Lab 3.
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Q1.

$Y|\theta \sim Bin(n, \theta)$.

$$L(\theta) = \prod_{i=1}^{n} \binom{n}{y_i} \theta^{y_i} (1-\theta)^{n-y_i}$$

$$l(\theta) = Log L(\theta) = \sum_{i=1}^{n} Log\binom{n}{y_i} + y_i Log\,\theta + (n-y_i) Log(1-\theta)$$

$$\frac{\partial l(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{y_i}{\theta} + \frac{(n-y_i)(-1)}{1-\theta}$$

$$= \sum_{i=1}^{n} \frac{y_i}{\theta} - \frac{y_i - n}{\theta - 1}$$

Setting $\frac{\partial l(\hat{\theta})}{\partial \hat{\theta}} = 0$:

$$\sum_{i=1}^{n} \frac{y_i}{\hat{\theta}} - \frac{y_i - n}{\hat{\theta} - 1} = 0$$

$$\frac{1}{\hat{\theta}} \sum_{i=1}^{n} y_i = \frac{1}{\hat{\theta}-1} \sum_{i=1}^{n} (y_i - n)$$

$$\frac{\hat{\theta}-1}{\hat{\theta}} \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_i - n^2$$

$$1 - \frac{1}{\hat{\theta}} = \frac{\sum_{i=1}^{n} y_i - n^2}{\sum_{i=1}^{n} y_i}$$

$$\frac{1}{\hat{\theta}} = \frac{\cancel{\sum_{i=1}^{n} y_i} - \cancel{\sum_{i=1}^{n} y_i} + n^2}{\sum_{i=1}^{n} y_i}$$

$$\frac{1}{\hat{\theta}} = n^2 / \sum_{i=1}^{n} y_i$$

$$\hat{\theta} = \frac{1}{n^2} \sum_{i=1}^{n} y_i = \frac{\bar{y}}{n} = \frac{118}{129} \approx 0.91.$$

To calculate the asymptotic variance of $\hat{\theta}$:

$$U(\theta) = \frac{\partial l(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{y_i}{\theta} - \frac{y_i - n}{\theta - 1}$$

$$I(\theta) = E\left[-\frac{\partial U(\theta)}{\partial \theta}\right] = -E\left[\sum_{i=1}^{n} -\frac{y_i}{\theta^2} + \frac{y_i - n}{(\theta-1)^2}\right]$$

$$= \frac{1}{\theta^2} E\left[\sum_{i=1}^{n} y_i\right] - \frac{1}{(\theta-1)^2} E\left[\sum_{i=1}^{n} y_i - n\right]$$

$$= \frac{n}{\theta^2} E\left[\frac{1}{n}\sum_{i=1}^{n} y_i\right] - \frac{n}{(\theta-1)^2}\left\{E\left[\frac{1}{n}\sum_{i=1}^{n} y_i\right] - n\right\}$$

$$= \frac{n\bar{y}}{\theta^2} - \frac{n(\bar{y} - n)}{(\theta-1)^2}$$

$$Var(\hat{\theta}) = \frac{1}{I(\hat{\theta})} = \frac{1}{\frac{129 \times 118}{0.91^2} - \frac{129(118-129)}{(0.91-1)^2}} \approx 5.166 \times 10^{-6}$$

$\therefore se(\hat{\theta}) = \sqrt{Var(\hat{\theta})} \approx 0.0023$

$\therefore$ 95% CI for $\hat{\theta}$: $\hat{\theta} \pm 1.96 \cdot se(\hat{\theta}) \approx (0.906, 0.914)$.

Q2.

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta')d\theta'}$$

$$= \frac{\binom{n}{y}\theta^y(1-\theta)^{n-y}\cdot 1}{\int_0^1 \binom{n}{y}\theta'^y(1-\theta')^{n-y}d\theta'}$$

$$= \frac{1}{Z}\theta^y\cdot(1-\theta)^{n-y} \quad \text{where } Z = \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}$$

$\therefore$ $\theta|y \sim \text{Beta}(y+1, n-y+1)$.

$$\therefore E[\theta|y] = \frac{y+1}{n+2} = \frac{118+1}{129+2} \approx 0.91.$$

To obtain the 95% credible interval $(\theta_{0.025}, \theta_{0.975})$ s.t

$p(\theta < \theta_{0.025}|y) = p(\theta > \theta_{0.975}|y) = 0.025$, we use R.

R computes $\theta_{0.025} \approx 0.85$, $\theta_{0.975} \approx 0.95$

$\therefore$ 95% credible interval is $(0.85, 0.95)$.

Q5.

$$p(x|y) = \frac{p(y|x)\,p(x)}{p(y)} \qquad p(x)=1.\ \forall x \in [0,1].$$

$$= \frac{\binom{n}{y} x^{y} (1-x)^{n-y} \cdot 1}{\int_0^1 \binom{n}{y} x'^{y} (1-x')^{n-y} dx'}$$

$$= \frac{1}{Z} \cdot x^{y} (1-x)^{n-y} \quad \text{where } Z = \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}$$

$\therefore\ x|y \sim \text{Beta}(y+1, n-y+1) = \text{Beta}(251528, 241946).$

$\therefore\ p(x \leq 0.5 \mid y) = 1.146058e-42$ (calculated in R).

$\approx 0.$