Automatic Differentiation Variational Inference

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VI Tutorial @ IST

https://vitutorial.github.io/tour/ist2019

What we know so far

- DGMs:

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  - Score function estimator: applicable to any model
  - Reparameterised gradients
    so far seems applicable only to Gaussian variables

Multivariate calculus recap

Let $x \in \mathbb{R}^K$ and let $\mathcal{T} : \mathbb{R}^K \to \mathbb{R}^K$ be differentiable and invertible

- $y = \mathcal{T}(x)$
- $x = \mathcal{T}^{-1}(y)$

Jacobian

The Jacobian matrix $J_{\mathcal{T}}(x)$ of $\mathcal{T}$
assessed at x is the matrix of partial derivatives

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Inverse function theorem

$$J_{\mathcal{T}^{-1}}(y) = \left(J_{\mathcal{T}}(x)\right)^{-1}$$

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- Scalar case

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- Multivariate case

$$\mathrm{d}y = |\det J_{\mathcal{T}}(x)|\mathrm{d}x$$

  the absolute value absorbs the orientation

Integration by substitution

We can integrate a function $g(x)$
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and then it follows that

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- $\mathcal{S}_\lambda(z)$ absorbs dependency on λ

Reparameterised expectations

If we are interested in

$$\mathbb{E}_{q(z|\lambda)}\left[g(z)\right]$$

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$$= \int \pi(\epsilon)$$

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$$= \int \pi(\epsilon) g(\mathcal{S}_\lambda^{-1}(\epsilon)) \mathrm{d}\epsilon = \mathbb{E}_{\pi(\epsilon)}\left[g(\mathcal{S}_\lambda^{-1}(\epsilon))\right]$$

Reparameterised gradients

For optimisation, we need tractable gradients

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$$\frac{\partial}{\partial \lambda}\mathbb{E}_{q(z|\lambda)}\left[g(z)\right] = \mathbb{E}_{\pi(\epsilon)}\left[\frac{\partial}{\partial \lambda}g(\mathcal{S}_{\lambda}^{-1}(\epsilon))\right]$$

$$\overset{\text{MC}}{\approx} \frac{1}{M}\sum_{\substack{i=1 \\ \epsilon_i \sim \pi(\epsilon)}}^{M} \frac{\partial}{\partial \lambda}g(\mathcal{S}_{\lambda}^{-1}(\epsilon_i))$$

Reparameterised gradients: Gaussian

We have seen one case, namely,
if $\epsilon \sim \mathcal{N}(0, I)$ and $Z \sim \mathcal{N}(\mu, \sigma^2)$

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and

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{\mathcal{N}(z|\mu,\sigma^2)} [g(z)]$$

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&= \mathbb{E}_{\mathcal{N}(0,I)}\left[\frac{\partial}{\partial z}g(z=\mu+\sigma\epsilon)\frac{\partial z}{\partial\lambda}\right]
\end{aligned}$$

Reparameterised gradients: Inverse cdf

Inverse cdf

- for univariate Z with pdf $f_Z(z)$ and cdf $F_Z(z)$

$$P \sim \mathcal{U}(0,1) \qquad Z \sim F_Z^{-1}(P)$$

where $F_Z^{-1}(p)$ is the *quantile function*

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Example: Kumaraswamy distribution

- $f_Z(z; a, b) = abz^{a-1}(1 - z^a)^{b-1}$
- $F_Z(z; a, b) = 1 - (1 - z^a)^b$
- $F_Z^{-1}(p; a, b) = (1 - (1 - p)^{1/b})^{1/a}$

Beyond

Many interesting densities cannot be easily reparameterised

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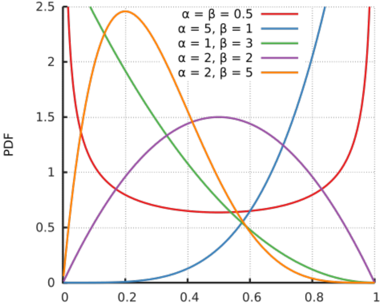
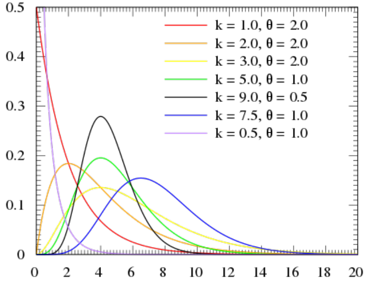
Beta



Beyond

Many interesting densities cannot be easily reparameterised

Gamma



Beyond

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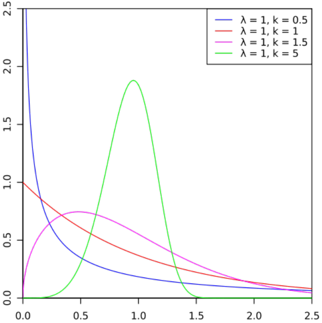
Weibull



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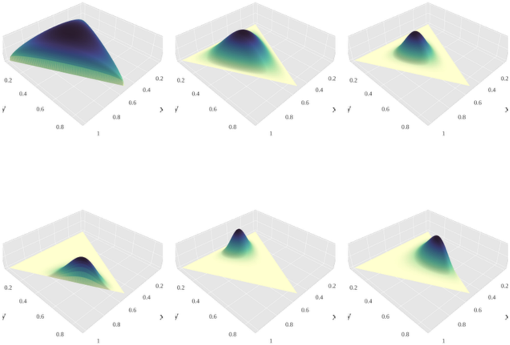
Dirichlet



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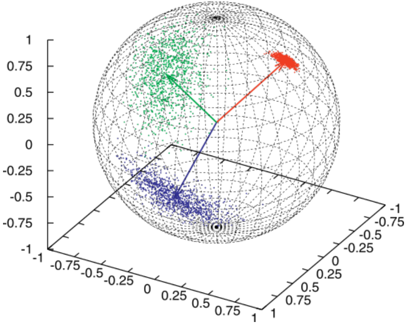
von Mises-Fisher



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Reparameterised gradients are a step towards automatising VI for differentiable models

- but not every model of interest employs rvs for which a reparameterisation is known

Example: Weibull-Poisson model

Suppose we have some ordinal data which we assume to be Poisson-distributed

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Suppose we have some ordinal data which we assume to be Poisson-distributed

$$\begin{aligned} z|r,k &\sim \mathrm{Weibull}(r,k) & r \in \mathbb{R}_{>0}, k \in \mathbb{R}_{>0} \\ X|z &\sim \mathrm{Poisson}(z) & z \in \mathbb{R}_{>0} \end{aligned}$$

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VI for Weibull-Poisson model

Generative model

$$p(x, z|r, k) = p(z|r, k)p(x|z)$$

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ELBO

$$\mathbb{E}_{q(z|\lambda)}\left[\log p(x, z|r, k)\right] + \mathbb{H}\left(q(z)\right)$$

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Can we make $q(z|\lambda)$ Gaussian?

No! $\text{supp}(\mathcal{N}(z|\mu, \sigma^2)) = \mathbb{R}$

Strategy

Build a change of variable into the model

$$p(x, z|r, k) = p(z|r, k)p(x|z)$$
$$= \text{Weibull}(z|r, k)\,\text{Poisson}(x|z)$$

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$$\mathbb{E}_{q(\zeta|\lambda)}\left[f(x, \zeta)\right] + \mathbb{H}\left(q(\zeta)\right)$$

Can we use a Gaussian approximate posterior?

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\end{aligned}
$$

ELBO

$$\mathbb{E}_{q(\zeta|\lambda)}[f(x, \zeta)] + \mathbb{H}(q(\zeta))$$

Can we use a Gaussian approximate posterior? Yes!

Differentiable models

We focus on *differentiable probability models*

$$p(x, z) = p(x|z)p(z)$$

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- members of this class have continuous latent variables z
- and the gradient $\nabla_z \log p(x, z)$ is valid within the *support* of the prior $\mathrm{supp}(p(z)) = \{z \in \mathbb{R}^K : p(z) > 0\} \subseteq \mathbb{R}^K$

Why do we need differentiable models?

Recall the gradient of the ELBO

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q(z;\lambda)} \left[\log p(x, z) \right] + \frac{\partial}{\partial \lambda} \mathbb{H} \left(q(z; \lambda) \right)$$

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VI optimisation problem

Let's focus on the design and optimisation of the variational approximation

$$\arg\min_{q(z)} \mathrm{KL}\left(q(z) \,||\, p(z|x)\right)$$

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- otherwise KL is not a real number
  $\mathrm{KL}\left(q \;||\; p\right) = \mathbb{E}_q\left[\log q\right] - \mathbb{E}_q\left[\log p\right] \stackrel{\text{def}}{=} \infty$

Support matching constraint

So let's constrain $q(z)$ to a family $\mathcal{Q}$ whose support is included in the support of the posterior

$$\arg\min_{q(z)\in\mathcal{Q}} \mathrm{KL}\left(q(z) \,||\, p(z|x)\right)$$

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$$\mathcal{Q} = \{q(z) : \mathrm{supp}(q(z)) \subseteq \mathrm{supp}(p(z|x))\}$$

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But what is the support of $p(z|x)$?

- typically the same as the support of $p(z)$
  as long as $p(x, z) > 0$ if $p(z) > 0$

Parametric family

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- a parameter vector λ picks out a member of the family

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We maximise the ELBO

$$\arg\max_{\lambda \in \Lambda} \mathbb{E}_{q(z;\lambda)} \left[\log p(x, z)\right] + \mathbb{H}\left(q(z;\lambda)\right)$$

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- Λ may be constrained to a subset of $\mathbb{R}^D$
  e.g. univariate Gaussian location lives in $\mathbb{R}$ but scale lives in $\mathbb{R}_{>0}$

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how can we get v from $\lambda_v \in \mathbb{R}^d$?

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It is typically possible to work with unconstrained parameters, it only takes an appropriate activation

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- support of $q(z;\lambda)$ depends on the choice of prior and thus may be a subset of $\mathbb{R}^K$

ADVI

A gradient-based black-box VI procedure

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   - basically, pick a Gaussian!
3. Intractable expectations
   - Reparameterised Gradients!

Joint model in real coordinate space

Let's introduce an invertible and differentiable transformation

$$\mathcal{T} : \mathrm{supp}(p(z)) \to \mathbb{R}^K$$

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VI in real coordinate space

We can design a posterior approximation whose support is $\mathbb{R}^K$

$$q(\zeta|\lambda) = \underbrace{\prod_{k=1}^{K} q(\zeta_k|\lambda)}_{\text{mean field}} = \prod_{k=1}^{K} \mathcal{N}(\zeta_k|\mu_k, \sigma_k^2)$$

where

- $\mu_k = \lambda_{\mu_k}$ for $\lambda_{\mu_k} \in \mathbb{R}^K$
- $\sigma_k = \text{softplus}(\lambda_{\sigma_k})$ for $\lambda_{\sigma_k} \in \mathbb{R}^K$

ELBO in real coordinate space

$\log p(x)$

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$$= \log \int p(x, \mathcal{T}^{-1}(\zeta))|\det J_{\mathcal{T}^{-1}}(\zeta)|\mathrm{d}\zeta$$

$$= \log \int q(\zeta) \frac{p(x, \mathcal{T}^{-1}(\zeta))|\det J_{\mathcal{T}^{-1}}(\zeta)|}{q(\zeta)}\mathrm{d}\zeta$$

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$$\log p(x) = \log \int p(x, z)\mathrm{d}z$$

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$$= \log \int q(\zeta) \frac{p(x, \mathcal{T}^{-1}(\zeta))|\det J_{\mathcal{T}^{-1}}(\zeta)|}{q(\zeta)}\mathrm{d}\zeta$$

$$\overset{\text{JI}}{\geq} \int q(\zeta) \log \frac{p(x, \mathcal{T}^{-1}(\zeta))|\det J_{\mathcal{T}^{-1}}(\zeta)|}{q(\zeta)}\mathrm{d}\zeta$$

ELBO in real coordinate space

$$
\begin{aligned}
\log p(x) &= \log \int p(x, z)\mathrm{d}z \\
&= \log \int p(x, \mathcal{T}^{-1}(\zeta))|\det J_{\mathcal{T}^{-1}}(\zeta)|\mathrm{d}\zeta \\
&= \log \int q(\zeta) \frac{p(x, \mathcal{T}^{-1}(\zeta))|\det J_{\mathcal{T}^{-1}}(\zeta)|}{q(\zeta)}\mathrm{d}\zeta \\
&\overset{\text{JI}}{\geq} \int q(\zeta) \log \frac{p(x, \mathcal{T}^{-1}(\zeta))|\det J_{\mathcal{T}^{-1}}(\zeta)|}{q(\zeta)}\mathrm{d}\zeta \\
&= \mathbb{E}_{q(\zeta)}\left[\log p(x, \mathcal{T}^{-1}(\zeta)) + \log\left|\det J_{\mathcal{T}^{-1}}(\zeta)\right|\right] + \mathbb{H}\left(q(\zeta)\right)
\end{aligned}
$$

Reparameterised ELBO

Recall that for Gaussians we have a standardisation procedure $\mathcal{S}_\lambda(\zeta) \sim \mathcal{N}(\epsilon|0, I)$

$$\mathbb{E}_{q(\zeta|\lambda)}\left[\log p(x, \mathcal{T}^{-1}(\zeta)) + \log\left|\det J_{\mathcal{T}^{-1}}(\zeta)\right|\right] + \mathbb{H}\left(q(\zeta|\lambda)\right)$$

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$$\mathbb{E}_{q(\zeta|\lambda)}\left[\log p(x, \mathcal{T}^{-1}(\zeta)) + \log\left|\det J_{\mathcal{T}^{-1}}(\zeta)\right|\right] + \mathbb{H}\left(q(\zeta|\lambda)\right)$$

$$= \mathbb{E}_{\mathcal{N}(\epsilon|0,I)}\left[\log p(x, \underbrace{\mathcal{T}^{-1}(\overbrace{\mathcal{S}_\lambda^{-1}(\epsilon)}^{\zeta})}_{z}) + \log\left|\det J_{\mathcal{T}^{-1}}(\mathcal{S}_\lambda^{-1}(\epsilon))\right|\right]$$

$$+ \mathbb{H}\left(q(\zeta|\lambda)\right)$$

Gradient estimate

For $\epsilon_i \sim \mathcal{N}(0, I)$

$$\frac{\partial}{\partial \lambda} \mathrm{ELBO}(\lambda)$$

Gradient estimate

For $\epsilon_i \sim \mathcal{N}(0, I)$

$$\frac{\partial}{\partial \lambda} \mathrm{ELBO}(\lambda) \overset{\text{MC}}{\approx} \frac{1}{M} \sum_{i=1}^{M} \frac{\partial}{\partial \lambda} \log \underbrace{p(x | \mathcal{T}^{-1}(\mathcal{S}_\lambda^{-1}(\epsilon_i)))}_{\text{likelihood}}$$

Gradient estimate

For $\epsilon_i \sim \mathcal{N}(0, I)$

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$$+ \frac{\partial}{\partial \lambda} \log \underbrace{p(\mathcal{T}^{-1}(\mathcal{S}_{\lambda}^{-1}(\epsilon_i)))}_{\text{prior}}$$
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$$+ \frac{\partial}{\partial \lambda} \underbrace{\mathbb{H}\left(q(\zeta; \lambda)\right)}_{\text{analaytic}}$$

Practical tips

Many software packages know how to transform the support of various distributions

- Stan
- Tensorflow `tf.probability`
- Pytorch `torch.distributions`

Weibull-Poisson model

Build a change of variable into the model

$$p(x, z|r, k) = p(z|r, k)p(x|\rho)$$
$$= \text{Weibull}(z|r, k)\,\text{Poisson}(x|z)$$

Weibull-Poisson model

Build a change of variable into the model

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\begin{aligned}
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\end{aligned}
$$

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$$
\begin{aligned}
p(x, z|r, k) &= p(z|r, k)p(x|\rho) \\
&= \text{Weibull}(z|r, k)\,\text{Poisson}(x|z) \\
&= \text{Weibull}(\underbrace{\log^{-1}(\zeta)}_{z}|r, k)\,\text{Poisson}(x|\underbrace{\log^{-1}(\zeta)}_{z})\left|\det J_{\log^{-1}}(\zeta)\right|
\end{aligned}
$$

Weibull-Poisson model

Build a change of variable into the model

$$
\begin{aligned}
p(x, z \mid r, k) &= p(z \mid r, k)\, p(x \mid \rho) \\
&= \text{Weibull}(z \mid r, k)\, \text{Poisson}(x \mid z) \\
&= \text{Weibull}(\underbrace{\log^{-1}(\zeta)}_{z} \mid r, k)\, \text{Poisson}(x \mid \underbrace{\log^{-1}(\zeta)}_{z}) \left|\det J_{\log^{-1}}(\zeta)\right| \\
&= p(x, z = \log^{-1}(\zeta)) \left|\det J_{\log^{-1}}(\zeta)\right|
\end{aligned}
$$

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\end{aligned}
$$

ELBO

$$
\mathbb{E}_{q(\zeta|\lambda)}[\dots] + \mathbb{H}(q(\zeta))
$$

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\end{aligned}$$

ELBO

$$\mathbb{E}_{q(\zeta \mid \lambda)} \left[\log p(x, z = \log^{-1}(\zeta)) \left|\det J_{\log^{-1}}(\zeta)\right|\right] + \mathbb{H}\left(q(\zeta)\right)$$

Weibull-Poisson model

Build a change of variable into the model

$$
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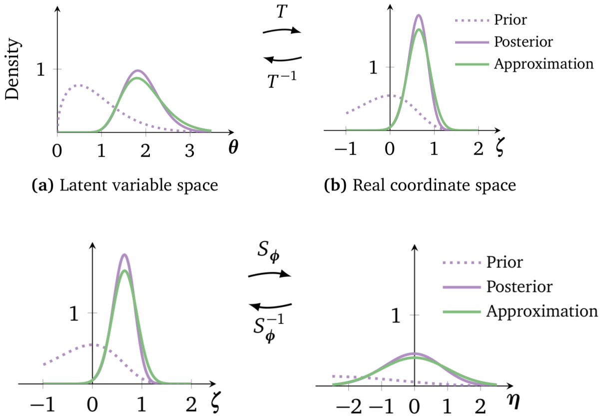
ELBO

$$
\mathbb{E}_{q(\zeta|\lambda)}\left[\log p(x, z = \log^{-1}(\zeta))\left|\det J_{\log^{-1}}(\zeta)\right|\right] + \mathbb{H}\left(q(\zeta)\right)
$$

$$
\mathbb{E}_{\phi(\epsilon)}\left[\log p(x, z = \log^{-1}(\mathcal{S}^{-1}(\epsilon)))\left|\det J_{\log^{-1}}(\mathcal{S}^{-1}(\epsilon))\right|\right] + \mathbb{H}\left(q(\zeta)\right)
$$

Visualisation



(a) Latent variable space **(b)** Real coordinate space

(a) Real coordinate space **(b)** Standardized space

Images from Kucukelbir et al. (2017)

Wait... no deep learning?

Sure! Parameters may well be predicted by NNs

- approximate posterior location and scale
- Weibull rate and shape

Everything is now differentiable, reparameterisable, and the optimisation is unconstrained!

Summary

ADVI is a big step towards blackbox VI

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What's left?

Summary

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- we knew how to map parameters to the unconstrained real coordinate space
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What's left? Our posteriors are still rather simple, aren't they?

Alp Kucukelbir, Dustin Tran, Rajesh Ranganath, Andrew Gelman, and David M. Blei. Automatic differentiation variational inference. *Journal of Machine Learning Research*, 18(14):1–45, 2017. URL http://jmlr.org/papers/v18/16-107.html.