Rowan Lochrin
MATH415B - Klaus Lux
2/5/18
Homework 3

16 Let $R = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. Prove that that the mapping $\phi : \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \to a$ is a valid homomorphism.

Proof. We seek to show

$\phi(a+b) = \phi(a) + \phi(b)$

$$\phi(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}) = \phi(\begin{bmatrix} a+a' & b+b' \\ 0 & c+c' \end{bmatrix}) = a + a' = \phi(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}) + \phi(\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix})$$

$\phi(ab) = \phi(a)\phi(b)$

$$\phi(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}) = \phi(\begin{bmatrix} aa' & ab' + bc' \\ 0 & cc' \end{bmatrix}) = aa' = \phi(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix})\phi(\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix})$$

□

27 Let R be a rind with unity and let ϕ be a ring homomorphism from R onto S where S has more than one element. Prove that S has unity.

Proof. For any element of $s \in S$ for some $r \in R$, $\phi(r) = s$. Meaning that

$$\phi(r) = \phi(1_R r) = \phi(1_R)\phi(r) = \phi(1_R)s = s$$

and

$$\phi(r) = \phi(r 1_R) = \phi(r)\phi(1_R) = s\phi(1_R) = s$$

so $\phi(1_R)$ is the unit in s. □

42 Determine all the ring homomorphisms from Q to Q.
For any homomorphism, ϕ, By the first property of ring homomorphisms

$$\phi(1) = \phi(1^2) = \phi(1)^2$$

So $\phi(1)$ must be an idempotent since there are only 2 idempotents in Q

$$\phi(1) = 0 \text{ or } \phi(1) = 1$$

If $\phi(1) = 0$ For all x in Q

$$\phi(x)\phi(1x) = \phi(1)\phi(x) = 0\phi(x) = 0$$

So ϕ must map every element of Q to zero.

If $\phi(1) = 1$ For all x in Q there exists a non-zero integer a such that ax is also an integer (e.g. Choose a to be the denominator of x). We can see that

$$\phi(a) = \phi(\overbrace{1 + ... + 1}^{a \text{ times}}) = \overbrace{\phi(1) + ... + \phi(1)}^{a \text{ times}} = \overbrace{1 + ... + 1}^{a \text{ times}} = a$$

By the same reasoning $\phi(ax) = ax$ from here we can see that

$$\phi(a)\phi(x) = ax$$

$$a\phi(x) = ax$$

$$\phi(x) = x$$

So ϕ must map every element of Q to itself.

44 Let R be a commutative ring of prime characteristic p. Show that the map $x \to x^p$ is a ring homomorphism from R to R.
By commutativity for any $a, b \in R$

$$\phi(ab) = (ab)^p = \overbrace{(ab)(ab)...(ab)}^{p \text{ times}} = \overbrace{(aa...a)}^{p \text{ times}}\overbrace{(bb...b)}^{p \text{ times}} = a^p b^p = \phi(a)\phi(b)$$

Also

$$\phi(a+b) = (a+b)^p = \sum_{k=0}^{p} \binom{p}{k} a^{n-k} b^k$$

Because p is prime, p divides $\binom{p}{k}$ for all values of k except $k = 0$ and $k = 1$. Meaning for some $x \in R$. $\phi(a+b) = a^p + b^p + px$ Because p is the characteristic of R, $px = 0$. Implying

$$\phi(a+b) = a^p + b^p = \phi(a) + \phi(b)$$

47 Suppose that R and S are commutative rings with unities. Let ϕ be a ring homomorphism from R onto S and let A be an ideal of S.

1. If A is prime in S, Show that $\phi^{-1}(A)$ is prime in R.
   Assume $ab \in \phi^{-1}(A)$, so $\phi(ab) = \phi(a)\phi(b) \in A$ because A is prime this means that either $\phi(a)$ or $\phi(b)$ is in A. So either a or b is in $\phi^{-1}(A)$.
2. If A is maximal in S, show that $\phi^{-1}(A)$ is maximal in R.
   Consider the homomorphism $\Phi : R \to S/A$ given by the mapping $\Phi(x) = \phi(x) + A$, we can see that $Ker\Phi = \phi^{-1}(A)$ so by the first isomorphism theorem.

   $$R/\phi^{-1}(A) \approx \Phi(R) \approx S/A$$

   Because A is maximal in S, S/A must be a filed and because it is isomorphic to $R/\phi^{-1}(A)$ that must also be a field, hence $\phi^{-1}(A)$ is maximal in R.

56 Show that the rings $Q[\sqrt{2}]$ and $Q[\sqrt{5}]$ are not isomorphic. Assume that there is an isomorphism ϕ from $Q[\sqrt{(2)}]$ to $Q[\sqrt{(5)}]$. Because ϕ must be onto we know by the 6th property of homomorphisms that

$$\phi(1) = 1$$

. Let $a = \phi(\sqrt{2}) \in Q[\sqrt{5}]$ then,

$$a^2 = \phi(\sqrt{2})^2 = \phi(\sqrt{2}^2) = \phi(2) = \phi(1+1) = \phi(1) + \phi(1) = 2$$

Since there is no element of $Q[\sqrt{5}]$ that squares to 2 this is a contradiction meaning the two rings are not isomorphic.

62 Give an example of a ring without unity that is contained in a field.
The ring $< 2 >$ in $\mathbb{C}$.

64 Suppose that $\phi : R \to S$ is a ring homomorphism and the image of ϕ is not $\{0\}$. If R has unity and S is an integral domain that ϕ carries the unity of R to the unity of S. Given an example to show that the preceding statement need not be true if S is not an integral domain.
Let $x = \phi(1) \in S$

$$\phi(1) = \phi(1^2) = \phi(1^2) = \phi(1)^2$$

So $x = x^2$ meaning that

$$\begin{aligned} 0 &= x^2 - x \\ &= x(x-1) \end{aligned}$$

Because S is an integral domain and has no zero divisors either $x = 0$ or $x = 1$. If $x = 0$ then for all $y \in R$,

$$\phi(y) = \phi(1y) = \phi(1)\phi(y) = 0\phi(y) = 0$$

so the image of ϕ is 0 meaning that $\phi(1) = 1$.
For an example of why this need not be the case if S is not an integral domain consider the homomorphism from Z_6 to Z_6 given by $x \to 3x$.