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# Problem 9

To see if this can prove that $P = NP$, we should first establish whether DNF (Disjunctive Normal Form) is NP-complete. We begin by proving the transformation:

$$\text{CNF} \leq_{\text{P}} \text{DNF}$$

For CNF, we have:

$$(X \vee Y) \wedge (Y^{\complement} \vee Z) = (X \wedge Y^{\complement}) \vee (X \wedge Z) \vee (Y \wedge Y^{\complement}) \vee (Y \wedge Z)$$

Hence, CNF can be reduced into DNF. However, assume there are $m$ clauses, and each clause contains $K_i$ terms $(i = 1, 2, \ldots, m)$. Then, the time to transform CNF to DNF is:

$$K_1 \times K_2 \times K_3 \times \ldots \times K_m$$

If all $K_i$ are equal, then the time complexity is $(K_i)^m$, which is exponential. Therefore, DNF is NP-hard, rather than NP-complete. Therefore, even if DNF is a P question, it cannot prove $P = NP$, as DNF is not NP-Complete.

# Problem 10

We first see the sum of subset problem, we define:

$$\sum_{i \subseteq t} W_i = W(\text{ t is the subset of the whole } w_i \text{ set })$$

$$\sum_{i=1}^{n} W_i = S(\text{S is the sum of all elements})$$

We can add two elements $W_i$ to the existing set, and the new set is represented as:

$$W_{\text{new}} = \{W_1, W_2, \ldots, W_n, W_x, W_y\}$$

Where:

$$W_x = 2S, \quad W_y = S + 2W$$

Now, the sum of all elements in $W_{new}$ is S+2S+S+2W=4S+2W. Therefore, the goal is to find a partition such that each part sums to:

$$\frac{4S + 2W}{2} = 2S + W$$

Then we turn to the part containing $W_x$, whose value is 2S. By excluding $W_x$, the remaining elements in this part sum up to $W$, and we can find the elements whose sum is W.

If there exists a valid partition of $W_{\text{new}}$ into two subsets such that each subset sums to $2S+W$, consider the subset that contains $W_x$ (value $2S$). The remaining elements in this subset must sum to:

$$(2S + W) - W_x = (2S + W) - 2S = W.$$

Since these remaining elements are a subset of $W$, we have found a subset $t \subseteq W$ such that:

$$\sum_{i \in t} W_i = W,$$

which solves the Subset Sum problem.

Thus, the reduction is valid because a solution to the Subset Sum problem exists if and only if a solution to the Fair Division problem exists.

## Why It Suffices to Prove for $p = 2$

In this reduction, $p = 2$ corresponds to dividing the set into two equal subsets. This is the core requirement of the Fair Division problem, and the reduction is specifically designed to work for this case. Extending the analysis to other values of $p$ (e.g., dividing into more subsets) would require different problem formulations and is outside the scope of this proof. Therefore, proving the reduction for $p = 2$ is sufficient to demonstrate the equivalence between the Subset Sum problem and the Fair Division problem.

## Polynomial-Time Complexity of the Reduction

To perform the reduction, we only need to compute $W_x = 2S$ and $W_y = S+2W$. These calculations involve summing the elements of $W$, which requires $O(n)$ time. Constructing the new set $W_{\text{new}}$ and calculating the target partition value $2S + W$ also take $O(n)$ time. Thus, the overall reduction process has a time complexity of $O(n)$, which is polynomial.

## Verifying Solutions in Polynomial Time

To complete the proof that the Fair Division problem is in NP, we must show that a solution can be verified in polynomial time. Given a proposed partition of $W_{\text{new}}$ into two subsets, we can compute the sums of the subsets and check whether they are equal. This verification requires summing the elements of each subset, which takes $O(n)$ time. Therefore, verifying a solution to the Fair Division problem can be done in polynomial time.

## Conclusion

We have shown that:

1. The Subset Sum problem can be reduced to the Fair Division problem in polynomial time.
2. The reduction is valid because a solution to the Subset Sum problem exists if and only if a solution to the Fair Division problem exists.
3. The Fair Division problem is in NP because its solutions can be verified in polynomial time.

Thus, we conclude that the Fair Division problem is NP-complete.