Deep Generative Models: Discrete Latent Variables

Wilker Aziz

VI Tutorial @ IST
https://vitutorial.github.io/tour/ist2019

Generative Models

Joint distribution over observed data x and latent variables Z.

$$p(x, z | \theta) = \underbrace{p(z)}_{\text{prior}} \underbrace{p(x | z, \theta)}_{\text{likelihood}}$$

The likelihood and prior are often standard distributions (Gaussian, Bernoulli) with simple dependence on conditioning information.

Deep generative models

Joint distribution with **deep observation model**

$$p(x, z|\theta) = \underbrace{p(z)}_{\text{prior}} \underbrace{p(x|z, \theta)}_{\text{likelihood}}$$

mapping from z to $p(x|z, \theta)$ is a NN with parameters θ

Deep generative models

Joint distribution with **deep observation model**

$$p(x, z|\theta) = \underbrace{p(z)}_{\text{prior}} \underbrace{p(x|z, \theta)}_{\text{likelihood}}$$

mapping from z to $p(x|z, \theta)$ is a NN with parameters θ

Marginal likelihood

$$p(x|\theta) = \int p(x, z|\theta) \, \mathrm{d}z = \int p(z) p(x|z, \theta) \, \mathrm{d}z$$

intractable in general

Goals

We want
- richer probabilistic models

Goals

We want

- richer probabilistic models
- complex observation models
  parameterised by NNs

Goals

We want

- richer probabilistic models
- complex observation models parameterised by NNs

but we can't perform gradient-based MLE

Goals

We want

- richer probabilistic models
- complex observation models parameterised by NNs

but we can't perform gradient-based MLE

We need approximate inference techniques!

1. **First Attempt: Wake-Sleep**

2. Neural Variational Inference

3. Score function estimator

4. Variance reduction

Wake-sleep Algorithm

- Generalise latent variables to Neural Networks
- Train generative neural model
- Use variational inference! (kind of)

Wake-sleep Architecture

2 Neural Networks:

Wake-sleep Architecture

2 Neural Networks:

- A generation network to model the data (the one we want to optimise) – parameters: θ

Wake-sleep Architecture

2 Neural Networks:

- A generation network to model the data (the one we want to optimise) – parameters: θ
- An inference (recognition) network (to model the latent variable) – parameters: λ

Wake-sleep Architecture

2 Neural Networks:

- A generation network to model the data (the one we want to optimise) – parameters: θ
- An inference (recognition) network (to model the latent variable) – parameters: λ
- Original setting: binary hidden units
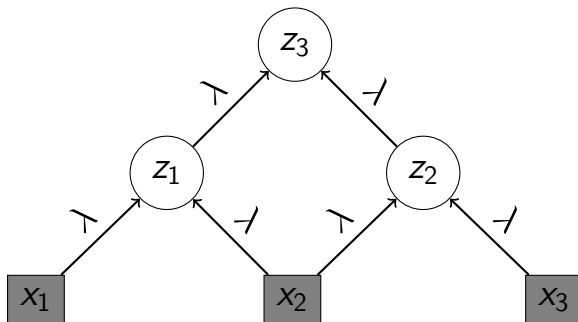
Wake-sleep Architecture

2 Neural Networks:

- A generation network to model the data (the one we want to optimise) – parameters: θ
- An inference (recognition) network (to model the latent variable) – parameters: λ
- Original setting: binary hidden units
- Training is performed in a "hard EM" fashion

Generator
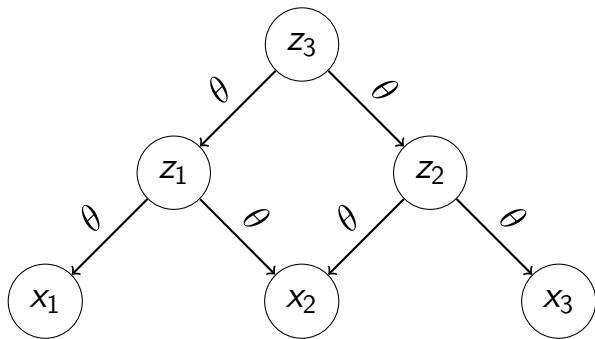
Inference Network

Wake-sleep Training

Wake Phase

- Use inference network to sample hidden unit setting z from $q(z|x, \lambda)$
- Update generation parameters θ to maximize joint log-liklelihood of data and latents $p(x, z|\theta)$

Wake-sleep Training

Wake Phase

- Use inference network to sample hidden unit setting z from $q(z|x, \lambda)$
- Update generation parameters θ to maximize joint log-liklelihood of data and latents $p(x, z|\theta)$

Sleep Phase

- Produce dream sample $\tilde{x}$ from random hidden unit z
- Update inference parameters λ to maximize probability of latent state $q(z|\tilde{x}, \lambda)$

Wake Phase Objective

Objective

$$\arg \min_{\theta} \mathrm{KL} \left(q(z|x, \lambda) \ || \ p(z|x, \theta) \right)$$

Wake Phase Objective

Objective

$$\arg\min_{\theta} \text{KL} \left(q(z|x, \lambda) \,||\, p(z|x, \theta) \right)$$

$$= \arg\max_{\theta} \underbrace{\mathbb{E}_{q(z|x,\lambda)} \left[\log p(z, x|\theta) \right] + \mathbb{H}[q(z|x, \lambda)]}_{\mathcal{G}(\theta)}$$

Wake Phase Objective

Objective

$$\arg\min_{\theta} \mathsf{KL}\left(q(z|x,\lambda) \,||\, p(z|x,\theta)\right)$$

$$= \arg\max_{\theta} \underbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(z,x|\theta)\right] + \mathbb{H}[q(z|x,\lambda)]}_{\mathcal{G}(\theta)}$$

Gradient estimate

$$\boldsymbol{\nabla}_{\theta}\mathcal{G}(\theta) = \boldsymbol{\nabla}_{\theta}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(z,x|\theta)\right] + \boldsymbol{\nabla}_{\theta}\mathbb{H}[q(z|x,\lambda)]$$

Wake Phase Objective

Objective

$$\arg\min_{\theta} \mathsf{KL}\left(q(z|x,\lambda) \,||\, p(z|x,\theta)\right)$$

$$= \arg\max_{\theta} \ \underbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(z,x|\theta)\right] + \mathbb{H}[q(z|x,\lambda)]}_{\mathcal{G}(\theta)}$$

Gradient estimate

$$\boldsymbol{\nabla}_{\theta}\mathcal{G}(\theta) = \boldsymbol{\nabla}_{\theta}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(z,x|\theta)\right] + \boldsymbol{\nabla}_{\theta}\mathbb{H}[q(z|x,\lambda)]$$

$$= \mathbb{E}_{q(z|x,\lambda)}\left[\boldsymbol{\nabla}_{\theta}\log p(z,x|\theta)\right]$$

Wake Phase Objective

Objective

$$\arg\min_{\theta} \mathrm{KL}\left(q(z|x,\lambda) \,||\, p(z|x,\theta)\right)$$

$$= \arg\max_{\theta} \underbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(z,x|\theta)\right] + \mathbb{H}[q(z|x,\lambda)]}_{\mathcal{G}(\theta)}$$

Gradient estimate

$$\boldsymbol{\nabla}_{\theta}\mathcal{G}(\theta) = \boldsymbol{\nabla}_{\theta}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(z,x|\theta)\right] + \boldsymbol{\nabla}_{\theta}\mathbb{H}[q(z|x,\lambda)]$$

$$= \mathbb{E}_{q(z|x,\lambda)}\left[\boldsymbol{\nabla}_{\theta}\log p(z,x|\theta)\right]$$

$$\stackrel{\mathrm{MC}}{\approx} \boldsymbol{\nabla}_{\theta}\log p(z,x|\theta) \quad \text{where } z \sim q(z|x,\lambda)$$

Wake Phase Objective

Assumes z to be fixed random draw from $q(z|x, \lambda)$.

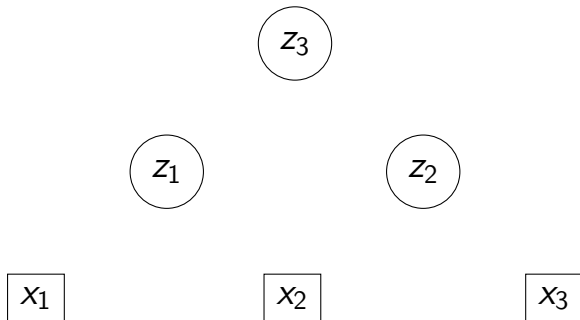$$\arg \min_{\theta} \mathrm{KL}\left(q(z|x, \lambda) \mid\mid p(z|x, \theta)\right)$$

$$\overset{\mathrm{MC}}{\approx} \arg \max_{\theta} \log p(z, x|\theta) \qquad \text{for } z \sim q(z|x)$$

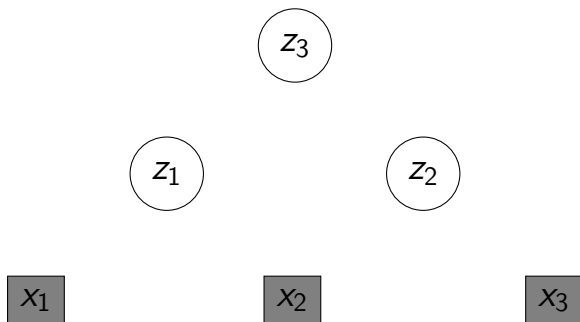This is simply supervised learning with imputed latent data!
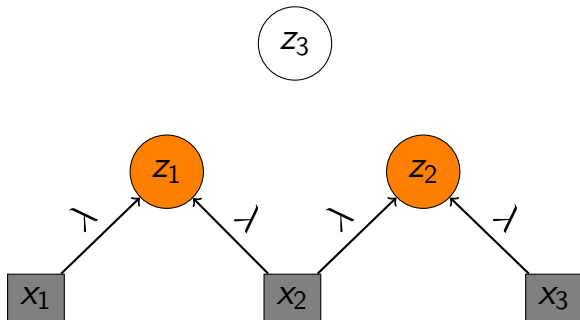
Wake Phase Sampling

Sampling $z \sim q(z|x, \lambda)$

Wake Phase Sampling

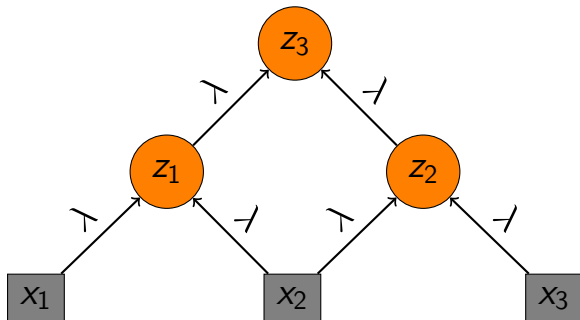Sampling $z \sim q(z|x, \lambda)$

Wake Phase Sampling

Sampling $z \sim q(z|x, \lambda)$

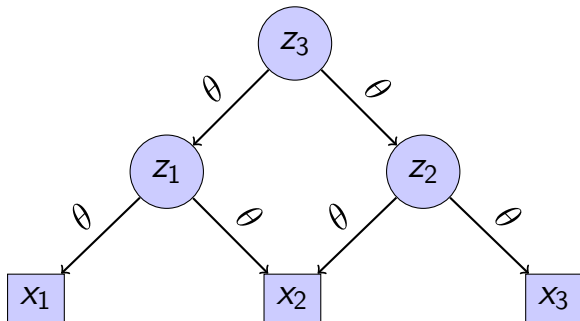# Wake Phase Sampling

Sampling $z \sim q(z|x, \lambda)$

Wake Phase Update

Compute $\log p(x, z|\theta)$ and update θ

Sleep Phase Objective

Objective

$$\arg\min_{\lambda} \mathrm{KL}\left(q(z|x, \lambda) \,||\, p(z|x, \theta)\right)$$

Sleep Phase Objective

Objective

$$\underset{\lambda}{\arg\min}\, \mathrm{KL}\left(q(z|x,\lambda)\,||\,p(z|x,\theta)\right)$$

$$= \underset{\lambda}{\arg\max}\,\, \underbrace{\mathbb{E}_{q(z|x,\lambda)}[\log p(z,x|\theta)] + \mathbb{H}[q(z|x,\lambda)]}_{\mathcal{R}(\lambda)}$$

Sleep Phase Objective

Objective

$$\arg\min_{\lambda} \mathsf{KL}\left(q(z|x,\lambda) \parallel p(z|x,\theta)\right)$$

$$= \arg\max_{\lambda} \underbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(z,x|\theta)\right] + \mathbb{H}[q(z|x,\lambda)]}_{\mathcal{R}(\lambda)}$$

Gradient estimate

$$\boldsymbol{\nabla}_{\lambda}\mathcal{R}(\lambda) = \boldsymbol{\nabla}_{\lambda}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(z,x|\theta)\right] + \boldsymbol{\nabla}_{\lambda}\mathbb{H}[q(z|x,\lambda)]$$

Sleep Phase Objective

Objective

$$\arg\min_{\lambda} \mathsf{KL}\left(q(z|x, \lambda) \,||\, p(z|x, \theta)\right)$$

$$= \arg\max_{\lambda} \underbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(z, x|\theta)\right] + \mathbb{H}[q(z|x, \lambda)]}_{\mathcal{R}(\lambda)}$$

Gradient estimate

$$\boldsymbol{\nabla}_{\lambda}\mathcal{R}(\lambda) = \boldsymbol{\nabla}_{\lambda}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(z, x|\theta)\right] + \boldsymbol{\nabla}_{\lambda}\mathbb{H}[q(z|x, \lambda)]$$

Let's change the objective!

Sleep Phase (Convenient) Objective

Flip the direction of the KL

$$\arg\min_{\lambda} \mathbb{E}_{p(x)} \left[\text{KL} \left(p(z|x, \theta) \ || \ q(z|x, \lambda) \right) \right]$$

Sleep Phase (Convenient) Objective

Flip the direction of the KL

$$\arg\min_{\lambda} \mathbb{E}_{p(x)} \left[\text{KL} \left(p(z|x, \theta) \,||\, q(z|x, \lambda) \right) \right]$$

$$= \arg\min_{\lambda} \mathbb{E}_{p(x|\theta)} \mathbb{E}_{p(z|x, \theta)} \left[\log p(z|x, \theta) - \log q(z|x, \lambda) \right]$$

Sleep Phase (Convenient) Objective

Flip the direction of the KL

$$\arg\min_\lambda \mathbb{E}_{p(x)}\left[\text{KL}\left(p(z|x,\theta) \,||\, q(z|x,\lambda)\right)\right]$$

$$= \arg\min_\lambda \mathbb{E}_{p(x|\theta)}\mathbb{E}_{p(z|x,\theta)}\left[\log p(z|x,\theta) - \log q(z|x,\lambda)\right]$$

$$= \arg\max_\lambda \underbrace{\mathbb{E}_{p(x,z|\theta)}\left[\log q(z|x,\lambda)\right]}_{\mathcal{R}(\lambda)} - \underbrace{\mathbb{E}_{p(x,z|\theta)}\left[\log p(z|x,\theta)\right]}_{\text{constant}}$$

Sleep Phase (Convenient) Objective

Flip the direction of the KL

$$\arg\min_{\lambda} \mathbb{E}_{p(x)} \left[\text{KL} \left(p(z|x,\theta) \,||\, q(z|x,\lambda) \right) \right]$$

$$= \arg\min_{\lambda} \mathbb{E}_{p(x|\theta)} \mathbb{E}_{p(z|x,\theta)} \left[\log p(z|x,\theta) - \log q(z|x,\lambda) \right]$$

$$= \arg\max_{\lambda} \underbrace{\mathbb{E}_{p(x,z|\theta)} \left[\log q(z|x,\lambda) \right]}_{\mathcal{R}(\lambda)} - \underbrace{\mathbb{E}_{p(x,z|\theta)} \left[\log p(z|x,\theta) \right]}_{\text{constant}}$$

Gradient

$$\boldsymbol{\nabla}_{\lambda} \mathcal{R}(\lambda) = \boldsymbol{\nabla}_{\lambda} \mathbb{E}_{p(x,z|\theta)} \left[\log q(z|x,\lambda) \right]$$

Sleep Phase (Convenient) Objective

Flip the direction of the KL

$$\arg\min_{\lambda} \mathbb{E}_{p(x)} \left[\text{KL} \left(p(z|x,\theta) \mid\mid q(z|x,\lambda) \right) \right]$$

$$= \arg\min_{\lambda} \mathbb{E}_{p(x|\theta)} \mathbb{E}_{p(z|x,\theta)} \left[\log p(z|x,\theta) - \log q(z|x,\lambda) \right]$$

$$= \arg\max_{\lambda} \underbrace{\mathbb{E}_{p(x,z|\theta)} \left[\log q(z|x,\lambda) \right]}_{\mathcal{R}(\lambda)} - \underbrace{\mathbb{E}_{p(x,z|\theta)} \left[\log p(z|x,\theta) \right]}_{\text{constant}}$$

Gradient

$$\boldsymbol{\nabla}_{\lambda} \mathcal{R}(\lambda) = \boldsymbol{\nabla}_{\lambda} \mathbb{E}_{p(x,z|\theta)} \left[\log q(z|x,\lambda) \right]$$

$$= \mathbb{E}_{p(x,z|\theta)} \left[\boldsymbol{\nabla}_{\lambda} \log q(z|x,\lambda) \right]$$

Sleep Phase (Convenient) Objective

Flip the direction of the KL

$$\arg\min_{\lambda} \mathbb{E}_{p(x)}\left[\text{KL}\left(p(z|x,\theta) \mid\mid q(z|x,\lambda)\right)\right]$$

$$= \arg\min_{\lambda} \mathbb{E}_{p(x|\theta)}\mathbb{E}_{p(z|x,\theta)}\left[\log p(z|x,\theta) - \log q(z|x,\lambda)\right]$$

$$= \arg\max_{\lambda} \underbrace{\mathbb{E}_{p(x,z|\theta)}\left[\log q(z|x,\lambda)\right]}_{\mathcal{R}(\lambda)} - \underbrace{\mathbb{E}_{p(x,z|\theta)}\left[\log p(z|x,\theta)\right]}_{\text{constant}}$$

Gradient

$$\boldsymbol{\nabla}_{\lambda}\mathcal{R}(\lambda) = \boldsymbol{\nabla}_{\lambda}\mathbb{E}_{p(x,z|\theta)}\left[\log q(z|x,\lambda)\right]$$

$$= \mathbb{E}_{p(x,z|\theta)}\left[\boldsymbol{\nabla}_{\lambda}\log q(z|x,\lambda)\right]$$

$$= \mathbb{E}_{p(z)p(x|z,\theta)}\left[\boldsymbol{\nabla}_{\lambda}\log q(z|x,\lambda)\right]$$

Sleep Phase (Convenient) Objective

Assumes fake data $\tilde{x}$ and latent variables z to be fixed random draws from $p(x, z|\theta)$.

$$\arg\max_{\lambda} \; \mathbb{E}_{p(x,z|\theta)} \left[\log q(z|x, \lambda) \right]$$

Sleep Phase (Convenient) Objective

Assumes fake data $\tilde{x}$ and latent variables z to be fixed random draws from $p(x, z|\theta)$.

$$\arg\max_{\lambda} \; \mathbb{E}_{p(x,z|\theta)} \left[\log q(z|x, \lambda) \right]$$

$$\overset{\text{MC}}{\approx} \arg\max_{\lambda} \; \log q(z|\tilde{x}, \lambda)$$

where $z \sim p(z)$ and $\tilde{x} \sim p(x|z)$ (fake data!)

Sleep Phase Sampling

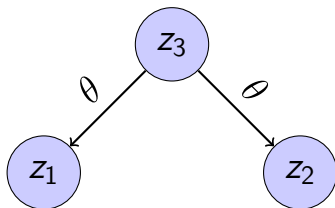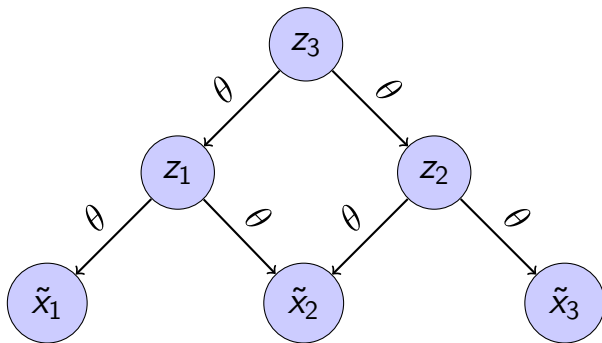Sampling $(z, \tilde{x}) \sim p(x, z | \theta)$

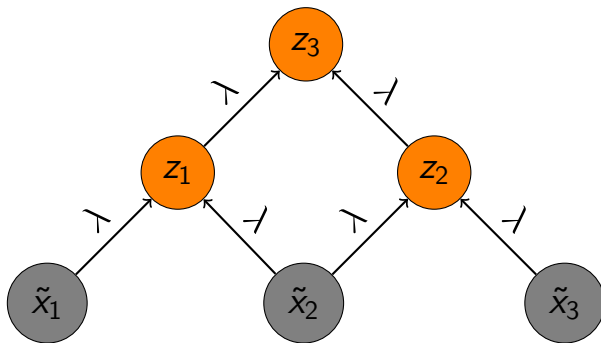# Sleep Phase Sampling

Sampling $(z, \tilde{x}) \sim p(x, z | \theta)$

Sleep Phase Sampling

Sampling $(z, \tilde{x}) \sim p(x, z|\theta)$

Sleep Phase Update

Compute $\log q(z|x, \lambda)$ and update λ

Wake-sleep Algorithm

Advantages

- Simple layer-wise updates
- Amortised inference: all latent variables are inferred from the same weights λ

Wake-sleep Algorithm

Advantages

- Simple layer-wise updates
- Amortised inference: all latent variables are inferred from the same weights λ

Drawbacks

- Inference and generative networks are trained on different objectives
- Inference weights λ are updated on fake data $\tilde{x}$
- Generative weights are bad initially, giving wrong signal to the updates of λ

1 First Attempt: Wake-Sleep

2 Neural Variational Inference

3 Score function estimator

4 Variance reduction

Variational Inference Learning (NVIL)

Generative model with NN likelihood

Variational Inference Learning (NVIL)

Generative model with NN likelihood

Let us consider a latent factor model for topic modelling:

Variational Inference Learning (NVIL)

Generative model with NN likelihood

Let us consider a latent factor model for topic modelling:

- a document $x = (x_1, \ldots, x_N)$ consists of n i.i.d. categorical draws from that model

Variational Inference Learning (NVIL)

Generative model with NN likelihood

Let us consider a latent factor model for topic modelling:

- a document $x = (x_1, \ldots, x_N)$ consists of n i.i.d. categorical draws from that model
- the categorical distribution in turn depends on binary latent factors $z = (z_1, \ldots, z_K)$ which are also i.i.d.
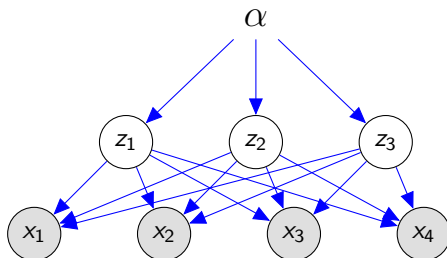
Latent factor model

$$Z_j \sim \text{Bernoulli}\,(\alpha) \qquad (1 \le k \le K)$$
$$X_i | z \sim \text{Categorical}\,(f(z;\theta)) \quad (1 \le i \le N)$$

Here $0 < \alpha < 1$ specifies a Bernoulli prior and $f(\cdot;\theta)$ is a function computed by a neural network with softmax output, e.g.
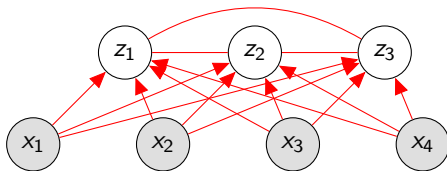
$$f(z;\theta) = \text{softmax}(Wz + b)$$
$$\theta = \{W, b\}$$

Example Model

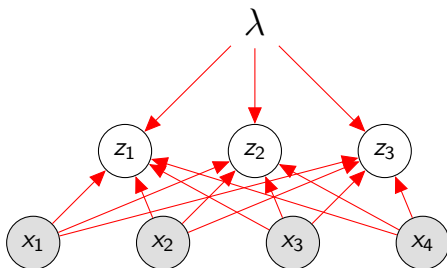Joint distribution: independent latent variables

Example Model

Posterior: latent variables are marginally dependent.

For our variational distribution we are going to assume that they are not (recall: mean field assumption).

Mean Field Inference

The inference network needs to predict K Bernoulli parameters b_1^K. Any neural network with sigmoid output will do that job.

Inference Network

$$q(z|x, \lambda) = \prod_{k=1}^{K} \text{Bern}(z_k|b_k)$$

$$\text{where } b_1^K = g(x; \lambda)$$

Example architecture

$$h = \frac{1}{N} \sum_{i=1}^{N} E_{x_i} \qquad b_1^K = \text{sigmoid}(Mh + c)$$

$$\lambda = \{E, M, c\}$$

Objective

$$\text{ELBO} = \mathbb{E}_{q(z|x,\lambda)}\left[\log p(x, z|\theta)\right] + \mathbb{H}\left(q(z|x, \lambda)\right)$$

Objective

$$\text{ELBO} = \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x, z|\theta) \right] + \mathbb{H} \left(q(z|x, \lambda) \right)$$
$$= \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z, \theta) \right] - \text{KL} \left(q(z|x, \lambda) \mid\mid p(z) \right)$$

Objective

$$\text{ELBO} = \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x, z|\theta) \right] + \mathbb{H} \left(q(z|x, \lambda) \right)$$
$$= \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z, \theta) \right] - \text{KL} \left(q(z|x, \lambda) \ || \ p(z) \right)$$

Parameter estimation

$$\underset{\theta,\lambda}{\arg\max} \ \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z, \theta) \right] - \text{KL} \left(q(z|x, \lambda) \ || \ p(z) \right)$$

KL

KL between K independent Bernoulli distributions is tractable

$$\text{KL}\left(q(z|x,\lambda) \mid\mid p(z|\alpha)\right) = \sum_{k=1}^{K} \text{KL}\left(q(z_k|x,\lambda) \mid\mid p(z_k|\alpha)\right)$$

KL

KL between K independent Bernoulli distributions is tractable

$$\text{KL}\left(q(z|x,\lambda) \mid\mid p(z|\alpha)\right) = \sum_{k=1}^{K} \text{KL}\left(q(z_k|x,\lambda) \mid\mid p(z_k|\alpha)\right)$$

$$= \sum_{k=1}^{K} \text{KL}\left(\text{Bernoulli}\left(b_k\right)\right) \mid\mid \text{Bernoulli}\left(\alpha\right))$$

KL

KL between K independent Bernoulli distributions is tractable

$$\text{KL}\left(q(z|x,\lambda) \,||\, p(z|\alpha)\right) = \sum_{k=1}^{K} \text{KL}\left(q(z_k|x,\lambda) \,||\, p(z_k|\alpha)\right)$$

$$= \sum_{k=1}^{K} \text{KL}\left(\text{Bernoulli}\left(b_k\right)\right) \,||\, \text{Bernoulli}\left(\alpha\right))$$

$$= \sum_{k=1}^{K} b_k \log \frac{b_k}{\alpha} + (1 - b_k) \log \frac{1 - b_k}{1 - \alpha}$$

Generative Network Gradient

$$\frac{\partial}{\partial \theta} \left(\mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z,\theta) \right] - \overbrace{\text{KL} \left(q(z|x,\lambda) \| p(z) \right)}^{\text{constant wrt } \theta} \right)$$

Generative Network Gradient

$$\frac{\partial}{\partial \theta} \left(\mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z,\theta) \right] - \overbrace{\mathsf{KL} \left(q(z|x,\lambda) \mid\mid p(z) \right)}^{\text{constant wrt } \theta} \right)$$

$$= \underbrace{\mathbb{E}_{q(z|x,\lambda)} \left[\frac{\partial}{\partial \theta} \log p(x|z,\theta) \right]}_{\text{expected gradient :)}}$$

Generative Network Gradient

$$\frac{\partial}{\partial \theta} \left(\mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z,\theta) \right] - \overbrace{\mathrm{KL} \left(q(z|x,\lambda) \,||\, p(z) \right)}^{\text{constant wrt } \theta} \right)$$

$$= \underbrace{\mathbb{E}_{q(z|x,\lambda)} \left[\frac{\partial}{\partial \theta} \log p(x|z,\theta) \right]}_{\text{expected gradient :)}}$$

$$\overset{\text{MC}}{\approx} \frac{1}{S} \sum_{s=1}^{S} \frac{\partial}{\partial \theta} \log p(x|z^{(s)},\theta) \quad \text{where } z^{(s)} \sim q(z|x,\lambda)$$

Inference Network Gradient

$$\frac{\partial}{\partial \lambda} \left(\mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z,\theta) \right] - \overbrace{\mathrm{KL}\left(q(z|x,\lambda) \,||\, p(z) \right)}^{\text{analytical}} \right)$$

Inference Network Gradient

$$\frac{\partial}{\partial \lambda} \left(\mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z,\theta) \right] - \overbrace{\text{KL} \left(q(z|x,\lambda) \mid\mid p(z) \right)}^{\text{analytical}} \right)$$

$$= \frac{\partial}{\partial \lambda} \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z,\theta) \right] - \underbrace{\frac{\partial}{\partial \lambda} \text{KL} \left(q(z|x,\lambda) \mid\mid p(z) \right)}_{\text{analytical computation}}$$

Inference Network Gradient

$$\frac{\partial}{\partial \lambda} \left(\mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z,\theta) \right] - \overbrace{\mathsf{KL} \left(q(z|x,\lambda) \,||\, p(z) \right)}^{\text{analytical}} \right)$$

$$= \frac{\partial}{\partial \lambda} \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z,\theta) \right] - \underbrace{\frac{\partial}{\partial \lambda} \mathsf{KL} \left(q(z|x,\lambda) \,||\, p(z) \right)}_{\text{analytical computation}}$$

The first term again requires approximation by sampling, but there is a problem

Inference Network Gradient

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q_\lambda(z|x)} \left[\log p_\theta(x|z) \right]$$

Inference Network Gradient

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q_\lambda(z|x)} \left[\log p_\theta(x|z) \right]$$

$$= \frac{\partial}{\partial \lambda} \sum_z q(z|x, \lambda) \log p(x|z, \theta)$$

Inference Network Gradient

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q_\lambda(z|x)} \left[\log p_\theta(x|z) \right]$$

$$= \frac{\partial}{\partial \lambda} \sum_z q(z|x, \lambda) \log p(x|z, \theta)$$

$$= \underbrace{\sum_z \frac{\partial}{\partial \lambda} (q(z|x, \lambda)) \log p(x|z, \theta)}_{\text{not an expectation}}$$

Inference Network Gradient

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q_\lambda(z|x)} \left[\log p_\theta(x|z) \right]$$

$$= \frac{\partial}{\partial \lambda} \sum_z q(z|x, \lambda) \log p(x|z, \theta)$$

$$= \underbrace{\sum_z \frac{\partial}{\partial \lambda} (q(z|x, \lambda)) \log p(x|z, \theta)}_{\text{not an expectation}}$$

- MC estimator is non-differentiable

Inference Network Gradient

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q_\lambda(z|x)} \left[\log p_\theta(x|z) \right]$$

$$= \frac{\partial}{\partial \lambda} \sum_z q(z|x, \lambda) \log p(x|z, \theta)$$

$$= \underbrace{\sum_z \frac{\partial}{\partial \lambda} (q(z|x, \lambda)) \log p(x|z, \theta)}_{\text{not an expectation}}$$

- MC estimator is non-differentiable
- Differentiating the expression does not yield an expectation: cannot approximate via MC

Score function estimator

We can again use the log identity for derivatives

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q_\lambda(z|x)} \left[\log p_\theta(x|z) \right]$$

$$= \sum_z \frac{\partial}{\partial \lambda} (q(z|x, \lambda)) \log p(x|z, \theta)$$

Score function estimator

We can again use the log identity for derivatives

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q_\lambda(z|x)} \left[\log p_\theta(x|z) \right]$$

$$= \sum_z \frac{\partial}{\partial \lambda} (q(z|x, \lambda)) \log p(x|z, \theta)$$

$$= \sum_z q(z|x, \lambda) \frac{\partial}{\partial \lambda} (\log q(z|x, \lambda)) \log p(x|z, \theta)$$

Score function estimator

We can again use the log identity for derivatives

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q_\lambda(z|x)} \left[\log p_\theta(x|z) \right]$$

$$= \sum_z \frac{\partial}{\partial \lambda} (q(z|x, \lambda)) \log p(x|z, \theta)$$

$$= \sum_z q(z|x, \lambda) \frac{\partial}{\partial \lambda} (\log q(z|x, \lambda)) \log p(x|z, \theta)$$

$$= \underbrace{\mathbb{E}_{q(z|x, \lambda)} \left[\log p(x|z, \theta) \frac{\partial}{\partial \lambda} \log q(z|x, \lambda) \right]}_{\text{expected gradient :)}}$$

Score function estimator: remarks

We can now build an MC estimator

$$\frac{\partial}{\partial\lambda}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right]$$

$$= \mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\frac{\partial}{\partial\lambda}\log q(z|x,\lambda)\right]$$

Score function estimator: remarks

We can now build an MC estimator

$$
\frac{\partial}{\partial \lambda} \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z, \theta) \right]
$$

$$
= \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z, \theta) \frac{\partial}{\partial \lambda} \log q(z|x, \lambda) \right]
$$

$$
\stackrel{\text{MC}}{\approx} \frac{1}{S} \sum_{s=1}^{S} \log p(x|z^{(s)}, \theta) \frac{\partial}{\partial \lambda} \log q(z^{(s)}|x, \lambda)
$$

where $z^{(s)} \sim q(z|x, \lambda)$

Score function estimator: remarks

We can now build an MC estimator

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z,\theta) \right]$$

$$= \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z,\theta) \frac{\partial}{\partial \lambda} \log q(z|x,\lambda) \right]$$

$$\overset{\text{MC}}{\approx} \frac{1}{S} \sum_{s=1}^{S} \log p(x|z^{(s)},\theta) \frac{\partial}{\partial \lambda} \log q(z^{(s)}|x,\lambda)$$

where $z^{(s)} \sim q(z|x,\lambda)$

- magnitude of $\log p(x|z,\theta)$ varies widely
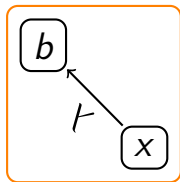
Score function estimator: remarks

We can now build an MC estimator

$$\frac{\partial}{\partial \lambda} \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z, \theta) \right]$$

$$= \mathbb{E}_{q(z|x,\lambda)} \left[\log p(x|z, \theta) \frac{\partial}{\partial \lambda} \log q(z|x, \lambda) \right]$$

$$\stackrel{\text{MC}}{\approx} \frac{1}{S} \sum_{s=1}^{S} \log p(x|z^{(s)}, \theta) \frac{\partial}{\partial \lambda} \log q(z^{(s)}|x, \lambda)$$
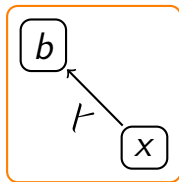
where $z^{(s)} \sim q(z|x, \lambda)$

- magnitude of $\log p(x|z, \theta)$ varies widely
- model likelihood does not contribute to direction of gradient
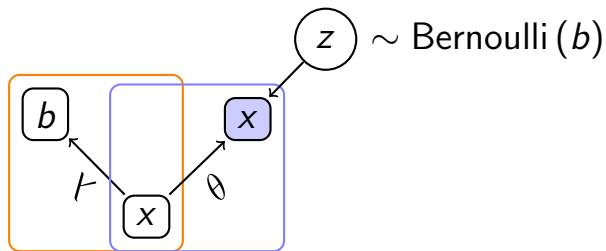
Computation Graph

inference model

Computation Graph
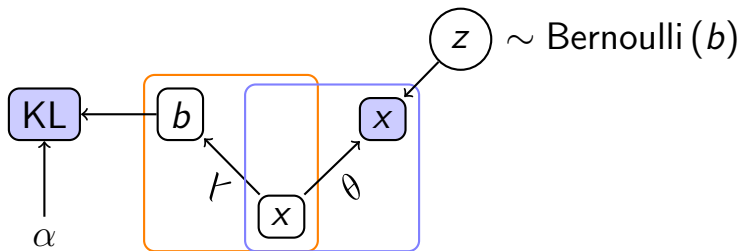

$$z \sim \text{Bernoulli}\,(b)$$

inference model

Computation Graph

Computation Graph

inference model   generation model

Computation Graph
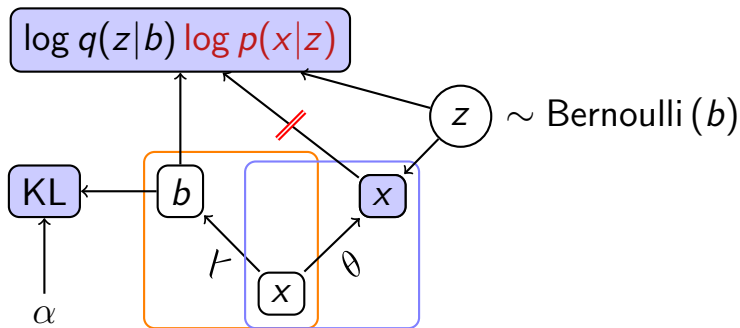
inference model   generation model

Pros and Cons

- Pros
  - Applicable to all distributions
  - Many libraries come with samplers for common distributions

Pros and Cons

- Pros
  - Applicable to all distributions
  - Many libraries come with samplers for common distributions
- Cons
  - High Variance!

When variance is high we can

When variance is high we can

- sample more

When variance is high we can

- sample more
- use variance reduction techniques (e.g. baselines and control variates)

Control variates

Intuition

To estimate $\mathbb{E}[f(z)]$ via Monte Carlo we compute the empirical average of $\hat{f}(z)$ where $\hat{f}(z)$ is chosen so that $\mathbb{E}[\hat{f}(z)] = \mathbb{E}[f(z)]$ and $\mathrm{Var}(f) > \mathrm{Var}(\hat{f})$.

Equivalent expectations

Let $\bar{f} = \mathbb{E}[f(z)]$ be an expectation of interest

Equivalent expectations

Let $\bar{f} = \mathbb{E}[f(z)]$ be an expectation of interest

- say we know $\bar{c} = \mathbb{E}[c(z)]$

Equivalent expectations

Let $\bar{f} = \mathbb{E}[f(z)]$ be an expectation of interest

- say we know $\bar{c} = \mathbb{E}[c(z)]$
- then for $\hat{f}(z) \triangleq f(z) - b(c(z) - \mathbb{E}[c(z)])$

Equivalent expectations

Let $\bar{f} = \mathbb{E}[f(z)]$ be an expectation of interest

- say we know $\bar{c} = \mathbb{E}[c(z)]$
- then for $\hat{f}(z) \triangleq f(z) - b(c(z) - \mathbb{E}[c(z)])$
  it holds that $\mathbb{E}[\hat{f}(z)] = \mathbb{E}[f(z)]$

Equivalent expectations

Let $\bar{f} = \mathbb{E}[f(z)]$ be an expectation of interest

- say we know $\bar{c} = \mathbb{E}[c(z)]$
- then for $\hat{f}(z) \triangleq f(z) - b(c(z) - \mathbb{E}[c(z)])$
  it holds that $\mathbb{E}[\hat{f}(z)] = \mathbb{E}[f(z)]$
- and $\text{Var}(\hat{f}) = \text{Var}(f) - 2b\,\text{Cov}(f, c) + b^2\,\text{Var}(c)$

Choosing the control variate

1. $\hat{f}(z) \triangleq f(z) - b(c(z) - \mathbb{E}[c(z)])$
2. $\text{Var}(\hat{f}) = \text{Var}(f) - 2b\,\text{Cov}(f, c) + b^2\,\text{Var}(c)$

How do we choose b and $c(z)$?

Choosing the control variate

1. $\hat{f}(z) \triangleq f(z) - b(c(z) - \mathbb{E}[c(z)])$
2. $\text{Var}(\hat{f}) = \text{Var}(f) - 2b \, \text{Cov}(f, c) + b^2 \, \text{Var}(c)$

How do we choose b and $c(z)$?

- If $f(z)$ and $c(z)$ are strongly correlated, then we improve on the original estimation problem

Choosing the control variate

1. $\hat{f}(z) \triangleq f(z) - b(c(z) - \mathbb{E}[c(z)])$
2. $\text{Var}(\hat{f}) = \text{Var}(f) - 2b\,\text{Cov}(f, c) + b^2\,\text{Var}(c)$

How do we choose b and $c(z)$?

- If $f(z)$ and $c(z)$ are strongly correlated, then we improve on the original estimation problem
- solving $\frac{\partial}{\partial b}\,\text{Var}(\hat{f}) = 0$

Choosing the control variate

1. $\hat{f}(z) \triangleq f(z) - b(c(z) - \mathbb{E}[c(z)])$
2. $\text{Var}(\hat{f}) = \text{Var}(f) - 2b \, \text{Cov}(f, c) + b^2 \, \text{Var}(c)$

How do we choose b and $c(z)$?

- If $f(z)$ and $c(z)$ are strongly correlated, then we improve on the original estimation problem
- solving $\frac{\partial}{\partial b} \text{Var}(\hat{f}) = 0$ yields $b^\star = {}^{\text{Cov}(f, c)}/_{\text{Var}(c)}$

Choosing the control variate

1. $\hat{f}(z) \triangleq f(z) - b(c(z) - \mathbb{E}[c(z)])$
2. $\mathsf{Var}(\hat{f}) = \mathsf{Var}(f) - 2b\,\mathsf{Cov}(f, c) + b^2\,\mathsf{Var}(c)$

How do we choose b and $c(z)$?

- If $f(z)$ and $c(z)$ are strongly correlated, then we improve on the original estimation problem
- solving $\frac{\partial}{\partial b}\mathsf{Var}(\hat{f}) = 0$ yields $b^\star = \mathsf{Cov}(f, c)/\mathsf{Var}(c)$

Of course, $\mathbb{E}[c(z)]$ must be known!

MC

We then use the estimate

$$\bar{f} \overset{\text{MC}}{\approx} \frac{1}{S} \left(\sum_{s=1}^{S} f(z^{(s)}) - bc(z^{(s)}) \right) + b\bar{c}$$

MC

We then use the estimate

$$\bar{f} \overset{\mathrm{MC}}{\approx} \frac{1}{S} \left(\sum_{s=1}^{S} f(z^{(s)}) - bc(z^{(s)}) \right) + b\bar{c}$$

And recall that for us

$$f(z) = \log p(x|z, \theta) \frac{\partial}{\partial \lambda} \log q(z|x, \lambda)$$

and $z^{(s)} \sim q(z|x, \lambda)$

Expected score

The Expectation of the score function is 0.

$$\mathbb{E}_{q(z|x,\lambda)} \left[\frac{\partial}{\partial \lambda} \log q(z|x, \lambda) \right]$$

Expected score

The Expectation of the score function is 0.

$$\mathbb{E}_{q(z|x,\lambda)} \left[\frac{\partial}{\partial \lambda} \log q(z|x,\lambda) \right]$$

$$= \int q(z|x,\lambda) \frac{\partial}{\partial \lambda} \log q(z|x,\lambda) \mathrm{d}z$$

Expected score

The Expectation of the score function is 0.

$$\mathbb{E}_{q(z|x,\lambda)} \left[\frac{\partial}{\partial \lambda} \log q(z|x,\lambda) \right]$$

$$= \int q(z|x,\lambda) \frac{\partial}{\partial \lambda} \log q(z|x,\lambda) \mathrm{d}z$$

$$= \int \frac{\partial}{\partial \lambda} q(z|x,\lambda) \mathrm{d}z$$

Expected score

The Expectation of the score function is 0.

$$\mathbb{E}_{q(z|x,\lambda)}\left[\frac{\partial}{\partial \lambda}\log q(z|x,\lambda)\right]$$

$$= \int q(z|x,\lambda)\frac{\partial}{\partial \lambda}\log q(z|x,\lambda)\mathrm{d}z$$

$$= \int \frac{\partial}{\partial \lambda}q(z|x,\lambda)\mathrm{d}z$$

$$= \frac{\partial}{\partial \lambda}\int q(z|x,\lambda)\mathrm{d}z$$

Expected score

The Expectation of the score function is 0.

$$\mathbb{E}_{q(z|x,\lambda)}\left[\frac{\partial}{\partial\lambda}\log q(z|x,\lambda)\right]$$

$$= \int q(z|x,\lambda)\frac{\partial}{\partial\lambda}\log q(z|x,\lambda)\mathrm{d}z$$

$$= \int \frac{\partial}{\partial\lambda}q(z|x,\lambda)\mathrm{d}z$$

$$= \frac{\partial}{\partial\lambda}\int q(z|x,\lambda)\mathrm{d}z$$

$$= \frac{\partial}{\partial\lambda}1 = 0$$

Baselines

With

$$f(z) = \log p(x|z, \theta) \frac{\partial}{\partial \lambda} \log q(z|x, \lambda)$$

and

$$c(z) = \frac{\partial}{\partial \lambda} \log q(z|x, \lambda)$$

we have

$$\hat{f}(z) =$$

Baselines

With
$$f(z) = \log p(x|z, \theta) \frac{\partial}{\partial \lambda} \log q(z|x, \lambda)$$
and
$$c(z) = \frac{\partial}{\partial \lambda} \log q(z|x, \lambda)$$
we have
$$\hat{f}(z) = (\log p(x|z, \theta) - b) \frac{\partial}{\partial \lambda} \log q(z|x, \lambda)$$

Baselines

With

$$f(z) = \log p(x|z, \theta) \frac{\partial}{\partial \lambda} \log q(z|x, \lambda)$$

and

$$c(z) = \frac{\partial}{\partial \lambda} \log q(z|x, \lambda)$$

we have

$$\hat{f}(z) = (\log p(x|z, \theta) - b) \frac{\partial}{\partial \lambda} \log q(z|x, \lambda)$$

b is known as *baseline* (Williams, 1992).

Examples of baselines

- Moving average of $\log p(x|z, \theta)$ based on previous batches

Examples of baselines

- Moving average of $\log p(x|z, \theta)$ based on previous batches
- A trainable constant b

Examples of baselines

- Moving average of $\log p(x|z, \theta)$ based on previous batches
- A trainable constant b
- A neural network prediction based on x e.g. $b(x; \omega)$

Examples of baselines

- Moving average of $\log p(x|z, \theta)$ based on previous batches
- A trainable constant b
- A neural network prediction based on x e.g. $b(x; \omega)$
- The likelihood assessed at a deterministic point, e.g. $b(x) = \log p(x|z^\star, \theta)$ where $z^\star = \arg\max_z q(z|x, \lambda)$

Trainable baselines

Baselines are predicted by a regression model (e.g. a neural net).

The model is trained using an L_2-loss.

$$\min_{\omega} \left(b(x; \omega) - \log p(x|z, \theta) \right)^2$$

Summary

- Wake-Sleep: train inference and generation networks with separate objectives

Summary

- Wake-Sleep: train inference and generation networks with separate objectives
- NVIL: a single objective (ELBO) for both models

Summary

- Wake-Sleep: train inference and generation networks with separate objectives
- NVIL: a single objective (ELBO) for both models
- Use score function estimator

Summary

- Wake-Sleep: train inference and generation networks with separate objectives
- NVIL: a single objective (ELBO) for both models
- Use score function estimator
- Always use baselines for variance reduction!

Implementation

Check one of our notebooks, e.g.

- inducing rationales for sentiment classification
  https://github.com/probabll/dgm4nlp/tree/
  master/notebooks/sst

Literature I

David M. Blei, Michael I. Jordan, and John W. Paisley. Variational bayesian inference with stochastic search. In *ICML*, 2012. URL http://icml.cc/2012/papers/687.pdf.

Karol Gregor, Ivo Danihelka, Andriy Mnih, Charles Blundell, and Daan Wierstra. Deep autoregressive networks. In Eric P. Xing and Tony Jebara, editors, *ICML*, pages 1242–1250, 2014. URL http://proceedings.mlr.press/v32/gregor14.html.

Literature II

G. E. Hinton, P. Dayan, B. J. Frey, and R. M. Neal. The wake-sleep algorithm for unsupervised neural networks. *Science*, 268:1158–1161, 1995. URL http://www.gatsby.ucl.ac.uk/~dayan/papers/hdfn95.pdf.

Andriy Mnih and Karol Gregor. Neural variational inference and learning in belief networks. In *Proceedings of the 31st International Conference on International Conference on Machine Learning - Volume 32*, ICML'14, pages II–1791–II–1799. JMLR.org, 2014. URL http://dl.acm.org/citation.cfm?id=3044805.3045092.

Literature III

Rajesh Ranganath, Sean Gerrish, and David Blei. Black
Box Variational Inference. In Samuel Kaski and Jukka
Corander, editors, *AISTATS*, pages 814–822, 2014.
URL http://proceedings.mlr.press/v33/
ranganath14.pdf.

Ronald J. Williams. Simple statistical gradient-following
algorithms for connectionist reinforcement learning.
Machine Learning, 8(3-4):229–256, 1992. URL
https://doi.org/10.1007/BF00992696.