Rowan Lochrin
MATH415B - Klaus Lux
2/26/18
Homework 6

1 Gallian

11 Trace the argument in example 7 to find q and r in $Z[i]$ such that $3-4i = (2+5i)q+r$ and $d(r) < d(2+5i)$
Note that in $Q[i]$,

$$\begin{aligned}(3-4i)(2+5i)^{-1} &= \frac{1}{29}(-14-23i)\\ &= (-i) + (\frac{-14}{29} + \frac{6}{29}i)\end{aligned}$$

So

$$\begin{aligned}(3-4i) &= (-i)(2+5i) + (\frac{-14}{29} + \frac{6}{29}i)(2+5i)\\ &= (-i)(2+5i) + (-2-2i)\end{aligned}$$

So $q = -i, r = -2-2i$, verify

$$d(r) = 2^2 + 2^2 = 8 < d(2+5i) = 2^2 + 5^2 = 29$$

20 Prove that $Z[\sqrt{-3}]$ is not a **PID**.

Proof. Because **PID** $\rightarrow$ **UFD** it will suffice to show $Z[\sqrt{-3}]$ is not a **UFD** Consider $4 = 2^2 = (1+\sqrt{-3})(1-\sqrt{-3})$
We can see that $d(2) = (2^2 + 0 = 4)$ so if x is a non-unit factor of $2, d(x) = 2$ meaning have $a^2+3b^2 = 2$ which has no solutions. So 2 is irreducible. $d(1+\sqrt{-3}) = 1^2+3(1^2) = 4$ so again if x is a factor of $(1+\sqrt{-3}, d(x) = 2$, implying $(1+\sqrt{-3})$ is also irreducible. so 4 has two factorizations in $Z[\sqrt{-3}]$. $\square$

26 In $Z[\sqrt{2}]$ show that any element of the form $(3+2\sqrt{2})^n$ is a unit.
Note that unity in $Z[\sqrt{2}]$ is 1.

$$(3+2\sqrt{2})(3-2\sqrt{2}) = 1$$

$$(3+2\sqrt{2})^n(3-2\sqrt{2})^n = 1^n = 1$$

That is to say that $(3-2\sqrt{2})^n$ is the additive inverse of $(3+2\sqrt{2})^n)$

32 Determine the units in $Z[i]$.
If x is a unit then we know that $N(x) = a^2+b^2 = 1$ which clearly only has the solutions, $a = \pm 1, b = 0$ and $a = 0b = \pm 1$. So $\pm i$ and ± 1 are the only units of $Z[i]$.

37 Show that an integral domain R satisfies the ascending chain condition iff every ideal of R is finitely generated.
Let I be a ideal generated by n elements if $I \subset I'$ then there must be an $x \notin I', x \in I$. So I' must be generated by $n+1$ element so if $(I_1 \subset I_2 \subset I_3 ...) \subset R$ by induction (and because I_1 is generated by at least one element) for any arbitrary k, I_k contains at least k elements.
Now suppose there is an ideal $I \subseteq R$ generated by an infinite number of elements $I =< i_1, i_2, i_3 ... >$ let $I_n =< i_1, i_2, ... i_n >$ we can see that $(I_1 \subset I_2 \subset I_3 ...) \subset R$ is an infinite chain of ideals.

40 Find the inverse of $(1+\sqrt{2})$ in $Z[\sqrt{2}]$, what is the multiplicative order of $1+\sqrt{2}$. $(1+\sqrt{2})(-1-\sqrt{2})=1$, however $(1+\sqrt{2})^n > 1 \forall n$ so it has an infinite multiplicative order.

2 GAP

18.1 2 reducible. 3 irreducible. 5 reducible. 7 irreducible. 11 irreducible. 13 reducible. 17 reducible. 19 irreducible. 23 irreducible. 29 reducible. 31 irreducible. 37 reducible. 41 reducible. 43 irreducible. 47 irreducible. 53 reducible. 59 irreducible.

18.2 $2 \mod 4 = 2$, $3 \mod 4 = 3$, $5 \mod 4 = 1$, $7 \mod 4 = 3$, $11 \mod 4 = 3$, $13 \mod 4 = 1$, $17 \mod 4 = 1$, $17 \mod 4 = 1$, $19 \mod 4 = 3$, $23 \mod 4 = 3$, $29 \mod 4 = 1$, $31 \mod 4 = 3$, $37 \mod 4 = 1$, $41 \mod 4 = 1$, $43 \mod 4 = 3$, $47 \mod 4 = 3$, $53 \mod 4 = 1$, $59 \mod 4 = 3$

18.3 A prime $p \in \mathbb{Z}$ is reducible in $\mathbb{Z}[i]$ iff $p \mod 4 = 1$.

18.4 $2 = 1^2 + 1^2$ $5 = 1^2 + 2^2$ $13 = 3^2 + 2^2$ $17 = 4^2 + 1^2$ $17 = 5^2 + 2^2$ $29 = 5^2 + 2^2$ $37 = 6^2 + 1^2$ $41 = 5^2 + 4^2$ $53 = 7^2 + 2^2$
All are irreducible in $Z[i]$.