $\Rightarrow$
$V(Z)$	by part a

so:

$$V(W_{X,Y,Z}) = \begin{cases} V(Y) & \text{if } V(X) = T \\ V(Z) & \text{if } V(X) = F \end{cases}$$

(c)

$$\mathcal{T}_{W_{P_1,Y,Z}} = \begin{cases} \mathcal{T}_Y & \text{if } V(P_1) = T \\ \mathcal{T}_Z & \text{if } V(P_1) = F \end{cases}$$

(d) *Proof.* Note that:

$$V(\phi \wedge \psi) = V(\sim (\phi \Rightarrow \sim \psi))$$
$$V(\phi \vee \psi) = V(\sim \phi \Rightarrow \psi)$$
$$V(\phi \Leftrightarrow \psi) = V((\phi \Rightarrow \psi) \wedge (\phi \Rightarrow \psi))$$

For any WFF's ϕ and ψ . Therefore by induction all WWF's can be expressed in terms of $\sim$ and $\Rightarrow$ $\square$

6. In $\mathbb{Z}_{11}$:

$$3x - y = 2$$
$$7x + 2y = 0$$

$$\Rightarrow 13x = 4$$

$$\Rightarrow 2x = 4$$

$$\Rightarrow x = 2$$

$$\Rightarrow y = 4$$

However in Z_{13}:

$$13x = 4$$

$$\Rightarrow 0x = 4$$

Which clearly has no solution so, the system does not have a solution in $\mathbb{Z}_{13}$

7. *Conjecture:* The only integer solution to $x^2 + 5y^2 = 3z^2$ is $(x, y, z) = (0, 0, 0)$

Proof. Suppose there exists an $(x, y, z) \neq (0, 0, 0)$ that solves $x^2 + 5y^2 = 3z^2$
If $gcd(x, y, z) \neq 1$. then (x, y, z) can be expressed as (gx', gy', gz') where $g = gcd(x, y, z)$ so:

$$(gx')^2 + 5(gy')^2 = 3(gz')^2 \Rightarrow g^2(x'^2 + 5y'^2) = 3g^2z'^2 \Rightarrow x' + 5y' = 3z'$$

Meaning there exists a solution (x, y, z) such that $gcd(x, y, z) = 1$.
Also note that:

$$x^2+5y^2 = 3z^2 \mod 4 \Rightarrow x^2+y^2 = -z^2 \mod 4 \Rightarrow x^2+y^2+z^2 = 0 \mod 4$$

Because in modulo 4:

$$x, y, z \in \{0, 1, 2, 3\} \Rightarrow x^2, y^2, z^2 \in \{0, 1\}$$

Meaning that:

$$x^2 + y^2 + z^2 = 0 \mod 4 \Rightarrow x^2 = y^2 = z^2 = 0 \mod 4$$

so $x = y = z = 0 \mod 2$ implying for any solution (x, y, z), $gcd(x, y, z) \geq 2$. Which is a contradiction. $\square$