Quiz 7.2 – Linear Algebra Review

Name: Key

Complex Conjugates

Give the complex conjugate of the following functions:

- e^{-ix} e^{ix}
- $\sin(\pi x) + i\cos(2\pi x)$ $\sin(\pi x) - i\cos(2\pi x)$
- $3x^2 + x - 2.4$ $3x^2 + x - 2.4$

Operators

Below are two linear algebra operators. Find the result when each operator operates on the function $f(x) = 2x^3 + \sin(x)$

- $\hat{A} = x^2$ $x^2(2x^3 + \sin(x)) = 2x^5 + x^2\sin(x)$
- $\hat{B} = -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}x}$ $-\frac{1}{c}\frac{d}{dx}(2x^3 + \sin(x)) = -\frac{6}{c}x^2 - \frac{1}{c}\cos(x)$

Eigenfunctions

Give an example of an eigenfunction (ϕ) and its associated eigenvalue for each of the three operators below:

- $\hat{C} = 2$ ϕ = Literally any function $\hat{C}\phi = 2\phi$

  $c = 2$
- $\hat{D} = \frac{\mathrm{d}}{\mathrm{d}x}$ $\phi = e^{4x}$ $\hat{D}\phi = 4\phi$

  $d = 4$
- $\hat{E} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ $\phi = \sin(3x)$ $\hat{E}\phi = -9\phi$

  $e = -9$

Orthogonality

Consider two even functions:

$$\phi_1(x) = \sqrt{\frac{3}{2}}x \qquad \phi_2(x) = \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right)$$

Show that these two functions are orthogonal over the interval $[-1, 1]$

If orthogonal, $\int_{-1}^{1} \phi_1 \phi_2 \, dx = \emptyset$

$$\int_{-1}^{1} \sqrt{\frac{525}{16}}\, x\left(x^3 - \frac{3}{5}x\right) dx = \sqrt{\frac{525}{16}} \int_{-1}^{1} x^4 - \frac{3}{5}x^2 \, dx = \sqrt{\frac{525}{16}} \left(\frac{1}{5}x^5 - \frac{1}{5}x^3\right)\Big|_{-1}^{1}$$

$$= \sqrt{\frac{525}{16}} \left[\left(\frac{1}{5} - \frac{1}{5}\right) - \left(-\frac{1}{5} + \frac{1}{5}\right)\right] = \emptyset$$

Commutators

Consider two operators: $\hat{A} = \frac{d^2}{dxdy}$ and $\hat{B} = x^2 + y$

Give the commutator $\left[\hat{A}, \hat{B}\right]$. You may assume that the operators will only be used with functions that are separable. i.e. You may use a trial function of $f(x)g(y) = \psi$

$$[\hat{A}, \hat{B}]\psi = \hat{A}\hat{B}\psi - \hat{B}\hat{A}\psi$$

$$[\hat{A}, \hat{B}]\psi = \frac{d^2}{dxdy}\left[(x^2+y) f(x) g(y)\right] - (x^2+y)\frac{d^2}{dxdy} f(x) g(y)$$

$$[\hat{A}, \hat{B}]\psi = \frac{d^2}{dxdy}\left[x^2 f(x) g(y) + y f(x) g(y)\right] - (x^2+y)\frac{d^2}{dxdy} f(x) g(y)$$

product rule

$$[\hat{A}, \hat{B}]\psi = \frac{d}{dx}\left[x^2 f(x) g'(y) + f(x) g(y) + y f(x) g'(y)\right] - (x^2+y)\frac{d}{dx} f(x) g'(y)$$

product rule

$$[\hat{A}, \hat{B}]\psi = 2x f(x) g'(y) + x^2 f'(x) g'(y) + f'(x) g(y) + y f'(x) g'(y) - (x^2+y) f'(x) g'(y)$$

$$[\hat{A}, \hat{B}]\psi = 2x f(x) g'(y) + f'(x) g(y) + \cancel{(x^2+y) f'(x) g'(y)} - \cancel{(x^2+y) f'(x) g'(y)}$$

$$[\hat{A}, \hat{B}]\psi = \left(2x \frac{d}{dy} + \frac{d}{dx}\right) f(x) g(y) \rightarrow [\hat{A}, \hat{B}] = \left(2x \frac{d}{dy} + \frac{d}{dx}\right)$$