Math 589 Homework2 Solution

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Question 1

Find the local trunction error of the difference equation
$$U_j^{n+1} = U_j^n - R(U_j^n - U_{j-1}^n) \tag{1}$$
for the PDE
$$u_t + au_x = 0, \quad 0 \le x \le 1, t > 0, \tag{2}$$
where $R = a\Delta t/\Delta x$ and $a > 0$ is a constant.

proof: By the definition of truncation error at grid point (x_j, t_n), denoted as T_j^n:
$$T_j^n := \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t} + \frac{a(u(x_j, t_n) - u(x_{j-1}, t_n))}{\Delta x}. \tag{3}$$

Then using Taylor expansion, we have the following two equations:
$$u(x_j, t_{n+1}) = u(x_j, t_n + \Delta t) = u(x_j, t_n) + u_t(x_j, t_n)\Delta t + \frac{1}{2}u_{tt}(x_j, \eta)(\Delta t)^2, \quad \eta \in (t_n, t_{n+1}).$$
$$u(x_{j-1}, t_n) = u(x_j - \Delta x, t_n) = u(x_j, t_n) - u_x(x_j, t_n)\Delta x + \frac{1}{2}u_{xx}(\xi, t_n)(\Delta x)^2, \quad \xi \in (x_{j-1}, x_j).$$

Plug the two equations above into truncation error equation (3), we get
$$\begin{aligned}
T_j^n &= u_t(x_j, t_n) + \frac{1}{2}u_{tt}(x_j, \eta)\Delta t + \frac{a}{\Delta x}\left(u_x(x_j, t_n)\Delta x - \frac{1}{2}u_{xx}(\xi, t_n)(\Delta x)^2\right) \\
&= u_t(x_j, t_n) + au_x(x_j, t_n) + \frac{1}{2}u_{tt}(x_j, \eta)\Delta t - \frac{a\Delta x}{2}u_{xx}(\xi, t_n) \\
&= \frac{1}{2}u_{tt}(x_j, \eta)\Delta t - \frac{a\Delta x}{2}u_{xx}(\xi, t_n) \quad \text{since (2)} \\
&= O(\Delta t) + O(\Delta x).
\end{aligned}$$

Question 2

proof: Suppose:
$$U_j^n = u(x_j, t_n) + e_j^n.$$
By definition, FTBS scheme converges if $e_j^n \to 0$ as $(\Delta x, \Delta t) \to 0$.

Plug the equation above into equation (1), we get
$$\begin{aligned}
&u(x_j, t_n) + e_j^n = u(x_j, t_{n+1}) + e_j^{n+1} + R\left(u(x_j, t_n) + e_j^n - u(x_{j-1}, t_n) - e_{j-1}^n\right) \\
&\Rightarrow e_j^{n+1} = (1 - R)e_j^n + Re_{j-1}^n - \left(u(x_j, t_{n+1}) - u(x_j, t_n) + R\left(u(x_j, t_n) - u(x_{j-1}, t_n)\right)\right) \\
&\Rightarrow e_j^{n+1} = (1 - R)e_j^n + Re_{j-1}^n - T_j^n\Delta t.
\end{aligned}$$

From the results of question 1, we know that

$$
\begin{aligned}
|T_j^n| &\le \frac{1}{2}M_{tt}\Delta t + \frac{a\Delta x}{2}M_{xx} \\
&= \left(\frac{1}{2}M_{tt} + \frac{a\Delta x}{2\Delta t}M_{xx}\right)\Delta t \\
&= \left(\frac{1}{2}M_{tt} + \frac{a^2}{2R}M_{xx}\right)\Delta t \\
&\le M\Delta t.
\end{aligned}
$$

where
$M_{tt} := \max_{0\le x\le 1, t>0} |u_{tt}(x,t)|$, $M_{xx} := \max_{0\le x\le 1, t>0} |u_{xx}(x,t)|$, $M := \frac{1}{2}M_{tt} + \frac{a^2}{2R}M_{xx} > 0$ is a constant.

Additionally, suppose $E^n := \max_j |e_j^n|$ and knowing that $R \in (0,1], 1-R \in [0,1)$. Then

$$
\begin{aligned}
|e_j^{n+1}| &\le E^{n+1} \le (1-R)E^n + RE^n + M(\Delta t)^2 \\
&\le E^{n-1} + 2M(\Delta t)^2 \le \cdots \le E^0 + (n+1)M(\Delta t)^2 \\
\Rightarrow \lim_{n\to\infty} E^n &\le \lim_{n\to\infty}(E^0 + nM(\Delta t)^2) \le \lim_{n\to\infty} t_F M\Delta t \to 0 \quad \text{as } (\Delta x, \Delta t) \to 0
\end{aligned}
$$

where t_F is the terminal time of FTBS scheme and also bounded. Thus convergence proved.

Question 3

proof: Assume the solution has the form $U_j^n = \lambda^n e^{ik(j\Delta x)}$.

Plug the solution above into FTBS scheme in equation (1):

$$
\begin{aligned}
&\lambda^{n+1}e^{ik(j\Delta x)} = \lambda^n e^{ik(j\Delta x)} - R(\lambda^n e^{ik(j\Delta x)} - \lambda^n e^{ik(j-1)\Delta x}) \\
&\Rightarrow \lambda = 1 - R + R\cos(k\Delta x) - iR\sin(k\Delta x) \\
&\Rightarrow |\lambda| = (1 - R + R\cos(k\Delta x))^2 + R\sin^2(k\Delta x) \\
&\Rightarrow |\lambda| = (2R^2 - 2R)(1 - \cos(k\Delta x)) + 1.
\end{aligned}
$$

To satisfy the stability condition, we have

$$
\begin{aligned}
&|\lambda| \le 1 \\
&\Rightarrow (2R^2 - 2R)(1 - \cos(k\Delta x)) \le 0 \\
&\Rightarrow 0 \le R \le 1
\end{aligned}
$$

Since by definition, $R := a\Delta t/\Delta x > 0$, the necessary stability condition of the FTBS scheme (1) is $R \in (0, 1]$.