Visualising Dirichlet Domains given Symmetric Polygons

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1 Introduction

Dirichlet domains are foundational in Fuchsian group theory and hyperbolic geometry, playing a pivotal role in understanding the geometry and topology of hyperbolic surfaces. These domains, associated with Fuchsian groups, are constructed as intersections of half-planes defined by perpendicular bisectors of geodesic segments.

The bounds on the possible number of sides of Dirichlet domains are shown in Theorem 10.5.1 [3], while Theorem 10.6.4 [3] demonstrates that for triangle groups with genus zero, the Dirichlet domains are either quadrilaterals or hexagons. The shapes of Dirichlet domains are further shown to depend on the location of base points in Theorem 24 [12].

In this paper, we visually investigate the relationship between the base point's location and the shape of Dirichlet domains for polygons with a small number of sides (3 and 4). This analysis aims to provide insights for future theoretical investigations.

We start by introducing fundamental concepts in hyperbolic geometry and defining Dirichlet domains. We then delve into their construction algorithm and complexity. Our experiments focus on visualising Dirichlet domains through a grid search algorithm, analysing their shapes and properties in the context of triangle and quadrilateral groups. Finally, we discuss future research directions, including the application of machine learning for analyzing and predicting the behavior of Dirichlet domains in more intricate Fuchsian groups.

2 Preliminaries

We give some definitions and properties relevant to discuss Dirichlet domains in this section. For more detailed information, see [7] and [3].

2.1 Hyperbolic Geometry

Let $\mathbb{C}$ be the complex plane. The *hyperbolic plane* is the metric space consisting of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ (similarly, $\mathbb{D} = z \in \mathbb{C} \mid |z| < 1)$

endowed with a metric ρ defined below. We use the usual notation for the real and imaginary parts of $z = x + iy \in \mathbb{C}$, where $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.

To define the hyperbolic metric ρ, we first introduce the concept of hyperbolic length for curves in $\mathbb{H}$.

Definition 2.1 (Hyperbolic Length) *Let $I = [0, 1]$ and $\gamma : I \to \mathbb{H}$ be a piecewise differentiable curve:*

$$\gamma = \{z(t) = x(t) + iy(t) \in \mathbb{H} \mid t \in I\}$$

The hyperbolic length of γ is given by

$$h(\gamma) = \int_I \frac{|dz|}{y(t)} = \int_0^1 \frac{\sqrt{(dx(t))^2 + (dy(t))^2}}{y(t)}\, dt.$$

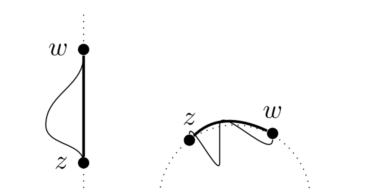
This hyperbolic length is independent of the parametrization of γ.

Definition 2.2 (Hyperbolic Distance) *The hyperbolic distance $\rho(z, w)$ between two points $z, w \in H$ is defined by the formula*

$$\rho(z, w) = \inf_{\gamma} h(\gamma),$$

where the infimum is taken over all γ joining z to w in $\mathbb{H}$.

Remark 2.1 *The above ρ is nonnegative, symmetric and satisfies the triangle inequality $\rho(z, w) \leq \rho(z, \xi) + \rho(\xi, w)$, that is, it is a distance function on H. And, among the curves joining z and w, the one with the shortest hyperbolic length (i.e., a **geodesic**) is a straight line or a semicircle orthogonal to the real axis $\mathbb{R} = \{z \in \mathbb{C} \mid Im(z) = 0\}$.*



Consider the group $SL(2, \mathbb{R})$ of real matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $\det(g) = ad - bc = 1$. Consider also the set of Möbius transformations of $\mathbb{C}$ onto itself. It contains a subgroup containing transformations of the form $\{z \to T(z) = \frac{az+b}{cz+d}, |a, b, c, d \in \mathbb{R}, ad - bc = 1\}$ such that the product of two transformations corresponds to the product of two corresponding matrices and the inverse of a transformation corresponds to the inverse matrix.

Note that a transformation of the form is represented by a pair of matrices $\pm g \in \mathrm{SL}(2, \mathbb{R})$. The group of such transformations is a subgroup in $\mathrm{PSL}(2, \mathbb{R})$ and it is isomorphic to the quotient group $\mathrm{SL}(2, \mathbb{R})/\{\pm I\}$.

Now, a transformation of H onto itself is called an isometry if it preserves the hyperbolic distance on H and is a homeomorphism. The set of all isometries of $\mathbb{H}$ forms a group; we shall denote it by $\mathrm{Isom}(\mathbb{H})$.

$\mathrm{PSL}(2,\mathbb{R})$ is also a topological space in which a transformation $T \in \mathrm{PSL}(2,\mathbb{R})$ can be identified with the point $(a,b,c,d) \in \mathbb{R}^4$. More precisely, we define a norm on $\mathrm{PSL}(2,\mathbb{R})$ induced from $\mathbb{R}^4$ such that for $T(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$, $\|T\| = \sqrt{a^2+b^2+c^2+d^2}$. $\mathrm{PSL}(2,\mathbb{R})$ is a topological group with respect to the metric $d(T,S) = \|T - S\|$, where $T, S \in \mathrm{PSL}(2,\mathbb{R})$.

Definition 2.3 (Discreteness) *A subgroup Γ of Isom$(\mathbb{H})$ is called **discrete** if the induced topology on Γ is a discrete topology, i.e., if Γ is a discrete set in the topological space Isom$(\mathbb{H})$.*

Definition 2.4 *A discrete subgroup Γ of PSL$(2,\mathbb{R})$ is called a **Fuchsian group**.*

Definition 2.5 (Fundamental domains) *Let Γ be a discrete group of isometries of $\mathbb{H}$. A closed region $F \subseteq \mathbb{H}$ (i.e., a closure of a non-empty open set F°, called the interior of F) is defined to be a **fundamental domain** for Γ if*

1. $\bigcup_{T\in\Gamma} T(F) = \mathbb{H}$,
2. $F^\circ \cap T(F^\circ) = \emptyset$ *for all* $T \in \Gamma - \{Id\}$.

*The set $\partial F = F - F^\circ$ is called the **boundary** of F. The family $\{T(F) \mid T \in \Gamma\}$ is called the **tessellation**.*

Now we will define the Dirichlet domains.

Definition 2.6 (Perpendicular bisectors) *A **perpendicular bisector** of the geodesic segment $[z_1, z_2]$ is the geodesic through w, the midpoint of $[z_1, z_2]$, orthogonal to $[z_1, z_2]$ such that it is a line given by $\{z \in \mathbb{H} \mid \rho(z, z_1) = \rho(z, z_2)\}$.*

Note that when the geodesic segment $[z_1, z_2]$ is given, the perpendicular bisector of $[z_1, z_2]$ is determined uniquely.

Definition 2.7 (Dirichlet domains) *Let Γ be an arbitrary Fuchsian group, and let $p \in \mathbb{H}$ be not fixed by any element of $\Gamma - \{Id\}$. We denote the perpendicular bisector of the geodesic segment $[p, T(p)]$ where $T \in \Gamma - \{Id\}$ by $L_p(T)$, i.e., $L_p(T) = \{z \in \mathbb{H} \mid \rho(z,p) = \rho(z, T(p))\}$, and denote the hyperbolic half-plane containing p by $H_p(T)$, i.e.,*

$$H_p(T) = \{z \in \mathbb{H} \mid \rho(z,p) \le \rho(z, T(p))\}.$$

*We define the **Dirichlet domain** for Γ centered at p to be the set*

$$D_p(\Gamma) = \bigcap_{T \in \Gamma - \{Id\}} H_p(T).$$

Let us introduce a couple of important notions to discuss the potential shapes of Dirichlet domains.

Definition 2.8 1. *We say $u, w \in D$ are **congruent** if they are in the same Γ-orbit. The congruence is an equivalence relation on D, in particular on the vertices of F. The equivalence classes of vertices are called **cycles**.*

2. *The **order of the cycle** C, denoted by $Ord(C)$, is the order of the stabilizer in Γ of any $v_i \in C$.*

Definition 2.9 (Conifite) *A Fuchsian group Γ is called **cofinite** if there exists a Dirichlet domain $D_p(\Gamma)$ whose hyperbolic area is finite.*

Definition 2.10 (Signature) *Let Γ be a finitely generated Fuchsian group. Let m_j represent the order of maximal elliptic cycles, s represent the number of conjugacy classes of maximal parabolic cyclic subgroups, and t represent the number of conjugacy classes of maximal hyperbolic cyclic subgroups. The symbol $(g; m_1, m_2, \ldots, m_r; s; t)$ is called the signature of Γ. Here each parameter is a non-negative integer and $m_j \geq 2$.*

Example 2.1 (Triangle groups [3]) *A Fuchsian group with the signature $(0; m_1, m_2, m_3)$, where $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1$, is called a hyperbolic triangle group. Note that the case where $m_i = \infty$ is allowed.*

In the context that follows, we examine the scenario where $g = 0$, indicating the absence of genus.

Theorem 2.1 ([3], Theorem 10.5.1) *Let Γ be a cofinite Fuchsian group with signature $(g = 0; m_1, \ldots, m_n)$, and let $D_p(\Gamma)$ be the Dirichlet polygon with centre p with N sides. Then, $2n - 2 \leq N \leq 4n - 6$.*

3 Visualisation of Dirichlet Domains

In this section, we describe the visualisation algorithm of the Dirichlet domains.

3.1 Algorithm to construct Dirichlet domains

We utilised a straightforward algorithm to construct the Dirichlet domains based on given base points. This algorithm, though simplistic in nature, effectively generates the fundamental regions associated with Fuchsian groups.

Given a n-sided polygon $P \subset \mathbb{H}$ with sides $s_1, s_2, \ldots, s_n$ with angles $0 \leq \frac{\pi}{m_i} \leq \infty$ Let us denote R_i the reflection of P in the side s_i, for instance, $R_1(P)$ represents P being reflected in s_1. Similarly for a point $p \in \mathbb{H}$.

3.1.1 Complexity analysis

Here we follow the Big-O notation as the convention to analyse algorithms.

1. Reflecting the base point twice in each side and its reflection requires $O(n)$ operations for each side, resulting in $O(n^2)$ operations overall.

Algorithm 1 Compute vertex of Dirichlet domain

Compute all perpendicular bisectors relevant to the Dirichlet domain.
$l_{bisec} := \{L_p(R_j R_i)\ \forall\ i, j \in \{1 \dots n\}; i \neq j\}\}$
Compute Intersections of l_{bisec} as vertex candidates: p_{cand}
$p_{cand} := \cup_{l,m \in l_{bisec}; l \neq m} \{p_l \in l \mid p_l \in m\}$
# Check if a point in p_{cand} lies in the exterior of each bisector.
for all p_c in p_{cand} **do**
  if $p_c \in H_p(R_j R_i)\ \forall\ i, j \in \{1 \dots n\}; i \neq j\}$ **then**
    return p_c as vertex of Dirichlet domain
  end if
end for

2. Computing all perpendicular bisectors between the reflected points requires comparing each pair of reflected points, resulting in $O(n^2)$ comparisons.

3. Computing intersections of bisectors as vertex candidates can result in up to $O(n^4)$ intersections in the worst case.

4. Checking if a candidate is exterior to all bisectors involves comparing the candidate with each bisector. In the worst case scenario where there are $O(n^4)$ candidate vertices, this step would require $O(n^2)$ operations for each candidate vertex.

Therefore, in the worst-case scenario where the number of candidate vertices is on the order of n^4, the overall computational complexity of the algorithm would be $O(n^6)$.

3.1.2 Implementations

Our algorithm was implemented using the SageMath package [10] in Python, leveraging its rich set of functions for intuitive implementation. The core of our implementation relies on the Hyperbolic Geometry module [1]. The source code for our implementation is available online[2].

3.2 Empirical Verification with Triangle groups

We verify our algorithm by reconstructing the argument for a Triangle group. In Theorem 2.1, by plugging in $n = 3$ (i.e., a triangle group), then N is bounded by $4 \leq N \leq 6$ and by the construction of Dirichlet domains, we know that the possible number of sides N is 4 or 6. For more details, See the Proof of Theorem 10.6.4 in [3] We will employ this example to verify our visualisation algorithm. To this end, we employ the grid search algorithm that is often utilised in the

[1] `https://doc.sagemath.org/html/en/reference/hyperbolic_geometry/index.html`
[2] `https://github.com/Rowing0914/sage-math-hyperbolic`

exploration of a space of the input variable of a black-box target function in Machine learning literature.

3.2.1 Grid search

Grid search is a technique used in optimisation to search for the optimal parameters of a function. It works by evaluating a given function on a grid of configuration space (e.g., parameter values of the function of interest) and selecting the point in the configuration space that achieves the required optimality.

Let us consider a simple example where we have $f(x) = x^2$ in $\mathbb{R}$ and we want to find the minimum value of this f. Now, we define a partition of possible parameter values for the line $\mathbb{R}$, such as $x \in \{1, 2, 3\}$ or $x \in \{0.1, 0.2, 0.3\}$. We then evaluate the function for each parameter value x. For example, we would evaluate the function at the points $x \in \{1, 2, 3\}$. Then we obtain $1, 4, 9$ for $f(x)$. Finally, we select the point in $\mathbb{R}$ that gives the minimum of f.

Grid search is a powerful yet computationally expensive method for parameter tuning, especially with large parameter spaces. It depends on two key factors: binning, which controls the granularity of the search, and range, which defines the scope of the search space. For example, a small binning over a large range allows for a broad exploration, while a large binning over a small range enables a more detailed investigation. In our experiment, we focused on the square $[-0.5, 0.5] \times [-0.5, 0.5] \subset \mathbb{H}$ for our grid search implementation in Python.

3.2.2 Result

We utilized a Grid search algorithm to compute Dirichlet domains within a Triangle group, focusing on the symmetric triangle group where each corner has an angle of $\pi/4$. The base polygon was centered at the origin $(0, 0)$ in the space.

Figure 1 illustrates the following observation.

Observation 3.1 *When the base point lies:*

1. *On the boundary,* $N = 4$.
2. *In the interior,* $N = 6$.

Note that, although difficult to discern, there are some bright orange points on the boundary indicating 4-sided shapes. If we had been able to compute Dirichlet domains for all boundary points, the boundary would have been colored in bright orange. Also, areas shaded in dark blue indicate regions outside the base triangle where the Dirichlet domain cannot be computed, resulting in 0 sides.

Note that our result obtained aligns with Theorem 24 of [12] that states the possible shapes of dirichlet domains for a hyperbolic triangle is either a quadrilateral or a hexagon based on the location of the base point accordingly.

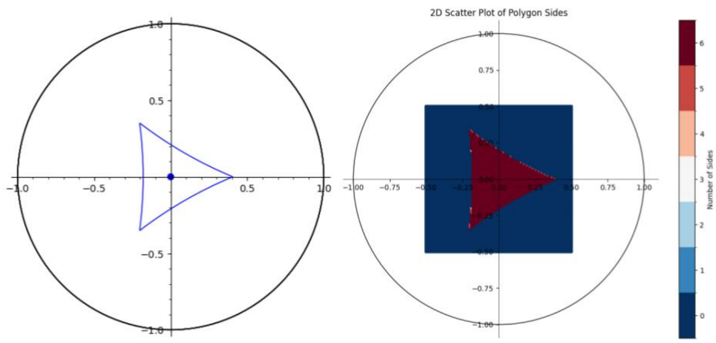


Figure 1: (Left) Base polygon to compute the dirichlet domain, (Right) **2D Scatter Plot** illustrating polygon sides within a square grid. The grid spans from -1.0 to 1.0 on both the x and y axes, with a circular boundary. The color bar indicates the number of sides computed at grid points.

3.3 Detailed Analysis of 4-gon Construction

We expand our investigation to include hyperbolic 4-gons.

3.3.1 Grid Search Overview

We now apply a Grid search for a symmetric 4-gon centered at the origin $(0, 0)$ with each corner being $\pi/4$. From Figure 2, we extracted the following observation.

Observation 3.2 *When the base point lies;*

1. *On the boundary, $N = 6$,*
2. *In the intersection of diagonals or on a diagonal, $N = 8$,*
3. *In the interior, $N = 10$.*

Note that areas shaded in dark blue indicate regions outside the base triangle where the Dirichlet domain is not computable, resulting in 0 sides.

3.3.2 Case study: Base point Variations

We study the behaviour of the dirichlet domains in the cases depending on the location of the base point. In the following, all plots follow the same format such that the red dots represent the vertices of the dirichlet domain, yellow and green geodesics are the perpendicular bisectors (See Algorithm 1).

(1) On Boundary: It is clear that there is a region that is constructed by reflecting the base 4-gon in each side. Thus, it posesses the same hyperbolic area as the base 4-gon. Therefore it is a dirichlet domain by Remark 11 [12].

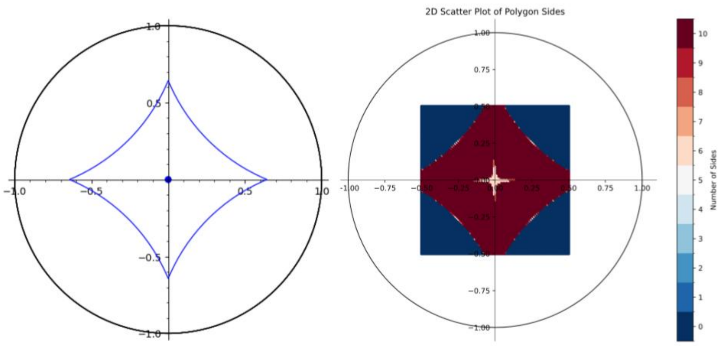
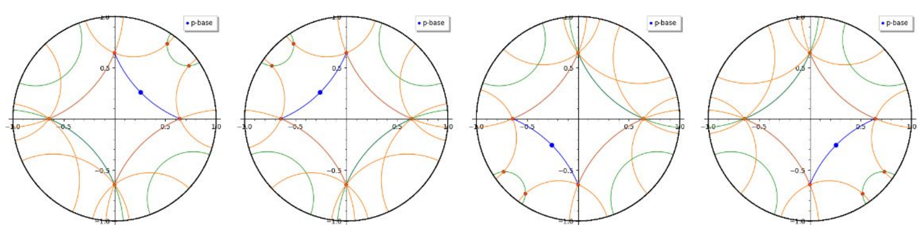
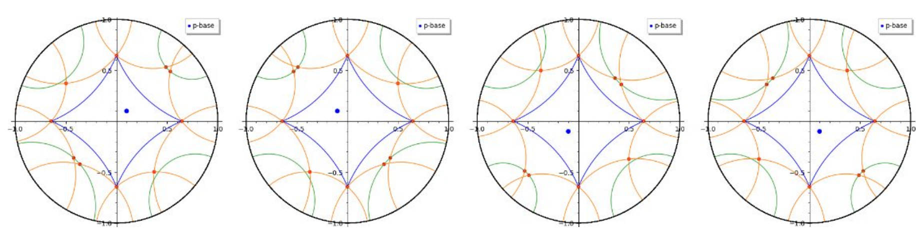


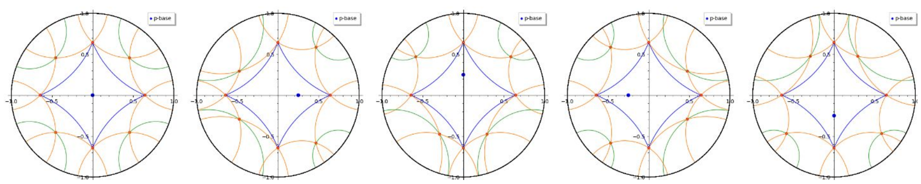
Figure 2: (Left) Base polygon to compute the dirichlet domain, (Right) Scatter plot of polygon sides at each grid point.



(2) Interior except on diagonals: This case can be considered as when the base point moves slightly towards the center but does not reach the diagonals. Let us focus on the leftmost case to begin. Compared to the leftmost case of (1), two things have occurred: (i) the perpendicular bisectors that were on the top-left and bottom-right sides of the base 4-gon have moved outwards to form two additional vertices of the Dirichlet domain, and (ii) the top-right and bottom-left *green* perpendicular bisectors have also moved outwards to form two more vertices. Thus, there are 10 sides in total. The same observation applies to the other cases compared to their corresponding ones in (1).



(3) On (a) diagonals: Let us reconsider the leftmost case (i.e., the base point lies at the intersection of the diagonals). Compared to the leftmost case of (2), we observe that the two vertices at the top-right corner have merged into one, as have those at the bottom-left corner. This results in a total of 8 vertices. In other cases, we can observe a similar transition of bisectors from their corresponding plots in (2).



3.4 Scaling Experiment: Grid Search for Larger N=5

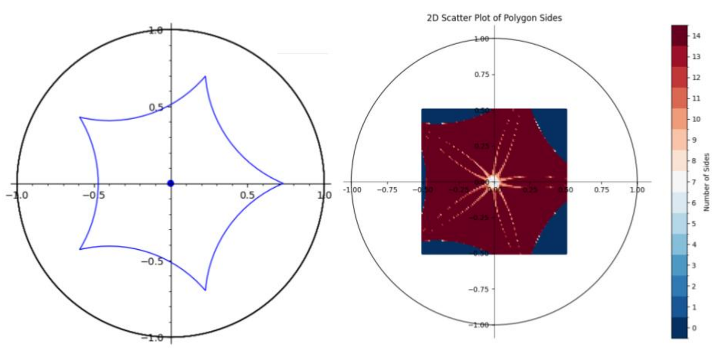


Figure 3: (Left) Base polygon to compute the dirichlet domain, (Right) Scatter plot of polygon sides at each grid point.

3.5 Future Directions

3.5.1 Application

There is a growing interest in the mathematics community regarding the utilization of machine learning models for exploring mathematical structures. Alessandretti et al. [2] employed data analysis techniques to analyze number-theoretic datasets. He et al. [6] investigated the performance of standard machine learning algorithms such as Bayesian or logistic classifiers, as well as some feed-forward neural networks, for predicting Arithmetic Curves. Gukov et al. [4] used the reinforcement learning algorithm Trust Region Policy Optimization (TRPO) [9] to find solutions to the problem of unknotting seemingly tangled ropes. Even for the direct generation of mathematical structures, Halverson et al. [5] explored statistical methods.

In our work, we employed a brute-force approach to explore the configuration space, specifically using Grid Search. However, we believe that more advanced machine learning methods could greatly benefit the analysis of Dirichlet domains' behavior.

3.5.2 Theory

In terms of theoretical advancements, to the best of our knowledge, only the case of triangle groups has been proven. Therefore, it would be interesting to pursue proofs for cases with a larger number of sides. We believe the proof of Theorem 10.6.4 in [3] could be helpful in this regard.

Investigating cases where $g \neq 0$ could also be interesting [8, 1]. Additionally, approximating the computation of Dirichlet domains can help scale the study of their behavior [11].

4 Acknowledgements

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