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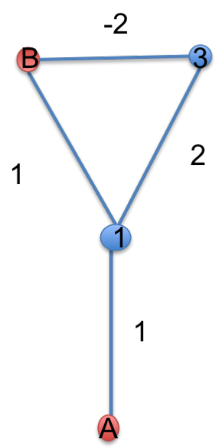
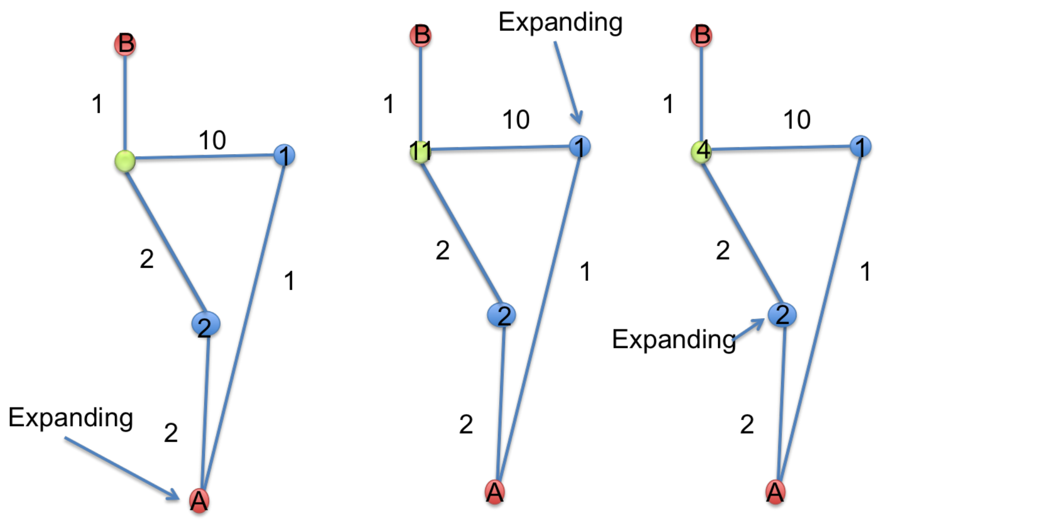
CSC445 - Alon Efrat
3/12/18
Homework 3

1. Pondered deeply.
2. Consider the following graph



Note that the minimum distance path from A to B is 1 (through edges of cost 1, 2 , -2) but Dijkstra's algorithm will not find this solution. This is because, after expanding the only vertex attached to A we will then choose the minimum distance vertex, B . Once at B the search terminates with a final path length of 2.

3. The following figure shows the first three steps of an execution of Dijkstra's algorithm to find a path from A to B. On the last step the distance value of the green vertex changes.



4. (a) Let $V = \{v_1, v_2 ... v_n\}$. To connect all n vertices $E = \{(v_1, v_2), (v_2, v_3), ..., (v_{n-1}, v_n)\}$. So $|E| = n - 1$.

(b) Let $V = \{v_1, v_2...v_n\}$. The maximum set of edges for an undirected graph with no parallel edges or cycles is

$$E = \{(v_i, v_j) | i < j\}$$

To see why this is, first consider a fully connected graph and notice how the requirement that our graph lacks parallel edges means we have to remove edges of the form (v_i, v_i). And the requirement that we lack cycles means we have to remove either all edges (v_i, v_j) such that $i > j$, or all edges such that $i < j$, As these connect back to previously connected components.
We can see that

$$|E| = \sum_{i=1}^{n} i = \frac{n(n-1)}{2}$$

(c) To do this we will use a hash set to keep track of nodes we've already seen, call it *seen*, start a counter i at $i = 0$ and run it till $i = n$. Our algorithm is as follows.

- Check node v_i is in *seen* (constant time). If it is increment i by one and repeat this step if it isn't continue.
- Increment the number of connected components by one.
- If node v_i is *seen*, advance to the next step, otherwise recurse [1] to all nodes with an edge to v_i and apply this step on them. Then add v_i to *seen*.
- Increment i by one and return to the top.

Continue this process until $i = n$. We can see that once we increment our counter of components we recursively add every node in the components to our seen table. In this way we ensure that whenever we come across an unseen node it belongs to a new component.
The runtime of the first step is $O(n)$ the runtime of the 3rd step is $O(n+m)$ as for all n initially unseen nodes it will need to recurse through all of there edges. Our runtime is

$$O(n) + O(n+m) = O(n+m)$$

5. **Algorithm** Use Dijkstra's algorithm by first constructing a graph where V is the set of paths and E the set of edges connecting every location (i, j) with $(i+1, j)(i-1, j), (i, j+1)$ and $(i, j-1)$. We can see that construction of this graph runs in linear time. From here we will consider the highest of the two points, the end point (reporting the path backwards at the end if need be), and set the costs on the edges to 0 for any positive elevation change, or the absolute value of the elevation change, if the change is negative. From there we can run Dijkstra's algorithm on the graph and report the minimum cost path.
Correctness The idea here is we want to use Dijkstra's algorithm to find most efficient path. To do this we must first consider what an inefficient move is.
Again we will swap the start and end points so that the path goes uphill. Let

[1] You could accomplish this without recursion by using the doubly linked list to backtrack instead. This would probably be a little faster but its on the same runtime and more difficult to describe.

$V = \{v_1, v_2, ..., v_n\}$ be the vertices of our path and let $e(v_i)$ be the elevation at v_i. We can see that f the amount of fuel used along our path is

$$f = \sum_{i=2}^{n} \max(0, e(v_i) - e(v_{i-1}))$$

and

$$\sum_{i=2}^{n} e(v_i) - e(v_{i-1}) = e(v_n) - e(v_1)$$

Which is constant regardless of the path taken between v_1 and v_n. So inefficiency along any path comes from making moves that decrease in elevation. That is moves from v_i to v_{i+1} such that $e(v_i) < e(v_{i+1})$. Put another way, going downhill is inefficient because we know we will have to make up any elevation we lose later on. So we assign a cost to these moves equal to how inefficient they are, $e(v_{i+1}) - e(v_i)$. [2]
From here correctness from Dijkstra's algorithm ensure that we will eventually find the most efficient path.
Runtime The runtime here is simply the runtime of Dijkstra's algorithm, plus the linear time construction of the graph.

$$O(m + n \log n) + O(n) = O(m + n \log n)$$

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6. **Algorithm** As we did above, create a start vertex with zero cost edges to every element of S_0. Here we will manufacture multiple end vertices for every S_i and run Dijkstra's algorithm until we find all of them (reporting every path without our constructed start and end vertices every time we find one).
Correctness Correctness of every individual path follows from correctness of the previous answer.
Runtime Note: in the worst case scenario Dijkstra's algorithm visits every vertex in the graph, so adding multiple endpoints won't impact the runtime as, excluding any overhead from reporting, the runtime to find all endpoints will be the same as the runtime to find only the farthest endpoint from S_0.

7. **Algorithm** Starting with a vertex $t \in V$. We've run Dijkstra's algorithm so we know $d[t]$ that is the total cost of the path from s to t. Let x be the node before t on this path. We can see that if c_x is the edge cost between x and t

$$d[t] = d[x] + c_x$$

So all we have to do is check nodes connected to x until we find a node that meets that condition. If there are two that do so, either one will work. Continue backtracking in this way till you get to s. Record the nodes you pass as you do so

[2] We could also define the least inefficient path as the path with the smallest number of positive elevation changes and make negative elevation changes have a cost of zero. We would still find the correct solution eventually and the runtime would still be the same if the map were finite. However we would first check every square that went down in elevation so we would probably spend a lot of time descending and looking for paths near the base of the mountain and only reluctantly start climbing it once we are out of other options.

then print them in reverse order for a path $s \to t$.

Correctness We let Dijkstra's algorithm fully execute, so for every node v $d[v]$ is the distance of the shortest path. Also note that there will be at least one node x such that $d[t] = d[x] + c_x$ and if $d[t] < d[x] + c_x$ then $\delta(s,t) < \delta(s,x) + \delta(x,t)$ implying x is not a vertex on the minimum path. By induction we can conclude that the elements we return represent the shortest path from x to t.

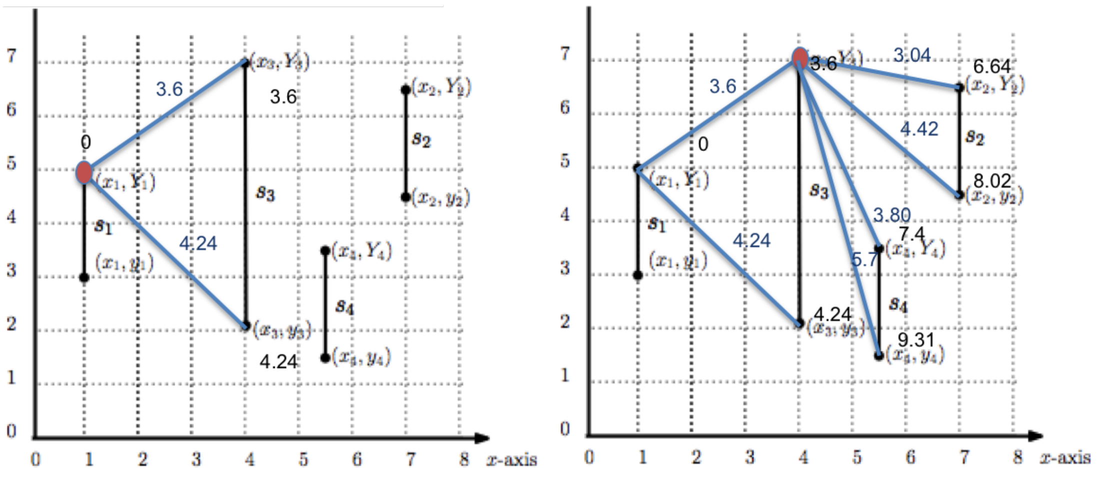
Runtime We can assume that checking all vertices connected to a particular vertex takes constant time, and we must do this for every node in the minimum path. Call the number of nodes in the minimum path k. Our runtime is

$$O(ck) = O(k)$$

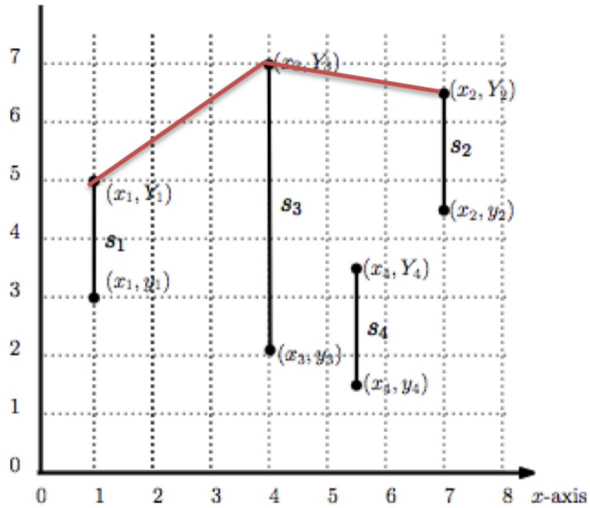
8. **Algorithm** Consider the set of all points to be the tops and bottoms of every vertical line. Also maintain $d[v]$ for all points on the grid in the same way you would using Dijkstra's algorithm.

   - Add (x_1, y_1) to a priority queue with a priority of zero.
   - Pop an element from the priority queue, call it (x_n, y_n), and consider every point (x_m, y_m) such that $x_n < x_m$. If you can draw a straight line from (x_n, y_n) to (x_m, y_m), without intersecting any of the given vertical lines, add it to the priority queue with priority equal to $d[(x_n, y_n)]$ plus the distance between it and (x_n, y_n) (updating $d[(x_m, y_m)]$ if need be). Repeat this step until you pop your endpoint from the queue.
   - Use the method described in question 8 to backtrack and find the shortest path.

Below is an illustration of the first two expansions of the algorithm on the sample data with the edge weights written in blue and the distance values for each node written in black.



We can see that the next node to be expanded is (x_3, y_3) as it has the lowest d value of any on the grid. Once there are no nodes on the fringe with d values above 6.64 we pop the final node from the fringe and are left with the following path.



Correctness First note that the shortest possible path through the vertical lines will consist of straight lines between points on the grid. While this makes sense intuitively, proving this rigorously would be difficult. However, the main idea here is induction on Euclid's *classic* axiom, the shortest path between any two points is a line.
Next consider a graph with a vertex representing every point on the grid, with edges representing where there is a straight line between the two points that does not intersect any of the given lines and a weight equal to the length of that line. Observe that the minimum cost path through this graph will correspond to the minimum distance path through the grid. Note that for every path between your start and end points consisting of straight lines between points on your grid there will be a path through your graph representing it. Once again the proof of correctness for Dijkstra's algorithm implies that this algorithm will find the shortest path on the graph/grid.
Runtime For every point you consider you will have to look at every other point on the grid with a greater x coordinate, and for all of those points you will have to check to see if the line you draw in between them intersects with any given vertical, to make sure you can draw a straight line between them. Because there will be $O(n)$ points of greater x coordinates for any point and $O(n)$ vertical lines to check for intersection of the line drawn between the two points the runtime of our algorithm is

$$nO(n)O(n) = O(n^3)$$