Rowan Lochrin
MATH415B - Klaus Lux
1/29/18
Homework 2

10 If A,B are ideals in a ring R show that $A+B = \{a+b | a \in Ab \in B\}$ is an ideal in R.
Consider $x \in (A+B)$, for any $r \in R$.

$$rx = r(a+b) = ra + rb \text{ for some } a \in A, b \in B$$

Because A, B are ideals $ra \in A$ and $rb \in B$, meaning $rx \in (A+B)$, $(A+B)$ is an ideal in R.

12 If A,B are ideals in a ring R show that $AB = \{a_1b_1 + a_2b_2 + ... + a_nb_n | a_i \in Ab_i \in B\}$ is an ideal.
First note that $a_1b_1 \in A$ (As $B \subseteq R$) so for any $x \in AB$, $r \in R$

$$\begin{aligned} rx &= r(a_1b_1 + a_2b_2 + ... + a_nb_n) \\ &= ra_1b_1 + ra_2b_2 + ... + ra_nb_n \\ &= a'_1 + a'_2 ... + a'_n \text{ for some } a_1, a_2 ..., a_n \in A \end{aligned}$$

And from question 10 we know that $a_1 + a_2 \in (A+A) = A$ so by induction:

$$rx = a'_1 + a'_2 + ... + a'_n \in A$$

23 Verify that R/I from example 12 has 16 elements.
Note that for any matrix $a \in R$.

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 2q_1 & 2q_2 \\ 2q_3 & 2q_4 \end{bmatrix} + \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}$$

For some q_i, r_i such that $a_i = 2q_i + r_i, 0 \le r_i < 2$ meaning

$$a + I = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} + I$$

so because there are 2 possible integer choices for each r value there are $2^4 = 16$ distinct cosets in the factor group R/I.

29 In $Z[x]$ the ring of polynomials with integer coefficients, let $I = \{f(x) \in Z[x] | f(0) = 0\}$. Prove that $I =< x >$.

Proof. We can see that for any polynomial $p(x) \in Z[x]$, $p(x)x$ evaluated at 0 is $p(0)0 = 0$ so $< x > \subseteq I$. We now seek to prove the other direction.
If $p(x) \in I$, then $p(0) = 0$, so we know that there are no constant terms in the polynomial meaning that either $p(0) = 0$ or for $a_1, a_2 ..., a_n \in \mathbb{Z}$ (for some postive n)

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x = (a_nxn - 1 + a_{n-1}x^{n-2} + ... + a_1)x$$

and $(a_nxn - 1 + a_{n-1}x^{n-2} + ... + a_1) \in Z[x]$ so $p(x) \in < x >$. Meaning $I \subseteq < x >$. □

34 In $Z[x]$ the ring of integers with polynomial with integer coefficients, let $I = \{f(x) \in Z[x] | f(0) = 0\}$. Prove that I is not a maximal ideal.

Proof. Consider the ideal $I' = \{f(x) \in Z[x] | f(n) = 0\}$. For at least one $n \in \mathbb{Z}$. We know that this is an ideal because for $f(x) \in I', g(x) \in R$

$$f(n)g(n) = 0g(n) = 0$$

So the polynomial $f(x)g(x)$ has a 0 at n and is also in I'

Note that I' contains all functions with at least one zero so $I' \neq R$ and $I' \neq I$ but $I \subset I'$. □

37 In $Z[x]$, the ring of polynomials with integer coefficients, let $I =< x, 2 >$ be an ideal.

1. Prove $< x, 2 >= \{f(x) \in Z[x] | f(0) \text{ is even}\}$

   Proof. If $f(x) \in< x, 2 >$, $f(x)$ has the form

   $$f(x) = g(x)x + 2h(x)$$

   For some $g(x), h(x) \in Z[x]$, so $f(0) = 2h(0)$, and because $h(0) \in Z$ we know that $f(0)$ is even so $f(x) \in I, < x, 2 >\subseteq I$
   If $f(x) \in I$

   $$\begin{aligned} f(x) &= a_1x + a_2x^2 + ...a_nx^n + 2b \\ &= (a_1 + a_2x^1 + ...a_nx^{n-1})x + 2b \end{aligned}$$

   So $f(x) \in< x, 2 >, I \subseteq< x, 2 >$ □

2. Is I prime? Yes. If $f(x) = g(x)h(x)$, for $f(x) \in I$, $g(x), h(x) \in Z[x]$

   $$f(0)|2 \Rightarrow g(0)|2 \text{ or } h(0)|2$$

   so either $h(x) \in I$ or $g(x) \in I$.

3. Is I maximal? Yes assume there is an ideal I' such that $I \subset I', I \neq I'$, so there is some $f(x) \notin I, f(x) \in I'$. From (1) we know $f(x)$ is not a polynomial that evaluates to an even integer at 0 so $f(x) + 1$ does. Implying $(f(x) + 1) \in I$ (and therefore in I') so

   $$(f(x) + 1) - f(x) = 1 \in I'$$

   meaning for all $h(x) \in Z[x]$

   $$1h(x) = h(x) \in I'$$

   Implying $I' = Z[x]$

41 For any ideal I of Z,let d be the smallest integer that divides every element of I, we can see that $I \subseteq< d >$ (by our definition of division). For any $x \in< d >$ then there exists some $q \in I$ such that $x = dq$. So by the definition of an ideal $x \in I$.

54 Let R be a commutative ring without unity, and let $a \in R$. Describe the smallest ideal I of R that contains a. Because $\forall x \in R, xa \in I$,

$$< a >\subseteq I$$

Now again by the properties of ideals we must include every element that is the product of an element in $< a >$ and an element in R. So

$$<< a >>\subseteq I$$

we must continue in this way until no new elements must be considered when we take the span of the elements in the previous set (e.g. If $<< a >> = <<< a >>>$) once this is the case the set we have is an ideal. Because we have never added elements that are not explicitly required by the properties of an ideal we know that this ideal is minimal. We can see from this construction of the minimal ideal that every element of the minimal ideal is a product of a. a product of a.

63 Let R be a commutative ring with unity and let $a, b \in R$. Show that $< a, b >$ the smallest ideal of R containing a and b. Because R has unity

$$a = 1_R a + 0b \in I$$

$$b = 0b + 1_R b \in I$$

If an ideal I contains a, b for any r, s in R

$$ra, (-1)bs \in I$$

By the ideal test we know that

$$ra - (-1)sb \in I \Rightarrow ra + sb \in I$$