Math 623 Homework 3 Yuan Yin

Problem 1 American options

1. According to the definition, exercise time is the first time that $V(S_\tau, \tau) = (K - S_\tau)^+$. Since knowing that $V(S,t) \geq (K-S)^+ \ \forall t \in [0,T]$. It means, Only when S_t is decreasing to a certain price, the option holder will exercise the put option at time t.

Since it's a put option, holders believe the price will be low in the future. So at the very beginning, only if S_t is low enough then the holders may exercise the option. That's why $Ex(t)$ is low at the beginning. When time goes by, holder' expection will based on current price S_t rather than initial price. So, S_t is closer to maturity time T and ~~thes~~ thus the expection range will be smaller. holders will not be that hopeful to believe price will change a lot in the future. Thus S_t is not that low but low enough and the holders may exercise the option. And when $t \to T$, as long as $S_t < K$, then $(K - S_t)^+ > 0$, the holders will gain money from this put option. That why $\lim_{t \to T} Ex(t) = K$

2. $u(x,t) := V(e^x, t)$.

$$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial x} \cdot e^x, \quad \frac{\partial u}{\partial t} = \frac{\partial V}{\partial t}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 V}{\partial x^2} e^{2x} + \frac{\partial V}{\partial x} \cdot e^x$$

replace $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}$ with $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x^2}$ in equation (1).

$$\Rightarrow \frac{\partial u}{\partial t} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - ru = 0$$ for all (e^x, t) satisfying $u(e^x,t) > (K - e^x)^+$

$\Rightarrow$ for $u(e^x, t) > (K - e^x)^+$,

$$0 = \frac{u(x,t) - u(x, t-\Delta t)}{\Delta t} + \frac{u(x+\Delta x, t) - u(x-\Delta x, t)}{2\Delta x}\left(r - \frac{1}{2}\sigma^2\right)$$

$$+ \frac{1}{2}\sigma^2 \cdot \frac{u(x+\Delta x, t) + u(x-\Delta x, t) - 2u(x,t)}{\Delta x^2} - ru(x,t).$$

$$\Rightarrow u(x,t-\Delta t) = (\alpha - r\Delta t)\,u(x,t) + \alpha^+ u(x+\Delta x,t) + \alpha^- u(x-\Delta x,t)$$

where $\alpha = 1 - \sigma^2 \frac{\Delta t}{\Delta x^2}$.

$$\alpha^{\pm} = \sigma^2 \frac{\Delta t}{2\Delta x^2} \pm (r - \tfrac{1}{2}\sigma^2)\frac{\Delta t}{2\Delta x}.$$

or else if $(K-e^x)^+ \geq u(e^x,t)$. we will choose

$u(e^x,t) = (K-e^x)^+$

In conclusion,

$$u(x,t-\Delta t) = \max\{K - e^x,\ (\alpha - r\Delta t)\,u(x,t) + \alpha^+ u(x+\Delta x,t) + \alpha^- u(x-\Delta x,t)\}$$

where α, $\alpha^{\pm}$ is as above.

3.

$$u(x,t) = e^{-r(T-t)} E[(K-S_T)^+ \mid S_t = e^x]$$

$$u(a,t) = e^{-r(T-t)} E[(K-S_T)^+ \mid S_t = e^a]$$

$$u(b,t) = e^{-r(T-t)} E[(K-S_T)^+ \mid S_t = e^b]$$

if $a \to -\infty$, $S_t \to 0$. $\mathbb{P}[S_T < K] \approx 1$.

$$u(a,t) \approx e^{-r(T-t)} E[(K-S_T) \mid S_t = e^a]$$

$$= e^{rt}\cdot\left(\frac{K}{e^{rT}} - \frac{S_t}{e^{rt}}\right)$$

$$= K\cdot e^{-r(T-t)} - e^a$$

We have $P[K < S_T \mid S_t = e^a] \approx 0$. , In order to achieve this.

BS model

$$\Rightarrow \mathbb{P}\left[W_T - W_t > \frac{\log K - a - (r - \frac{\sigma^2}{2})(T-t)}{\sigma}\right] \approx 0.$$

$$\text{LHS} \leq \mathbb{P}\left[\xi > \frac{\log K - a - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right] \quad \text{where } \xi \sim N(0,1).$$

Thus, if we choose $\frac{\log K - a - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = 3$ i.e.

$a = \log K - 3\sigma\sqrt{T} - (r - \frac{\sigma^2}{2})T$, Then $P[\xi > 3] \approx 0.003 \to 0. \Rightarrow \text{LHS} \approx 0$

2.

With the same reason.

If $b \to +\infty$, $S_t \to +\infty$ $\mathbb{P}[K < S_T] \approx 1$.

$$u(b,t) \approx e^{-r(T-t)} E[0 \mid S_t = e^b] = 0$$

We have $\mathbb{P}[K > S_T \mid S_t = e^b] \approx 0$. In order to achieve this,

$$\xRightarrow{\text{BS model}} \mathbb{P}\left[W_T - W_t < \frac{\log K - b - (r - \frac{\sigma^2}{2})(T-t)}{\sigma}\right] \approx 0.$$

$$LHS = \mathbb{P}\left[\xi < \frac{\log K - b - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right] \text{ where } \xi \sim N(0,1).$$

we choose $b = \log K + 3\sigma\sqrt{T} + |r - \frac{\sigma^2}{2}|T$ so that

$$\frac{\log K - b - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \leq -3.$$

$$\Rightarrow LHS \leq \mathbb{P}[\xi < -3] = 0.003 \to 0 \Rightarrow LHS \approx 0.$$

In conclusion, $a = \log K - 3\sigma\sqrt{T} - (r - \frac{\sigma^2}{2})T$

$u(a,t) = Ke^{-r(T-t)} - e^a$.

$b = \log K + 3\sigma\sqrt{T} + |r - \frac{\sigma^2}{2}|T$

$u(b,t) = 0$.

4. & 5. (See the code ~~behind~~ on the last page).

Problem 2. Barrier Option.

1. $a = \log L$. $u(x,t) = P(e^x, t)$

$$\frac{\partial u}{\partial x} = \frac{\partial P}{\partial x} e^x, \quad \frac{\partial u}{\partial t} = \frac{\partial P}{\partial t}, \quad \frac{\partial^2 P}{\partial x^2} = e^{-2x}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} + (r - \frac{1}{2}\sigma^2)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - ru = 0.$$

Terminal & Boundary conditions:

$$u(a,t) = P(L,t) = 0.$$

$$u(x,T) = P(e^x, T) = (S_T - K)^+$$

2. $$u(x,t) = e^{-r(T-t)} E[(S_T-K)^+ \mid S_t = e^x].$$

$$u(b,t) = e^{-r(T-t)} E[(S_T-K)^+ \mid S_t = e^b]$$

if $b \to +\infty$, $S_t \to +\infty$. $\mathbb{P}[S_T > K] \approx 1$.

$$u(b,t) \approx e^{-r(T-t)} E[S_T - K \mid S_t = e^b]$$

$$= e^b - K \cdot e^{-r(T-t)}$$

In order to achieve $\mathbb{P}[S_T < K \mid S_t = e^b] \approx 0$.

$$\xrightarrow{\text{BS model}} \mathbb{P}\left[W_T - W_t < \frac{\log K - b - (r - \frac{\sigma^2}{2})(T-t)}{\sigma}\right] \approx 0.$$

$$LHS = \mathbb{P}\left[\xi < \frac{\log K - b - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right], \quad \xi \sim N(0,1)$$

we choose $b = \log K + 3\sigma\sqrt{T} + |r - \frac{\sigma^2}{2}| \cdot T$ so that

$$\frac{\log K - b - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \le -3$$

$$\Rightarrow LHS \le \mathbb{P}[\xi \le -3] = 0.003 \to 0. \quad \Rightarrow LHS \approx 0.$$

In conclusion, $$b = \log K + 3\sigma\sqrt{T} + |r - \frac{\sigma^2}{2}| T$$

$$u(b,t) = e^b - K \cdot e^{-r(T-t)}$$

3. & 4. (see the code on the last page).

Problem 3.

1. $e_0^m = u_0^m - u(x_0, t_m) = 0 - u(a, \frac{m}{M}T) = 0 - 0 = 0$

$e_N^m = u_N^m - u(x_N, t_m) = 0 - u(b, \frac{m}{M}T) = 0 - 0 = 0$

$e_n^M = u_n^M - u(x_n, t_M) = g(x_n) - u(x_n, T) = g(x_n) - g(x_n) = 0.$

2. $h = \frac{\Delta t}{2\Delta x^2} = \frac{u_n^{m-1} - u_n^m}{u_n^m + u_{n-1}^m - 2u_n^m}$

$$\begin{aligned}
Sta_n^{m-1} + Con_n^{m-1} &= h(u_{n-1}^m - u(x_{n-1}, t_m)) + h\,u(x_{n-1}, t_m) \\
&\quad + (1-2h)(u_n^m - u(x_n, t_m)) + (1-2h)\,u(x_n, t_m) \\
&\quad + h(u_{n+1}^m - u(x_{n+1}, t_m)) + h\,u(x_{n+1}, t_m) \\
&\quad - u(x_n, t_{m-1}). \\
&= u_n^m + h(u_{n-1}^m + u_{n+1}^m - 2u_n^m) - u(x_n, t_{m-1}). \\
&= u_n^{m-1} - u(x_n, t_{m-1}) = e_n^{m-1} \quad \blacksquare
\end{aligned}$$

3. Stability condition:

$$\begin{cases} \alpha \geq 0 \\ \alpha^{\pm} \geq 0 \end{cases}$$

compare $u(x, t-\Delta t) = \alpha u(x,t) + \alpha_+ u(x+\Delta x, t) + \alpha_- u(x-\Delta x, t)$

where $\alpha = 1 - \frac{\Delta t}{\Delta x^2}$

$\alpha_{\pm} = \frac{\Delta t}{2\Delta x^2}$

Then stability condition: $\begin{cases} 1 \geq \frac{\Delta t}{\Delta x^2} \\ \frac{\Delta t}{2\Delta x^2} \geq 0 \end{cases} \Rightarrow \underline{\Delta t \leq \Delta x^2}$

4. According to Taylor's expansion.

4.

$$u(x_{n-1},t_m) = u(x_n,t_{m-1}) + \Delta t\cdot\frac{\partial u}{\partial t} - \Delta x\frac{\partial u}{\partial x} + \frac{1}{2}\Delta x^2\cdot\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\cdot\Delta t^2 - \Delta x\,\Delta t\cdot\frac{\partial^2 u}{\partial x\partial t} + \frac{1}{2}\frac{\partial^3 u}{\partial x^2\partial t}(\Delta x)^2\Delta t + O(\Delta x^2\Delta t)$$

$$u(x_{n+1},t_m) = u(x_n,t_{m-1}) + \Delta x\cdot\frac{\partial u}{\partial x} + \Delta t\frac{\partial u}{\partial t} + \frac{1}{2}(\Delta x)^2\cdot\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\cdot\Delta t^2 + \Delta x\cdot\Delta t\cdot\frac{\partial u^2}{\partial x\partial t} + \frac{1}{2}\frac{\partial^3 u}{\partial x^2\partial t}(\Delta x)^2\Delta t + O(\Delta x^2\Delta t)$$

$$u(x_n,t_m) = u(x_n,t_{m-1}) + \Delta t\cdot\frac{\partial u}{\partial t} + \frac{1}{2}(\Delta t)^2\cdot\frac{\partial^2 u}{\partial t^2} + O(\Delta t^2)$$

$$\Rightarrow Con_n^{m-1} = h(u(x_{n-1},t_m) + u(x_{n+1},t_m)) + (1-2h)u(x_n,t_m) - u(x_n,t_{m-1})$$

$$= h\left(2u(x_n,t_{m-1}) + 2\Delta t\cdot\frac{\partial u}{\partial t} + \Delta x^2\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^2\partial t}(\Delta x)^2\cdot\Delta t + \frac{\partial^2 u}{\partial t^2}\cdot\Delta t^2 + 2O(\Delta x^2\Delta t)\right) + (1-2h)u(x_n,t_m) - u(x_n,t_{m-1})$$

$$= \Delta t\cdot\frac{\partial u}{\partial t} + h\cdot\Delta x^2\cdot\frac{\partial^2 u}{\partial x^2} + h(\Delta x)^2\cdot\Delta t\cdot\frac{\partial^3 u}{\partial x^2\partial t} + 2hO(\Delta x^2\Delta t) + \frac{1}{2}\cancel{(1-2h)}\cdot\Delta t^2\cdot\frac{\partial^2 u}{\partial t^2} + (1-2h)O(\Delta t^2)$$

since $h = \frac{\Delta t}{2\Delta x^2}$. $\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial^2 u}{\partial x^2}$

$$Con_n^{m-1} = \frac{1}{2}\Delta t^2\cdot\frac{\partial^3 u}{\partial x^2\partial t} + \frac{1}{2}\Delta t^2\cdot\frac{\partial^2 u}{\partial t^2} + 2hO(\Delta x^2\Delta t) + (1-2h)O(\Delta t^2)$$

$$= 2hO(\Delta x^2\Delta t) + (1-2h)O(\Delta t^2)$$

According to 3. $h \le \frac{1}{2}$.

Thus $|Con_n^{m-1}| \le A(\Delta t^2 + \Delta t\Delta x^2)$ for some constant A.

5. Since $0 \le h \le \frac{1}{2} \Rightarrow 1-2h \ge 0$.

$$\|e_n^{m-1}\| = \|Sta_n^{m-1} + Con_n^{m-1}\|$$
$$\le \|Sta_n^{m-1}\| + \|Con_n^{m-1}\|$$
$$\le |h|\,\|e_{n-1}^m\| + |1-2h|\,\|e_n^m\| + |h|\,\|e_{n+1}^m\| + O(\Delta t^2 + \Delta t\Delta x^2)$$

$$\Rightarrow \|e^{m-1}\| \le \|e^m\| + O(\Delta t^2 + \Delta t\Delta x^2)$$

By induction,

$$\|e^{m-1}\| \le \underbrace{\|e^M\|}_{=0} + (M+1-m)A(\Delta t^2 + \Delta t\Delta x^2)$$
$$\le AT(\Delta t + \Delta x^2) \qquad (\because \Delta t = \tfrac{T}{M})$$

MATH 623 Homework 3

Yuan Yin

November 8, 2018

Problem 1

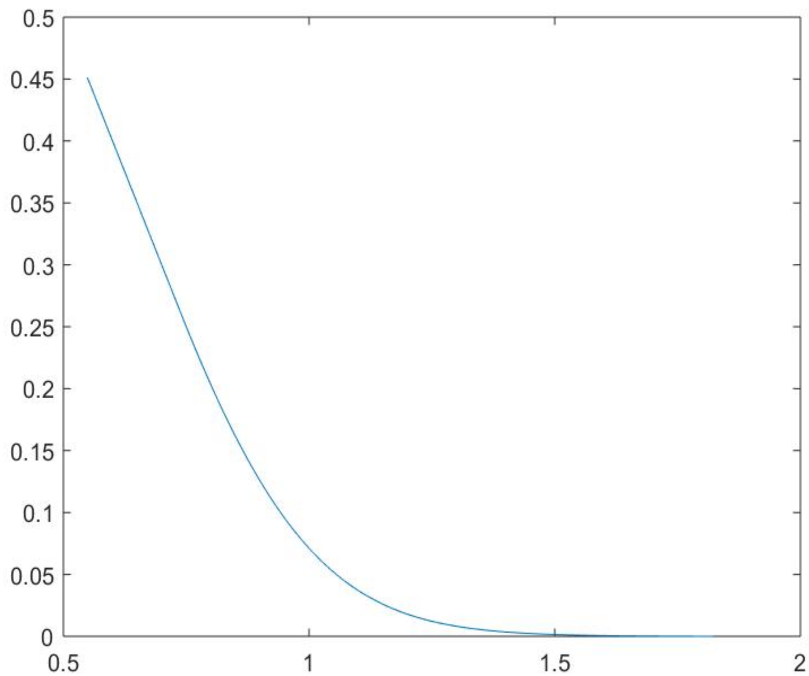
4. & 5.

The main code:

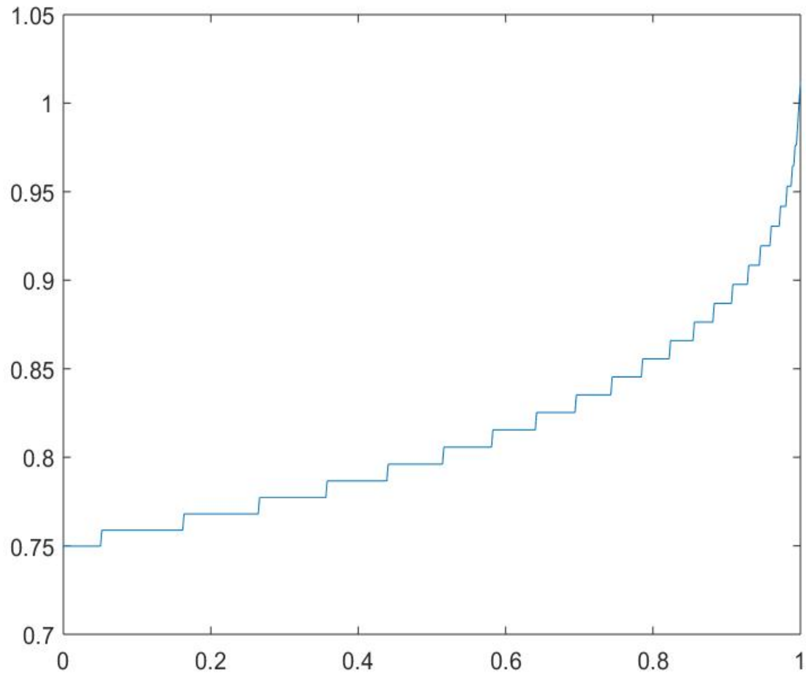
```
r = 0.02; sigma = 0.2; K = 1; T = 1; N = 100;
a = log(K) - 3 * sigma * T^0.5 - (r - sigma^2 / 2) * T;
b = log(K) + 3 * sigma * T^0.5 + abs(r - sigma^2 / 2) * T;
M = 556; dt = T / M; dx = (b - a) / N; u = zeros(M + 1, N + 1);
alpha0 = 1 - sigma^2 * dt / dx^2;
alpha1 = sigma^2 * dt / (2 * dx^2) + (r - sigma^2 / 2) * dt / dx / 2;
alpha2 = sigma^2 * dt / (2 * dx^2) - (r - sigma^2 / 2) * dt / dx / 2;
for i = 1 : (N + 1)
   u(1, i) = max(0, K - exp(a + (i - 1) * dx));
end
for i = 1 : (M + 1)
   u(i, 1) = K * exp(- r * (T - dt * (i - 1))) - exp(a);
   u(i, N + 1) = 0;
end
for i = 2 : (M + 1)
   for j = 2 : N
      u(i, j) = max(K - exp(a + dx * (j - 1)), (alpha0 - r * dt) * u(i - 1, j) + alpha1 *
          u(i - 1, j + 1) + alpha2 * u(i - 1, j - 1));
   end
end
x = a: dx : b; s = exp(x);
figure(1)
plot(s, u(M + 1, :))
Ex = [];
for i = (M + 1) : -1 : 1
   p = [];
   for j = 1 : (N + 1)
      if (u(i, j) > (K - exp(a + dx * (j - 1))))
```

```
            p = [p, exp(a + dx * (j - 1))];
        end
    end
    Ex = [Ex, min(p)];
end
t = 0 : dt : T;
figure(2)
plot(t, Ex)
```

the plot of $V(S, 0)$ is as below:



the plot of $(Ex_m)_{0 \leq m \leq M}$ is as below:



Problem 2

3. & 4.

The main code:

```
r = 0.02; sigma = 0.2; K = 1; T = 1; L = 0.5; N = 100; M = 556;
a = log(L); b = log(K) + 3 * sigma * T^0.5 + abs(r - sigma^2 / 2) * T;
dt = T / M; dx = (b - a) / N;
alpha0 = 1 - sigma^2 * dt / dx^2;
alpha1 = sigma^2 * dt / (2 * dx^2) + (r - sigma^2 / 2) * dt / dx / 2;
alpha2 = sigma^2 * dt / (2 * dx^2) - (r - sigma^2 / 2) * dt / dx / 2;
for i = 1 : (N + 1)
   u(1, i) = max(0, exp(a + (i - 1) * dx) - K);
end
for i = 1 : (M + 1)
   u(i, N + 1) = exp(b) - K * exp(- r * (T - dt * (i - 1)));
   u(i, 1) = 0;
end
for i = 2 : (M + 1)
   for j = 2 : N
      u(i, j) = (alpha0 - r * dt) * u(i - 1, j) + alpha1 * u(i - 1, j + 1) + alpha2 * u(i -
          1, j - 1);
```

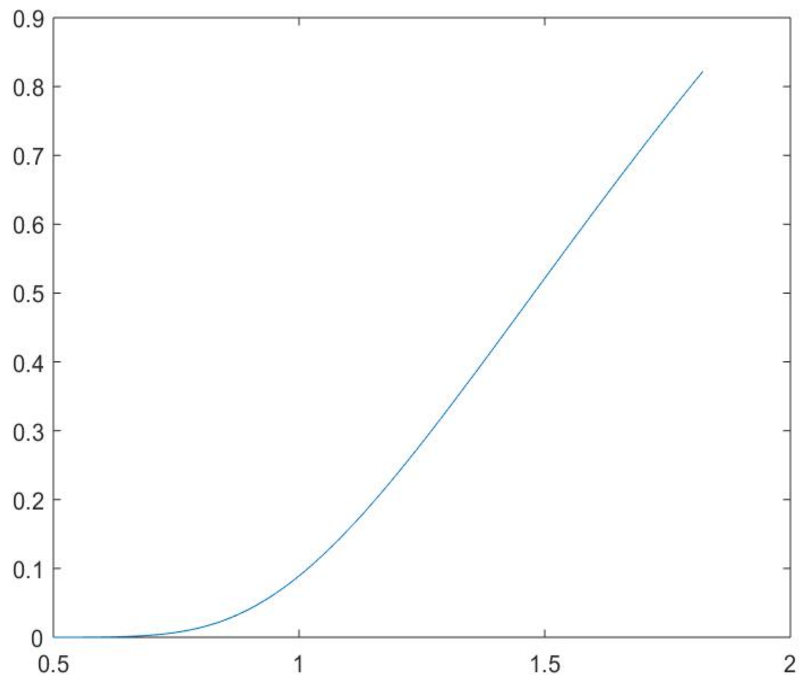
```
    end
end
x = a: dx : b; s = exp(x);
figure(1)
plot(s, u(M + 1, :))

[call_1,put_1] = blsprice(1, K, r, T, sigma);
[call_L2, put_L2] = blsprice(L^2, K, r, T, sigma);
real_p = call_1 - L^(2 * r / sigma^2 - 1) * call_L2;
m = 1056 : -1 : 556;
delta_t = []; error = [];
for i = 1 : length(m)
    delta_t = [delta_t, T / m(i)];
    est_p = est_of_p(T / m(i));
    error = [error, abs(real_p - est_p)];
end
figure(2)
plot(delta_t, error)
```

and the function code:

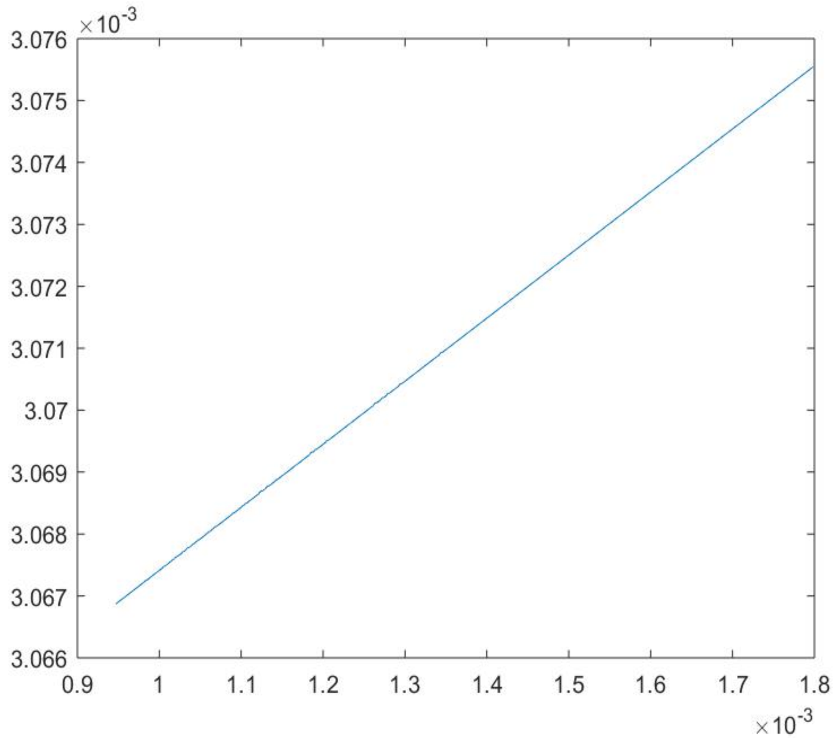
```
function y = est_of_p(x)
r = 0.02; sigma = 0.2; K = 1; T = 1; L = 0.5; N = 100; M = fix(T / x);
a = log(L); b = log(K) + 3 * sigma * T^0.5 + abs(r - sigma^2 / 2) * T;
dt = T / M; dx = (b - a) / N;
alpha0 = 1 - sigma^2 * dt / dx^2;
alpha1 = sigma^2 * dt / (2 * dx^2) + (r - sigma^2 / 2) * dt / dx / 2;
alpha2 = sigma^2 * dt / (2 * dx^2) - (r - sigma^2 / 2) * dt / dx / 2;
for i = 1 : (N + 1)
    u(1, i) = max(0, exp(a + (i - 1) * dx) - K);
end
for i = 1 : (M + 1)
    u(i, N + 1) = exp(b) - K * exp(- r * (T - dt * (i - 1)));
    u(i, 1) = 0;
end
for i = 2 : (M + 1)
    for j = 2 : N
        u(i, j) = (alpha0 - r * dt) * u(i - 1, j) + alpha1 * u(i - 1, j + 1) + alpha2 * u(i -
            1, j - 1);
    end
end
for i = 1 : (N + 1)
    if ((a + dx * (i - 1) < 0) & (a + dx * i >= 0))
        q = i;
```

```
    end
end
y = u(M + 1, q + 1);
end
```

the plot of $V(S, 0)$ is as below:



the plot of error is as below:



×10^-3
3.076
3.075
3.074
3.073
3.072
3.071
3.07
3.069
3.068
3.067
3.066
0.9
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
×10^-3