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MATH3961
Homework 1
27/4/17

1. (a) The map $I : (X, d) \to (X, d')$ is a homomorphism. As every Cauchy sequence in (X, d) is a Cauchy sequence in (X, d') and vice versa (part b) and I is the identity function.

   (b) if (x_n) is a Cauchy sequence in (X, d), then:

   $$\forall \epsilon > 0 \exists N_\epsilon | n, m > N_\epsilon \Rightarrow d(x_n, x_m) < \epsilon$$

   Because $\forall x_n, x_m : d'(x_n, x_m) \leq d(x_n, x_n)$, $d'(x_n, x_m) < \epsilon$.

   If (x_n) is a Cauchy sequence in (X, d'), then:

   $$\forall \epsilon > 0 \exists N_\epsilon : n, m > N_\epsilon \Rightarrow d'(x_n, x_m) < \epsilon$$

   Meaning

   $$\exists N_1 : n, m > N_1 \Rightarrow d'(x_n, x_m) < 1$$

   So for any ϵ chose $N_{\epsilon'} = max(N_1, N_\epsilon)$, then:

   $$n, m > N_\epsilon, \Rightarrow d'(x_n, x_m) < \epsilon$$

   and because $d(x_n, x_m) < 1$:

   $$n, m > N_1 \Rightarrow d(x_n, x_m) = d'(x_n, x_m)$$

   so

   $$n, m > N_{\epsilon'} \Rightarrow d(x_n, x_m) < \epsilon$$

   (c) Yes. If (X, d) is complete, every Cauchy sequence in (X, d) converges to an element of X. Because the set of all Cauchy sequences in (X, d) and (X, d') is the same (part b), (X, d') must also be complete. If (X, d) is incomplete then there must be some Cauchy sequence in (X, d) and (X, d') that does not converge to an element of X so (X, d') is also incomplete.

2. (a)

   $$f_n(x) = (-1)^n |x| - |2x| + 3$$

   Under the uniform metric $(f_n) \to 3$ on the interval [-1,1]. However by the L_1 metric it alternates between 3 and 5 forever and thus doesn't converge.

(b)

$$f_n(x) = \begin{cases} n - xn^3 & \text{if } x < 1/n^2 \\ 0 & \text{if } x \geq 1/n^2 \end{cases}$$

On the interval [0,1] f_n converges to 0 but L_1 metric. However by the uniform metric $(f_n) \to \infty$ because $(f_n(0)) \to \infty$ so it diverges.

3.

4. (a) d is a metric on $\mathbb{R}^2$ because $\forall x, y, z \in \mathbb{R}$:

   i. $d(x, y) = max(|x_1 - y_1|, |x_2 - y_2|) \geq 0$. and:

   $$max(|x_1 - y_1|, |x_2 - y_2|) = 0 \Leftrightarrow x_1 = y_1, x_2 = y_2 \Leftrightarrow x = y$$

   ii. $max(|x_1 - y_1|, |x_2 - y_2|) = max(|y_1 - x_1|, |y_2 - x_2|) \Rightarrow d(x, y) = d(y, x)$

   iii. Without loss of generality let $|x_1 - z_1| \geq |x_2 - z_2|$:

   $$\begin{aligned} d(x, z) &= max(|x_1 - z_1|, |x_2 - z_2|) \\ = |x_1 - z_1| &= |x_1 - y_1 + y_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1| \\ &\leq max(|x_1 - y_1|, |x_2 - y_2|) + max(|y_1 - z_1|, |y_2 - z_2|) \\ &= d(x, y) + d(y, z) \end{aligned}$$

   The ball $B(0; 1)$ is a open square of side length 2 around the origin.

   (b) All balls centered at 0 have one of the following forms

   $$[0, n) : 0 < n < 1$$

   $$[0, 1]$$

   $$[0, 1] \cup [2, n) : 2 < n < 3$$

   $$[0, 1] \cup [2, 3]$$

5. If $A = B$:

   (a)

   $$int(A \cup B) = int(A) = int(A) \cup int(A) = int(A) \cup Int(B)$$

   (b)

   $$\overline{A \cup B} = \overline{A \cup A} = \overline{A} = \overline{A} \cup \overline{A} = \overline{A} \cup \overline{B}$$