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CSC445 - Alon Efrat
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Homework 2

1. Use MID5 to create characteristic hashes for every file on your system. MID5 is ideal for this as it's used as a checksum against data corruption. This means that small differences in files will result in large differences in hashes, so two similar files are unlikely to have the same hash. We will use the characteristic hashes as keys for a hash table where the checksums are keys and the values are paths to files they represent. We then go through every file on the system and test duplicates when they collide in the table. We don't have to continue to probe after two entries collide in the table s when collisions happen we simply look up both files by there file paths.
   It will be necessary to generate a longer hash for any perspective duplicate and only report the pair if those hashes match. This is because the range of output of your hashing algorithm will be much larger then the size of the table.
   This algorithm is based on the idea that if two files are identical every hash of those two files will be identical. The core comparison stage of this algorithm, where every file is hashed uses a hash table so it runs in constant time per file. with $n$ files this algorithm is on the order of $O(n)$ and avoids most of the monotony that would have come from reading whole files to compare them with hashing.

2. (a) The hashing algorithm $h(k) = k \bmod 32$ where $k = 0 \bmod 8$ is week. The range of $h$ is only $\{0, 8, 16, 24\}$ (values in $Z_{32}$, 0 mod 8). So there are only 4 possible hashes that this function can output. E.g. If you attempted to use this to fill a 32 row hash table you'd only every get entries on the 0th, 8th, 16th, and 24th row. This will result in a slow table.
   (b) Same thing as above, noting that $k = 0 \bmod 32$ implies $Ak = 0 \bmod 32$. So the same problem applies.
   (c) No, the multiplication method described in class involved hashing by taking the module mod $2^n - 1$ not $2^n$. The flaw in the previous methods arises because $gcd(kA, m) = gcd(8, 32) = 8$, so we restrict our range by a factor of 8. We can see that $gcd(kA, m) = gcd(8, 31) = 1$, meaning that all integers from $0, 31$ are in the range of the function.
   (d) No. To see why, consider $A \approx 7/8$. We can see that $Ak \mod 8 = 7$ (provided $k \neq 0$). And because $\gcd(7, 32) = 1$ that $A$ will give us anything in the range. In general, we want to choose values for $A$ such that $\gcd(Ak \mod 8, m) = 1$. The downside of this hashing algorithm is that because of the approximate nature of floating point numbers we may not get the same hash every time. It's difficult to say how often this will happen but it could be massively problematic if it does.

3. Let $\{k_1, k_2, k_3, k_4\} = \{77, 147, 217, 287\}$.
   We can see that $k_1 = k_2 = k_3 = k_4 = 7 \mod 10$. And

$$k_1 + i(x \mod 10) \mod m = 77 + 0 \mod 21 = 14$$

   So the first key will be inserted into index 14.

$$147 + 0 \mod 21 = 0$$

The second key will be inserted into index 0.

$$217 + 0 \mod 21 = 7$$

The third key will be inserted into index 7. However we can see that

$$287 + 0 \mod 21 = 14$$

Which is full so we try $i = 1$

$$287 + 1(287 \mod 10) \mod 21 = 287 + 7 \mod 21 = 0$$

Which is also full so try $i = 2$

$$287 + 2(287 \mod 10) \mod 21 = 287 + 14 \mod 21 = 7$$

Again full trying $i = 3$

$$287 + 3(287 \mod 10) \mod 21 = 287 + 21 \mod 21 = 287 + 0 \mod 21 = 14$$

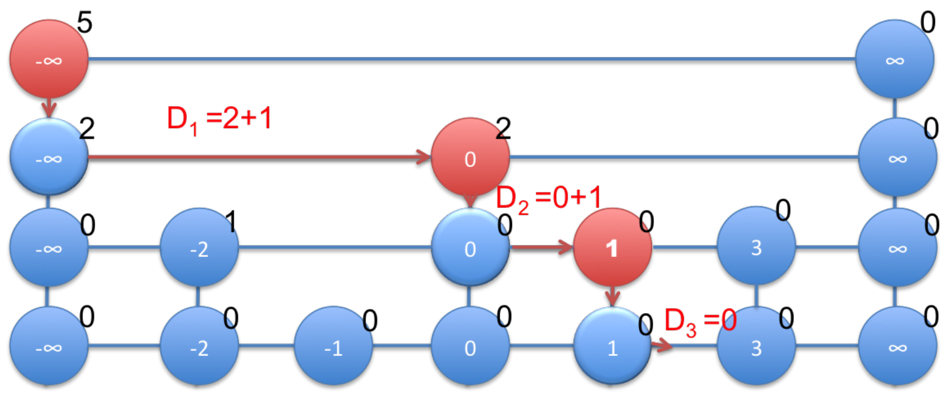
We can see that we will continue in this way forever, and the insertion fails.

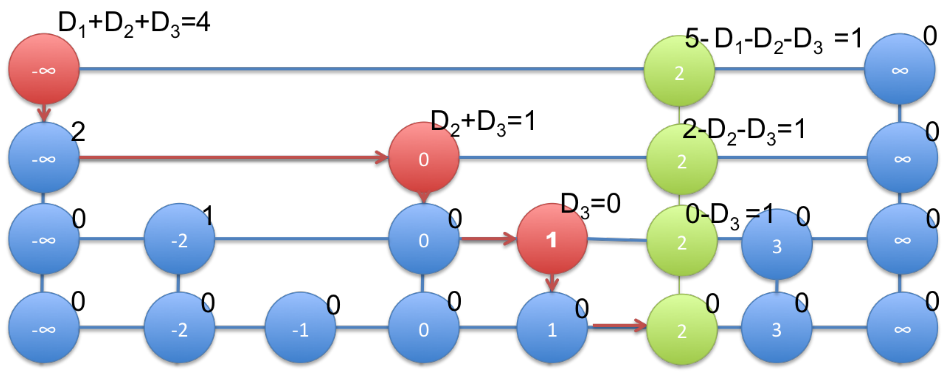
4. To do this I would give every node an attribute that can be used to count the number of nodes between it and the next node on the current level (keeping in mind nodes on lower levels). We will initialize this value to zero and will have to maintain this counter on insertion and deletion.

   **Insertion** Every time we use the down pointer on a node we must increment the counter value. This is because if we go down the structure, we know that the node we're inserting is less then the next node on the current level. That is to say it will end up somewhere between them. If we only have to insert the node at the lowest level of the structure this is the only change we need to make here.
   However if we are inserting a node that will need to be promoted to higher levels me must keep a list of all the nodes that we took the down pointer on and a list of the number of total nodes we passed till we get to the next down pointer (or to the location of our insertion for the base level). For levels above the base we can accomplish this without going to the base of the structure by simply counting the nodes we pass on the current level along with the sum of the number of nodes in between them. We will call the number of total nodes we passed on level $n$ to get to the next down pointer $d_n$. In this way we can see that the number of nodes between our final insertion and the last node before the insertion, on level $n$, for a skiplist of height $m$ will be

$$d_n + d_{n+1} + ... + d_m$$

   or the sum of the number of nodes we passed on every lower level. Let $v_n$ be the current value of the counter on the last node on level $n$ before the insertion. Because we split the distance between the next node on the level when we are inserting we must subtract $d_n + d_{n+1} ... + d_m$ from $v_m$ to get the distance between the insertion node on that level and the next node on that level. We must also modify the counter to the left of the node by changing it to $d_n + d_n + 1 + ... + d_m$ (or the total distance traveled before the insertion after we hit that node). A 4 level insert is illustrated below with the counter on every node drawn in it's top left corners.





**Deletion** Deletion works in much the same way when we are deleting an element on the base layer, except that by decrementing every node we took the left pointer on instead of incrementing it. Deleting a multi-level entry can be done by adding it's counter on every level to the previous node before removing the node. This works because the number of nodes between the previous node and the new node it points to (that is the node past the deleted node) must be increased by the number of nodes between the deleted node and node it pointed to.

**LesserThen** To create the operation we can modify the find function to count how many total nodes we pass on the way to the node we are trying to find. This is done by counting the nodes on every level and adding the number of nodes in between them. This will be the return value of our function.

Note that none of these changes impact the manner in which we traverse the tree, so insert and delete still run on $O(\log(n))$. Because LesserThen is a modification of the find function and also doesn't impact traversal, we know it will also run on the order of $O(\log(n))$.

5. (a) The expected time for *find*$(x)$ will decrease when $p = 0.01$. To see why, consider a skiplist with $n$ elements on the base level. If $p = \frac{1}{2}$ then there will be an

   - An average of $\frac{n}{2}$ elements on the first level.
   - $\frac{n}{4}$ elements on the second level.
   - $\frac{n}{8}$ elements on the third level.
   - ....

- 2 element on the $h-1$ level.
- 1 element on the $h = \log_2(n)$ level. Call the element on level $h$, $P$ (for pivot). We will first ask the question: How many elements are there on level $h-1$ less $p$? Because there are 2 elements on $h-1$ (on average) one will be less then $p$ and one element will be greater then $p$. So we only need to make one move on level $h$. Now we can consider every element less then $p$ and every element greater then $p$ as a skiplists of height $h-1$, and by induction conclude that we will have will on average to make one move per level. So the total number of moves will be

$$\sum_{1}^{\log_2(n)} 2 = \log_2(n)$$

If $p = 0.01$

- An average of $\frac{n}{100}$ elements on the first level.
- $\frac{n}{100^2}$ elements on the second level.
- $\frac{n}{100^3}$ elements on the third level.
- ....
- 100 elements on the $h-1$ level.
- 1 element on the $h = \log_{100}(n)$ level.

To analyze the runtime of *find* we note that there will be an average of 50 to the left and right of $P$ meaning that the average number of nodes we need to touch on this level is 25 as earlier we can conclude by induction that we will need to touch an average of 25 nodes on every level. So the total number of moves will be.

$$\sum_{1}^{\log_{100}(n)} 25 = 25\log_{100}(n)$$

And

$$25\log_{100}(n) = \frac{25\log_2(n)}{\log_{100}(2)} \approx 3.76\log_2(n)$$

So on average when $P = 0.01$ the search takes 3.76 times as long as it normally would.

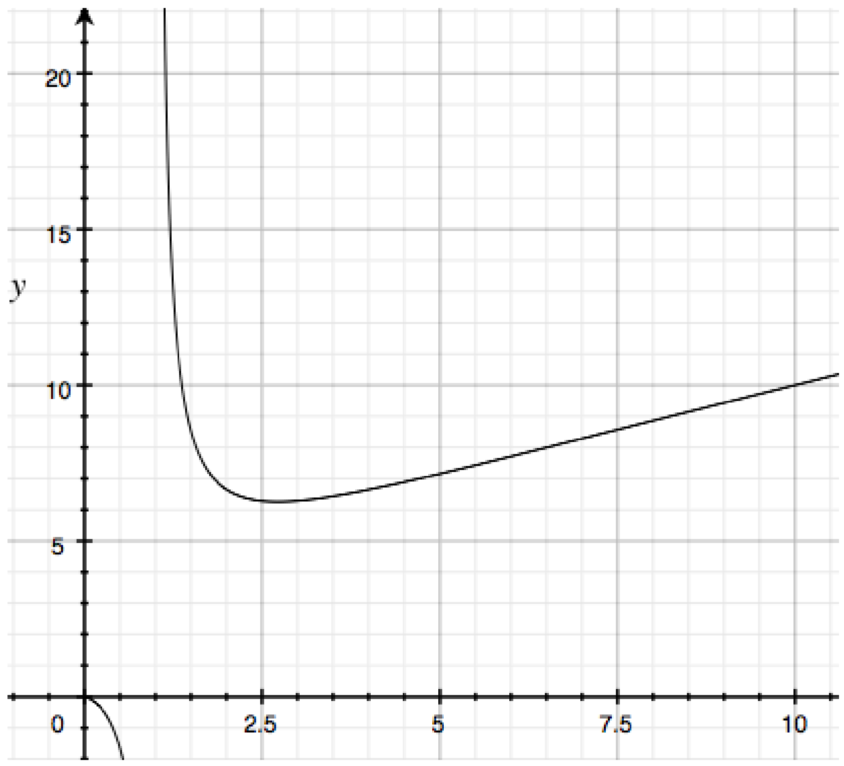
(b) We can generalize the formula we found by noting that every level of our skiplist partitions the level below it into $d/n$ partitions of size $d$. Meaning we only have to do on average $d/2$ comparisons on every level. From here we have to figure out the height of our list by consider that there will be $1/d$ times the keys in the base level in the first level, $1/d$ times the keys in the second level in the first level so $1/d^2$ keys on. So on the $m$th level there will be $1/d^m$ keys. We know there will only be 1 key on the last level so the height of our tree is $h$ is

$$\frac{n}{d^h} = 1 \to n = d^h \to h = \log_d(n)$$

So because we will have to do $d/2$ comparisons on every level we can write our formula for the runtime of *find*$(x)$ as

$$\sum_{1}^{\log_d(n)} \frac{d}{2} = \frac{d\log_d(n)}{2}$$

Graphing our function we see



It has a minimum at $e \approx 2.71$.

6. For every key is on level 1 there is a 1/2 probability that any key will be not promoted to the level above it. Call key $k$. The probability of all $n$ keys not being promoted to the second level is the product of the individual probabilities

$$\prod_{1}^{n} \frac{1}{2} = \frac{1}{2^n}$$

7. Note that whenever the top level of a skip list is empty, it is deleted and that the height of any particular insertion is not dependent on the number of nodes in the tree. So after removing $2^n - n$ nodes we can expect the height of the skip list to be the same as it would have been if we had not made the inserts in the first place. This means that we can use the formula from class and say, the probability that the skiplist is over height $Z \log_2(n)$ is $1/n^{Z-1}$.

8. For $L$ our prioritized list of names, let

$$L = [f_1, f_2, ..., f_n]$$

   We will use a hash table with open addressing to solve this. For our purposes the keys will be the names on the list and the values will be their respective indices (e.g. 0 for the highest priority on the list). That is to say every name $f_i$ will be hashed to an index of the table and the value stored at that index will be the priority, $i$, of that name. Then when given two names $a, b$ we simply look them up in the hash table and choose the one that returns the higher priority.
   Inserting every name into a hash table takes $n$ time (constant time for every name) and searching the table runs in constant time. So creating a table for all $n$ males

takes will take $n^2$ time. The next goal is to figure out how many times every individual table is quarried. If hypothetically every female had the same exact priorities list, then on every round all but one female would be rejected and they would all move down to their next choice on the list. In this way on the first round there are $n$ meetings, on the second round there are $n-1$ meetings and so on until there are no unmatched females. In this way we can see that there will be $n+n-1+...+1 = n(n+1)/2$ meetings and therefore $n(n+1)/2$ total constant time look ups so the runtime of our algorithm is on the order of $O(n^2)$.

9. (a) Let $p(x)$ be the preference list for a male or female $x$. We can see we only have to define

$$\begin{aligned} p(f_1) &= \{m_1, ...\} \\ p(f_2) &= \{m_2, ...\} \\ &... \\ p(f_n) &= \{m_n, ...\} \end{aligned}$$

And

$$\begin{aligned} p(m_1) &= \{f_1, ...\} \\ p(m_2) &= \{f_2, ...\} \\ &... \\ p(m_n) &= \{f_n, ...\} \end{aligned}$$

And we can see that everyone will go to their first preference on the first round, meaning no female will reject any male and all will be one string. So the algorithm terminates after one round with the set of pairings

$$\{(f_1, m_1), (f_2, m_2), ..., (f_3, m_3)\}$$

(b) The preferences

$$\begin{aligned} p(f_1) &= \{m_n, ...\} \\ p(f_2) &= \{m_{n-1}, ...\} \\ &... \\ p(f_n) &= \{m_1, ...\} \end{aligned}$$

And

$$\begin{aligned} p(m_1) &= \{f_n, ...\} \\ p(m_2) &= \{f_{n-1}, ...\} \\ &... \\ p(m_n) &= \{f_1, ...\} \end{aligned}$$

Will produce the set of pairings

$$\{(f_1, m_n), (f_2, m_{n-1}), ..., (f_n, m_1)\}$$

10. First we will make an observation about the problem

$$\Delta(p1, p2, p3) = d(p_1, p_2) + d(p_2, p_3) + d(p_3, p_1)$$

and

$$d(p_2, p_3) = \sqrt{d(p_1, p_2)^2 + d(p_1, p_3)^2}$$

So

$$\Delta(p1, p2, p3) = d(p_1, p_2) + d(p_1, p_3) + \sqrt{d(p_1, p_2)^2 + d(p_1, p_3)^2}$$

We can minimize this function by minimizing $d(p_1, p_2)^2$ and $d(p_1, p_3)^2$. So for every point, the minimum triangle connected to the point can be determined by finding the closest two points. Thus the minimum perimieter triangle can be found by going through every point calculating it's minimum triangle, remembering the size and location of the current smallest one as you go.
Now we can begin our description of the algortihm. The main problem we hope to solve is to find the two nearest neighbors of any point in expected constant time. To do this we will use the randomized method of finding the closest pair of points approach discussed in class. This involves taking random permutations of points and partitioning by the smallest distance between random pairs. The only modification we need to make to this algortihm is that, when visiting each point, instead of remembering the current closest two sets of points we need to remember both the perimieter of the current mimimum triangle, and the points it contains. At the end we will report the points it contains. Because the expected runtime of the original algorithm is linear and the only modifications we are making to it is finding 2 closest points for every point, and doing a slightly more complicated arithmatic calculation, we know that this runtime will also be linear.

11. Let $M_i$,$x_i$ denote the values of $M$ and $x$ on step $i$

    (a) Because $x_1 > 0$, $M_0 = -\infty$ so $x_0 > M_0$ x $M$'s value changes.

    (b) because $M$ changed last step we know that $M_1 = x_0$. And if $x_2 > M_1$ the value of $M$ changes. So if $\max(M_1, x_2) = \max(x_1, x_2) = x_2$ then $M$'s value changes. Since both our random variables are in the same range the chance of this will be 1/2

    (c) From the earlier step we know $M_2 = \max(x_1, x_2)$ and $M$ will change if

    $$M_3 = \max(M_2, x_3) = \max(\max(x_1, x_2), x_3) = \max(x_1, x_2, x_3) = x_3$$

    And the probablity of that happening is 1/3.

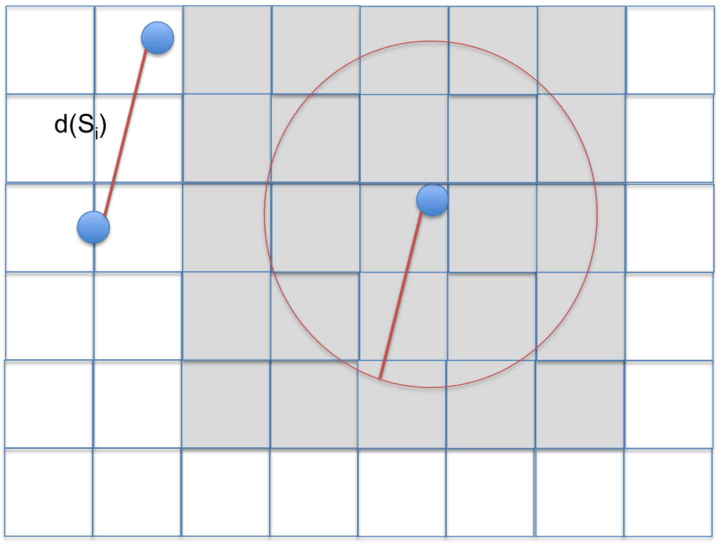
    (d) Because count is incremented on every successful change of $M$ and we can see from the earlier step that the odds of that happening are $1/m$ on step $m$, we can see for steps one thorugh $n$ that the average amount the count changes is

    $$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i} \approx 0.577 \log(n)$$

12. Assuming the unrandomized approach.

    (a) We first insert two points in the hash table $p_1, p_2$ and calculate the distance between them, then partition the space into a grid of size $d(p_1, p_2)/2$, add a

third point $p_3$ and partition around it to determine if there is a point closer then the current mimimum distance $d(p_1, p_2)$. Because every square has size $d(p_1, p_2)/2$ this means that any partition with any point closer to $p_3$ than $p_1$ is to $p_2$ will be under two partitions away from the parition containting $p_3$. Meaning we only have to check the $5x5$ grid centered around the partition $p_3$ contains, or 25 partitions total. This is illustrated below with the checked partion in gray. Note that we check all corners as calculating weather or not to include them would probably cost about as much computation.
Because we have to do this for all points except $p_1, p_2$ we will have to make $25(n-2)$ lookups.



(b) Doing this in one dimension, the only modification we have to make is to consider that the radius of two partitions around the partition containing the insertion point will simply be the line containing $5^1$ partitions.

(c) In three dimensions we must look at the cube containg $5^3 = 125$ partitions around the point. In general, in $n$ dimensions we must look at the $5^n$ partitions around the insertion partitions.