Math 623 Homework 3 Yuan Yin

Problem 1 American options

1. According to the definition, exercise time is the first time that $V(S_\tau, \tau) = (K - S_\tau)^+$. Since knowing that $V(S,t) \geq (K-S)^+ \ \forall t \in [0,T]$. It means, Only when $S_t$ is decreasing to a certain price, the option holder will exercise the put option at time $t$.

Since it's a put option, holders believe the price will be low in the future. So at the very beginning, only if $S_t$ is low enough then the holders may exercise the option. That's why $Ex(t)$ is low at the beginning. When time goes by, holder' expection will based on current price $S_t$ rather than initial price. So, $S_t$ is closer to maturity time $T$ and ~~thes~~ thus the expection range will be smaller. holders will not be that hopeful to believe ~~&~~ price will change a lot in the future. Thus $S_t$ is not that low but low enough and the holders may exercise the option. And when $t \to T$, as long as $S_t < K$, then $(K - S_t)^+ > 0$, the holders will gain money from this put option. That why $\lim_{t \to T} Ex(t) = K$

2. $u(x,t) := V(e^x, t)$.

$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial x} \cdot e^x$, $\frac{\partial u}{\partial t} = \frac{\partial V}{\partial t}$, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 V}{\partial x^2} e^{2x} + \frac{\partial V}{\partial x} \cdot e^x$

replace $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}$ with $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x^2}$ in equation (1).

$$\Rightarrow \frac{\partial u}{\partial t} + (r - \tfrac{1}{2}\sigma^2)\frac{\partial u}{\partial x} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - ru = 0$$ for all $(e^x, t)$ satisfying $u(e^x,t) > (K - e^x)^+$

$\Rightarrow$ for $u(e^x,t) > (K-e^x)^+$,

$$0 = \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} + \frac{u(x+\Delta x, t) - u(x-\Delta x, t)}{2\Delta x}(r - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2}\sigma^2 \cdot \frac{u(x+\Delta x,t) + u(x-\Delta x,t) - 2u(x,t)}{\Delta x^2} - ru(x,t).$$

$$\Rightarrow u(x,t-\Delta t) = (\alpha - r\Delta t)\,u(x,t) + \alpha^+ u(x+\Delta x,t) + \alpha^- u(x-\Delta x,t)$$

where $\alpha = 1-\sigma^2 \frac{\Delta t}{\Delta x^2}$.

$$\alpha^{\pm} = \sigma^2 \frac{\Delta t}{2\Delta x^2} \pm (r-\tfrac{1}{2}\sigma^2)\frac{\Delta t}{2\Delta x}.$$

or else if $(K-e^x)^+ \geq u(e^x,t)$. we will choose

$u(e^x,t) = (K-e^x)^+$

In conclusion,

$$u(x,t-\Delta t) = \max\{K-e^x,\ (\alpha-r\Delta t)u(x,t) + \alpha^+ u(x+\Delta x,t) + \alpha^- u(x-\Delta x,t)\}$$

where $\alpha$, $\alpha^{\pm}$ is as above.

3.

$$u(x,t) = e^{-r(T-t)} E[(K-S_T)^+ \mid S_t = e^x]$$

$$u(a,t) = e^{-r(T-t)} E[(K-S_T)^+ \mid S_t = e^a]$$

$$u(b,t) = e^{-r(T-t)} E[(K-S_T)^+ \mid S_t = e^b]$$

if $a \to -\infty$, $S_t \to 0$. $\mathbb{P}[S_T < K] \approx 1$.

$$u(a,t) \approx e^{-r(T-t)} E[(K-S_T) \mid S_t = e^a]$$

$$= e^{rt}\cdot\left(\frac{K}{e^{rT}} - \frac{S_t}{e^{rt}}\right)$$

$$= K\cdot e^{-r(T-t)} - e^a$$

We have $P[K < S_T \mid S_t = e^a] \approx 0$. , In order to achieve this.

BS model

$$\Rightarrow \mathbb{P}\left[W_T - W_t > \frac{\log K - a - (r-\frac{\sigma^2}{2})(T-t)}{\sigma}\right] \approx 0.$$

$$\text{LHS} \leq \mathbb{P}\left[\xi > \frac{\log K - a - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right] \quad \text{where } \xi \sim N(0,1).$$

Thus, if we choose $\frac{\log K - a - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = 3$ i.e.

$a = \log K - 3\sigma\sqrt{T} - (r-\frac{\sigma^2}{2})T$ , Then $P[\xi > 3] \approx 0.003 \to 0. \Rightarrow \text{LHS} \approx 0$

2.

With the same reason.

If $b \to +\infty$, $S_t \to +\infty$ $\mathbb{P}[K < S_T] \approx 1$.

$$u(b,t) \approx e^{-r(T-t)} E[0 \mid S_t = e^b] = 0$$

We have $\mathbb{P}[K > S_T \mid S_t = e^b] \approx 0$. In order to achieve this,

$$\xRightarrow{\text{BS model}} \mathbb{P}\left[W_T - W_t < \frac{\log K - b - (r - \frac{\sigma^2}{2})(T-t)}{\sigma}\right] \approx 0.$$

$$LHS = \mathbb{P}\left[\xi < \frac{\log K - b - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right] \quad \text{where } \xi \sim N(0,1).$$

we choose $b = \log K + 3\sigma\sqrt{T} + |r - \frac{\sigma^2}{2}|T$ so that

$$\frac{\log K - b - (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \le -3.$$

$$\Rightarrow LHS \le \mathbb{P}[\xi < -3] = 0.003 \to 0 \Rightarrow LHS \approx 0.$$

In conclusion, $a = \log K - 3\sigma\sqrt{T} - (r - \frac{\sigma^2}{2})T$

$u(a,t) = Ke^{-r(T-t)} - e^a$.

$b = \log K + 3\sigma\sqrt{T} + |r - \frac{\sigma^2}{2}|T$

$u(b,t) = 0$.

4. & 5. (See the code ~~behind~~ on the last page).

Problem 2. Barrier Option.

1. $a = \log L$. $u(x,t) = P(e^x, t)$

$$\frac{\partial u}{\partial x} = \frac{\partial P}{\partial x} e^x, \quad \frac{\partial u}{\partial t} = \frac{\partial P}{\partial t}, \quad \frac{\partial^2 P}{\partial x^2} = e^{-2x}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} + (r - \tfrac{1}{2}\sigma^2)\frac{\partial u}{\partial x} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - ru = 0.$$

Terminal & Boundary conditions:

$$u(a,t) = P(L,t) = 0.$$

$$u(x,T) = P(e^x,T) = (S_T - K)^+$$

2. $u(x,t) = e^{-r(T-t)} E[(S_T-K)^+ \mid S_t = e^x]$.

$$u(b,t) = e^{-r(T-t)} E[(S_T-K)^+ \mid S_t = e^b]$$

if $b \to +\infty$, $S_t \to +\infty$. $\mathbb{P}[S_T > K] \approx 1$.

$$u(b,t) \approx e^{-r(T-t)} E[S_T - K \mid S_t = e^b]$$

$$= e^b - K \cdot e^{-r(T-t)}$$

In order to achieve $\mathbb{P}[S_T < K \mid S_t = e^b] \approx 0$.

$$\xrightarrow{\text{BS model}} \mathbb{P}\left[W_T - W_t < \frac{\log K - b - (r - \frac{\sigma^2}{2})(T-t)}{\sigma}\right] \approx 0.$$

$$LHS = \mathbb{P}\left[\xi < \frac{\log K - b - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right], \quad \xi \sim N(0,1)$$

we choose $b = \log K + 3\sigma\sqrt{T} + |r - \frac{\sigma^2}{2}| \cdot T$ so that

$$\frac{\log K - b - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \le -3$$

$$\Rightarrow LHS \le \mathbb{P}[\xi \le -3] = 0.003 \to 0. \quad \Rightarrow LHS \approx 0.$$

In conclusion, $b = \log K + 3\sigma\sqrt{T} + |r - \frac{\sigma^2}{2}| T$

$$u(b,t) = e^b - K \cdot e^{-r(T-t)}$$

3. & 4. (see the code on the last page).

Problem 3.

1. $$e_0^m = u_0^m - u(x_0, t_m) = 0 - u(a, \tfrac{m}{M}T) = 0 - 0 = 0$$

$$e_N^m = u_N^m - u(x_N, t_m) = 0 - u(b, \tfrac{m}{M}T) = 0 - 0 = 0$$

$$e_n^M = u_n^M - u(x_n, t_M) = g(x_n) - u(x_n, T) = g(x_n) - g(x_n) = 0.$$

2. $$h = \frac{\Delta t}{2\Delta x^2} = \frac{u_n^{m-1} - u_n^m}{u_n^m + u_{n-1}^m - 2u_n^m}$$

$$\begin{aligned}
Sta_n^{m-1} + Con_n^{m-1} &= h(u_{n-1}^m - u(x_{n-1}, t_m)) + h\,u(x_{n-1}, t_m) \\
&\quad + (1-2h)(u_n^m - u(x_n, t_m)) + (1-2h)\,u(x_n, t_m) \\
&\quad + h(u_{n+1}^m - u(x_{n+1}, t_m)) + h\,u(x_{n+1}, t_m) \\
&\quad - u(x_n, t_{m-1}). \\
&= u_n^m + h(u_{n-1}^m + u_{n+1}^m - 2u_n^m) - u(x_n, t_{m-1}). \\
&= u_n^{m-1} - u(x_n, t_{m-1}) = e_n^{m-1} \quad \blacksquare
\end{aligned}$$

3. Stability condition:

$$\begin{cases} \alpha \geq 0 \\ \alpha^{\pm} \geq 0 \end{cases}$$

compare $u(x, t-\Delta t) = \alpha u(x,t) + \alpha_+ u(x+\Delta x, t) + \alpha_- u(x-\Delta x, t)$

where $\alpha = 1 - \frac{\Delta t}{\Delta x^2}$

$\alpha_{\pm} = \frac{\Delta t}{2\Delta x^2}$

Then stability condition: $\begin{cases} 1 \geq \frac{\Delta t}{\Delta x^2} \\ \frac{\Delta t}{2\Delta x^2} \geq 0 \end{cases} \Rightarrow \underline{\Delta t \leq \Delta x^2}$

4. According to Taylor's expansion.

4.
$$u(x_{n-1}, t_m) = u(x_n, t_{m-1}) + \Delta t \cdot \frac{\partial u}{\partial t} - \Delta x \frac{\partial u}{\partial x} + \frac{1}{2}\Delta x^2 \cdot \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2} \cdot \Delta t^2$$
$$- \Delta x \Delta t \cdot \frac{\partial^2 u}{\partial x \partial t} + \frac{1}{2}\frac{\partial^3 u}{\partial x^2 \partial t}(\Delta x)^2 \Delta t + O(\Delta x^2 \Delta t)$$

$$u(x_{n+1}, t_m) = u(x_n, t_{m-1}) + \Delta x \cdot \frac{\partial u}{\partial x} + \Delta t \frac{\partial u}{\partial t} + \frac{1}{2}(\Delta x)^2 \cdot \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u}{\partial t^2} \cdot \Delta t^2$$
$$+ \Delta x \cdot \Delta t \cdot \frac{\partial u^2}{\partial x \partial t} + \frac{1}{2}\frac{\partial^3 u}{\partial x^2 \partial t}(\Delta x)^2 \Delta t + O(\Delta x^2 \Delta t)$$

$$u(x_n, t_m) = u(x_n, t_{m-1}) + \Delta t \cdot \frac{\partial u}{\partial t} + \frac{1}{2}(\Delta t)^2 \cdot \frac{\partial^2 u}{\partial t^2} + O(\Delta t^2)$$

$$\Rightarrow Con_n^{m-1} = h(u(x_{n-1}, t_m) + u(x_{n+1}, t_m)) + (1-2h)u(x_n, t_m) - u(x_n, t_{m-1})$$
$$= h(2u(x_n, t_{m-1}) + 2\Delta t \cdot \frac{\partial u}{\partial t} + \Delta x^2 \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial x^2 \partial t}(\Delta x)^2 \cdot \Delta t + \frac{\partial^2 u}{\partial t^2} \cdot \Delta t^2$$
$$+ 2O(\Delta x^2 \Delta t)) + (1-2h)u(x_n, t_m) - u(x_n, t_{m-1})$$
$$= \Delta t \cdot \frac{\partial u}{\partial t} + h \cdot \Delta x^2 \cdot \frac{\partial^2 u}{\partial x^2} + h(\Delta x)^2 \cdot \Delta t \cdot \frac{\partial^3 u}{\partial x^2 \partial t} + 2hO(\Delta x^2 \Delta t)$$
$$+ \frac{1}{2}(\cancel{1-2h}) \cdot \Delta t^2 \cdot \frac{\partial^2 u}{\partial t^2} + (1-2h)O(\Delta t^2)$$

since $h = \frac{\Delta t}{2\Delta x^2}$. $\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial^2 u}{\partial x^2}$

$$Con_n^{m-1} = \frac{1}{2}\Delta t^2 \cdot \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{1}{2}\Delta t^2 \cdot \frac{\partial^2 u}{\partial t^2} + 2hO(\Delta x^2 \Delta t) + (1-2h)O(\Delta t^2)$$
$$= 2hO(\Delta x^2 \Delta t) + (1-2h)O(\Delta t^2)$$

According to 3. $h \le \frac{1}{2}$.

Thus $|Con_n^{m-1}| \le A(\Delta t^2 + \Delta t \Delta x^2)$ for some constant $A$.

5. Since $0 \le h \le \frac{1}{2} \Rightarrow 1-2h \ge 0$.

$$\|e_n^{m-1}\| = \|Sta_n^{m-1} + Con_n^{m-1}\|$$
$$\le \|Sta_n^{m-1}\| + \|Con_n^{m-1}\|$$
$$\le |h|\,\|e_{n-1}^m\| + |1-2h|\,\|e_n^m\| + |h|\,\|e_{n+1}^m\| + O(\Delta t^2 + \Delta t \Delta x^2)$$
$$\Rightarrow \|e^{m-1}\| \le \|e^m\| + O(\Delta t^2 + \Delta t \Delta x^2)$$

By induction,
$$\|e^{m-1}\| \le \underbrace{\|e^M\|}_{=0} + (M+1-m)A(\Delta t^2 + \Delta t \Delta x^2)$$
$$\le AT(\Delta t + \Delta x^2) \qquad (\because \Delta t = \frac{T}{M})$$