OVERVIEW OF RIEMANN SURFACES

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By
Norio Kosaka
Supervisor: Dr. Ben Fairbairn
Department of Economics, Mathematics and Statistics
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Abstract

BIRKBECK, UNIVERSITY OF LONDON

ABSTRACT OF DISSERTATION submitted by **Norio Kosaka** and entitled **Overview of Riemann Surfaces**.

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A Riemann surface is a fundamental mathematical concept in complex analysis and algebraic geometry. It represents a smooth, one-dimensional complex manifold, allowing us to encode the geometric information of an underlying function.

In Chapter 1, we will see that Riemann surfaces provide a framework for exploring the behaviour of complex functions, such as branch points, singularities, and analytic continuation, making them indispensable in various areas of mathematics. This abstract surface serves as a bridge between complex analysis and topology, shedding light on the intricate connections between algebraic and geometric properties of complex functions. The classification of Riemann surfaces allows us to investigate those functions by grouping points based on a notion of equivalence of points in a surface that is much simpler than the original surface. This classification result, called the Uniformisation theorem (Theorem 1.7.53), states that Riemann surfaces are equivalent to one of $\mathbb{C}, \mathbb{C} \cup \{\infty\}$ or $\mathbb{H}$ (Upper half complex plane).

It turns out that the Riemann surfaces being equivalent to $\mathbb{H}$ requires depth investigation in the hyperbolic space. Thus, in Chapter 2, we start with studying the basics of hyperbolic space and finish with exploring this class of Riemann surfaces as well as taking a look at a classical yet novel way to express algebraic functions.

Declaration

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Chapter 1

Riemann surfaces and Maps between them

1.1 Meromorphic functions and Singularities

Analytic functions play a central role in this work.

Definition 1.1.1 (Analytic and holomorphic functions)**.** Let $D \subset \mathbb{C}$ be a domain. A function $f : D \to \mathbb{C}$ is called holomorphic or analytic if either of the following two equivalent definitions is satisfied; (i) f is $\mathbb{C}$-differentiable at every $z_0 \in D$, or (ii) for any $z_0 \in D$, there is $r > 0$ such that f has a power-series expansion;

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for any $z_0 \in D(z_0, r) \subset D$.

The notion of zeroes plays an important role in this work so we, first, establish it here. Note that an *extended complex plane* is denoted as $\Sigma = \mathbb{C} \cup \{\infty\}$.

Theorem 1.1.2 (Identically zero)**.** *Let f be an analytic function on a region R of σ. If f has zeros at an infinite sequence of points z_n in R with a limit $z^* = \lim_{n\to} z_n$ in R, then f is identically zero on R.*

Proof. Theorem 1.3.1 of [JS87]. □

In this section, we will define the singularity and the pole of analytic functions. Singularities arise when analytic functions are defined on punctured discs.

Definition 1.1.3 (Isolated singularity)**.** An analytic function $f : D_*(z_0, r) \to \mathbb{C}$ is said to have an isolated singularity at z_0.

Just as holomorphic functions have Taylor series, analytic functions have Laurent series at their singularities.

Proposition 1.1.4 (Laurent series)**.** *If an analytic function f has an isolated singularity at z_0, then;*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

on $D_*(z_0, r)$ for some $r > 0$.

Proof. Proposition 1.5 of [Wil20]. □

Definition 1.1.5 (Multiplicities of the solution)**.** Suppose that f is analytic at $a \in \mathbb{C}$, with $f(a) = c \in \mathbb{C}$; If f is not constant then $f^{(k)}(a) \neq 0$ for some $k \geq 1$, and we call the least such k the multiplicity of the solution of $f(z) = c$ at $z = a$. Thus;

$$f(z) = c + \sum_{j=k}^{\infty} \frac{f^{(j)}(a)}{j!}(z - a)^j$$

The multiplicity of solutions deduces the following definition of an analytic function.

Definition 1.1.6 (Poles)**.** A function f is said to have a *pole* at z = j, when j in the above analytic form is a negative number, i.e., a fraction of polynomials. Also, the terms $\sum_{j=-k}^{-1} a_j(z - a)^j$ of the above are called the *principal part* of f. Finally, a is a *simple point* for f if it has multiplicity $k = 1$, and a *multiple point* if $k > 1$.

Poles help us to define the important class of functions, intuitively speaking, a function whose only singularities in a region R are poles is called meromorphic in R.

Definition 1.1.7 (Meromorphic function)**.** Let D be a domain in $\mathbb{C}$. If there is a discrete subset A of D, and f is a holomorphic function on $D \backslash A$ with poles at the points of A, then f is said to be a *meromorphic function* on D.

Below, we equip the definition of poles with its multiplicity;

Definition 1.1.8 (Order of pole)**.** If f is meromorphic at $a \in \mathbb{C}$, with a pole of order k at a, we say that $f(a) = \infty$ with multiplicity k; then $f(z) = \sum_{j=-k}^{\infty} a_j(z - a)^j$ near $z = a$, with constants a_j such that $a_{-k} \neq 0$

Example 1.1.9. 1. $f(z) = (z^2 + 1)^{-1}$ is analytic at ∞ with a zero of order 2;

2. $f(z) = z^3$ is meromorphic at ∞, with a pole of order 3 there;
3. $\sin(z)$ has an isolated essential singularity at ∞, and is therefore not analytic at ∞.

Now, we can classify the singularities by the leading coefficient a_n as follows;

Definition 1.1.10 (Singularities)**.** 1. If $a_n = 0$ for $n < 0$ then z_0 is a *removable singularity* In this case, f can be extended to an analytic function $g(z)$ defined on a neighbourhood of z_0.

2. If there is $m > 0$ such that $a_m = 0$ for all $n < -m$ but $a_{-m} \neq 0$, then f is said to have a *pole of order* m at z_0. In this case, $f(z) = (z - z_0)^{-m} g(z)$ on a neighbourhood of z_0, for some analytic function g with $g(z_0) \neq 0$.
3. Otherwise (that is, if $a_n \neq 0$ for infinitely many $n < 0$), then f is said to have an *essential singularity* at z_0, e.g., $f(z) = e^{1/z_0} = 1 + \frac{1}{z_0} + \frac{1}{z_0^2} + \cdots$.

1.1.1 Results of functions near singularities

Now, we consider the characteristics of the behaviour of analytic functions evaluated in the vicinity of an essential singularity. First, let us define;

A set $Y \subset X$ is called *dense* in X if for every $x \in X$ and every $\epsilon > 0$, there exists $y \in Y$ such that $d(x, y) < \epsilon$. An example is the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

The following result shows that the distance is bounded amongst all points except essential singularities.

Theorem 1.1.11 (Casorati–Weierstrass)**.** *An analytic function f on a domain D has an essential singularity at z_0 if and only if $f(D_*(z_0, r))$ is dense in $\mathbb{C}$, for any $r > 0$ such that $D_*(z_0, r) \subset D$.*

Proof. Theorem 1.8 of [Wil20]. □

The sum of the multiplicities of the solutions of $f(z) = c$ is found to be finite as follows;

Corollary 1.1.12 (Finite multiplicities)**.** *A non-constant meromorphic function $f : \Sigma \to \Sigma$ takes any given value $c \in \Sigma$ only finitely many times, counting multiplicities.*

Proof. Corollary 1.3.2 of [JS87]. □

One result says that functions with the same poles/principle parts have a similar form;

Theorem 1.1.13. *Let f and g be meromorphic functions on Σ with poles at the same points in Σ and with the same principal parts at these points. Then $f(z) = g(z) + c$ for some constant*

$c \in \mathbb{C}$. (Thus meromorphic functions on Σ are determined, puto additive constants, by their principal parts).

Proof. Theorem 1.3.3 of [JS87]. □

1.2 Rational functions

A *rational function* is a function of the form $f(z) = p(z)/q(z)$ where $p(z)$ and $q(z)$ are polynomials with complex coefficients and $q(z)$ is not identically zero (See Theorem 1.1.2). And the *degree (or order)* $\deg(f)$ is the maximum of the degrees of p and q. Thus f is said to be constant if and only if $\deg(f) = 0$.

As in the case of Sec1.3 of [JS87], the rational functions form a field which we denote by $\mathbb{C}(z)$ that contains a subfield isomorphic to $\mathbb{C}$, so $\mathbb{C}(z)$ can be considered as an extension field of $\mathbb{C}$, namely $\{a + bz \mid a, b \in \mathbb{C}\}$.

Definition 1.2.1 (Möbius transformations)**.** It is a complex rational function of the form; $f(z) = \frac{az+b}{cz+d}$ for $z, a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Remark 1. Möbius transformations are in $\mathrm{PSL}(2, \mathbb{C})$ and they have at least one fixed-point on Σ (by the fundamental theorem of algebra).

Example 1.2.2. Note that Möbius transformations can take various forms and their compositions can express a variety of functions;

- *Translation*: $f(z) = b + z$ (a = 1, c = 0, d = 1),
- *Rotation*: $f(z) = az$ (b = 0, c = 0, d = 1),
- *Inversion*: $f(z) = 1/z$ (a = 0, b = 1, c = 1, d = 0),

An automorphism is an isomorphism from a mathematical object to itself. Theorem 2.1.1 of [JS87] shows that Möbius transformations are contained in a group of automorphisms ($\mathrm{Aut}(\Sigma)$). Also, Theorem 2.1.3 of [JS87] shows that $\mathrm{Aut}(\Sigma) \simeq \mathrm{PGL}(2, \mathbb{C}) = \mathrm{PSL}(2, \mathbb{C})$. Thus, Möbius transformations are in $\mathrm{PSL}(2, \mathbb{C})$ (This comes particularly useful when we discuss the Fuchsian group in Sec.2).

By the fundamental theorem of Algebra, we can express every polynomial as a product of linear factors; $f(z) = c(z - \alpha_i)^{m_1} \dots c(z - \alpha_r)^{m_r} (z - \beta_1)^{-m_1} \dots (z - \beta_s)^{-m_s}$. The next result shows that the algebraic definition of a rational function is equivalent to an analytic condition:

Theorem 1.2.3 (Relation of Rational function and Meromorphic function)**.** *A function $f : \Sigma \to \Sigma$ is rational if and only if it is meromorphic on Σ.*

Proof. Theorem 1.4.1. of [JS87]. □

1.3 Simply periodic functions

A periodic function is a function that repeats its values after every particular interval;

Definition 1.3.1 (Finite / Infinite periodic)**.** Let f be a function defined on the complex plane $\mathbb{C}$, then a complex number ω is called a *period* of f if $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$, and f is called *periodic* if it has a period $\omega \neq 0$. If there is no such ω exists, then f is said to have an infinite period.

Example 1.3.2.

1. $\sin t$ and $\cos t$ have period 2π, e^z has period $2\pi i$, and $\sin(2\pi z)/\omega$ has period ω for $\omega \in \mathbb{C}\backslash\{0\}$.
2. In terms of transformations, the period (or order) of a transformation T is the least positive integer m such that $T^m = I$ where I is the identity element.

As we find out later, certain transformations form a cyclic group, thus we can consider the order of a transformation as the order of an element (e.g., permutations) in the cyclic group as well.

We now aim to classify the periodic functions by considering the algebraic structure of the set of periods. A subset δ of a topological space is called *discrete* if every $x \in \delta$ has a neighbourhood U such that $U \cap \delta = \{x\}$. The integers $\mathbb{Z}$ form a discrete subset of $\mathbb{R}$ or, more generally, any subset of $\mathbb{R}^{\ltimes}$ is discrete.

Theorem 1.3.3 (Discrete subset is indeed a subgroup)**.** *Let Ω_f be the set of periods of a function f defined on $\mathbb{C}$; then Ω_f is a subgroup of the additive group $\mathbb{C}$.*

Proof. Thm 3.1.1 of [JS87]. □

Remark 2. If it is a *constant* meromorphic function the set of periods is trivial.

To recap, the set of periods of a non-constant meromorphic function is a discrete (additive) subgroup of $\mathbb{C}$. Now, we show the classification of types of such subgroups of $\mathbb{C}$;

Theorem 1.3.4 (Classification of periodic functions: Simply and doubly periodic functions)**.** *If a function f has its set Ω of periods, then;*

1. *Trivial (or non-periodic):* $\Omega = \{0\}$,
2. *Simply periodic* ($f(z+\omega) = f(z)$)*:* $\Omega = \{n\omega_1 \mid n \in \mathbb{Z}\}$ *for some fixed* $\omega_1 \in \mathbb{C}\backslash\{0\}$. *Note that in this case* Ω *can be proven to be isomorphic to* $\mathbb{Z}$,
3. *Doubly periodic* ($f(z+\omega_1) = f(z+\omega_2) = f(z)$)*:* $\Omega = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$ *for some fixed* $\omega_1, \omega_2 \in \mathbb{C}\backslash\{0\}$ *where both are mutually linearly independent over* $\mathbb{R}$ *(i.e.,* $\omega_1/\omega_2 \notin \mathbb{R}$*; otherwise the two periods point in the same direction), Note that in this case* Ω *can be proven to be isomorphic to* $\mathbb{Z} \times \mathbb{Z}$,

Proof. Theorem 3.1.3 of [JS87]. □

1.4 Doubly periodic functions

So far, we studied meromorphic functions on Σ (the extended complex plane), but we now turn our attention to another compact surface, the *torus*, and its meromorphic functions. First, we discuss the periodicity in general, and then, having obtained the torus as a quotient space of $\mathbb{C}$, we consider the important properties of elliptic functions (regarded as functions from the torus to the Riemann sphere Σ). It is not simple to construct non-constant elliptic functions, so we will study the construction of the Weierstrass functions $\wp$-function to deduce further properties of elliptic functions in general.

1.4.1 Lattices and fundamental regions

A group Ω of type (3) in Theorem 1.3.4 is called a *lattice* and is denoted by $\Omega(\omega_1, \omega_2)$ where $\{\omega_1, \omega_2\}$ is a *basis* for Ω, that is, a pair of generators. We can find other bases for Ω such as $\{\omega_1, \omega_1 + \omega_2\}$.

In general, if $\omega_1', \omega_2' \in \Omega(\omega_1, \omega_2)$ then for $a, b, c, d \in \mathbb{Z}$;

$$\begin{aligned} \omega_2' &= a\omega_2 + b\omega_1 \\ \omega_1' &= c\omega_2 + d\omega_1 \end{aligned} \tag{1.1}$$

Theorem 1.4.1 (Basis of the set of periods)**.** *Eq.1.1 define a basis* $\{\omega_1', \omega_2'\}$ *for* $\Omega(\omega_1, \omega_2)$ *if and only if* $ad - bc = \pm 1$.

Proof. Theorem 3.4.2 of [JS87]. □

Remark 3. Any lattice has infinitely many bases.

We can view lattices from two different points;

(1) *algebraic*: given a lattice Ω, we define $z_1, z_2 \in \mathbb{C}$ to be *congruent* mod Ω, written $z_1 \sim z_2$, if $z_1 - z_2 \in \Omega$. Notice that the congruence modulo Ω is an equivalence relation on $\mathbb{C}$, and the equivalence classes are the cosets $z + \Omega$ of Ω in the additive group $\mathbb{C}$.

(2) *geometric*: alternatively, we can regard Ω as acting on $\mathbb{C}$ as a group of translations (e.g., Möbius transformations), each $\omega \in \Omega$ induces $t_\omega : z \mapsto z + \omega$ of $\mathbb{C}$. Since the composition $t_{\omega_1+\omega_2} = t_{\omega_1} \circ t_{\omega_2}$, we have a group isomorphism $\Omega \simeq \{t_\omega \mid \omega \in \Omega\}$. Thus, two points $z_1, z_2 \in \mathbb{C}$ are congruent mod Ω if and only if they lie in the same orbit under this action of Ω.

Formally lattices can be characterised by a principle component;

Definition 1.4.2 (Fundamental regions)**.** A closed, connected subset P of $\mathbb{C}$ is defined to be a *fundamental region* for Ω if;

1. for each $z \in \mathbb{C}$, P contains at least one point in the same Ω-orbit as z (i.e., every point $z \in \mathbb{C}$ is congruent to some point in P,
2. no two points in the interior of P are in the same Ω-orbit (i.e., no pair of points in the interior of P are congruent).

This type of covering is called *Tessellation* (a.k.a tiling). The above conditions (1) and (2) ensure that if P is any fundamental region for a lattice Ω, then P and its images under the action of Ω (that is, it translates $P + \omega, \omega \in \Omega$ cover the plane $\mathbb{C}$ completely, overlapping only at their boundaries; this type of covering is known as a *tessellation* of $\mathbb{C}$.

Theorem 1.4.3 (All fundamental regions share the same area)**.** *Let P_1 and P_2 be fundamental regions for a lattice Ω. Then $\mu(P_1) = \mu(P_2)$ where μ computes the area.*

Proof. Theorem 3.4.6 of [JS87]. □

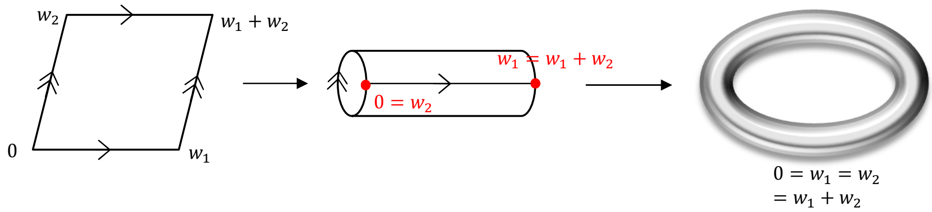
1.4.2 Application of Lattices: Bravais lattices in Physics

In our study, we are mainly interested in the surface of Riemann surface though, lattices found their applications outside the mathematics community. For instance, a *Bravais lattice* is an infinite array of points that can be generated by translating a set of basic points into three dimensions. In condensed matter physics, a Bravais lattice refers to the 14 different ways that atoms can be arranged in a crystal. The smallest group of atoms that can be repeated to create the entire crystal is called a *unit cell* (i.e., a fundamental region).

1.4.3 Torus

Topologically speaking, it is intuitive that a torus can be constructed by tiling a tiny parallelogram a certain amount of times in vertical and horizontal directions and smoothing out the surface, i.e., the tessellation of the fundamental region over the surface of a torus. Now we concretise our insight with a tool from group theory.

If a function f is doubly periodic concerning a lattice $\Omega = \Omega(\omega_1, \omega_2)$, then its behaviour on a fundamental region P for Ω is important to see as we can observe the same pattern on other areas of $\mathbb{C}$ as the tessellation; suppose a parallelogram with vertices $\{0, \omega_1, \omega_2, \omega_1 + \omega_2\}$, then P is tessellated by the translation of $P + \omega (\omega \in \Omega)$. So, by considering f on P, we can identify f on the resulting space of Tessellation T (a *torus*).



Now, we will characterise f as a doubly periodic function on T. By the definition of a fundamental region, we see that for every Ω-orbit on $\mathbb{C}$ there is one point of T. So, T can be considered as the set of Ω-orbits, that is, as the quotient set $\mathbb{C}/\Omega$ of cosets of Ω in $\mathbb{C}$. Since there is a continuous function from the closed bounded set P onto T (by identifying boundary points), we can see that T is indeed compact.

Definition 1.4.4 (Complex tori)**.** Let $\omega_1, \omega_2 \in \mathbb{C}$ be complex numbers that are linearly independent over $\mathbb{R}$ ($\omega_i \in \mathbb{C}$ but $\omega_2/\omega_1 \notin \mathbb{R}$), that is, they are a set of basis. Let $\Omega = \langle \omega_1, \omega_2 \rangle$, the additive subgroup generated by them, T is the quotient group $\mathbb{C}/\Omega$, and $\pi : \mathbb{C} \to T (\simeq \mathbb{C}/\Omega)$ is the quotient map.

Borrowing the terminology of Sec.1.6, $\mathbb{C}$ is a covering space of T, and π is a covering map. As the order of the additive subgroup Ω is infinite, there are infinitely many sheets, and no branch points.

1.4.4 Properties of Elliptic functions

Definition 1.4.5 (Elliptic function)**.** A meromorphic function $f : \mathbb{C} \to \Sigma$ is *elliptic* with respect to a lattice $\Omega \subseteq \mathbb{C}$ if f is doubly periodic with respsect to Ω, that is, if $f(z + \omega) = f(z) \forall z \in \mathbb{C}, \omega \in \Omega$, so that each $\omega \in \Omega$ is a period of f.

Remark 4. Note that a rational function for a sphere is the close analogue of an elliptic function for a Torus.

As an example construction of elliptic functions, we will take a look at the famous Weierstrass $\wp$-function.

Weierstrass $\wp$-function: We aim to construct a non-constant function that is doubly periodic concerning periods $\Omega = \langle \omega_1, \omega_2 \rangle$ defined on a complex torus $\mathbb{C}/\Omega$. With the Riemann-Hurwitz formula (describing the relationship of the Euler characteristics of surfaces; 1.7.49), it can be seen that such a function has a degree greater than 2 (Corollary 13.11 of [Wil20]). This suggests that as the simplest case, we should look into an approximation of $1/z^2$ in a small disc about 0.

For instance, consider the following function of a series;

$$S(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

To analyse the behaviour of this function, we check its convergence. The series $\sum_{n=1}^{\infty} z^2/n^2$ *converges normally* on all compact subsets of $\mathbb{C}$ by comparison with the convergent series $\sum n^{-2}$, so the above product *converges normally* on all compact subsets (See Theorem 3.8.4 of [JS87]), and hence $S(z)$ is *analytic* on $\mathbb{C}$ (Theorem 3.8.6 of [JS87] states that an infinite product of analytic function is also analytic).

Now, we will derive the famous expression of *Weierstrass $\wp$-function*. First, we transform $S(z)$ further;

$$S(z) = z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2}) = S(z) = z \prod_{n=1}^{\infty} (1 - \frac{z}{n})(1 + \frac{z}{n}) \tag{1.2}$$

And by the smoothness of the logarithmic function, we can define another expression;

$$\begin{aligned} Z(z) = \frac{d}{dz}\log(S(z)) = \frac{S'(z)}{S(z)} &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{z}{z-n} + \frac{1}{z+n}\right) \end{aligned} \tag{1.3}$$

This series converges uniformly on all compact subsets of $\mathbb{C}$. We can differentiate term by term to obtain a more concise form of meromorphic function;

$$\begin{aligned} P(z) &= -Z'(z) \\ &= \frac{1}{z^2} + \sum_{n=1}^{\infty} \left(\frac{z}{(z-n)^2} + \frac{1}{(z+n)^2}\right) = \sum_{n=1}^{\infty} \frac{1}{(z-n)^2} \end{aligned} \tag{1.4}$$

Remark 5. This last step is valid since the series is absolutely convergent (shown at the end of Sec 3.7 of [JS87]).

It turns out that we obtain $P(z)$ a simply periodic meromorphic function, with $\mathbb{Z}$ as its group of periods. In fact, in the exercises (3J and 3K of [JS87]) we shall see the exact expressions for S, Z and P; $P(z) = \pi^2 \text{cosec}^2 \pi z, Z(z) = \pi \cot \pi z$, and $S(z) = \pi \sin \pi z$.

Remark 6. We do not discuss the functions S and Z further in this work but together with P, they form a famous three functions of Weierstrass, namely the Weierstrass sigma function and the Weierstrass zeta function.

This derivation leads to a rather general notion of an elliptic function;

Theorem 1.4.6. *For each integer $N \geq 3$, the function $F_N(z) = \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^N}$ is elliptic of order N with respect to Ω.*

Proof. Theorem 3.9.3 of [JS87]. □

We were able to see a form nearly Weierstrass $\wp$-function yet there is an issue to reach there. The issue is that the two-dimension analogue of Rieman Zeta-function $\sum_{\omega \in \Omega} |\omega|^{-s}$ converges if and only if $s > 2$ (by Theorem 3.9.2 of [JS87]). Thus, we cannot prove the convergence of $\sum (z-\omega)^{-2}$. To work around this, we make the terms of this series smaller by replacing $(z-\omega)^{-2}$ with $(z-\omega)^{-2} - \omega^{-2}$ for each $\omega \neq 0$ to get the Weierstrass function as follows;

Definition 1.4.7 (Weierstrass $\wp$-function)**.** Let $\Omega = \langle \omega_1, \omega_2 \rangle$ be a lattice. The associated *Weierstrass $\wp$-function* is defined by;

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

representing an elliptic function of order 2.

Remark 7. The Weierstrass $\wp$-function is elliptic with the periods Ω by Theorem 3.9.8 of [JS87].

In fact, in the detailed discussion of Chapter 3 of [JS87], we can summarise the following properties of $\wp$-function;

1. $\wp$-function is meromorphic with periods Ω;
2. $\wp$-function has poles only at Ω (See Eq.1.5);
3. $\lim_{z \to 0}(\wp(z) - 1/z^2) = 0$.

These properties uniquely characterise $\wp$-function. Suppose $f(z)$ is any other function satisfying (1) and (2), then $f(z) - \wp(z)$ is also an elliptic function on $\mathbb{C}$ with poles only at Ω (lattice points). But (3) implies that $f(z) - \wp(z) \to 0$ as $z \to 0$, so the poles are removable singularities, and $f(z) - \wp(z)$ is constant by Corollary 13.10 of [Wil20]. Notice that this constant turns out to be 0 by (3), thus, $f = \wp$.

1.4.5 Differential equations of $\wp$-function

Our aim for this section is to investigate the branching behaviour of $\wp$-function. To analyse the roots of the algebraic relation, we will investigate the derivative of $\wp$-function. Again, term-by-term differentiation yields

$$\wp'(z) = -\frac{2}{z^3} - \sum_{\omega \in \Omega} \frac{2}{(z-\omega)^3} \quad = -2 \sum_{\omega \in \Omega} (z-\omega)^{-3} \tag{1.5}$$

As (2) of List of characteristics (See 1.4.4), we can see that $\wp'$ has poles precisely at the lattice points in Ω and these poles are of order 3, so is the periodicity (i.e., $\wp'$ is an odd function indeed). Therefore, for any $\omega \in \Omega$, $\wp'(\omega/2) = -\wp'(-\omega/2) = -\wp'(\omega/2)$ by oddness and periodicity, hence, $\wp'(\omega/2) = 0$ (corresponds to the result of Theorem 3.9.9 of [JS87]).

Although we derived $\wp'$ from $\wp$ by a direct differentiation, it turns out that there is an algebraic relation between the two functions;

Theorem 1.4.8 (Algebraic relation of $\wp$ and $\wp'$)**.** *There are constants $g_2, g_3 \in \mathbb{C}$, depending only on Ω, such that*

$$(\wp')^2 \equiv 4\wp^3 - g_2\wp - g_3$$

Remark 8. It is customary to write $g_2 = 60 \sum \omega^{-4}$, $g_3 = 140 \sum \omega^{-6}$.

Proof. Theorem 3.10.4 of [JS87] or Proposition 14.7 of [Wil20]. □

The constants g_2 and g_3 are related to the branch points that we saw above ($e_1 = \omega_1/2$, $e_2 = \omega_2/2$, and $e_3 = (\omega_1 + \omega_2)/2$ of $\wp$). Recall that e_i are the images under $\wp$ which are the zeroes of $\wp' = 0$. Therefore, the cubic equation has three distinct zeroes: $4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$. Similarly, the relation of Theorem 1.4.8 can be rewritten as $(\wp')^2 \equiv 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$.

In the previous section, we saw the formulation of the torus as quotients $\mathbb{C}/\Omega$. And, in the future section, we will see that by glueing the function elements defined for the analytic continuation we can construct the torus. The next result shows that every torus constructed as a quotient also arises as a compactification of a graph (algebraic relation).

Corollary 1.4.9. *Let $\mathbb{C}/\Omega$ be a complex torus. There are constants $g2, g3$ such that $\mathbb{C}/\Omega$ is biholomorphic to a one-point compactification of the graph $X := \{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2x - g_3\}$.*

Proof. Corollary 15.1 of [Wil20]. □

We conclude this section by classifying the elliptic functions. If f and g are elliptic, then so are $f + g, f - g$ and fg by linearity, and if g is not identically zero (1.1.2) then $1/g$ is elliptic. Thus, the set of all elliptic functions naturally forms a *field*, $E(\Omega)$. This field contains the subfield $E_1(\Omega)$ consisting of the *even* elliptic functions (e.g., $\wp$-function). Since $E_1(\Omega)$ contains $\wp(z) = \wp(z, \Omega)$ it contains all rational functions of $\wp$; these rational functions form a field $\mathbb{C}(\wp) = \{a + b\wp \mid a, b \in \mathbb{C}\}$, the smallest field containing $\wp$ and the constant functions $\mathbb{C}$. Similarly, $E(\Omega)$ contains $\wp$ and $\wp'$, and hence contains the field $\mathbb{C}(\wp, \wp') = \{a+b(\wp+\wp') \mid a, b \in \mathbb{C}\}$ of rational functions of $\wp$ and $\wp'$, the smallest field containing $\wp, \wp'$, and $\mathbb{C}$.

The next result classifies all meromorphic functions on a complex torus, and indeed they can all be written in terms of $\wp$;

Theorem 1.4.10 (Classification of elliptic functions)**.** *1. If f is an even elliptic function, then $f = R_1(\wp)$ for some rational function R_1; thus $E_1(\wp) = \mathbb{C}(\wp)$,*

2. If f is any elliptic function, then $f = R_1(\wp) + \wp' R_2(\wp)$ where R_1 and R_2 are rational functions; thus $E(\Omega) = \mathbb{C}(\wp, \wp')$.

Proof. Theorem 3.11.1 of [JS87]. □

1.5 Continuation of functions

1.5.1 Meromorphic and analytic continuation

So far, we consider the surface suitable for a function. But, in this subsection, we take a function (possibly many-valued like $log(z)$) and find the most natural surface to regard as its domain.

We will ask ourselves that If f is meromorphic or analytic on some region $D \subset \Sigma$, can we extend f to a function which is meromorphic or analytic on some larger region $E \supset D$?

If two analytic functions on a domain agree reasonably often, then they are considered to be equal. Here, *reasonably often* means, precisely, on a non-discrete subset.

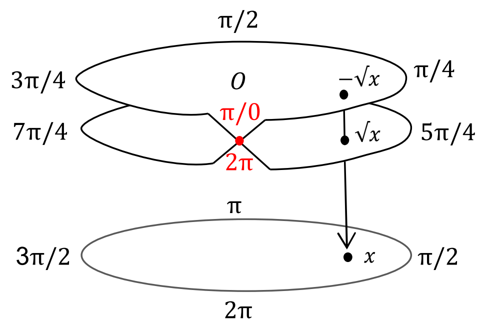
Corollary 1.5.1 (Identity principle)**.** *Let f, g be analytic functions defined on a domain D in $\mathbb{C}$. Unless the set $\{z \in D \mid f(z) = g(z)\}$ is discrete, $f \equiv g$ on D.*

Proof. Corollary 1.3 of [Wil20]. □

This principle allows us to define that, given analytic f on an open disc $D_1 = D(z_1, r_1)$, and another disc D_2 intersecting D_1, then we can analytically extend the domain of f across D_2.

Definition 1.5.2 (Analytic Continuation, Function element, and direct analytic continuation)**.** *A function element* on $D \subset \mathbb{C}$ is a pair (f, U), where $U \subset D$ as well as f is analytic on U. If (g, V) is another function element on D, then $(f, U) \sim (g, V)$ means that $U \cap V \neq \phi$ and $f|_{U\cap V} = g|_{U\cap V}$. In this case, (g, V) is said to be *a direct analytic continuation* of (f, U). If there is a finite sequence of direct analytic continuations $(f, U) = (f_1, U_1) \sim \cdots \sim (f_{n-1}, U_{n-1}) \sim (f_n, U_n) = (g, V)$ then (g, V) is said to be an analytic continuation of (f, U), and we write $(f, U) \approx (g, V)$.

Example 1.5.3. Consider the curve $y^2 = x$ with solutions $y = \pm\sqrt{x}$ above x in the unit disc of the x-plane. The values above x are pinched together in a region of 0. Note that the line of self-intersection is only a consequence of representing the relation $y^2 = x$ in a few dimensions.



Also, we can group the function elements to form an equivalence class.

Definition 1.5.4 (Complete analytic function)**.** A ≈-equivalence class F of function elements on a domain D is called a complete analytic function on D.

Note that the notation C_* denotes the punctured complex plane $\mathbb{C}\backslash\{0\}$.

Example 1.5.5 (Complex logarithm)**.** The complex logarithm ($\mathrm{Log}(z) = \log|z| + i\arg(z)$) arises from attempting to invert the exponential function $\exp : \mathbb{C} \to \mathbb{C}_*$. This isn't globally well defined as exp is not injective (e.g., $e^0 = 1 = e^{2\pi i}$). Thus, to define "log" as a function, we need to pick a function element that we are working on, namely *branch cut*. Below, we will see that in total, although "log" isn't strictly speaking a function, this collection of function elements defines "log" as a complete analytic function.

Given any interval $(a, b) \subset \mathbb{R}$ with $|a - b| < 2\pi$, define $U_{(a,b)} = \{re^{i\theta} \mid r > 0, a < \theta < b\}$, for instance, $U_{(0,2\pi)} = \mathbb{C}\backslash\mathbb{R}_{\geq 0}$ where $\mathbb{R}_{\geq 0}$ denotes the positive x-axis on the real line. For $z \in U_{(a,b)}$, which we can think of as $z = re^{i\theta}$ (Euler's formula) with $\theta \in (a, b)$, define; $f_{(a,b)}(z) = \log r + i\theta$ where log is the real logarithm.

Now, let us check if the complex logarithm is analytic. Let $u = \log(r)$ and $v = \theta$. Notice that the real part is independent of θ and the imaginary part is independent of r, the *Cauchy-Riemann equations* in polar coordinates ($e^{i\theta} = r(\cos\theta + i\sin\theta)$) reduce to $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ which is true in this case. So, together with the corresponding domain, we can construct the function elements $F_{(a,b)} = \left(f_{a,b}, U_{(a,b)}\right)$, and for all $n \in \mathbb{Z}$, we can cover $\mathbb{R}$ by the intervals in the form of $I(n) = ((n-1)\pi/2, (n+1)\pi/2)$.

Next, we will examine the cases when these intervals overlap to determine when $F_{I(m)} \sim F_{I(n)}$ (i.e., the direct continuation defined above) by considering the congruence modulo. For simplicity, let us take the case where $m - n \mod 4$.

1. If $m \equiv n \mod 4$ then $U_{I(m)} = U_{I(n)}$, but $I(m)$ and $I(n)$ are disjoint unless $m = n$. Therefore, $F_{I(m)} \sim F_{I(n)}$ if and only if $m = n$,
2. If $m \equiv n+1 \mod 4$ then $U_{I(m)} \cup U_{I(n)}$ is a quadrant of $\mathbb{C}$, but $I(m)$ and $I(n)$ are disjoint unless $m = n + 1$. Therefore, $F_{I(m)} \sim F_{I(n)}$ if and only if $m = n + 1$,
3. If $m \equiv n+2 \mod 4$ then $U_{I(m)}$ and $U_{I(n)}$ are disjoint, so $F_{I(m)} \not\sim F_{I(n)}$,
4. If $m \equiv n+3 \mod 4$ then as in the second case, $F_{I(m)} \sim F_{I(n)}$ if and only if $m = n - 1$.

In conclusion, $F_{I(m)} \sim F_{I(n)}$ if and only if $|m - n| \leq 1$. Applying these direct analytic continuations iteratively, all of the function elements $F_{I(n)}$ are in the same $\equiv$-equivalence class, so they all define the same complete analytic function. This is the complex logarithm.

1.5.2 Natural boundary

When we studied torus, we observed the possibility of infinitely many sheets (function element). Thus, when we try to analytically continue, we may build very large complete analytic functions, taking infinitely many values at any point. On the other hand, sometimes we simply cannot analytically continue very far. For instance, for $z = 0$ or ∞, $\log(z)$ has no *single-valued* meromorphic branch near either of these points (similarly points $\{0, 1, \infty\}$ for $f(z) = z^n$ for some $n \in \mathbb{N}$).

Let us introduce some new notations to analyse these observations. Consider a power series (i.e., analytic);

$$f(z) = \sum_{n \geq 0} a_n z^n$$

with radius of convergence 1. In particular, the series converges absolutely and uniformly on any closed disc $D \subseteq \mathbb{D}$.

Definition 1.5.6 (Regular and Singular points)**.** A point $z_0 \in \partial D$ (the boundary of the unit disc) is called *regular for f* if there is an open neighbourhood U of z_0 and an analytic function g on U such that $g \equiv f$ on $U \cap \partial D$, namely, there is a direct analytic/meromorphic continuation. Otherwise, z_0 is called *singular for f*.

Remark 9. The set of regular points in ∂D is open by definition; hence, the set of singular points is closed.

The point of this definition is that we may analytically continue to the regular points, but not to the singular points and this allows us to concretise the notion of such boundary;

Definition 1.5.7 (Natural boundary)**.** If all points $c \in \partial D$ are singular, then ∂D is called the *natural boundary* for f.

Example 1.5.8. Consider the power series;

$$f(z) = \frac{1}{1 - z} = \sum_{n \geq 0} z^n$$

Clearly, every point of $\partial D \backslash \{1\}$ is regular. Note that the power series does not converge at $z_0 = -1$, so the convergence is not required to distinguish between regular and singular.

Every branch point is inherently located adjacent to points with multiple values. Consequently, at a branch point, a function lacks analyticity, implying that all branch points are singularities. However, it's important to note that the reverse isn't always accurate: not all singularities are required to be branch points.

We will close by examining an important case. If a power series has a radius of convergence 1 such that every point $c \in C$ (a unit circle in $\mathbb{C}$) is regular: $\sum_{n=0}^{\infty} z^n$. Then at a point $c = 1$, analytic continuation is impossible. Indeed, Theorem 4.3.3 of [JS87] and Proposition 2.5 of [Wil20] show that this is a common situation such that there is at least one point on a domain where the analytic continuation is impossible. So the above case is the best possible case where there is just such a point.

1.5.3 Glueing function elements of Analytic continuation

In this section, we take a step towards defining the covering space based on our study so far. In Example 1.5.5, we saw that the complex logarithm consists of a collection of function elements. These collections can be glued together for the complex logarithm by taking the quotient of the disjoint union of the domains of the function elements and this resultant surface encodes the geometry of the function. Here, we briefly outline the construction but the detailed discussion can be found later.

In the notation of Example 1.5.5, let

$$R = \Big(\bigsqcup_{n \in \mathbb{Z}} U_{I(n)} \Big) / \sim$$

where $\sim$ identifies $z_1 \in U_{I(m)}$ and $z_2 \in U_{I(n)}$ if and only if $z_1 = z_2$ in $\mathbb{C}$ and, furthermore $f_{I(m)}(z_1) = f_{I(n)}(z_2)$, namely $F_{I(n)}$ is a direct analytic continuation of $F_{I(m)}$ in the previous context. Notice that by definition, the union of the open subsets (a domain of function element) yields the topological space and this quotient gives R the quotient topology. So the points in this space are a *coset* of points distributed across different domains with the same image value on f, or *fibre* (preimages).

R is equipped with two functions;

1. Let $F_A = (f_A, U_A)$ and $F_B = (f_B, U_B)$ (such that $A = I(n), B = J(n)$ for some $n \in \mathbb{N}$), then f_A and f_B agree when points of U_A and U_B are in the same coset of R. So, we have a well-defined map $f : R \to \mathbb{C}$ such that $f([z]) = f_A(z)$ for an arbitrary representative of U_A. Note that f is the "log".

2. Natural inclusion map $\pi : U_i \hookrightarrow \mathbb{C}$ such that simply $[z] \mapsto z$. Similar to (1), we obtain a well-defined map $\pi : R \to \mathbb{C}$.

Since each f_i is defined as an inverse to the exponential map, it follows that $\exp \circ f \equiv \pi$.

Thus, we constructed a quotient topological space R that fitted into a commutative diagram of maps;

$$\begin{tikzcd} R \arrow[r, "\log"] \arrow[dr, "\pi"'] & \mathbb{C} \arrow[d, "\exp"] \\ & \mathbb{C}_* \end{tikzcd}$$

In the next section, we will revise the terminologies from the topology to formally discuss our construction above to define the covering spaces.

1.6 Covering space theory

A *path* γ is a continuous function $\gamma : I \to \Sigma$, where I is the closed unit interval $[0, 1]$. If $a = \gamma(0)$ and $b = \gamma(1)$ then we say that γ is a *path from a to b*. γ is a *closed* path if $a = b$, and *simple* if $\gamma(s) = \gamma(s')$ implies either $s = s'$ or else $s = 0$ and $s' = 1$.

A path-component of X is an equivalence class of X under the equivalence relation which makes x equivalent to y if there is a path from x to y. The space X is said to be path-connected if there is exactly one path-component, i.e. if there is a path joining any two points in X.

Definition 1.6.1 (Covering and Regular covering maps)**.** Let X and $\tilde{X}$ be path-connected, Hausdorff, topological spaces. A *covering map* $\pi : X \to \tilde{X}$ is a local homeomorphism: that is, each $\tilde{x} \in \tilde{X}$ has an open neighbourhood $\tilde{U}$ such that $\pi|_{\tilde{U}}$ is a homeomorphism onto its image.

Clearly, the exponential map in Sec.1.5.3 was an example of a covering map.

We aim to resolve the non-uniqueness issue observed in Example 1.5.5 by studying how paths are lifted to a covering space via a map. First, we introduce a new notion.

Definition 1.6.2 (Analytic/Meromorphic continuation along a path)**.** Let (f, U) be a function element in the domain D, and consider an analytic continuation $(f, U) \equiv (g, V)$, exhibited by a sequence of direct continuations as in the definition.

$$(f, U) = (f_1, U_1) \ \ldots \ (f_{n-1}, U_{n-1}) \ (f_n, U_n) = (g, V)$$

Let $\gamma : I \to D$ be a continuous path. If there is a dissection

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

such that $\gamma([t_{i-1}, t_i]) \subseteq U_i$ for each $1 \leq i \leq n$, then (g, V) is an *analytic continuation of* (f, U) *along* γ, and we write $(f, U) \equiv_\gamma (g, V)$.

Definition 1.6.3 (Lift of path along π)**.** Let $\pi : \tilde{X} \to X$ be a covering map and $\gamma : I \to X$ be a path. A *lift of* γ *along* π is a path $\tilde{\gamma} : I \to \tilde{X}$ such that $\pi \circ \tilde{\gamma} = \gamma$.

For instance, $\exp : \mathbb{C} \to \mathbb{C}_*$ in Example 1.5.5, does not lift the paths uniquely. Let us consider the anticlockwise loop around the unit circle $\gamma(t) = e^{2\pi i t}$. Then both of lifted paths $\tilde{\gamma_1}(t) = 2\pi i t$ and $\tilde{\gamma_2}(t) = 2\pi i(t + 1)$ are valid.

The following results clarify this issue by paying attention to the base points of paths.

Proposition 1.6.4 (Uniqueness of Lifts)**.** *Suppose $\tilde{\gamma}_1, \tilde{\gamma}_2$ are both lifts of γ along a covering map π. If $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$ then $\tilde{\gamma}_1 \equiv \tilde{\gamma}_2$.*

Proof. Proposition 7.3 of [Wil20]. □

Finally, we show that the meromorphic continuation along a path is unique.

Theorem 1.6.5. *Let $(D, f) \sim (D_1, f_1) \sim, \dots, \sim (D_m, f_m)$ and $(D, f) \sim (E_1, g_1) \sim, \dots, \sim (E_n, g_n)$ be meromorphic continuations of (D, f) along a path γ from a to b, and let $0 = t_0 < t_1 < \cdots < t_n = 1$ be the corresponding subdivisions of the interval $I = [0, 1]$. Then $(D_i, f_i) \sim (E_j, g_j)$ whenever $[s_{i-1}, s_i] \cap [t_{j-1}, t_j] \neq \emptyset$, and $f_m(b) = g_n(b)$.*

Proof. Theorem 4.4.1 of [JS87]. □

1.6.1 Fundamental group and monodromy theorem

We continue developing the idea of continuations of functions along a path with a notion of homotopy and define a monodromy. To start with, we consider the case where we can deform curves into one another.

Definition 1.6.6 (Homotopy)**.** Let X be a topological space and $\alpha, \beta : [0, 1] \to X$ paths with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. This pair of paths is *homotopic* (written $\alpha \simeq \beta$) if there exists a family of paths $(\alpha_s)_{s \in [0,1]}$ such that:

1. $\alpha_0 \equiv \alpha$ and $\alpha_1 \equiv \beta$;
2. $\alpha_s(0) = \alpha(0)$ and $\alpha_s(1) = \alpha(1) \forall s$;
3. the map $(t, s) \to \alpha_s(t)$ is a continuous map $[0, 1]^2 \to X$.

And $\simeq$ is an equivalence relation, and the equivalence classes are called *homotopy classes*.

Example 1.6.7 (Linear homotopy)**.** Let $D \subseteq \mathbb{C}$ be a convex domain. The formula $\alpha_s(t) = (1-s)\alpha(t) + s\beta(t)$ defines a homotopy between any two paths α, β in D with equal endpoints, so D is simply connected (See below).

Remark 10. Theorem 4.5.1 of [JS87] shows that homotopy paths yield the equivalence of meromorphic continuations.

In particular, homotopy enables us to make rigorous sense of the notion of a space with no holes.

Definition 1.6.8 (Simply connected)**.** Let $\gamma_{(a)}$ denote the constant path $\gamma_{(a)}(s) == a \forall s \in I$; then a closed path γ from a to $a \in X$ is said to be *null-homotopic* if it is homotopic in X to $\gamma_{(a)}$, and X is said to be *simply connected* if it is path-connected and all closed paths in X are null-homotopic.

Proposition 1.6.9 (Property of simply connected)**.** *A topological space X is simply connected if and only if for each pair of points $a, b \in X$ there is a single homotopy class of paths from a to b in X.*

Proof. Theorem 4.5.2 of [JS87]. □

So, the continuation of a function element onto a simply connected region E is independent of the path of continuation, so we have a single-valued meromorphic function on E. The *monodromy theorem* states this more precisely;

Theorem 1.6.10 (Monodromy theorem)**.** *Let E be a simply connected region in Σ, and let (D, f) be a function element with $D \subseteq E$. If (D, f) can be continued meromorphically along all paths in E starting at some point $a \in D$, then there is a direct meromorphic continuation $(E, g) \sim (D, f)$.*

Proof. Theorem 4.5.3 of [JS87] or Theorem 9.4 of [Wil20]. □

1.7 Abstract Riemann surface

A Riemann surface is a surface that can be made up of many small pieces, each of which is topologically equivalent to the complex plane. This means that the pieces can be smoothly stitched together without any tears or overlaps.

Definition 1.7.1 (Surface is a 2-manifold)**.** A *surface S* is a Hausdorff topological space such that every point $s \in S$ has an open neighbourhood U homeomorphic to an open subset of $\mathbb{C}$; thus S has the same local topological properties as the plane. Or, more generally, an *n-manifold* is a Hausdorff space in which every point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$; thus a surface is a *2-manifold*.

The notion of systems of coordinates is made precise in the next definition;

Definition 1.7.2 (Charts)**.** A *chart* on R is a pair (ϕ, U), where $U \subset R$ is open and $\phi : U \to D(\subset \mathbb{C})$ is a homeomorphism to an open subset of $\mathbb{C}$

Definition 1.7.3 (Atlas of charts)**.** A set of charts $\mathcal{A}$ is called an atlas on R if the following hold:

1. $\bigcup_{(\phi,U)\in\mathcal{A}} U = R$;
2. If $(\phi_1, U_1), (\phi_2, U_2) \in \mathcal{A}$ and $U_1 \cap U_2 \neq \emptyset$, then the transition function $(\phi_1 \circ \phi_2^{-1})$ is analytic on $\phi_2(U_1 \cap U_2)$.

Remark 11. An atlas on S is called *analytic* if all its transition functions are analytic. Intuitively, an analytic atlas is a collection of charts that cover a (Riemann) surface S, such that the transition functions between any two charts are analytic.

Definition 1.7.4 (Compatible atlases)**.** $\mathcal{A} = \{(U_i, \phi_i)\}$ and $\mathcal{B} = \{(V_i, \psi_i)\}$ are be compatible if, whenever $(U_i, \phi_i) \in \mathcal{A}$ and $(V_i, \psi_i) \in \mathcal{B}$ satisfy $U_i \cap V_j \neq \emptyset$, then

$$\phi_i \circ \psi_j^{-1}(U_i \cap V_j) \to \phi_i(U_i \cap V_j)$$

is analytic, namely the atlas $\mathcal{A} \cup \mathcal{B}$ is analytic.

Remark 12. The compatibility of atlases forms an equivalence relation (Exercise 4F of [JS87]).

The equivalence class of atlases forms the complex structure on S as follows;

Definition 1.7.5 (Conformal Structure)**.** A *conformal structure* on R is an atlas $\mathcal{A}$ on R which is maximal in the following sense: if (ψ, V) is a chart on R such that, for any $(\phi, U) \in \mathcal{A}$, the transition function $\phi \circ \psi^{-1}$ is analytic, then $(\psi, V) \in \mathcal{A}$.

Now, we formally define abstract Riemann surfaces;

Definition 1.7.6 (Abstract Riemann surface)**.** A Riemann surface is a pair $(R, \mathcal{A})$, where $\mathcal{A}$ is a conformal structure on R.

The previously discussed examples (log or torus) were indeed abstract Riemann surfaces that focused on specific functions but here the notion of the abstract Riemann surface does not restrict itself to a specific function. To distinguish it from the previous construction, let us see some examples;

Example 1.7.7. Here are some commonly studied examples of Riemann Surfaces;

- Any open set $U \subset \mathbb{C}$ is a Riemann surface with just one chart $\phi : U \to \mathbb{C}$ given by inclusion.
- *Riemann Sphere*: A sphere $S^2 = \Sigma := \mathbb{C} \cup \{\infty\}$ with stereographic projection is a Riemann surface with an atlas of two charts: the identity map ($z \mapsto z$ on $\mathbb{C}$) and the reciprocal map ($z \mapsto 1/z$ on $\Sigma \setminus \{0\}$).
- *Torus*: Here, we construct the torus as follows;

  Theorem 1.7.8 (Lattice is a Riemann surface)**.** *If Ω is a lattice in $\mathbb{C}$ then $\mathbb{C}/\Omega$ is a Riemann surface.*

  Proof. Theorem 4.11.1 of [JS87]. □

  The following is the sketch of the construction of the Torus described in the above theorem by Jones.

  Recall that $p : \mathbb{C} \to \mathbb{C}/\Omega$ is the projection map $z \mapsto [z] = z + \Omega$, and $U \subseteq \mathbb{C}/\Omega$ is defined to be open if and only if $p^{-1}(U)$ is open in $\mathbb{C}$; thus p is open and continuous. Now, consider the restriction p_V of p to V, this is clearly continuous and open as p is. Thus, p_V turns out to be a homeomorphism of V onto its image $p(V)(= U_V$, say). So, by definition, we can define a homeomorphism $\phi_V = p^{-1} : U_V \to V$ and the set of charts form an atlas $\mathcal{A} = \{(U_V, \phi_V) \mid V \in \mathcal{V}\}$ where $\mathcal{V}$ is a collection of open subsets of the domain. Therefore, $\mathbb{C}/\Omega$ is a Riemann surface.

We will come back to this point to connect the previous discussion of $\log(z)$ and the abstract Riemann surfaces in the subsequent section.

1.7.1 Branch point and the monodromy group

In this sub-section, we shall see that the monodromy theorem (Theorem 1.6.10) can be used to define an invariant of regular covering maps, the monodromy group. The examples considered so far may have given the impression that at each branch point of a Riemann surface, all the sheets come together at a single point. However, the following example shows that this is not generally so.

Let $\pi : \tilde{X} \to X$ be a regular covering map, and let's pick a base point $x_0 \in X$. Now consider a loop $\gamma : [0, 1] \to X$ based at x_0; that is, a path with $\gamma(0) = \gamma(1) = x_0$. Path lifting enables us to associate to γ a *self-map* (permutation) σ_γ of the preimage (*fibre*) $\pi^{-1}(x_0)$.

Definition 1.7.9 (Self-map)**.** Let $\tilde{x} \in \pi^{-1}(x_0)$, and let $\tilde{\gamma}_{\tilde{x}}$ be the unique lift of γ starting at $\tilde{x}$. Since it is a lift, $\pi(\tilde{\gamma}_{\tilde{x}}(1)) = \gamma(1) = x_0$, so $\tilde{\gamma}_{\tilde{x}}(1) \in \pi^{-1}(x_0)$. Therefore, we can define $\sigma_\gamma : \pi^{-1}(x_0) \to \pi^{-1}(x_0)$ by $\sigma_\gamma(\tilde{x}) := \tilde{\gamma}_{\tilde{x}}(1)$ for any $\tilde{x} \in \pi^{-1}(x_0)$.

Let us look at some properties;

1. For the constant loop $i : t \mapsto x_0$, the corresponding map σ_i is the identity.
2. Let $\bar{\gamma}$ be the loop $\bar{\gamma}(t) := \gamma(1-t)$. Using the uniqueness of lifts (Proposition1.6.4), it is easy to see that $\sigma_{\bar{\gamma}} = \sigma_\gamma^{-1}$, the inverted loop from the end to the start. In particular, σ_γ is always a bijection, i.e., a *permutation*.
3. If a and b are both loops based at x_0, the product of paths is;

$$(\alpha \cdot \beta)(t) := \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

which is also a loop based on x_0. If $\tilde{x}_1 \in \pi^{-1}(x_0)$ and the lift $\tilde{\alpha}_{\tilde{x}_1}$ ends at $\tilde{x}_2$ then,

$$\widetilde{(\alpha \cdot \beta)}_{x_1} = \tilde{\alpha}_{\tilde{x}_1} \cdot \tilde{\beta}_{\tilde{x}_2}$$

by uniqueness of lifts. In particular, $\sigma_{\alpha \cdot \beta} = \sigma_b(\tilde{x}_2) = \sigma_b \circ \sigma_a(\tilde{x}_1)$.

Taking all those properties together, we see that the set of all permutations σ_γ of $\pi^{-1}(x_0)$ corresponding to loops based at x_0 form a subgroup of the symmetric group $\mathrm{Sym}(\pi^{-1}(x_0))$. This is the *monodromy group* of the regular covering map π (Def.1.6.1). We now show that this monodromy group is not dependent on the choice of the base point x_0.

Example 1.7.10 (Permutation of the sheets of the covering space)**.** Consider the power map $p_n(z) = z^n$ defining a regular covering map $\mathbb{C}_* \to \mathbb{C}_*$, for any $n \in \mathbb{N}$ (Natural number). Let $\gamma(t) = \exp 2\pi i t$, the standard clockwise loop around the unit circle in $\mathbb{C}_*$. Let also $\zeta_n = \exp 2\pi i/n$ be a primitive n-th root of unity and $\tilde{\gamma}_k$ be the unique lift of γ at $\zeta_n^k == \exp 2\pi i k/n$. By uniqueness of lifts, $\tilde{\gamma}_k(t) = \exp 2\pi i(k+t)/n$; In particular, $\tilde{\gamma}_k(1) = \zeta_n^{k+1}$. Therefore, γ naturally defines a permutation $\sigma_\gamma \in \mathrm{Sym}(n)$ via $\tilde{\gamma}_k(1) = \zeta_n^{\sigma_\gamma(k)}$. Any loop in $\mathbb{C}_*$ starting and ending at 1 is homotopic to γ^n for some $n \in \mathbb{Z}$. Therefore, the regular covering map p_n and the choice of base point 1 defines a subgroup of $\mathrm{Sym}(n)$, namely $\langle \sigma_\gamma \rangle$.

1.7.2 Functions on Riemann Surfaces

The fact that the transition functions in an atlas are analytic allows us to import notions from the complex plane to the Riemann surface. Thus, we introduce the concept of an analytic map between Riemann surfaces.

Definition 1.7.11 (Holomorphic map)**.** Let R and S be Riemann Surfaces. A continuous map $f : R \to S$ is *analytic* or *holomorphic* if, for all charts (ϕ, U) on R and (ψ, V) on S, the map $\psi \circ f \circ \phi^{-1}$ is analytic on $\phi(U \cup f^{-1}V)$.

Using covering maps, we can "pull back" a conformal structure from the range to the domain;

Lemma 1.7.12. *If $\pi : \tilde{R} \to R$ is a covering map and R is a Riemann surface then there is a unique conformal structure on $\tilde{R}$ such that π is analytic.*

Proof. Lemma 4.1 of [Wil20]. □

We can scale the notion of the holomorphic map between the Riemann surfaces to the multiple.

Theorem 1.7.13 (Transitivity)**.** *If $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are holomorphic functions between Riemann surfaces, then $g \circ f : S_1 \to S_3$ is holomorphic.*

Proof. Theorem 4.12.1 of Jones. □

1.7.3 The space (sheaf) of germs

In Sec.1.7, we discussed the construction of abstract Riemann surfaces of certain many-valued functions. The natural question to ask, here, is if there is any globally defined Riemann surface. So, we shall construct a single abstract Riemann surface $\mathcal{T}$ which is so large that it has, among its subspaces, the Riemann surface of every many-valued meromorphic function.

One of the main difficulties is that of describing the points in these surfaces, that is, of defining precisely the concept of "a single-valued branch of a many-valued function $w = f(z)$ at a point $z \in \Sigma$ ".

Definition 1.7.14 (Equivalence class of functions)**.** Let (f, U) and (g, V) be function elements on $D \subseteq \mathbb{C}$. For any $z \in D \cap E$, write $(f, U) \equiv_z (g, V)$ if f and g agree on a neighbourhood of z.

Definition 1.7.15 (Germ)**.** Let (f, U) be a function element and $z \in U$. The equivalence class of (f, U) under $\equiv_z$ is called the *germ* of f at z, and is denoted by $[f]_z$. Also, two germs $[f]_z$ and $[g]_w$ are equal if and only if $z = w$ and $f = g$ on a neighbourhood of $z = w$. This germ is *analytic* if f is analytic at z, that is, $f(z) \neq \infty$.

We can think of the germs as representing single-valued meromorphic functions defined in the neighbourhood of z.

Remark 13. When integrated with the concept of covering spaces, the space of germs offers a concise explanation of analytic continuation.

Theorem 1.7.16 (Analytic continuation revisited)**.** *Let* (f, U) *and* (g, V) *be function elements on a domain* $D \subseteq \mathbb{C}$*, and let* $\gamma : [0, 1] \to D$ *be a path starting in* U *and ending in* V*. Then* $(f, U) \equiv_\gamma (g, V)$ *if and only if the lift* $\tilde{\gamma}$ *to (a component of)* $\mathcal{G}$ *starting at* $[f]_{\gamma(0)}$ *exists, and ends at* $[g]_{\tilde{\gamma}(1)}$*.*

Proof. Theorem 9.1 of [Wil20] □

The insight of the theorem is that it provides a criterion for when two function elements are considered equivalent along a given path in a specific space (potentially a covering space or Riemann surface).

Definition 1.7.17 (Space/Sheaf of germs)**.** The space of germs over D is $\mathcal{G} := \{[f]_z \mid z \in D, (f, U) \text{ a function element with } z \in U\}$ as a set.

Now, we will show how to establish the conformal structure on $\mathcal{G}$ to define the global Riemann surface. Let us introduce the short-hand notation; $[f]_U := \{[f]_z \mid z \in U\}$ for any function element (correspondingly, [JS87] used U-neighbourhood $U(m)$ with $m = [U, f]_z$). The open sets of the topology are all unions of all sets of the form $[f]_U$, for all function elements (f, U) on D.

Lemma 1.7.18. *Unions of sets of the form* $[f]_U$ *define a topology on* $\mathcal{G}$*.*

Proof. Lemma 8.8 of [Wil20] □

Riemann surfaces also require the Hausdorff property in the components of G.

Lemma 1.7.19. *The space of germs* $\mathcal{G}$ *is Hausdorff.*

Proof. Lemma 8.9 of [Wil20] □

To define a conformal structure on each component of $\mathcal{G}$, we will employ the following map;

Definition 1.7.20 (Forgetful map)**.** Let $\mathcal{G}$ be the space of germs over a domain D. The *forgetful map* is defined as $\pi : \mathcal{G} \to D$ is defined as $\pi([f]_z) = z$ to extract the base point.

Now we will show that this is indeed a covering map on D with $\mathcal{G}$.

Lemma 1.7.21 (Forgetful map is a covering map)**.** *For each component $G \subseteq \mathcal{G}$, the restriction of the forgetful map $\pi : G \to D$ is a covering map.*

Proof. Lemma 8.11 of [Wil20] □

Let us employ the notion of [JS87] (U-neighbourhood) below to establish the construction of a global abstract Riemann surface. It is clear that the restriction $\pi_{U,m}$ of π to a U-neighbourhood $U(m)$ of m maps $U(m)$ homeomorphically onto the open set $U \subseteq \Sigma$. So, with this atlas of charts, G is an abstract Riemann surface of interest.

As well as the forgetful map, the space of germs carries another naturally defined map.

Definition 1.7.22 (Evaluation map)**.** Let $\mathcal{G}$ be the space of germs on a domain D. The *evaluation map* $\mathcal{E} : \mathcal{G} \to \mathbb{C}$ (not just D!) is defined as $\mathcal{E}([f]_z) = f(z)$.

1.7.4 The Riemann surface of algebraic functions

We say that w is an algebraic function of z if the relationship between w and z has the form $A(z, w) = 0$ for some polynomial $A(z, w)$. For instance, $A(z, w) = w^4 - 2w^2 + 1 - z$ corresponds to the many-valued function $w = \sqrt{(1 + \sqrt{z})}$ (by the recursive use of squared-root; See Sec 4.10 of [JS87] for more details). In this subsection, we shall prove that the Riemann surface of an algebraic function is always compact.

Let us start with defining the unbranched Riemann surface as follows. In the following, we will reuse the notations in Sec.1.7.3. If $A(z, w)$ is a single-valued function of z and w, then the *unbranched Riemann surface $\mathcal{G}_A$* of the equation $A(z, w) = 0$ is defined to be the largest open subset of $\mathcal{G}$ on which $A(\psi(m), \phi(m))$ for some $m \in \mathcal{G}$ where ψ is the forgetful map $[f]_z \mapsto z$ and ϕ is the evaluation map $[f]_z \mapsto f(z)$.

$$\begin{tikzcd} \mathcal{G} \supseteq [f]_U \arrow[r, "\phi"] \arrow[d, "\psi"] & \Sigma \\ \mathbb{C} \supseteq D \arrow[ur, "\phi\circ\psi^{-1}"'] & \end{tikzcd} \quad , \begin{tikzcd} {[f]_z} \arrow[r, mapsto] \arrow[d, mapsto] & \Sigma \\ \mathbb{C} \supseteq D \arrow[ur] & \end{tikzcd}$$

Figure 1.1: The relationship of components. Note that $[f]_U := \{[f]_z \mid z \in U\}$ (U-neighbourhood).

In Sec.1.7, we consider globally large Riemann surfaces, similarly, we aim to construct the Riemann surface of algebraic functions. Notice that we can indeed factorise $A(z, w)$ uniquely as a product of powers of finitely many distinct irreducible polynomials $A_i(z, w) (1 \leq i \leq r)$.

Since $A_i = 0$ implies $A = 0$, the unbranched Riemann surface $\mathcal{G}_A$ contains each $\mathcal{G}_{A_i}$. By induction based on the topological property, each A_i needs to be compact if A is compact. So, we will look further into an irreducible component A_i below by assuming A *is irreducible* to simplify the notation.

Collecting powers of w, we can write; $A(z, w) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z)$ with $a_i(z)$ is a polynomial in z, and $a_0(z) \neq 0$ such that the degree of $A(z, w)$ is $n \in \mathbb{N}$.

By the Fundamental theorem of algebra, for a fixed $z \in \Sigma$, $A(z, w) = 0$ is a polynomial with n distinct roots w; we call such values of z *regular points* for $A(z, w)$ and the others form the set C_A of *critical points*. Thus, C_A satisfies one or more of the following conditions;

1. $z = \infty$,
2. $a_n(z) = 0$,
3. $A(z, w) = 0$ has a repeated root w

It is clear that there are only finitely many points in Σ satisfying (1) or (2). Since A is assumed to be irreducible (has degree n), its partial derivative with respect to w $\partial A/\partial w$ has degree $n-1$, and in fact, degrees of A and $\partial A/\partial w$ are co-prime. Then, Theorem A.14 of [JS87] shows that they have a common root w for only finitely many $z \in \mathbb{C}$, so C_A is finite.

We shall show that the branch-points of $\mathcal{G}_A$ all lie above critical points, so they are finite in number. At a regular point $a \in \Sigma \backslash C_A$, $A(z, w) = 0$ has *distinct simple roots* $w = w_1, \ldots, w_n$ and the following result shows that for z in the neighbourhood of a, the roots of $A = 0$ remain simple and distinct, and vary analytically with respect to z.

Lemma 1.7.23. *If $a \in \Sigma \backslash C_A$ then there exist $D \in \mathcal{D}_A$ (the domain of A) and analytic function element (D, f_i) for $1 \leq i \leq n$ satisfying;*

1. *$f_i(a) = w_i$ for $i = 1, \ldots, n$,*
2. *for each $z \in D$ the solutions of $A = 0$ are $w = f_i(z)(i = 1, \ldots, n)$, all simple and distinct.*

Proof. Lemma 4.14.2 of Jones □

Thus, at each $a \in \Sigma \backslash C_A$ we have n distinct analytic *germs* representing the solutions of $A(z, w) = 0$ for z near a (these germs are the elements of $\mathcal{G}$ projected onto a by ψ, the forgetful map).

We can now show the following result on algebraic functions;

Theorem 1.7.24. *If $A(z, w)$ is an irreducible polynomial then $\mathcal{G}_A$ is connected.*

Proof. Theorem 4.14.5 of Jones. □

More importantly,

Theorem 1.7.25. *If $A(z, w)$ is a polynomial then $\mathcal{G}_A$ is compact.*

Proof. Theorem 4.14.8 of Jones □

Together with the preceding results, we can find the following rather powerful result connecting topological and algebraic objects;

Theorem 1.7.26. *Any compact abstract Riemann surface can be identified with the Riemann surface $\mathcal{G}_A$ of some algebraic function $A(z, w) = 0$.*

Proof. The proof is quite difficult and requires concepts beyond the scope of this work. So it is omitted. □

Remark 14.

- By "can be identified with" we meant "is conformally equivalent to" which is a concept we shall discuss in the subsequent section.
- The result shows compactness corresponds to polynomials, and connected components to irreducible factors.

1.7.5 Orientable and non-orientable surfaces

Surfaces can be classified into two classes, *orientable* and *non-orientable* ones.

Roughly speaking, a surface is *orientable* if it is possible to choose a consistent orientation (e.g., clockwise or anti-clockwise) at any point on the surface. So, if we take any closed path γ based at a point p in the surface, and cover γ by a finite number of discs (as in analytic continuation), then discs share the orientation and we can return to the original orientation at p by following the sequence of discs. If, on the other hand, we end up with a different orientation at p then the surface is *non-orientable* (Fig.1.2).

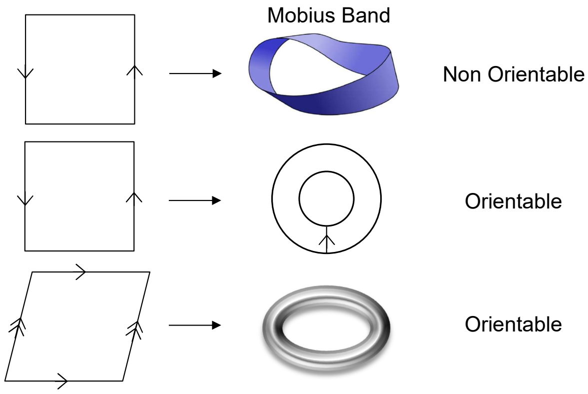


Figure 1.2: Orientable surfaces.

Our aim in this subsection is to show that all Riemann surfaces are orientable. We acknowledge that there are many rigorous definitions of *orientability* in the community, but here we choose the most convenient one for us to discuss the orientability of Riemann surfaces, namely the *smoothness* of surfaces.

Definition 1.7.27 (Smooth surfaces)**.** Let a surface in $\mathbb{R}^2$ by identifying $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$. An atlas of charts is *smooth* (or $\mathbb{C}^\infty$-differentiable) if all its coordinate transition functions f are smooth, that is, the partial derivatives $\frac{\partial^n f}{\partial x^k \partial y^{(n-k)}}$ all exist.

Remark 15. As with analytic atlases discussed before, two smooth atlases A and B are called *compatible* if the atlas $A \cup B$ is smooth. And compatibility of atlases forms an equivalence relation with an equivalence class of smooth atlases (called *smooth structure*). Finally, a *smooth surface* is a surface with a smooth structure, that is, a surface on which the atlas is smooth.

In the remark, we notice that since every analytic function is smooth, it is clear that every Riemann surface is smooth indeed. Now, we will see how to connect the smoothness and the orientability by formulating the differentiation more concretely.

Definition 1.7.28 (Jacobian)**.** If U and V are open subsets of $\mathbb{R}^2$ and $f : (x, y) \mapsto (u, v)$ is a smooth function $U \to V$, then the *Jacobian* of f is

$$J_f = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Remark 16. If f has a smooth inverse function $g = f^{-1} : V \to U$ (as the coordinate transition functions of a smooth surface), then the chain rule $J_g J_f = J_{g \circ f} = J_{identity} = 1 > 0$ implies that $J_f \neq 0$ at all points of U (Sec 4.15 of Jones).

Definition 1.7.29 (Orientation-preserving)**.** If $J_f > 0$ at all points of U then f is said to be *orientation-preserving*.

Example 1.7.30 (Reflection vs Rotations and Translations)**.** *Reflections* is not preserving the orientation as $f(x, y) = (x, -y)$ (i.e., $f(z) = \bar{z}$ in complex coordinates) does *NOT* preserve orientation, since $J_f \equiv -1$ on $\mathbb{R}^2$. On the other hand, *Rotations/Translations* of $\mathbb{R}^2$ have $J_f \equiv 1$ so they preserve the orientation.

Now formally, the orientability is defined as follows;

Definition 1.7.31 (Orientability)**.** A smooth atlas is said to be *orientable* if all its coordinate transition functions preserve orientation; a smooth surface is *orientable* if its smooth structure contains an *orientable* atlas, that is, its atlas of charts is compatible with an *orientable* atlas.

Theorem 1.7.32. *Every analytic atlas is orientable.*

Proof. Theorem 4.15.2 of [JS87]. □

Thus, we conclude this subsection with the orientability in the complex structure;

Corollary 1.7.33. *Every Riemann surface is orientable.*

Proof. Corollary 4.15.3 of [JS87]. □

1.7.6 The genus of compact Riemann surface

In the last section, we studied that if $A(z, w) = 0$ is an irreducible polynomial then its Riemann surface ($S = \mathcal{G}_A$) is compact, connected, and orientable (Theorem 1.7.26 and Corollary1.7.33). Such surfaces can be classified topologically by the number of holes.

Theorem 1.7.34. *Each compact, connected, orientable surface is homeomorphic to a surface S_g formed by attaching g handles to a sphere for some $z \in \mathbb{Z}$.*

Proof. Theorem 4.16.1 of [JS87]. □

Remark 17. We call such g the genus of the surface. For instance, the sphere S^2 is the surface of genus 0, and the torus is the surface of genus 1.

In this section, we aim to calculate g using polygonal subdivision from Algebraic topology.

Definition 1.7.35 (Polygonal subdivision)**.** A *polygonal subdivision* M of a surface S consists of a finite set of points of S, called *vertices*, and a finite set of simple (that is, non-self-intersecting) paths on S, called *edges*, such that

1. Every edge has two end-points, these points being vertices;
2. Edges can intersect only at their end-points;
3. The union of the edges (which we also denote by M) is connected;
4. The components of the complement $S\backslash M$ are homeomorphic to open discs. The components are called *faces*.

Remark 18. Since each edge is incident with at most two faces, there are only finitely many faces.

Example 1.7.36. When we project the vertices and edges of a tetrahedron onto a sphere S that encompasses the tetrahedron, we create a polygonal subdivision of S. This subdivision comprises four vertices, six edges, and four triangular faces, with each face having three sides.

In fact, it can be shown that every compact, connected, orientable Riemann surface has a polygonal subdivision (Sec. 4.16 of [JS87]).

Definition 1.7.37 (Euler characteristic)**.** Euler characteristic of a compact, connected surface S is

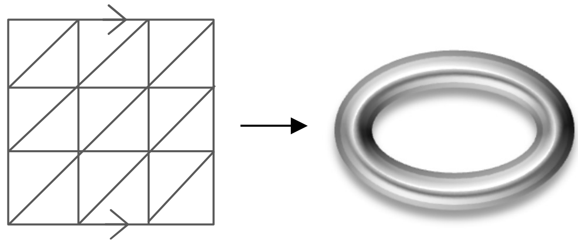
$$\chi(S) = \chi(M) = V - E + F$$

where M is a polygonal subdivision of S with V vertices, E edges, and F faces.

Remark 19. It is clear that homeomorphic surfaces will have the same Euler characteristic, provided we can show that $\chi(S)$ is well defined, that is, independent of the choice of M.

Example 1.7.38.

1. The *Riemann sphere* is homeomorphic to a tetrahedron. Therefore $\chi(\Sigma) = 4 - 6 + 4 = 2$ (also known as Euler's theorem).
2. Subdividing the square into nine squares, then dividing each of these into two triangles yields the grids with small triangles. Now, by identifying opposite sides of the square in the usual way, we get a triangulation of the torus $S^1 \times S^1$. Counting the number of faces, edges and vertices, it follows that $\chi(\mathbb{C}/\Omega) = 9 - 18 + 9 = 0$, for any complex torus $\mathbb{C}/\Omega$. See Fig.2.



We can now calculate the Euler characteristic $\chi(S_g)$ of a compact, connected, orientable surface S_g of genus g.

Theorem 1.7.39 (Topological classification of Riemann surfaces)**.** *The Euler characteristic of a compact, connected, orientable Riemann surface S_g of genus g is given by $\chi(S_g) = 2 - 2g$.*

Proof. Theorem 4.16.2 of [JS87]. $\square$

1.7.7 Branching and Multiplicity

One advantage of working with Riemann surfaces is that, in a suitable chart, any analytic function can be put into a very simple local form. The proof of this uses the inverse function theorem. First, let us recall the inverse function theorem as follows;

Theorem 1.7.40 (Inverse function theorem)**.** *Let f be an analytic function on a domain $D \subseteq \mathbb{C}$. If $f'(z_0) \neq 0$ for $z_0 \in D$, then there are open neighbourhoods U of z_0 and V of $f(z_0)$ such that f restricts to a biholomorphism $U \to V$.*

Now, we show the powerful result;

Proposition 1.7.41 (Exponential form of Riemann Surface)**.** *Let f be a non-constant analytic function on a Riemann surface S and let $p \in S$ be a zero of f. There is a chart (ϕ, U) about p with $\phi(p) = 0$ such that*

$$f \circ \phi^{-1}(z) = z^m$$

for some integer $m > 0$.

Proof. Proposition 4.6 of [Wil20]. $\square$

Compactifying Riemann surfaces with added points sometimes leads to meromorphic functions that no longer covering maps. For instance, although the power map $p^k(z) = z^k$ is a covering map $\mathbb{C}_* \to \mathbb{C}_*$, its natural extension to the Riemann sphere $\tilde{p}^k : \Sigma \to \Sigma$ fails to be a local homeomorphism at 0 and ∞ if $k \geq 2$.

Definition 1.7.42 (Multiplicity revisited)**.** Let $f : R \to S$ be an analytic map of Riemann surfaces and let $p \in R$. By Proposition.1.7.41, there are choices of charts (ϕ, U) about $p \in U$ and (ψ, V) about $f(p) \in V$, with $\phi(p) = 0$, such that

$$\psi \circ f \circ \phi^{-1}(z) = z^{m_f(p)}(\text{change of coordinates})$$

for some integer $m_f(p) \geq 0$. Note that $m_f(p)$ is equal to the number of preimages (the size of fibre) in a sufficiently small neighbourhood of $f(p)$, and so is independent of the choice of charts. This is the *multiplicity* of f at p.

Now, we introduce an additional concept to discuss the branch-point.

Definition 1.7.43. If $m_f(p) > 1$ then p is called *ramification point* (or *critical point*) and $f(p)$ is called a *branch-point* of f. in this case, the multiplicity $m_f(p)$ is also called the *ramification index* of p.

Example 1.7.44. Let $\hat{p}_k : \Sigma \to \Sigma$ be the power map $z \mapsto z^k$ for $k \geq 2$. As we have seen in the properties of the set of critical points in Sec.1.7.4, the only points with multiplicity greater than 1 are 0 and ∞, so these are the ramification points, each of which has ramification index equal to k. And, the branch points are their images, which are also 0 and ∞.

The behaviour of multiplicity under the composition of maps is;

Lemma 1.7.45. *If $f : R \to S$ and $g : S \to T$ are analytic functions of Riemann surfaces then;*

$$m_{g \circ f}(p) = m_g(f(p))m_f(p)$$

for any point $p \in R$.

Proof. Lemma 10.7 of [Wil20]. □

The subsequent theorem asserts that when dealing with non-constant mappings between compact Riemann surfaces, they exhibit a specific behaviour: they become n-to-1 mappings, where n is a well-defined integer, taking into account the concept of multiplicities.

Theorem 1.7.46 (Valency theorem)**.** *Suppose that $f : R \to S$ is a non-constant, analytic map between compact Riemann surfaces. The function $n : S \to \mathbb{N}$ defined by;*

$$n(q) := \sum_{p \in f^{-1}(q)} m_f(p)$$

is constant on S.

Proof. Theorem 11.1 of [Wil20]. ☐

Remark 20. This generally does not hold true on non-compact Riemann surfaces (e.g., $\mathbb{C}$).

So, we can see that the constant n is an important invariant property of f and it has a dedicated name;

Definition 1.7.47 (Valency)**.** The constant n that the Valency theorem associates to a non-constant analytic map f of compact Riemann surfaces is called the *degree* or *valency* of f, and denoted by $\deg(f)$.

An easy corollary of the valency theorem is;

Theorem 1.7.48 (Fundamental theorem of algebra)**.** *Any polynomial f of degree d has exactly d zeroes (roots), counted with multiplicity.*

Proof. Corollary 11.3 of [Wil20]. ☐

We are now prepared to present the Riemann-Hurwitz formula. This formula serves as a fundamental tool, allowing us to extract topological insights from the branching characteristics of an analytic mapping.

Theorem 1.7.49 (Riemann-Hurwitz)**.** *If S is the compact, connected, orientable Riemann surface $\mathcal{G}_A$ of an irreducible algebraic equation $A(z, w) = 0$ of degree n in w, and if the branch-points have orders $n_1, \ldots, n_r$, then the genus g of S is given by;*

$$g = 1 - n + \frac{1}{2} + \sum_{i=1}^{r} n_i.$$

Proof. Theorem 4.16.3 of Jones. ☐

Remark 21 (Applications of Riemann–Hurwitz: Higher-genus Riemann surfaces)**.** Up until this point in our study, we haven't come across Riemann surfaces with a high genus. In Example 12.6 of [Wil20], the author considers the *Fermat curve* F of degree d, namely the graph

$$F_d := \{(x, y) \in \mathbb{C}^2 \mid x^d + y^d = 1\}$$

Rational points on this graph correspond to integer solutions to the famous Fermat equation $x^d + y^d = z^d$. The author's construction illustrated compact Riemann surfaces of arbitrarily large genera.

1.7.8 Conformal equivalence and automorphisms of Riemann surface

In this section, we will progress our understanding of mappings between Riemann surfaces and introduce the concept of equivalence between them. Then, we will present the central outcome of this study, which contributes to the classification of Riemann surfaces.

Let's begin by establishing the necessary tools.

Definition 1.7.50 (Conformal Equivalence)**.** A *conformal equivalence* or *biholomorphism* is an analytic bijection of Riemann surfaces $f : R \to S$ with any analytic inverse $f^{-1} : S \to R$.

Example 1.7.51. 1. Complex conjugation $z \mapsto \bar{z}$ defines analytic bijections in both directions between the Riemann surfaces C and C. Thus, these two Riemann surfaces are conformally equivalent.

2. We shall show that the disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ are *conformally equivalent*. Consider a Möbius transformation $T : \mathbb{H} \to \mathbb{C}$ by

$$T(z) = \frac{z - i}{z + i}(a = 1, b = i, c = 1, d = i)$$

Since $T \in PSL(2, \mathbb{C})$ and $-i \in \mathbb{H}$, T defines a holomorphic homeomorphism of $\mathbb{H}$ onto $T(H)$. For all $z \in \mathbb{H}$ we have

$$\begin{aligned} |T(z)|^2 = T(z)T(\bar{z}) &= \frac{(z - i)(\bar{z} + i)}{(z + i)(\bar{z} - i)} \\ &= \frac{|z|^2 - 2\text{Im}(z) + 1}{|z|^2 + 2\text{Im}(z) + 1} < 1 \end{aligned}$$

So, T maps $\mathbb{H}$ into D. Similarly, with $\text{Im}(z) = 0$ T maps the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$ to the unit circle, which bounds D, so $T(\mathbb{H})$, which is a disc by the property of the Möbius transformation, must coincide with D. Hence, $\mathbb{H}$ and D are conformally equivalent as required.

3. As a *non-conformally equivalent* case, we see that although homeomorphic, $\mathbb{C}$ and D are not conformally equivalent. If $f : \mathbb{C} \to D$ is holomorphic, then being bounded f is constant by *Liouville's theorem*, so f cannot be a homeomorphism.

Now, we state the important result which classifies all simply connected open subsets of $\mathbb{C}$ up to conformal equivalence.

Theorem 1.7.52 (Riemann Mapping Theorem)**.** *If S is a simply connected open subset of $\mathbb{C}$, then either $S = \mathbb{C}$ or else $S \simeq D$.*

Proof. Corollary 16.15 of [Wil20]. □

The following rather general result is the famous *uniformisation theorem* which allows us to uniformise (that is, parameterise) compact Riemann surfaces by means of single-valued functions.

Theorem 1.7.53 (Uniformisation Theorem)**.** *Every simply connected Riemann surface is conformally equivalent to one of* $\Sigma = \mathbb{C} \cup \{\infty\}$ *(Riemann sphere),* $\mathbb{C}$ *(Complex plane), or* $\mathbb{D}$ *(Disc).*

Example 1.7.54. Suppose that $A(x, y) = 0$ is an irreducible algebraic function with Riemann surface S of genus greater than one. In Sec.1.7.4, we saw that the forgetful map ϕ and the evaluation map ψ are meromorphic functions defined on S such that $\phi(s) = x$ and $\psi(s) = y$. For instance, $x^2 + y^2 = 1$ (a circle in $\mathbb{C}$ and as a covering space we know that it consists of multiple branches of angle θ; $2\pi n \leq \theta \leq 2\pi(n + 1)$ for some $n \in \mathbb{Z}$) can be uniformised by $x = \sin t, y = \cos t$ or $x = (2t)/(1 + t^2), y = (1 - t^2)/(1 + t^2)$.

Remark 22. The three Riemann surfaces ($\Sigma, \mathbb{C}$, or $\mathbb{D}$) are **not** *conformally equivalent* to each other. Indeed, Σ is compact, so not even homeomorphic to the other two. $\mathbb{C}$ and $\mathbb{D}$ are homeomorphic, but $\mathbb{D}$ is bounded as a disc and by Liouville's theorem any analytic map $\mathbb{C} \to \mathbb{D}$ is constant. So, they are not conformally equivalent.

1.7.9 Revisit: Quotients of Riemann surfaces

The construction of complex tori presented the importance of a quotient group. So, in this section, we will concretise this idea with new notions of the discreteness of underlying group actions.

Definition 1.7.55 (Properly discontinuous actions)**.** Let a group G act by homomorphism on a space X. The action is said to be *properly discontinuous* if, for every compact $K \subseteq X$, the set

$$\{g \in G \mid g(K) \cap K \neq \emptyset\}$$

is finite. If, for each $x \in X$, the stabiliser $\mathrm{Stab}_G(x)$ is trivial, then the action is said to be *free*.

Remark 23. Every discontinuous group acts properly discontinuously.

- the finite group of homeomorphisms of $\mathbb{C}$ generated by $z \mapsto ze^{2\pi i/n}$ ($n = 2, 3, \dots$) acts properly discontinuously but not discontinuously.

- If Ω is a lattice in $\mathbb{C}$ then the action of Ω on $\mathbb{C}$ by *translation* is properly discontinuous and free.

Despite its name proper discontinuity is a weaker condition than discontinuity. However, the term *proper discontinuity* is now widely used in the literature.

Now, we show that the quotient space is Hausdorff and the quotient map is a covering map with this new definition;

Lemma 1.7.56. *Let G be a group acting freely and properly discontinuously by homeomorphisms on a Riemann surface R. The quotient space $G\backslash R$ is Hausdorff and the quotient map $\pi : R \to G\backslash R$ is a regular covering map.*

Proof. Lemma 15.5 of [Wil20]. □

As in the case of lattices on $\mathbb{C}$, free and properly discontinuous actions are a convenient way of constructing conformal structures on the quotients.

Proposition 1.7.57. *Let R be a Riemann surface, and let G be a group acting freely and properly discontinuously by conformal equivalences on R. Then the quotient $S = G\backslash R$ is a Riemann surface, and the quotient map $\pi : R \to S$ is analytic and a regular covering map.*

Proof. Proposition 15.6 of [Wil20]. □

As an example of the application of these ideas, we will show a simple case of a famous theorem of Hurwitz about the automorphisms of Riemann surfaces.

Theorem 1.7.58 (Hurwitz theorem)**.** *Let R be a compact Riemann surface of genus $g_R \geq 2$, and suppose that a group G acts freely and properly discontinuously on R by conformal equivalences. Then G is finite, and indeed $|G| \leq g_R - 1$.*

Proof. Theorem 15.7 of [Wil20]. □

1.7.10 Applications of Uniformisation theorem

With the uniformisation theorem in hand, we can make progress in classifying all Riemann surfaces. Let's start with the case of surfaces of genus 0.

Corollary 1.7.59 (Classification of Riemann surface with genus 0)**.** *The conformal structure on S^2 is conformally equivalent to Σ.*

Proof. Corollary 16.3 of [Wil20]. □

For Riemann surfaces of genus $g \geq 1$, the following results associate the properly discontinuous actions and the covering map;

Theorem 1.7.60. *Every Riemann surface R has a regular covering map $\pi : \tilde{R} \to R$ such that $\tilde{R}$ is simply connected. Furthermore, there is a group G acting freely and properly discontinuously by conformal equivalences on $\tilde{R}$, and the covering map π descends to a conformal equivalence $G\backslash\tilde{R} \simeq R$.*

Proof. Theorem 16.4 of [Wil20]. □

With the uniformisation theorem, we now obtain a general classification result of all Riemann surfaces based on quotients.

Corollary 1.7.61. *Every Riemann surface R is conformally equivalent to a quotient $R \simeq G\backslash\tilde{R}$ where $\tilde{R}$ is one of $\Sigma, \mathbb{C}$ or D, and G is a properly discontinuous group of conformal equivalences of $\tilde{R}$.*

Proof. Corollary 16.5 of [Wil20] or Theorem 4.19.5 of [JS87]. □

Remark 24.
- We say that R is *uniformised by* $\tilde{R}$.
- G can be defined as the group of automorphisms; $G = \{\phi \in \mathrm{Aut}(\tilde{R}) \mid \pi \circ \phi = \pi\}$ where $\mathrm{Aut}(\tilde{R})$ is the group of conformal equivalences $\tilde{R} \to \tilde{R}$.

This leads us to the idea of examining regular coverings as a means to classify Riemann surfaces. Specifically, we focus on properly discontinuous groups of conformal equivalences of $\tilde{R} = \Sigma, \mathbb{C}$, and $\mathbb{D}$. We will start with the simplest case is $\tilde{R} = \Sigma$.

Proposition 1.7.62. *If a Riemann surface R is uniformised by $\tilde{R} = \Sigma$ then R is conformally equivalent to Σ.*

Proof. Proposition 16.7 of [Wil20]. □

Remark 25. Every Möbius transformation $g \in \mathrm{PSL}(2, \mathbb{C})$ has at least one fixed-point on Σ (Remark 1). Therefore, no non-trivial subgroup of $\mathrm{PSL}(2, \mathbb{C})$ can act freely on Σ. So $G = \{1\}$ and $R \simeq \Sigma$.

We now identify properties of automorphism groups to uniformise a Riemann surface R by $\mathbb{C}$;

Proposition 1.7.63 (Classification of automorphism groups under the uniformisation with $\mathbb{C}$)**.** *If a Riemann surface R is uniformised by $\mathbb{C}$ then one of the following holds;*

1. $G = \{1\}$ *and* $R \simeq \mathbb{C}$,
2. $G \simeq \mathbb{Z}$ *and* $R \simeq \mathbb{C}_*$,
3. $G \simeq \mathbb{Z}^2$ *and* $R \simeq \mathbb{C}/\Omega$ *for some lattice* Ω.

Proof. Proposition 16.8 of [Wil20]. □

It is important to determine the automorphism groups of the three simply connected Riemann surfaces in the Uniformisation theorem (Theorem 1.7.53);

Theorem 1.7.64.

1. $Aut(\Sigma) = PSL(2, \mathbb{C})$,
2. $Aut(\mathbb{C}) = \{z \mapsto az + b \mid a, b \in \mathbb{C}, a \neq 0\}$,
3. $Aut(\mathbb{H}) = PSL(2, \mathbb{R})$ *where* $\mathbb{H}$ *is the upper half-plane* ($\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$).

Proof. Theorem 4.17.3 of [JS87]. □

Remark 26.

- $\mathbb{D}$ (Disc) and $\mathbb{H}$ are conformally equivalent via a Möbius transformation $T \in \mathrm{PSL}(2, \mathbb{C}) : \mathbb{H} \to \mathbb{C}$ such that $z \mapsto \frac{z-i}{z+i}$ (Example (1) of Sec4.17 of [JS87]).
- In each of the three cases, the automorphism group consists entirely of Möbius transformations.

So far, we have considered the cases where the Riemann surfaces have been uniformised by Σ or $\mathbb{C}$. Everything else is supposed to be uniformised by $\mathbb{D}$. Now, we aim to revisit Remark 22 to concretise the idea that the three different uniformising spaces are mutually exclusive. To this end, we need the following result;

Lemma 1.7.65. *Let $f : R \to S$ be an analytic map of Riemann surfaces. Suppose that R is simply connected, and let $\pi : \tilde{S} \to S$ be the uniformising map (covering map) of S. Then there is an analytic map $F : R \to \tilde{S}$ such that $f = \pi \circ F$.*

Proof. Lemma 16.9 of [Wil20]. □

The mutual exclusivity suggests;

Proposition 1.7.66. *A Riemann surface R is uniformised by at most one of $\Sigma, \mathbb{C}$ and $\mathbb{D}$.*

Proof. Proposition 16.10 of [Wil20]. □

In all except four simple cases, the universal covering space $\tilde{S}$ is $\mathbb{D}$, rather than Σ or $\mathbb{C}$. This motivates us to focus on the upper half-plane and the action on it of various subgroups of its automorphisms group $\mathrm{PSL}(2,\mathbb{R})$ in the next section.

1.7.11 Proofs of Uniformistaion theorem

As the proof goes beyond the scope of this work, let us briefly touch upon the literature that helps readers study it.

[Cha04] provide three methods of proof; (i) Construct a global analytic function by minimizing to find where it would land on, (ii) Triangulate the surface by making the set of triangles larger and larger, and (iii) Although it seems incomplete it is based on sheaf cohomology.

Although not complete, [Tao16] discusses the proof (centred around solving Laplace's equation by using Perron's method and Green's functions) in Sec.5 of his lecture note based on the work of [Mar05]. Similarly, [Pic11] shows the proof based on the last chapter of [Gam03]. Finally, paired with the full-brown lecture series[1], [Sai16] shows the historical development of the uniformisation problem as well as the proof.

[1] https://www.youtube.com/playlist?list=PLo4jXE-LdDTTdCSceqIRlrM6UdhX_bFDw

Chapter 2

Advanced concepts

2.1 Fuchsian groups

2.1.1 Introduction to Hyperbolic geometry

Recall that the arc length of a curve in $\mathbb{C}^2$ $\gamma : [a, b] \to \mathbb{R}^2$, where $\gamma(t) = (x(t), y(t))$ is defined as;

$$s = \int_b^a \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

by the application of Pythagorean theorem. Thus, the Riemannian metric (or Standard Euclidean metric) is

$$(ds)^2 = (dx)^2 + (dy)^2$$

Also, the distance-minimizing curves are called geodesics.

Definition 2.1.1. If (M, d) is a metric space with the distance metric d, an *isometry* is a function $f : M \to M$ such that $d(f(x), f(y)) = d(x, y) \forall x, y \in M$.

It immediately follows that Isometries send geodesics to geodesics.

Definition 2.1.2 (Hyperbolic plane)**.** The upper half-plane (Poincare upper half-plane) is defined as $\mathbb{H} = \{(x, y) \in \mathbb{C}^2 \mid y > 0\} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ wit the Riemann metric defined as

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

.

Example 2.1.3. 1. Consider a horizontal line segment from (a, y) to (b, y). Note that $dy = 0$ since the curve is horizontal. The length of this line segment is

$$\int_b^a ds = \frac{(dx)^2 + (dy)^2}{y^2} = \int_a^b \frac{dx}{y} = \frac{b-a}{y}$$

2. Consider a vertical line segment from (x, a) to (x, b). Note that $dx = 0$ since the curve is vertical. The length of this line segment is

$$\int_b^a ds = \frac{(dx)^2 + (dy)^2}{y^2} = \int_a^b \frac{dy}{y} = \log(b) - \log(a)$$

Also, recall that the set of *linear fractional transformations* of $\mathbb{H}$, also known as *Möbius transformations* are of the form

$$\{z \mapsto \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1\}$$

. We can also define the matrix form of the transformations as follows; let $T(z) = \frac{az+b}{cz+d}$, then

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is the matrix associated with T with the inverse of T;

$$T^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 2.1.4. For instance,

$$T = \frac{z+1}{z+2} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} z$$

, or

$$T = z + 1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} z$$

are clearly transformations in $\mathrm{PSL}(2, \mathbb{R})$.

Next, we will discuss the properties of hyperbolic triangles. A *hyperbolic triangle* in $\mathbb{H}$ is a topological triangle whose edges are hyperbolic geodesic segments. The Gauss-Bonnet formula shows that the Sum of the angles of a hyperbolic triangle is less than π.

Theorem 2.1.5 (Gauss-Bonnet)**.** *Let Δ be a hyperbolic triangle with angles a, bc. Then*

$$\mu(\Delta) = \pi - a - b - c$$

.

Proof. Theorem 5.5.2 of [JS87]. □

There are hyperbolic analogues of reflections and rotations. Lines are geodesics in Euclidean space, and just as we can define a reflection across a line, we can define a hyperbolic reflection across a geodesic.

Definition 2.1.6. Let L be an H-line (a geodesic). Then an *H-reflection* over L is an H-isometry of $\mathbb{H}$, other than the identity, which fixes every point of L.

Example 2.1.7. For instance, the reflection across the imaginary axis L_0 is $R_0 : z \to -\bar{z}$, and all other reflections are conjugate to this one. That is, given an arbitrary geodesic L in $\mathbb{H}$, then H-reflection $R_L = M \circ R_0 \circ M^{-1}$ where $M \in \mathrm{PSL}(2, \mathbb{R})$ is an isometry ($M(L_0) = L$).

Remark 27. Note that reflections are *anti-conformal*, they preserve angles but reverse orientation. Thus, a reflection is not holomorphic, in other words, the reflection R can be written in the form

$$R(z) = \frac{a\bar{z} + b}{c\bar{z} + d}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc = -1$.

Just as for Euclidean space, the composition of two reflections is a rotation. If R_1 and R_2 are reflections fixing geoedesics L_1 and L_2, then $R_1 \circ R_2$ is a rotation about the point of intersection of L_1 and L_2, and if the angle between L_1 and L_2 is θ, then $R_1 \circ R_2$ is a rotation by 2θ. For instance, the imaginary axis L_1 and the unit circle L_2 intersect at a 90° angle at i. As mentioned above, then $R_1 : z \mapsto -\bar{z}$ and $R_2 : z \mapsto 1/\bar{z}$, so

$$R_2 \circ R_1 : z \mapsto -\bar{z} \mapsto \frac{1}{\bar{(-z)}} = -1/z$$

, and the corresponding matrix is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix}$$

In fact, the reflection in the hyperbolic geometry can be characterised by the Euclidean inversion.

Theorem 2.1.8. *An H-reflection in L (geodesic) is the restriction of a Euclidean inversion in L to $\mathbb{H}$.*

Proof. Theorem 5.6.6 of [JS87]. □

2.1.2 Algebraic properties of PSL(2, R)

The notion of transitivity helps us to understand the behaviour of functions;

Theorem 2.1.9. *1. $PSL(2, \mathbb{R})$ is transitive on $\mathbb{H}$,*

2. $PSL(2, \mathbb{R})$ is doubly transitive on $\mathbb{R} \cup \{\infty\}$.

Proof. Theorem 5.2.1 of [JS87]. □

We now discuss the conjugacy classes in $\mathrm{PSL}(2, \mathbb{R})$. Recall let G be a group and $a, b \in G$, then these two elements are called *conjugate* if there exists an element $g \in G$ such that $gag^{-1} = b$, in which case b is called a *conjugate* of a. The equivalence class that contains the element $a \in G$ is $\mathrm{Cl}(a) = \{gag^{-1} : g \in G\}$ is called *conjugacy class*. Also recall that we say that $a \in \mathbb{R}$ is a *fixed-point* of some $T \in \mathrm{PSL}(2, \mathbb{R})$ if $T(a) = a$, then $U(a)$ is a fixed-point of the conjugate transformation $UTU^{-1} \in \mathrm{PSL}(2, \mathbb{R})$.

Thus, we consider the conjugacy class based on the fixed points. The fixed-points can be found by simply solving $z = \frac{az+b}{cz+d}$. As a, b, c, d are real, we see that this transformation has either two fixed-points in $R \cup \{\infty\}$, one fixed-point in $\mathbb{R} \cup \{\infty\}$, or a pair of complex conjugate fixed-points.

The geometric classification of Möbius transformations of Sec 2.10 of [JS87] states that for $\mathrm{PSL}(2, \mathbb{C})$ the Möbius transformations can be classified into either of *elliptic, parabolic, hyperbolic, or loxodromic* each of which exhibits different behaviours in the fixed-points in transformation. Here, we analyse the transformations in $\mathrm{PSL}(2, \mathbb{R})$ following this classification;

Definition 2.1.10. For a transformation $f \in \mathrm{PSL}(2, \mathbb{R})$,

1. If the trace $|tr(f)| < 2$, then f is elliptic.
2. If the trace $|tr(f)| = 2$, then f is parabolic.
3. If the trace $|tr(f)| > 2$, then f is hyperbolic.

Remark 28. By computing the discriminant of the resulting quadratic, one can show that a hyperbolic transformation will have 2 fixed points, a parabolic transformation will fix a single point, and an elliptic transformation has a fixed point $p \in \mathbb{H}$.

As the most relevant example for us, we will take a look at the elliptic elements (other examples can be found in Sec.5.2 of [JS87]).

Example 2.1.11 (Elliptic elements)**.** Let $T(z) = \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}, \theta \in (0, \pi)$, then $T \in \mathrm{PSL}(2, \mathbb{R})$ as $ad - bc = \cos^2\theta + \sin^2\theta = 1$. T is an elliptic element as the trace is $2 \mid \cos\theta \mid < 2$, if $\theta \in (0, \pi)$,. Also, T fixes one point only in $\mathbb{H}$ because by the quadratic equation to solve $z = T(z)$ we get $z = \frac{\pm\sqrt{-4\sin^2\theta}}{2\sin\theta} = \pm 1$ and hence only $i \in \mathbb{H}$ by definition. Similarly, $T(z) = \frac{\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}$ can be proven to be elliptic.

2.1.3 Fuchsian groups

A group structure helps us to associate two kinds of structure with a set. In Remark.**??**, we naively saw that like the Möbius transformations, automorphisms form a group under composition. Here, we will concretise this by the notion of a *topological group* that is both a group and topological space, the two structures being related by the requirement that both the group multiplication and the taking of inverses are continuous operations.

Definition 2.1.12 (Topological Group)**.** Let G be a group under the multiplicative operation $*$. A topological space (X, τ) on G said to be a *topological group* if:

1. $m : G \times G \to G$ such that $(x, y) \mapsto xy$
2. $i : G \to G$ such that $x \mapsto x^{-1}$

are continuous.

Here are a couple of examples;

Example 2.1.13.
1. The complex plane $\mathbb{C}$ with its additive group structure is a topological group since the group operations $m(z, w) = z + w$ and $i(z) = -z$ are continuous.
2. The circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, with multiplication of complex numbers as the group operation.

Lemma 2.1.14. *$PSL(2, \mathbb{R})$ is a topological group.*

Proof. First of all, it is clear that $\mathrm{PSL}(2, \mathbb{R})$ is a topological space and the fact $\mathrm{PSL}(2, \mathbb{R}) \simeq \mathrm{SL}(s, \mathbb{R})/\{\pm I\}$ where I is an identity matrix in $GL(2, \mathbb{R})$ tells us that by construction, it is also a group (the detailed treatment can be found Chapter.2 of [JS87]). Now, we will examine the two axioms of the topological group. (1) Let $m : G \times G \to G$ given by $(T, S) \mapsto T \circ S$ for $T, S \in \mathrm{PSL}(2, \mathbb{R})$. Since $\mathrm{PSL}(2, \mathbb{R})$ is a group under composition, the composition stays in $\mathrm{PSL}(2, \mathbb{R})$, thus it is continuous as Möbius transformations preserve the continuity of a set that is to be transformed. (2) Similarly, the inverse is continuous as

$$T^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

which also stays in $\mathrm{PSL}(2,\mathbb{R})$. □

Remark 29. $\mathrm{PSL}(2,\mathbb{C})$ is found to be a *Galois group* (See Sec.2.15 of [JS87]).

A topological group G is called a *discrete* group if there is no limit point in it. Since any group can be equipped with a discrete topology, an arbitrary group can be regarded as a discrete topological group. For instance, the lattice is found to be a discrete subgroup of $\mathbb{C}$ (Theorem 1.3.4). With this notion of a discrete subgroup, we define the Fuchsian group;

Definition 2.1.15. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Example 2.1.16. For instance, it is easily seen that $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$ as $\mathrm{SL}(2,\mathbb{R})$ can be viewed as $\mathbb{R}^4$ under the standard Euclidean metric, and $\mathrm{SL}(2,\mathbb{R})$ is then a subspace of $\mathbb{Z}^4$, which is a discrete subspace of $\mathbb{R}^4$. Thus, $\mathrm{PSL}(2,\mathbb{Z})$ (the modular group) is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and hence is a Fuchsian group. Similarly, the set of integer translations $T(z) = \{z+n \mid n \in \mathbb{N}\}$ is a Fuchsian group. On the other hand, $\mathrm{PSL}(2,\mathbb{Q})$ is NOT a Fuchsian group, thus, the set of all translations $T(z) = \{z + a \mid a \in \mathbb{R}\}$ as it is not discrete.

We study the similarity of lattices to Fuchsian groups. Lattices are discrete groups of Euclidean isometries (i.e., translation, rotation, reflection and glide reflection) and their quotients are compact Riemann surfaces homeomorphic to the torus. Similarly, Fuchsian groups are discrete groups of Hyperbolic isometries (e.g., Möbius transformations) and their quotient-spaces are also Riemann surfaces (See Theorem 5.9.1 of [JS87]). Yet, by the uniformisation theorem (Theorem 1.7.53) all orientable surfaces other than the four are quotients of Fuchsian groups acting on $\mathbb{H}$ without fixed-points. For this reason, Fuchsian groups vary more widely.

Following the classification of elements, we will refine our previous example. Firstly, there are cyclic groups; *Hyperbolic cyclic groups* are generated by a hyperbolic element, for instance, $z \mapsto \lambda z (\lambda > 1)$ generates a Fuchsian group consisting of hyperbolic elements with the identity. *Parabolic cyclic groups* are generated by a parabolic element, for instance, $z \mapsto z+1$ generates such a group. *Elliptic cyclic groups* are generated by an elliptic element, and they are Fuchsian if and only if they are finite. A Fuchsian group with a more complicated structure is the *modular group* $\mathrm{PSL}(2,\mathbb{Z})$ as seen before.

Lattices have a discontinuity in their actions on $\mathbb{C}$ in the sense that every point of $\mathbb{C}$ has a neighbourhood which is translated to the outside of the fundamental region by all elements except the additive identity. Whereas, Fuchsian groups do not have such a discontinuous behaviour because if elliptic elements are included then they fix points and these fixed-points cannot have such a neighbourhood.

Recall the definition of properly discontinuous actions (Def.1.7.55). Now, we show that Fuchsian groups act properly discontinuously on $\mathbb{H}$.

Theorem 2.1.17. 1. *Let Γ be a subgroup of $PSL(2,\mathbb{R})$. Then Γ is a Fuchsian group if and only if Γ acts properly discontinuously on $\mathbb{H}$.*

2. *Let Γ be a Fuchsian group and let $p \in \mathbb{H}$ be fixed by some element of Γ. Then there is a neighbourhood W of p such that no other point of W is filed by an element of Γ other than the identity.*

Proof. Theorem 5.6.3 of [JS87]. □

A word of warning that not all Fuchsian groups have properly discontinuous actions, meaning whether a discrete group acts discontinuously or not depends very much on the underlying space that the group acts on. For example, the modular group ($\mathrm{PSL}(2,\mathbb{Z})$) does not act properly discontinuously on $\mathbb{R} \cup \{\infty\}$ as the orbit of 0 is the set $\mathbb{Q} \cup \{\infty\}$ ($\mathbb{Q}$ is the set of rationals) which is dense in $\mathbb{R} \cup \{\infty\}$, thus not properly discontinuous. Similarly, the *Picard modular group* $\mathrm{PSL}(2,\mathbb{Z}[i])$ is the ring of Gaussian integers and does not have a properly discontinuous action on hyperbolic 3-space (See Sec 5.6 of Fuchsian groups of [JS87]).

2.1.4 Fundamental regions of Fuchsian groups

The fundamental region for a Fuchsian group can be defined in the same way as for the lattices (See Sec 3.4 of [JS87]).

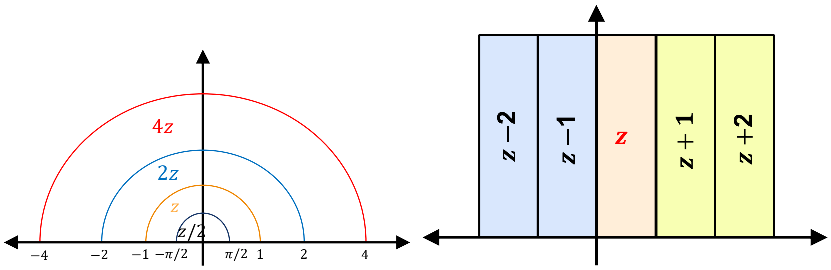
Definition 2.1.18 (Fundamental regions of Fuchsian groups)**.** F is a fundamental region for Γ if F is a closed set such that

1. $\bigcup_{T\in\Gamma} T(F) = \mathbb{H}$ (Generates the entire half-plane)
2. $F^{\circ} \cap T(F^{\circ}) = \emptyset \; \forall T \in \Gamma \backslash \{I\}$ where F° is the interior of F (The interior and the translated one are disconnected).

Remark 30. $\mathbb{H}$ is tessellated by the Γ-images of F, which overlap only at boundary points.

Example 2.1.19. 1. (Fig.2.1.19 (Left)) Let Γ be the cyclic group generated by the transformation $T(z) = 2z \in \mathrm{PSL}(2, \mathbb{R})$, $\Gamma = \langle 2z \rangle = \{2^n z \mid n \in \mathbb{Z}\}$. Then the semi-annulus shown below is easily seen to be a fundamental region for Γ.

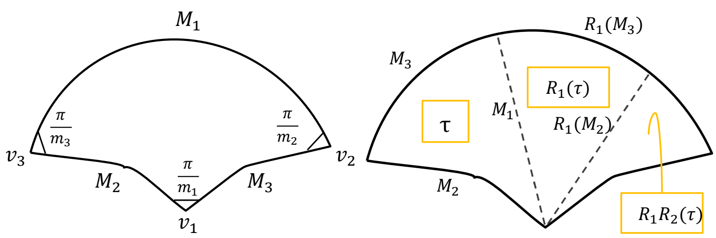
2. (Fig.2.1.19 (Right)) Let Γ be the cyclic group generated by the transformation $T(z) = z + 1 \in \mathrm{PSL}(2, \mathbb{R})$, $\Gamma = \langle z + 1 \rangle = \{z + n \mid n \in \mathbb{Z}\}$. Then the semi-annulus shown below is easily seen to be a fundamental region for Γ.



Next, as another and more complicated example, we describe one particular Fuchsian group that is studied well in the community and for subsequent use.

Let τ be an *H-triangle* with vertices v_1, v_2, v_3, angles $\pi/m_1, \pi/m_2, \pi/m_3$ at these vertices and sides M_1, M_2, M_3 opposite these vertices as illustrated in Fig.2.1.4 (Left).

Next, let R_i be the *H-reflection* over the H-line containing $M_i \forall i = \{1, 2, 3\}$, and let Γ^* be the group generated by those reflections. The image of τ under R_1 is the hyperbolic triangle $R_1(\tau)$ with sides $R_1(M_1) = M_1, R_1(M_2), R_1(M_3)$. As $R_1 R_2 R_1^{-1}$ fixes $R_1(M_2)$ pointwise, it is the H-reflection over $R_1(M_2)$. By this reflection $R_1(\tau)$ is transformed into $R_1 R_2 R_1^{-1}(R_1(\tau)) = R_1 R_2(\tau)$, as shown in Fig.2.1.4 (Right).



Continuing, we can see that $\{R(\tau) \mid R \in \Gamma^*\}$ forms a *tessellation* of $\mathbb{H}$, that is, no two Γ^*-images of τ overlap and every point of $\mathbb{H}$ belongs to some Γ^*-images. Now, let p be some point of τ, then the Γ^*-images of p are the corresponding points of the other triangles of the tessellation and thus they form a *discrete set*. Therefore, the Γ-orbit of each point of $\mathbb{H}$ is also a discrete set.

As a corollary of Theorem.2.1.17, we can show that our discussion regarding the triangles above leads to a Fuchsian group;

Corollary 2.1.20. *Let* Γ *be a subgroup of* $PSL(2, \mathbb{R})$*. Then* Γ *is a Fuchsian group if and only if for all* $z \in \mathbb{H}$, Γz *(*Γ*-orbit of z) is a discrete subset of* $\mathbb{H}$*.*

Proof. Corollary 5.6.4 of Jones. □

Remark 31. A Fuchsian group built in this way is called a *triangle group*.

We characterise the Triangle group as follows;

Definition 2.1.21 (Triangle group)**.** Given $a, b, c \in \mathbb{Z} \geq 2$ such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$, the *triangle group* $\Delta(a, b, c)$ is the subgroup (generated by the rotations with prescribed angles) $\langle R_a, R_b, R_c\rangle$ of $\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{PSL}(2, \mathbb{R})$.

Remark 32 (Presentation of Triangle group)**.** Let $X = R_1R_2, Y = R_2R_3$ then $\Gamma = \langle X, Y \mid X^{m_3} = Y^{m_1} = (XY)^{m_2} = I$ is a presentation of Γ (See Appendix.2 of Jones).

Definition 2.1.22 (Dirichlet region)**.** Let Γ be an arbitrary Fuchsian group and let $p \in \mathbb{H}$ be not fixed by any element of $\Gamma \backslash \{I\}$ (Identity element). Such points exist by Theorem.2.1.17(ii). The *Dirichlet region* for Γ centred at p is the set

$$D_p(\Gamma) = \{z \in \mathbb{H} \mid \rho(z, p) \leq \rho(T(z), p) \ \forall T \in \Gamma\}$$

where ρ is the hyperbolic distance metric (See Example.2.1.3).

Example 2.1.23. Let $\Gamma = \langle T \mid T(z) = 2z\rangle$ and let $p = 3i$. The Dirichlet region centred at p is visualised in Fig.2.1.4.

Theorem 3.4.5 of [JS87] shows that for the lattices the Dirichlet region is a fundamental region. Similarly, here, we study the connection between them for the hyperbolic geometry.

Theorem 2.1.24. *If* p *is not fixed by any element of* $\Gamma \backslash \{I\}$*, then* $D_p(\Gamma)$ *is a connected fundamental region for* Γ*.*

Proof. Theorem 5.8.3 of [JS87]. □

We shall study the tessellation of $\mathbb{H}$ formed by a fundamental region (called *a Dirichlet tessellation*) and all its images. First, we introduce a new concept;

Definition 2.1.25 (Locally finite)**.** A fundamental region F for a Fuchsian group Γ is called *locally finite* if every point $a \in F$ has a neighbourhood $V(a)$ such that $V(a) \cap T(F) \neq \emptyset$ for only finitely many $T \in \Gamma$.

$D_p(\Gamma)$ is an intersection of hyperbolic half-planes and is thus a *hyperbolically convex region* (H-convex). H-convex region F is bounded by a union of H-lines (geodesics), and the intersection of F with one of these H-lines is either a single point or a segment of an H-line. These segments are called *sides* of F.

As in the case of lattices, we consider the congruence of sides.

Definition 2.1.26 (Congruence of sides)**.** Let s be a side of a Dirichlet region F for a Fuchsian group. If $T \in \Gamma\backslash\{I\}$ and $T(s)$ is a side of F then s and $T(s)$ are called *congruent sides*.

Now $T(s)$ is also a side of $T(F)$ so that $T(s) \subseteq F \cap T(F)$. As $T(F)$ is a neighbouring F in the Dirichlet tessellation must have $T(s) = F \cap T(F)$. There cannot be more than two sides in a congruent set for suppose that $T_1(s)$ is also a side of $F(T_1 \in \Gamma)$. Then $T_1(s) = F \cap T_1(F)$ and thus $s = T^{-1}(F) \cap F = T_1^{-1}(F) \cap F$ so that $T = T_1$. Hence, the sides of a Dirichlet region can be considered as congruent pairs.

Finally, in the cases of lattices, the opposite sides of a fundamental parallelogram are congruent in pairs and the transformations which pair them generate the lattice. Similarly, by identifying the sides in each pair we obtain the quotient space. In the next subsection, we consider the quotient space $\mathbb{H}/\Gamma$ and we will show that we can also obtain it by identifying the congruent sides of a Dirichlet region.

2.1.5 Quotient-space of $\mathbb{H}/\Gamma$

We let $[z]$ denote the Γ-orbit of z and $\prod : \mathbb{H} \to \mathbb{H}/\Gamma$ the natural projection such that $z \mapsto [z]$. And, a set $V \subseteq \mathbb{H}/\Gamma$ is said to be *open* if $\prod^{-1}(V) = \{z \in \mathbb{H} \mid \prod(z) \in V\}$ is open in $\mathbb{H}$. With this, $\prod$ is a continuous and open map.

Theorem 2.1.27. *$\mathbb{H}/\Gamma$ is a connected Riemann surface and $\prod : \mathbb{H} \to \mathbb{H}/\Gamma$ is a holomorphic map.*

Proof. Theorem 5.9.1 of [JS87]. □

Now, we know that the quotient-spaces of Fuchsian groups are Riemann surfaces, and by Theorem 1.7.64, the universal covering space of a Riemann surface X not homeomorphic to the sphere, plane or torus is $\mathbb{H}$, and $X = \mathbb{H}/\Gamma$ where Γ is a properly discontinuous group of automorphisms of $\mathbb{H}$ acting without fixed-points (i,e., Γ is the group of covering transformations). Thus, by Theorem 2.1.17 Γ is a Fuchsian group and as it acts without fixed-point and elliptic elements fix a single point (See Example 2.1.11), Γ contains no elliptic elements.

We want to emphasise the importance of excluding the elliptic elements. Suppose we include Fuchsian groups with elliptic elements then *every* Riemann surface can be represented as the quotient-space of $\mathbb{H}$ by a Fuchsian group. For example, we shall see that the quotient-space for a triangle group is a sphere and as every Riemann surface homeomorphic to the sphere is *only conformally equivalent* to the Riemann sphere Σ by Theorem 1.7.53 it follows that Σ is conformally equivalent to $\mathbb{H}/\Gamma$ where Γ is a triangle group. Also, Theorem 5.9.9 of [JS87] shows that if $\mathbb{H}/\Gamma$ is compact then Γ contains no *parabolic elements*.

2.1.6 Results about automorphism groups

In the previous sections, we considered the automorphisms of the Rieman surface of genus 0 as well as 1 (e.g., Möbius transformations in $\mathrm{PSL}(2, \mathbb{C})$). In these cases, we found that the automorphism groups were infinite, by contrast, in this section, we study the groups of automorphisms of compact Riemann surfaces of genus $g \geq 2$ are necessarily finite.

Theorem 2.1.28. *Let S be a compact Riemann surface of genus $g \geq 2$. Then $|Aut(S)| \geq 84(g-1)$.*

Proof. Theorem 5.11.1 of [JS87] or Exercise 2.5 of [Har13]. □

Let us briefly mention the case when the upper bound is met, namely, a group of $84(g-1)$ automorphisms of a compact Riemann surface of genus $g \geq 2$ is called a *Hurwitz group*.

Theorem 2.1.29. *A finite group H is a Hurwitz group if and only if H is non-trivial and has two generators x, y following the relations; $x^2 = y^3 = (xy)^7 = 1$.*

Proof. Theorem 5.11.2 of [JS87]. □

More interesting results can be found in Sec 5.10 of [JS87] or [Con90].

2.1.7 Automorphic functions and uniformisation

An automorphic function generally is a function on a space that is invariant under the action of some group, in other words a function on the quotient space. Here, we define one tailored for $\mathbb{H}$ to simplify our discussion.

Definition 2.1.30 (Automorphic functions)**.** Automorphic functions are meromorphic functions defined on $\mathbb{H}$ which are invariant under transformations of a Fuchsian group.

Remark 33. Elliptic functions are meromorphic functions defined on $\mathbb{C}$ invariant under transformations of a lattice.

Automorphic functions are generalizations of trigonometric functions and elliptic functions and we can specify an underlying group of transformations as follows;

Definition 2.1.31 (Γ-automorphic functions)**.** Let Γ be a Fuchsian group; for simplicity, we shall assume that $\mathbb{H}/\Gamma$ is compact. A function f which is meromorphic on $\mathbb{H}$ is called Γ-*automorphic* if $f(T(z)) = f(z)\ \forall T \in \Gamma$.

Example 2.1.32. An automorphic function is a rather general concept. Thus, we can find some examples by replacing the space of interest from $\mathbb{H}$ to; (i) Σ: Γ are finite and the curves Σ/Γ are algebraic curves of genus 0, consequently the automorphic functions generate a field of rational functions, (ii) $\mathbb{C}$: automorphic functions are periodic functions (e.g., $e^{2n\pi i}$ for $n \in \mathbb{Z}$) is invariant under a group of translations $\{n \mapsto n + m \mid m \in \mathbb{Z}\}$ or doubly periodic like elliptic functions, (iii) $\mathbb{H}$: $\mathbb{H}/\Gamma$ is compact and an example of Γ as we saw is a group of Möbius transformations in $\mathrm{PSL}(2, \mathbb{R})$.

Remark 34. Γ-automorphic functions are known to form a field which we denote by $\mathcal{F}(\Gamma)$ as rational (or meromorphic) functions form a field (See Sec.1.2).

Finally, we shall briefly revisit the uniformisation theorem and how the Fuchsian group fits in. Similarly to Example.1.7.54, we suppose that $A(x, y) = 0$ is an irreducible algebraic function with Riemann surface S of genus greater than one. Then we have the forgetful and the evaluation maps as discussed in the example. But, now, by Theorem 4.19.8 of [JS87], $S = \mathbb{H}/\Gamma$ for some Fuchsian group Γ without elliptic elements and then ϕ, ψ give rise to Γ-automorphic functions $\bar{\phi}, \bar{\psi}$ such that $\bar{\phi}(z) = \phi([z]_\Gamma), \bar{\psi}(z) = \psi([z]_\Gamma)$. Thus, $A(\bar{\phi}(z), \bar{\psi}(z)) \equiv 0$. Therefore, from the algebraic relation $A(x, y) = 0$ expressing y as a multi-valued function of x we found single-valued functions $\bar{\phi} = x, \bar{\psi} = y$.

2.2 Dessins d'Enfants: Geometric representation of Riemann surface

Dessin d'enfants was first discussed by Felix Klein in 1879 in relation to an 11-fold cover of $\mathbb{P}^1(\mathbb{C})$ using itself with a monodromy group of $\mathrm{PSL}(2, \mathbb{Z}_{11})$. The modern development of these

"Children's Drawings" was advanced by Alexander Grothendieck during his exploration of the absolute Galois group. This exposition is quite deep and due to the limited space, we do not include in this work. Note that $\mathbb{P}^1(\mathbb{C})$ is the one-dimensional projective plane over the complex number (i.e., Σ in our context).

In this section, we will study the notion of the dessins which represent algebraic functions in a visually interpretable space and conclude with a simple example. We start by introducing the tools from Graph theory.

A graph G can be regarded as a pair $(V; E)$ consisting of a set V of vertices and a set E of edges. We will suppose the graph to be connected and finite. A *planar graph* is a graph that can be drawn on a plane without any edges crossing each other except at their endpoints (vertices). Not all graphs are planar; some graphs, due to their intricate connections, cannot be drawn on a plane without crossings. *Graph embedding* refers to the act of placing a graph onto a surface in such a way that the edges do not cross except at their endpoints. The surface could be a plane, a sphere, a torus, or any other topological space.

To develop insight during the subsequent, it is helpful to establish the notion of Dessins informally.

Definition 2.2.1 (Dessins d'Enfants)**.** A *dessins d'Enfants* is a compact oriented topological surface X with a graph D embedding in it, such that the following holds;

- D is connected,
- D is bi-coloured, that is, each vertex is assigned one of two colours (black or white) and two vertices connected by an edge share the same colours,
- Each connected component $X \backslash D$ is homeomorphic to a topological disc, and called *faces* of the dessin (also known as a bipartite map).

Example 2.2.2. Any tree contained in the sphere is a dessin with one face, as a tree can always be bicoloured.

2.2.1 Belyi's Theorem and dessin d'enfants

When we say that a compact Riemann surface S is defined over $\bar{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$) the corresponding irreducible, non-singular algebraic curve is defined over $\bar{\mathbb{Q}}$. Similarly, a curve C is defined over $\mathbb{Q}$ if it can be described as the zero set of polynomials whose coefficients lie in $\mathbb{Q}$.

Theorem 2.2.3 (Belyi's theorem)**.** *A compact Riemann surface S can be defined over $\bar{\mathbb{Q}}$ if and only if there exists a covering $f : S \to \mathbb{P}^1(\mathbb{C})$ unramified outside of $\{0, 1, \infty\}$.*

Proof. Theorem 5.1 of [Pos+14] and Sec.2 of [Pér18]. □

This result gives us the definition of a Belyi map with the notion of morphism (A morphism is a rather general map between two objects);

Definition 2.2.4 (Belyi morphism)**.** A morphism $\beta : S \to \mathbb{P}^1(\mathbb{C})$ whose critical values lie in $\{0, 1, \infty\}$ is called a *Belyi morphism*. We call β a *pre-clean* Belyi morphism if all the ramification orders over 1 are less than or equal to 2, and *clean* if they are all exactly equal to 2.

As an immediate result, we can simplify Theorem 2.2.3 as follows;

Corollary 2.2.5. *A compact Riemann surface S can be defined over $\bar{\mathbb{Q}}$ if and only if there exists a clean Belyi morphism.*

Proof. Corollary 6.2 of [Pos+14]. □

Definition 2.2.6 (Belyi pair)**.** If S is a compact Rieman surface and β is a (clean) Belyi morphism defined on S, then we call (S, β) a (clean) *Belyi pair*.

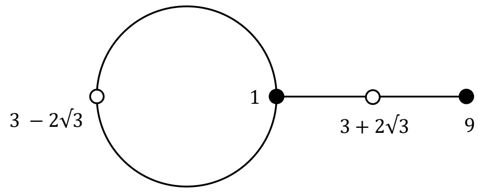
Remark 35. The pair (S_D, β_D) constructed above is called the *Belyi pair associated with* (X, D).

In graph theory, a *hypermap* is a map whose vertices are coloured black and white under the condition that each edge connects two vertices of different colours. Now, we employ this notion to formally define the Dessins d'Enfants.

Definition 2.2.7 (Revisit: Dessins d'Enfants)**.** A dessin d'enfant D is a hypermap considered as a representation of a particular Belyi pair (S, β).

Example 2.2.8 (Example 2.3.1 of [LZZ04])**.** Consider a simple graph in the Figure below. For the moment, let's ignore the white vertices and suppose that we do not know the value 9 for the right-most vertex (we call 9 a in the following). Then, there is an imaginary vertex located far right to the vertex a, at $x = \infty$, the centre of the face of degree one (the left circle), at $x = 0$, and the vertex of degree 3, at $x = 1$. Finally, we denote the vertex a as $x = 1$ (an unknown parameter that we will compute). The Belyi function f for this map is rational of degree 4 as the vertices of the map are its roots. From the connectivity, we can infer the expression of $f(x) = K\frac{(x-1)^3(x-a)}{x}$ with a constant K. To compute K, we leverage the fact that the function $f - 1$ has two double roots as we have the two white vertices. Thus, we conclude that

$\frac{K(x-1)^3(x-a)}{x} - 1 = \frac{K(x-1)^3(x-a)-x}{x} = \frac{K(x^2+bx+c)^2}{x}$ with additional unknown constants b and c. The differentiation yields $f'(x) = \frac{K(x-1)^2(3x^2-2ax-a)}{x^2}$. By solving the equations $3x^2 - 2ax - a$ and x^2+bx+c, we get $b = -2a/3$ and $c = -a/3$. Therefore, we get $a = 9$ and $K = -1/64$, so the dessin d'enfant with corresponding Belyi function (morphism) $f(x) = -(x-1)^3(x-9)/64x$, embedded in the complex projective plane is in the following figure.



Remark 36 (Dessin to Belyi pair)**.** The map {dessins} $\to$ {Belyi pairs} which sends (X, D) to (S_D, β_D) induces a well-defined map from equivalence classes of Dessins d'enfants to equivalence classes of Belyi pairs.

The converse direction (i.e., Belyi pair being mapped to a dessin) has been proven in Proposition 4.22 of [GG12] or Proposition 3.7 of [Pér18]. Thus, we have the following result without proof.

Corollary 2.2.9 (Grothendieck correspondence)**.** *There is a one-to-one correspondence between Dessins and Belyu pairs.*

Proof. Theorem 6.7 of [Pos+14]. □

Definition 2.2.10 (Morphism between dessins)**.** A morphism between the dessin d'enfants (S_1, D_1) and (S_2, D_2) is a morphism $\phi : S_1 \to S_2$ such that the following commutes;

$$\begin{tikzcd} S_1 \arrow[r, "f_{D_1}"] \arrow[d, "\phi"'] & \mathbb{P}^1(\mathbb{C}) \arrow[d, "\mathrm{Id}"] \\ S_2 \arrow[r, "f_{D_2}"] & \mathbb{P}^1(\mathbb{C}) \end{tikzcd}$$

One of the consequences of this characterisation is that the genus of the surface X is encoded in the graph D (and its embedding to X);

Proposition 2.2.11. *Let (S, D) with genus of S being g be a dessin with v vertices, e edges, and f faces. Then, Euler formula $2 - 2g = \chi(S) = v - e + f$ holds.*

Proof. Proposition 3.11 of [Pér18]. □

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