Math 589 Homework 1 Solution

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Question 1

(a)

$f(h) = \sqrt{(h)} + h$, $g(h) = \sqrt{(h)}$.

Then $\exists h_0 > 0$ is a constant and $C = 1 + \sqrt{h_0}$, as $0 < h < h_0$,

$$\frac{f(h)}{g(h)} = 1 + \sqrt{h} \le C.$$

Then by definition, it satisfies

$$|f(h)| \le C|g(h)|.$$

We conclude that

$$f(h) = O(g(h)) \text{ as } h \to 0.$$

(b)

$f(h) = h \log h + h^2$, $g(h) = h \log(h)$.

Then,

$$\lim_{h \to 0} \frac{f(h)}{g(h)} = \lim_{h \to 0}(\frac{\log h}{\sqrt{h}} + \sqrt{h}) = -\infty.$$

Then by definition, the right hand side is neither bounded by a constant nor a function $\varepsilon(h)$ such that $\lim_{h \to 0} \varepsilon(h) = 0$.

We conclude that $f(h)$ is neither $O(g(h))$ nor $o(g(h))$.

(c)

$f(h) = h \log h + h^2$, $g(h) = h^p$ with $1 < p < 2$.

Then,

$$\lim_{h \to 0} \frac{f(h)}{g(h)} = \lim_{h \to 0}(\frac{\log h}{h^{p-1}} + h^{p-2}) = -\infty.$$

Then by definition, the right hand side is neither bounded by a constant nor a function $\varepsilon(h)$ such that $\lim_{h \to 0} \varepsilon(h) = 0$

We conclude that $f(h)$ is neither $O(g(h))$ nor $o(g(h))$.

Question 2

(a)

By Taylor Theorem

$$u_{i+1} = u_i + (u_x)_i h + \frac{(u_{xx})_i}{2!} h^2 + \frac{(u_{xxx})_i}{3!} h^3 + O(h^4)$$

$$= u_i + (u_x)_i h + \frac{(u_{xx})_i}{2} h^2 + \frac{(u_{xxx})_i}{6} h^3 + O(h^4)$$

$$u_{i+2} = u_i + (u_x)_i (2h) + \frac{(u_{xx})_i}{2!} (2h)^2 + \frac{(u_{xxx})_i}{3!} (2h)^3 + O(h^4)$$

$$= u_i + 2(u_x)_i h + 2(u_{xx})_i h^2 + \frac{4(u_{xxx})_i}{3} h^3 + O(h^4)$$

where $(u_{xx})_i$ is the second derivative of u(x) at the grid point x_i and $(u_{xxx})_i$ is the third derivative of u(x) at the grid point x_i.

Then by computation

$$4u_{i+1} - u_{i+2} = 3u_i + 2(u_x)_i h - \frac{2(u_{xxx})_i}{3} h^3 + O(h^4)$$

Rewrite the equation above, we get

$$(u_x)_i = \frac{4u_{i+1} - u_{i+2} - 3u_i + \frac{2(u_{xxx})_i}{3} h^3 + O(h^4)}{2h}$$

$$= \frac{4u_{i+1} - u_{i+2} - 3u_i}{2h} + \frac{(u_{xxx})_i}{3} h^2 + O(h^3) \qquad (1)$$

$$= \frac{1}{2h}(-3u_i + 4u_{i+1} - u_{i+2}) + O(h^2),$$

Proved.

(b)

Knowing that

$$u(x) = e^{-x} \cos(3x),$$

we can easily compute the first, second and third derivative of $u(x)$ as the following results:

$$u_x = -e^{-x} \cos(3x) - 3e^{-x} \sin(3x)$$
$$u_{xx} = -8e^{-x} \cos(3x) + 6e^{-x} \sin(3x)$$
$$u_{xxx} = 26e^{-x} \cos(3x) + 18e^{-x} \sin(3x)$$

To find the upper bound of u_{xxx}, we can first compute it's first derivative:

$$u_{xxxx} = 28e^{-x} \cos(3x) - 96e^{-x} \sin(3x)$$

and we can easily check that $u_{xxxx} > 0$ as $x = 0$, $u_{xxxx} < 00$ as $x \to \infty$. Then the maximum of u_{xxx} is attained when $U_{XXXX} = 0$. Solving this equation and we get:

$$28e^{-x} \cos(3x) - 96e^{-x} \sin(3x) = 0$$

$$\Rightarrow \tan(3x) = \frac{7}{24}$$

$$\Rightarrow x = \frac{\arctan{(7/24)}}{3} \approx 0.0946.$$

Thus, max $u_{xxx} \approx 27.2922$

Thus, using equation (1) from (a) above, to satisfy the condition statement in question, we get

$$Approx\,Error = (u_x)_i - u'(x) = \frac{(u_{xxx})_i}{3}h^2 + O(h^3) \le 10^{-4}$$

$$\Rightarrow \frac{27.2922}{3}h^2 \le 10^{-4}$$

$$\Rightarrow h \le 3.3154 \times 10^{-3}$$

Question 3

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
```

```
In [2]: # initialization

        J = int(100)
        dx = 1/J
        x = np.array([dx*j for j in range(J+1)])

        U = np.ones(len(x))
        U[0] = 0
        U[-1] = 0
        mu = .498
        # mu = .503
        dt=mu*dx**2
        T = 0.05478
        # T = 0.0503
        nt = int(T/dt)

        # iteration
        for n in range(nt):
            U = U[1:-1]+mu*(U[:-2]-2*U[1:-1]+U[2:])
            U = np.insert(U,0,0)
            U = np.append(U,0)
```
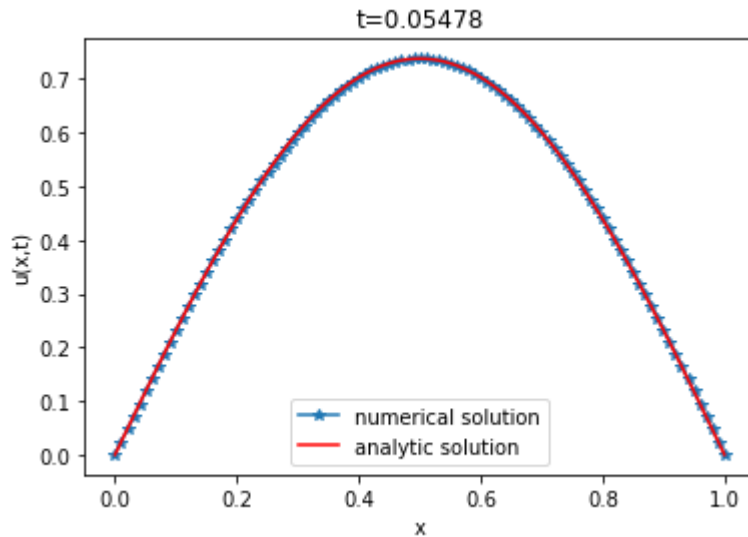
```
In [3]: nterm = 100
        u = np.zeros(len(x))
        for n in range(nterm):
            u += 4/np.pi/(2*n+1)*np.exp(-np.pi**2*T*(2*n+1)**2)*np.sin((2*n+1)*np.pi*x)
```

(a)

Compare the plots between the numerical and analytical results, we see

In [4]:
```python
plt.plot(x, U, '*-', label='numerical solution')
plt.plot(x, u, 'r', label='analytic solution')
plt.legend()
plt.xlabel('x')
plt.ylabel('u(x,t)')
plt.title('t=0.05478')
```

Out[4]: Text(0.5, 1.0, 't=0.05478')

Since the two plots are close to each other, we can conclude that using the parameter settings in question (a), the numerical converges to the analytical solution of the PDE in the statement.

(b)

Compare the plots between the numerical and analytical results, we see

In [5]:
```python
plt.plot(x, U, '*-', label='numerical solution')
plt.plot(x, u, 'r', label='analytic solution')
plt.legend()
plt.xlabel('x')
plt.ylabel('u(x,t)')
plt.title('t=0.0503')
```

Out[5]: Text(0.5, 1.0, 't=0.0503')
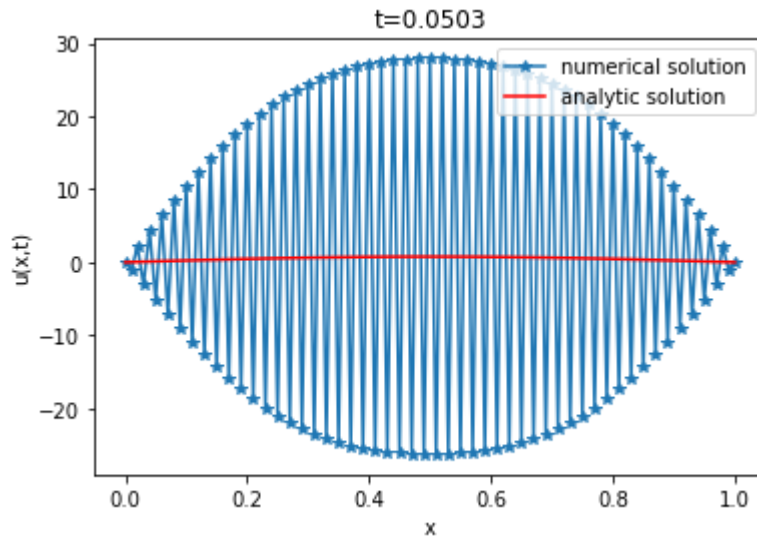
The plots above shows that there is a dramatic difference between the numerical and analytical results when using the parameter settings in question (b). And we can see that the numerical result is oscillating, which indicates it's violating the stability condition.